ASYMPTOTICS OF TWISTED ALEXANDER POLYNOMIALS AND HYPERBOLIC VOLUME

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ABSTRACT. For a hyperbolic knot and a natural number n, we consider the Alexander polynomial twisted by the n-th symmetric power of a lift of the holonomy. We establish the asymptotic behavior of these twisted Alexander polynomials evaluated at unit complex numbers, yielding the volume of the knot exterior. More generally, we prove this asymptotic behavior for cusped hyperbolic manifolds of finite volume. The proof relies on results of Müller, and Menal-Ferrer and the last author. Using the uniformity of the convergence, we also deduce a similar asymptotic result for the Mahler measures of those polynomials.

1. INTRODUCTION

Alexander polynomials are Laurent polynomials associated to a free abelian cover of a three-dimensional manifold M. Classically M is a knot or link exterior in the three-dimensional sphere S^3, and the associated free abelian cover is the maximal one. The Alexander polynomial carries only metabelian information on the fundamental group, and the idea behind the *twisted* Alexander polynomial is to associate a Laurent polynomial invariant to a free abelian cover of M together with a linear representation of the fundamental group $\pi_1(M)$.

Twisted Alexander polynomials of knots have been defined by Lin [23] and Wada [44]. Kitano [21] showed that they are Reidemeister torsions, generalizing Milnor's theorem on the (untwisted) Alexander polynomial [30].

In this article we are interested in orientable, *hyperbolic* three-dimensional manifolds of finite volume since such a manifold has a natural representation of its fundamental group into $\mathrm{PSL}_2(\mathbb{C})$, the hyperbolic holonomy, that is unique up to conjugation. The holonomy representation lifts to $\mathrm{SL}_2(\mathbb{C})$ and a lift is unique up to multiplication with a representation into the center of $\mathrm{SL}_2(\mathbb{C})$ (see [8]).

The corresponding twisted Alexander polynomial has been considered, among others, by Dunfield, Friedl and Jackson in [10]. Here we compose the lift of the holonomy representation with the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ in $\mathrm{SL}_n(\mathbb{C})$, the $(n-1)$-th symmetric power, and study its asymptotic behavior as n tends to $+\infty$.

Examples of twisted Alexander polynomials associated to a lift of the holonomy representation into $\mathrm{SL}_2(\mathbb{C})$ can be found in [10]. More examples, also for the n-th symmetric power are presented in http://dunfield.info/torsion.

For illustration we have included in the Appendix A the twisted polynomials for the figure-eight knot associated for the n-th symmetric power for $n \leq 9$ (following our normalisation which is different from [14]).

Before considering non-compact, orientable, hyperbolic three-manifolds of finite volume in general, we discuss first the case of a hyperbolic knot complement $S^3 \setminus K$. Let $\rho_n \colon \pi_1(S^3 \setminus K) \to \mathrm{SL}_n(\mathbb{C})$ be the composition of a lift of the holonomy with the $(n-1)$-th symmetric power $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$. Let $\Delta_K^{\rho_n}$ denote the Alexander polynomial of K twisted by ρ_n, which equals Wada's definition for n even, but it is Wada's polynomial divided by $(t-1)$ when n is odd, so that its evaluation at $t = 1$ does not vanish. The set of unit complex numbers is denoted by $\mathbb{S}^1 = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$. The following is a particular case of the main result of this paper.

Theorem 1.1. *For any $\zeta \in \mathbb{S}^1$,*
$$\lim_{n \to \infty} \frac{\log |\Delta_K^{\rho_n}(\zeta)|}{n^2} = \frac{1}{4\pi} \mathrm{vol}(S^3 \setminus K)$$
uniformly on ζ.

For a knot exterior there are two lifts ρ of the holonomy, Theorem 1.1 holds true for both choices of lift. Based on Yamaguchi's work ([48]), Goda showed in [14] that $|\Delta_K^{\rho_n}(1)|$ is equal to the Reidemeister torsions with coefficients in ρ_n and so Theorem 1.1 generalizes results of Müller [35] and of Menal-Ferrer and the last author [29].

Theorem 1.1 is a particular case of Theorem 1.6 below. To extend the definition of twisted Alexander polynomial to general cusped manifolds (Definition 2.6), we need to make some assumptions, that are always satisfied for hyperbolic knot exteriors. Let M be an orientable, non-compact, connected, finite volume hyperbolic three-manifold. It admits a compactification $\overline{M}$ by adding $l \geq 1$ peripheral tori, one for each end:
$$\partial \overline{M} = T_1^2 \sqcup \cdots \sqcup T_l^2.$$
Let
$$\alpha \colon \pi_1(M) \twoheadrightarrow \mathbb{Z}^r$$
be an epimorphism.

In this paper we require the two following hypotheses, that are always satisfied in the case of a knot complement.

Assumption 1.2. For each peripheral torus T_i^2, $\alpha(\pi_1(T_i^2)) \cong \mathbb{Z}$.

This assumption is a necessary condition for the acyclicity (over the field of rational fractions) of the $\mathbb{Z}^r$-cover associated with α, so that the twisted Alexander polynomials do not vanish. Assumption 1.2 holds true for the abelianization map of a knot in a homology sphere, or more generally for the abelianization map of a link in a homology sphere having the property that the linking number of pairwise different components vanish. Furthermore, for any cusped, oriented, hyperbolic 3-manifold M, there exists an epimorphism $\alpha \colon \pi_1(M) \twoheadrightarrow \mathbb{Z}$ satisfying Assumption 1.2 (compose the abelianization map with a generic surjection of $H_1(M, \mathbb{Z})$ onto $\mathbb{Z}$).

Let $l_i \in \pi_1(T_i^2)$ be a generator of $\ker(\alpha|_{\pi_1(T_i^2)})$, we say that l_i is a *longitude for* α, and we use the terminology α-*longitude*. We choose $\rho \colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ a lift of the hyperbolic holonomy satisfying the following:

Assumption 1.3. The lift of the holonomy $\rho \colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ satisfies
$$\mathrm{tr}(\rho(l_i)) = -2.$$
for each α-longitude l_i, $i = 1, \dots, l$.

For a knot exterior in a homology sphere and the abelianization map, Assumption 1.3 is satisfied for *every* lift of the holonomy. More generally, it is also satisfied for *every* lift of the holonomy for a link exterior in a homology sphere with the property that the linking number of pairwise different components vanish. In general, for any M and α satisfying Assumption 1.2, there exists at least one lift of the holonomy satisfying Assumption 1.3 [29, Proposition 3.2]. In terms of spin structures, this is the condition for a spin structure to extend along certain Dehn fillings which we will consider in Section 3.

For $n \geq 2$, recall that the unique n-dimensional irreducible holomorphic representation of $\mathrm{SL}_2(\mathbb{C})$ is given by the natural action on the $(n-1)$-th symmetric power of $\mathbb{C}^2$. We denote it by $\mathrm{Sym}^{n-1} \colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$. For a lift of the holonomy representation $\rho \colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$, we denote the composition with the $(n-1)$-th symmetric power by

$$\rho_n \colon \pi_1(M) \xrightarrow{\rho} \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\mathrm{Sym}^{n-1}} \mathrm{SL}_n(\mathbb{C}).$$

Remark 1.4. This convention follows the notation of [29], but it differs from [35], it is shifted by 1.

To construct the twisted Alexander polynomial, we consider the polynomial representation associated to α:
$$\begin{aligned} \bar{\alpha} \colon \pi_1(M) &\to \mathbb{C}[t_1^{\pm 1}, \dots, t_r^{\pm 1}] \\ \gamma &\mapsto t_1^{\alpha_1(\gamma)} \cdots t_r^{\alpha_r(\gamma)} \end{aligned}$$

where $\alpha = (\alpha_1, \dots, \alpha_r)$ are the components of α. We define the twisted Alexander polynomial $\Delta_M^{\alpha,n}$ in Definition 2.6 as the inverse of the Reidemeister torsion of the pair $(M, \bar{\alpha} \otimes \rho_n)$, after removing some factors $(t_1^{\beta_1} \cdots t_r^{\beta_r} - 1)$ when n is odd (one factor for each peripheral torus). It is a Laurent polynomial with variables $t_1^{\pm 1}, \dots, t_r^{\pm 1}$ defined up to sign and up to multiplicative factors $t_i^{\pm 1}$.

Remark 1.5. We stress out the fact that, for $\zeta_1, \dots, \zeta_r \in \mathbb{S}^1$, only the modulus $|\Delta_M^{\alpha,n}(\zeta_1, \dots, \zeta_r)|$ is well defined.

The main result of this paper is:

Theorem 1.6. *Let M be an oriented, hyperbolic three-manifold of finite volume. We suppose that $\alpha \colon \pi_1(M) \twoheadrightarrow \mathbb{Z}^r$ is an epimorphism and that $\rho \colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ is a lift of the holonomy representation.*
If M is closed*, or if the Assumptions 1.2 and 1.3 are satisfied, then for any $\zeta_1, \dots, \zeta_r \in \mathbb{S}^1$,*
$$\lim_{n \to \infty} \frac{\log |\Delta_M^{\alpha,n}(\zeta_1, \dots, \zeta_r)|}{n^2} = \frac{\mathrm{vol}(M)}{4\pi}$$

uniformly on the $\zeta_1, \ldots, \zeta_r$.

Remark 1.7. Assumptions 1.2 and 1.3 are vacuous for closed manifolds.

The *logarithmic Mahler measure* of a Laurent polynomial $P(t_1, \ldots, t_r) \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$ is defined as

$$\mathrm{m}(P) = \frac{1}{(2\pi)^r} \int_0^{2\pi} \cdots \int_0^{2\pi} \log|P(e^{i\theta_1}, \ldots, e^{i\theta_r})| \, d\theta_1 \cdots d\theta_r.$$

With Theorem 1.6, as the convergence is uniform on $\zeta_1, \ldots, \zeta_r \in \mathbb{S}^1$, we also prove:

Theorem 1.8. *Under Assumptions 1.2 and 1.3,*

$$\lim_{n \to \infty} \frac{\mathrm{m}(\Delta_M^{\alpha,n})}{n^2} = \frac{\mathrm{vol}(M)}{4\pi}.$$

Assume now that M is fibered over the circle and let $\alpha \colon \pi_1(M) \to \mathbb{Z}$ be induced by the fibration $M \to \mathbb{S}^1$. We chose a representative $\Delta_M^{\alpha,n}(t)$ so that $\Delta_M^{\alpha,n}(t) \in \mathbb{C}[t]$ (it is a polynomial in one variable) and $\Delta_M^{\alpha,n}(0) \neq 0$.

Corollary 1.9. *We have*

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{\lambda \in \mathrm{Spec}(\Delta_M^{\alpha,n})} \big| \log|\lambda| \big| = \frac{1}{2\pi} \mathrm{vol}(M),$$

where $\mathrm{Spec}(\Delta_M^{\alpha,n}) = \{\lambda \in \mathbb{C} \mid \Delta_M^{\alpha,n}(\lambda) = 0\}$.

Moreover, the maximum of the modulus of the roots grows at least exponentially with n.

Proof. If $p(t) = a \prod_{i=1}^d (t - \alpha_i) \in \mathbb{C}[t]$ is a polynomial without zeros on the unit circle and $p(0) \neq 0$, then Jensen's formula implies that

$$\log|a| + \sum_{i=1}^d \log\big(\max(|\alpha_i|, 1)\big) = \frac{1}{2\pi} \int_0^{2\pi} \log|p(e^{i\theta})| \, d\theta = \mathrm{m}(p).$$

We aim to apply this formula to the normalized polynomial $\Delta_M^{\alpha,n}(t)$. From the fiberness of M it follows that $\Delta_M^{\alpha,n}(0) = \pm 1 \neq 0$. Moreover, $\Delta_M^{\alpha,n}(t)$ is symmetric which implies for all $\lambda \in \mathbb{C}^*$ that $\Delta_M^{\alpha,n}(\lambda) = 0$ if and only if $\Delta_M^{\alpha,n}(1/\lambda) = 0$. Finally, by Theorem 1.11 the Alexander polynomial has no root on the unit circle. Hence we can apply the above formula for $\Delta_M^{\alpha,n}(t)$, and we obtain:

$$\sum_{\lambda \in \mathrm{Spec}(\Delta_M^{\alpha,n})} \big| \log|\lambda| \big| = 2\,\mathrm{m}(\Delta_M^{\alpha,n}),$$

and the first statement follows directly from Theorem 1.8.

Finally notice that $\deg \Delta_M^{\alpha,n}(t)$ is linear on n, so the second statement follows directly from the first. $\qquad \square$

Remark 1.10. When M is not fibered, it may happen that $\Delta_M^{\alpha,n}$ is not monic and we must take into account $|\Delta_M^{\alpha,n}(0)|$ in Jensen's formula.

Given $\zeta_1, \ldots, \zeta_r \in \mathbb{S}^1$, we compose $\alpha \colon \pi_1(M) \twoheadrightarrow \mathbb{Z}^r$ with the homomorphism

$$\mathbb{Z}^r \to \mathbb{S}^1$$
$$(n_1, \ldots, n_r) \mapsto \zeta_1^{n_1} \cdots \zeta_r^{n_r}$$

and we denote the composition by $\chi \colon \pi_1(M) \to \mathbb{S}^1$. Namely, we evaluate $\bar{\alpha}$ at $t_j = \zeta_j$: if $\alpha = (\alpha_1, \ldots, \alpha_r)$ are the components of α, then

(1)
$$\chi \colon \pi_1(M) \to \mathbb{S}^1$$
$$\gamma \mapsto \zeta_1^{\alpha_1(\gamma)} \cdots \zeta_r^{\alpha_r(\gamma)}.$$

In fact, Theorem 1.6 is a theorem on Reidemeister torsions, as $|\Delta_M^{\alpha,n}(\zeta_1, \ldots, \zeta_r)|$ is the inverse of the modulus of the Reidemeister torsion of M twisted by the representation $\chi \otimes \rho_n$ (in some cases perhaps up to some factor independent of n or after the choice of basis in homology), see Section 2.

The definition of twisted Alexander polynomial as a Reidemeister torsion requires a vanishing theorem in cohomology, Theorem 2.3. Its proof mimics the classical vanishing theorem on L^2-cohomology of Matsushima–Murakami, as we explain in Appendix C . As a direct consequence of this vanishing theorem, we obtain that the twisted Alexander polynomials have no roots on the unit circle:

Theorem 1.11. *Under Assumptions 1.2 and 1.3, for any $\zeta_1, \ldots, \zeta_r \in \mathbb{S}^1$,*

$$\Delta_M^{\alpha,n}(\zeta_1, \ldots, \zeta_r) \neq 0.$$

We apply this theorem to study the dynamics of a pseudo-Anosov diffeomorphism on the variety of representations. Let Σ be a compact orientable surface, possibly with boundary and with negative Euler characteristic. For a pseudo-Anosov diffeomorphism $\phi \colon \Sigma \to \Sigma$, consider its action on the relative variety of (conjugacy classes of) representations $\phi^* \colon \mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C})) \to \mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C}))$. The mapping torus $M(\phi)$ is a hyperbolic manifold of finite volume and its holonomy restricts to a representation of $\pi_1(\Sigma)$ in $\mathrm{SL}_2(\mathbb{C})$ whose conjugacy class is fixed by ϕ^*. In particular the conjugacy class of the composition $[\rho_n] = [\mathrm{Sym}^{n-1} \circ \mathrm{hol}|_{\pi_1(\Sigma)}]$ in $\mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C}))$ is fixed by ϕ^*. In Appendix D we prove:

Theorem 1.12. *The tangent map of ϕ^* at $[\rho_n]$ on $\mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C}))$ has no eigenvalues of norm one.*

For $n = 2$ and $\partial\Sigma = \emptyset$, this was proved by M. Kapovich in [19]. Kapovich answered thereby a question of McMullen [26], namely, the point $[\rho_2]$ is a hyperbolic fixed point of ϕ^*.

The relation with the rest of the paper comes from the formula (Proposition D.3)

$$\det\left((d\phi^*)_{[\rho_n]} - t\,\mathrm{Id}\right) = \prod_{k=1}^{n-1} \Delta_{M(\phi)}^{\alpha,2k+1}(t),$$

where $\alpha \colon \pi_1(M(\phi)) \twoheadrightarrow \mathbb{Z}$ is induced by the natural fibration of the mapping torus over $\mathbb{S}^1$ with fiber Σ. Then the result follows from Theorem 1.11.

Summary of the proof. Most of the paper is devoted to prove Theorem 1.6 for $\zeta_1, \ldots, \zeta_r \in e^{i\pi\mathbb{Q}}$. For that purpose, we consider sequences of closed manifolds $M_{p/q}$ obtained by Dehn filling, that converge geometrically to M by Thurston's hyperbolic Dehn filling theorem. The assumption $\zeta_1, \ldots, \zeta_r \in e^{i\pi\mathbb{Q}}$ allows us to chose the Dehn fillings $M_{p/q}$ so that the twist $\chi \colon \pi_1(M) \to \mathbb{S}^1$ in (1) factors through $\pi_1(M_{p/q})$. Then the strategy is to apply Müller's theorem [35] to the asymptotic behavior of the torsion of closed Dehn fillings $M_{p/q}$. Even if in Müller's paper there is no twist, as χ is unitary we can modify the proof of Müller's theorem by considering Ruelle functions twisted by χ. The key ingredient is a proof of a version of Fried's theorem (Theorem 4.2) relating the value of the Ruelle zeta function with the Reidemeister torsion of any closed manifold. Then we prove our main theorem for $\zeta_1, \ldots, \zeta_r \in e^{i\pi\mathbb{Q}}$ by analyzing the behavior of those twisted Ruelle zeta functions and the arguments of Müller's proof under limits of Dehn fillings, as in [29]. To conclude the proof for arbitrary $\zeta_1, \ldots, \zeta_r \in \mathbb{S}^1$, we establish an intermediary result (Corollary 6.6) where Dehn fillings do not appear anymore. As Corollary 6.6 holds for $\zeta_1, \ldots, \zeta_r \in e^{i\pi\mathbb{Q}}$, we use continuity and a density argument to extend it to any unitary $\zeta_1, \ldots, \zeta_r \in \mathbb{S}^1$.

Organization of the paper. In Section 2 we define (a normalized version of) the twisted Alexander polynomial $\Delta_M^{\alpha,n}$, in particular we establish the basic results in cohomology required for that, based on Appendix C and we prove Theorem 1.11. Section 3 is devoted to construct Dehn fillings that approximate M, so that the character in (1) extends to them, as we assume that $\zeta_1, \ldots, \zeta_r \in e^{i\pi\mathbb{Q}}$. In Section 4 we prove a twisted version (Theorem 4.2) of Fried's Theorem and some properties of the Ruelle zeta functions that will be used later. Essentially this section extends the results of [35] to the case where a unitary twist is added. Section 5 discusses the behavior of Reidemeister torsion and Ruelle zeta functions under sequences of approximating Dehn fillings, and the proof of the main theorem is completed in Section 6.

The paper contains four appendices. In Appendix A we present for small values of n the polynomials $\Delta_{4_1}^{\alpha,n}(t)$ for the canonical surjection α from the knot group to the integers. In Appendix B we recall the main properties of combinatorial torsion. The results in L^2-cohomology needed in Section 2 are established in Appendix C. Finally, in Appendix D we establish Theorem 1.12.

2. Reidemeister torsion and twisted Alexander polynomials

In this section we define the twisted Alexander polynomial for a cusped hyperbolic manifold M, equipped with an epimorphism $\alpha \colon \pi_1(M) \twoheadrightarrow \mathbb{Z}^r$ satisfying Assumption 1.2, and a lift ρ of the holonomy satisfying Assumption 1.3. Before defining the polynomial from the Reidemeister torsion of the pair $(M, \bar{\alpha} \otimes \rho_n)$, we need to consider homology and cohomology of $(M, \chi \otimes \rho_n)$, as the representation $\chi \otimes \rho_n$ is a specialization of $\bar{\alpha} \otimes \rho_n$.

In Subsection 2.1 we study (co)-homology of $(M, \chi \otimes \rho_n)$ and $(M, \bar{\alpha} \otimes \rho_n)$. In Subsection 2.1.3 we consider the Reidemeister torsion of $(M, \chi \otimes \rho_n)$. Then we define the twisted Alexander polynomial in Subsection 2.2 from the Reidemeister torsion of $(M, \bar{\alpha} \otimes \rho_n)$. We express evaluations of the twisted Alexander polynomial at unit complex numbers as Reidemeister torsions of the representations $\chi \otimes \rho_n$ and we prove Theorem 1.11.

Preliminary constructions and results on homology, cohomology and Reidemeister torsion are gathered in Appendix B, where we also recall some properties of Sym^{n-1}. This section also relies on results on L^2-cohomology from Appendix C.

2.1. **Cohomology of** $(M, \chi \otimes \rho_n)$ **and** $(M, \bar{\alpha} \otimes \rho_n)$**.** When χ is trivial, the results of this subsection on cohomology twisted by $\chi \otimes \rho_n = \rho_n$ can be found in [29, Section 4].

In Corollary C.7 (in Appendix C) we prove that the inclusion $\partial \overline{M} \hookrightarrow \overline{M}$ induces a monomorphism

$$(2) \qquad 0 \to H^1(\overline{M}, \chi \otimes \rho_n) \to H^1(\partial \overline{M}, \chi \otimes \rho_n).$$

Thus to understand the cohomology of M we need to understand the cohomology of the peripheral tori T_i^2, $i = 1, \ldots, l$, where

$$\partial \overline{M} = T_1^2 \sqcup \cdots \sqcup T_l^2$$

is the decomposition in connected components. In particular l is the number of cusps of M.

2.1.1. *Peripheral cohomology.*

Lemma 2.1. *If Assumptions 1.2 and 1.3 hold, then for any peripheral torus T_i^2*

(a) $\dim_{\mathbb{C}} H^0(T_i^2, \chi \otimes \rho_n) = \begin{cases} 0 & \text{if } n \text{ even or } \chi(\pi_1(T_i^2)) \neq \{1\}, \\ 1 & \text{if } n \text{ odd and } \chi(\pi_1(T_i^2)) = \{1\}. \end{cases}$

(b) $\dim_{\mathbb{C}} H^0(T_i^2, \chi \otimes \rho_n) = \dim_{\mathbb{C}} H^2(T_i^2, \chi \otimes \rho_n) = \frac{1}{2} \dim_{\mathbb{C}} H^1(T_i^2, \chi \otimes \rho_n).$

Proof. (a) To compute its dimension, we view $H^0(T_i^2, \chi \otimes \rho_n)$ as the space of invariants $(\mathbb{C}^n)^{\chi \otimes \rho_n(\pi_1(T_i^2))}$. For any non-trivial element γ in $\pi_1(T_i^2)$ its image $\rho(\gamma)$ by the holonomy is parabolic, with trace $2\epsilon_\gamma$, for some $\epsilon_\gamma = \pm 1$. Hence $\chi(\gamma) \rho_n(\gamma)$ has only one eigenspace, with dimension one and eigenvalue $\chi(\gamma)\epsilon_\gamma^{n-1}$ (see Remark B.7 in Appendix B).

By Assumptions 1.2 and 1.3, when n is even or when $\chi(\pi_1(T_i^2)) \neq \{1\}$, there is always an element $\gamma \in \pi_1(T_i^2)$ that satisfies $\chi(\gamma)(\epsilon_\gamma)^{n-1} \neq 1$, thus $\dim_{\mathbb{C}}(H^0(T_i^2, \chi \otimes \rho_n)) = 0$. In case n is odd and $\chi(\pi_1(T_i^2)) = \{1\}$, then $\chi(\gamma)(\epsilon_\gamma)^{n-1} = 1$ for every $\gamma \in \pi_1(T_i^2)$.

(b) Poincaré duality induces a nondegenerate pairing, see Remark B.6:

$$H^0(T_i^2, \chi \otimes \rho_n) \times H^2(T_i^2, \overline{\chi} \otimes \rho_n) \to \mathbb{C}.$$

Notice that we use χ and its inverse $\overline{\chi}$ (its complex conjugate). Hence by (a):

$$\dim_{\mathbb{C}}(H^0(T_i^2, \chi \otimes \rho_n)) = \dim_{\mathbb{C}}(H^0(T_i^2, \bar{\chi} \otimes \rho_n)) = \dim_{\mathbb{C}}(H^2(T_i^2, \chi \otimes \rho_n)).$$

Then the assertion follows from these computations and vanishing of the Euler characteristic of T^2. □

To compute further cohomology groups, we discuss L^2-forms, in particular de Rham cohomology. Let $E_{\chi \otimes \rho_n}$ denote the flat bundle on M (or on any submanifold) twisted by the representation $\chi \otimes \rho_n$, see Appendix C. For each peripheral torus T_i^2, let $T_i^2 \times [0, \infty) \subset M$ denote the cusp, which is an end of M, and consider the space of forms on the cusp valued on the bundle $E_{\chi \otimes \rho_n}$, $\Omega^*(T_i^2 \times [0, \infty), E_{\chi \otimes \rho_n})$. It is equipped with a metric as in Appendix C, in particular we may talk about L^2-forms, as forms with a finite norm. A cohomology class is called L^2 if represented by an L^2-form, and the subspace of L^2-cohomology classes in $H^i(T_j^2 \times [0, +\infty), E_{\chi \otimes \rho_n})$ is denoted by $H^i(T_j^2 \times [0, +\infty), E_{\chi \otimes \rho_n})_{L^2}$.

Lemma 2.2. *Assume that n is odd and that the restriction of the character $\chi(\pi_1(T_j^2))$ is trivial. Then:*

(a) Every class in $H^i(T_j^2 \times [0, +\infty), E_{\rho_n})_{L^2}$ is represented by a form $v \otimes \omega$, where ω is an i-form on T^2 and $v \in (\mathbb{C}^n)^{\rho(\pi_1(T^2))}$.

(b) $\dim_{\mathbb{C}} H^0(T^2 \times [0, +\infty), E_{\rho_n})_{L^2} = \dim_{\mathbb{C}} H^1(T^2 \times [0, +\infty), E_{\rho_n})_{L^2} = 1$ and $H^2(T^2 \times [0, +\infty), E_{\rho_n})_{L^2} = 0$.

Proof. In [27, Lemma 3.3] the same statement is proved for the composition with the adjoint representation on the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, $\mathrm{Ad} \circ \mathrm{Sym}^{n-1}$. Recall that from Lemma 2.1 the space of invariants $(\mathbb{C}^n)^{\rho_n(\pi_1(T_i^2))}$ is one-dimensional, for n odd. Then the lemma follows from Clebsch-Gordan formula, see Equation (33) in Appendix B. □

2.1.2. *Cohomology of M.* In this paragraph we prove the properties of the cohomology of M required for our definition of twisted Alexander polynomial.

Theorem 2.3. *Let M, ρ, and α satisfy Assumptions 1.2 and 1.3.*

(a) If n is even or if χ is non trivial on every peripheral subgroup, then $H^*(M, \chi \otimes \rho_n) = 0$.

(b) If n is odd, then $\dim_{\mathbb{C}} H^1(M; \chi \otimes \rho_n) = \dim_{\mathbb{C}} H^2(M; \chi \otimes \rho_n)$ is the number of peripheral subgroups to which the restriction of χ is trivial.

Proof. For both (a) and (b), first notice that M has the homotopy type of a 2–complex, hence $H^i(M, \chi \otimes \rho_n) = 0$ for any $i \geq 3$. In addition, the space of invariants $H^0(M, \chi \otimes \rho_n) \cong (\mathbb{C}^n)^{\chi \otimes \rho_n(\pi_1(M))}$ also vanishes since ρ_n is irreducible.

To prove (a), the vanishing of $H^1(M, \chi \otimes \rho_n)$ is a consequence of Lemma 2.1 and the monomorphism in (2). We conclude that $H^2(M, \chi \otimes \rho_n) = 0$ because the Euler characteristic $\chi(M)$ is zero.

For (b) assume that n is odd. We use that $H^1(M, \chi \otimes \rho_n)$ has no L^2-forms, by Theorem C.1, hence by Lemmas 2.1 and 2.2, the map

$$H^1(\overline{M}, \chi \otimes \rho_n) \to H^1(T_i^2, \rho_n)$$

has rank at most one if $\chi|_{\pi_1(T_i^2)}$ is trivial, and 0 otherwise. Thus, if s is the number of peripheral tori T_i^2 where χ restricts trivially, by (2),

$$\dim_{\mathbb{C}} H^1(M; \chi \otimes \rho_n) \leq s.$$

On the other hand, using duality twice (Poincaré and homology/cohomology) $H^3(\overline{M}, \partial\overline{M}; \chi \otimes \rho_n) = 0$ and, by the long exact sequence of the pair, $H^2(\overline{M}, \chi \otimes \rho_n) \twoheadrightarrow H^2(\partial\overline{M}, \chi \otimes \rho_n)$ is a surjection. Hence by Lemma 2.1:

$$\dim_{\mathbb{C}} H^2(M; \chi \otimes \rho_n) \geq s.$$

Finally, as $\chi(M) = 0$, $\dim_{\mathbb{C}} H^1(M; \chi \otimes \rho_n) = \dim_{\mathbb{C}} H^2(M; \chi \otimes \rho_n) = s$. $\qquad\square$

We need to precise the bases for the cohomology groups. It is easier to describe them for the homology groups. For a torus T_i^2 such that $\chi(\pi_1(T_i^2))$ is trivial, if $h_i \in \mathbb{C}^n$ is invariant by $\pi_1(T_i^2)$, then the class of $h_i \otimes T_i^2$ is a well defined element in $H_2(T_i^2, \chi \otimes \rho_n)$, and so is $h_i \otimes l_i$ in $H_1(T_i^2, \chi \otimes \rho_n)$.

Lemma 2.4. *Assume that χ is trivial precisely on $\pi_1(T_1^2), \ldots, \pi_1(T_s^2)$. Let $h_i \in \mathbb{C}^n$ be non-zero and invariant by $\pi_1(T_i^2)$, for $i = 1, \ldots, s$. Let i_* denote the map induced by inclusion in homology. Then:*

(a) $\{i_*(h_1 \otimes T_1^2), \ldots, i_*(h_s \otimes T_s^2)\}$ is a basis for $H_2(M, \chi \otimes \rho_n)$.
(b) $\{i_*(h_1 \otimes l_1), \ldots, i_*(h_s \otimes l_s)\}$ is a basis for $H_1(M, \chi \otimes \rho_n)$.

Proof. For $i = 1, \ldots, s$, since χ is trivial on T_i^2, (a) follows from the isomorphisms

$$H_2(T_i^2, \rho_n) \cong H^0(T_i^2, \rho_n) \cong (\mathbb{C}^n)^{\rho_n(\pi_1(T^2))},$$

and from the isomorphism

$$0 \to H_2(T_1^2, \rho_n) \oplus \cdots \oplus H_2(T_s^2, \rho_n) \xrightarrow{i_*} H_2(\overline{M}, \chi \otimes \rho_n) \to 0$$

coming from the long exact sequence in homology.

For (b) we claim first that $h_j \otimes l_j$ is non-zero in $H_1(T_j^2, \rho_n)$, for $j = 1, \ldots, s$. We prove the claim by computing cellular homology explicitly. For this purpose, chose a cell decomposition of the torus with one 0-cell, one 2-cell and two 1-cells, that are loops, and assume that one of these loops represents l_j. Furthermore, using the description of $\rho_n(\pi_1(T_j^2))$, a straightforward computation shows that $h_j \otimes l_j$ is not a boundary, see (7) below. Alternatively, as in the proof of Lemma 2.2, an equivalent statement is proved in [27, Lemma 3.4] for $\mathrm{Ad} \circ \mathrm{Sym}^{n-1}$, and our claim follows from Clebsch-Gordan formula (33).

From the proof of Theorem 2.3 we have an injection

$$0 \to H^1(M, \overline{\chi} \otimes \rho_n)) \xrightarrow{i^*} H^1(T_1^2, \rho_n) \oplus \cdots \oplus H^1(T_s^2, \rho_n)$$

and a surjection

$$H_1(T_1^2, \rho_n) \oplus \cdots \oplus H_1(T_s^2, \rho_n) \xrightarrow{i_*} H_1(M, \chi \otimes \rho_n) \to 0.$$

We also have naturality with the pairing between homology and cohomology (see Appendix B):

$$\langle i_*(-), - \rangle = \langle -, i^*(-) \rangle$$

where the pairing on $\partial \overline{M}$ is understood to be the sum of pairings on each component T_i^2. Thus, by Poincaré duality,

$$(3) \qquad \qquad \ker(i_*) = \operatorname{im}(i^*)^{\perp}.$$

By Remark B.7, $h_j \in (\mathbb{C}^n)^{\rho_n(\pi_1(T_i^2))}$ is isotropic for the ρ_n-invariant bilinear form. Hence by Lemma 2.2(a), the pairing between $h_j \otimes l_j$ and any L^2-class in $H^1(T_j^2 \times [0, \infty), \rho_n)$ vanishes. Thus, by dimension considerations:

$$(4) \qquad \langle h_1 \otimes l_1, \ldots, h_s \otimes l_s \rangle = \left(H^1(\partial \overline{M} \times [0, \infty), \overline{\chi} \otimes \rho_n)_{L^2} \right)^{\perp}.$$

Furthermore, by Theorem C.1:

$$\operatorname{im}(i^*) \cap H^1(\partial \overline{M} \times [0, \infty), \overline{\chi} \otimes \rho_n)_{L^2} = 0.$$

As $\dim \operatorname{im}(i^*) = \dim H^1(\partial \overline{M} \times [0, \infty), \overline{\chi} \otimes \rho_n)_{L^2} = \frac{1}{2} \dim H^1(\partial \overline{M}, \overline{\chi} \otimes \rho_n) = s$,

$$(5) \qquad \operatorname{im}(i^*) \oplus H^1(\partial \overline{M} \times [0, \infty), \overline{\chi} \otimes \rho_n)_{L^2} = H^1(\partial \overline{M}, \overline{\chi} \otimes \rho_n).$$

Finally, (5), (3) and (4) yield

$$\ker(i_*) \oplus \langle h_1 \otimes l_1, \ldots, h_s \otimes l_s \rangle = H_1(\partial \overline{M}, \chi \otimes \rho_n),$$

in particular $\langle h_1 \otimes l_1, \ldots, h_s \otimes l_s \rangle \cap \ker(i_*) = 0$. Thus $\{i_*(h_1 \otimes l_1), \ldots, i_*(h_s \otimes l_s)\}$ are linearly independent, hence a basis.   □

When χ is trivial, Lemma 2.4 is [29, Proposition 4.6].

2.1.3. *Reidemeister torsion.* We use the convention of [31] and [43] for Reidemeister torsion, so that it is compatible with the standard convention for analytic torsion but it is the reciprocal to the twisted Alexander polynomial. See Appendix B.

As a consequence of Theorem 2.3, we have that

$$|\operatorname{tor}(M, \chi \otimes \rho_n)| \in \mathbb{R}_{>0}$$

is well defined when n is even or when n is odd and the restriction of χ to every peripheral torus is nontrivial. The absolute value in $|\operatorname{tor}(M, \chi \otimes \rho_n)|$ is needed, because χ introduces an indeterminacy of the argument, more precisely $\operatorname{tor}(M, \chi \otimes \rho_n)$ is only defined up to multiplication by a unit complex number.

In the non-acyclic case we shall consider

$$|\operatorname{tor}(M, \chi \otimes \rho_n; b_1, b_2)|,$$

where b_1 and b_2 are the basis of the homology provided by Lemma 2.4. Notice that $|\operatorname{tor}(M, \chi \otimes \rho_n; b_1, b_2)|$ is independent on the vectors h_i in Lemma 2.4. This follows since $(\mathbb{C}^n)^{\pi_1(T_i)}$ is one-dimensional and $|\operatorname{tor}(M, \chi \otimes \rho_n; b_1, b_2)|$ does not change if we replace the vector h_i in Lemma 2.4 by a multiple, since in the formula for the torsion the additional factors cancel out.

2.2. **Twisted Alexander polynomials.** In this subsection we introduce the twisted Alexander polynomial for a finite volume 3-manifold M (connected and orientable) and an epimorphism $\alpha\colon \pi_1(M) \twoheadrightarrow \mathbb{Z}^r$ as in the introduction. We define it as the inverse of a Reidemeister torsion of $\bar\alpha \otimes \rho_n$, where $\bar\alpha(\gamma) = t_1^{\alpha_1(\gamma)} \cdots t_r^{\alpha_r(\gamma)}$, $\forall \gamma \in \pi_1(M)$.

We use the notation $\mathbb{C}[t^{\pm 1}] = \mathbb{C}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$ for the ring of Laurent polynomials and $\mathbb{C}(t) = \mathbb{C}(t_1, \dots, t_r)$ for its field of fractions.

Lemma 2.5. *If Assumptions 1.2 and 1.3 hold, then $H^*(M, \bar\alpha \otimes \rho_n) = H_*(M, \bar\alpha \otimes \rho_n) = 0$.*

Proof. Choose $\zeta_1, \dots \zeta_r \in \mathbb{S}^1$ generic so that the corresponding homomorphism $\chi\colon \pi_1(M) \to \mathbb{S}^1$ in (1) has non trivial restriction on each peripheral subgroup $\pi_1(T_i^2)$. Then by Theorem 2.3 we have $H^*(M, \chi \otimes \rho_n) = 0$. Using combinatorial cohomology, notice that the matrices used to compute $H^*(M, \chi \otimes \rho_n)$ are the evaluation at $(t_1, \dots, t_r) = (\zeta_1, \dots, \zeta_r)$ of the matrices used to compute $H^*(M, \bar\alpha \otimes \rho_n)$. In addition, the $\mathbb{C}(t)$-rank of a matrix with coefficients in $\mathbb{C}[t^{\pm 1}]$ is larger than or equal to its $\mathbb{C}$-rank after evaluation at $(t_1, \dots, t_r) = (\zeta_1, \dots, \zeta_r)$. Thus, by acyclicity of $\chi \otimes \rho_n$, the $\mathbb{C}$-rank of the matrices used to compute cohomology is maximal, hence the $\mathbb{C}(t)$-rank of these matrices before evaluation at $(t_1, \dots, t_r) = (\zeta_1, \dots, \zeta_r)$ is also maximal, and therefore $\bar\alpha \otimes \rho_n$ is acyclic. $\square$

For each peripheral torus T_i^2 chose m_i so that $\pi_1(T_i^2) = \langle l_i, m_i \rangle \cong \mathbb{Z}^2$, where l_i is an α-longitude. Writing $\alpha(m_i) = (\alpha_1(m_i), \dots, \alpha_r(m_i)) \in \mathbb{Z}^r$, we denote

$$t^{\alpha(m_i)} = \bar\alpha(m_i) = t_1^{\alpha_1(m_i)} \cdots t_r^{\alpha_r(m_i)}.$$

Definition 2.6. The *twisted Alexander polynomial* of $(M, \bar\alpha \otimes \rho_n)$ is

$$\Delta_M^{\alpha,n}(t_1, \dots, t_r) := \begin{cases} \dfrac{1}{\mathrm{tor}(M, \bar\alpha \otimes \rho_n)} & \text{for } n \text{ even,} \\[2ex] \dfrac{1}{\mathrm{tor}(M, \bar\alpha \otimes \rho_n)(t^{\alpha(m_1)} - 1) \cdots (t^{\alpha(m_l)} - 1)} & \text{for } n \text{ odd.} \end{cases}$$

It is an element of $\mathbb{C}(t) = \mathbb{C}(t_1, \dots, t_r)$, a quotient of polynomials in the variables $t_1, \dots, t_r$, defined up to sign and up to multiplication by monomials $t_1^{n_1} \cdots t_r^{n_r}$. In Corollary 2.9 we prove that it is a Laurent poynomial, an element of $\mathbb{C}[t^{\pm 1}] = \mathbb{C}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$.

Remark 2.7. For even dimensional representations, this is the same as Wada's polynomial [44], using Kitano's Theorem [21]. For odd dimensional representations, it is a normalization of the latter.

We view $\mathbb{C}[t^{\pm 1}]^n \cong \mathbb{C}[t^{\pm 1}] \otimes \mathbb{C}^n$ as a $\pi_1(M)$-module via $\bar\alpha \otimes \rho_n$, and denote it by $\mathbb{C}[t^{\pm 1}]_{\bar\alpha \otimes \rho_n}^n$. For the definition of the order of a $\mathbb{C}[t^{\pm 1}]$-module, see [43].

Lemma 2.8. *(a) Up to units in $\mathbb{C}[t^{\pm 1}]$:*

$$\frac{1}{\mathrm{tor}(M, \bar\alpha \otimes \rho_n)} = \mathrm{order}_{\mathbb{C}[t^{\pm 1}]} H_1(M, \mathbb{C}[t^{\pm 1}]_{\bar\alpha \otimes \rho_n}^n).$$

(b) For n odd, $\mathrm{order}_{\mathbb{C}[t^{\pm 1}]} H_1(M, \mathbb{C}[t^{\pm 1}]_{\bar\alpha \otimes \rho_n}^n) \in (t^{\alpha(m_1)} - 1) \cdots (t^{\alpha(m_l)} - 1)\mathbb{C}[t^{\pm 1}]$.

Corollary 2.9. *The twisted Alexander polynomial is a Laurent polynomial:*

$$\Delta_M^{\alpha,n} \in \mathbb{C}[t^{\pm 1}].$$

Before proving Lemma 2.8 we need the following lemma:

Lemma 2.10. *Assume that n is odd.*
(a) *For each peripheral torus T_j^2,*

$$H_1(T_j^2, \mathbb{C}[t^{\pm 1}]_{\bar{\alpha} \otimes \rho_n}^n) \cong \mathbb{C}[t^{\pm 1}]/(t^{\alpha(m_j)} - 1).$$

In addition, it is generated by the image of $h_j \otimes l_j$ via the natural map

$$(\mathbb{C}^n)^{\rho_n(l_j)} \otimes H_1(S_j^1; \mathbb{Z}) \to H_1(T_j^2, \mathbb{C}[t^{\pm 1}]_{\bar{\alpha} \otimes \rho_n}^n),$$

where $(\mathbb{C}^n)^{\rho_n(l_j)}$ is the (1-dimensional) subspace invariant by $\rho_n(l_j)$, $0 \neq h_j \in (\mathbb{C}^n)^{\rho_n(l_j)}$, and S_j^1 is a circle representing l_j.
(b) *The inclusion induces a monomorphism*

$$H_1(\partial \overline{M}, \mathbb{C}[t^{\pm 1}]_{\bar{\alpha} \otimes \rho_n}^n) \hookrightarrow H_1(\overline{M}, \mathbb{C}[t^{\pm 1}]_{\bar{\alpha} \otimes \rho_n}^n).$$

Proof. To prove (a) consider $S_j^1 \times \mathbb{R} \to T_j^2$ the infinite cyclic covering corresponding to $\bar{\alpha}$ with deck transformation group $\mathbb{Z}$ generated by $\tau \colon S_j^1 \times \mathbb{R} \to S_j^1 \times \mathbb{R}$. There is a long exact sequence in homology [32, 38]

$$\cdots \to H_i(S_j^1 \times \mathbb{R}, \mathbb{C}[t^{\pm 1}]_{\bar{\alpha} \otimes \rho_n}^n) \xrightarrow{\tau_* - 1} H_i(S_j^1 \times \mathbb{R}, \mathbb{C}[t^{\pm 1}]_{\bar{\alpha} \otimes \rho_n}^n)$$
$$\to H_i(T_j^2, \mathbb{C}[t^{\pm 1}]_{\bar{\alpha} \otimes \rho_n}^n) \to \cdots$$

As $\bar{\alpha}(\pi_1(S_j^1)) = 1$, $H_i(S_j^1 \times \mathbb{R}, \mathbb{C}[t^{\pm 1}]_{\bar{\alpha} \otimes \rho_n}^n) \cong \mathbb{C}[t^{\pm 1}] \otimes H_i(S_j^1, \mathbb{C}_{\rho_n}^n)$ and the action of τ_* on $\mathbb{C}[t^{\pm 1}]$ corresponds to multiplication by $t^{\alpha(m_j)}$. Furthermore $H_i(S_j^1, \mathbb{C}_{\rho_n}^n) = 0$ for $i \neq 0, 1$ and $H_1(S_j^1, \mathbb{C}_{\rho_n}^n) \cong H^0(S_j^1, \mathbb{C}_{\rho_n}^n) \cong (\mathbb{C}^n)^{\rho_n(l_j)}$. Then (a) follows from these considerations.

In the proof of (b), we need two preliminary steps. The first one is to show that

(6) $$H_1(\partial \overline{M}, \mathbb{C}[t^{\pm 1}]^n) \otimes_\chi \mathbb{C} \to H_1(\overline{M}, \mathbb{C}[t^{\pm 1}]^n) \otimes_\chi \mathbb{C}$$

is injective for every $\chi \in \mathrm{Hom}(\mathbb{Z}^r, \mathbb{S}^1)$. By Lemma 2.4 (b) and Assertion (a) in this lemma, the composition

$$H_1(\partial \overline{M}, \mathbb{C}[t^{\pm 1}]_{\bar{\alpha} \otimes \rho_n}^n) \otimes_\chi \mathbb{C} \to H_1(\overline{M}, \mathbb{C}[t^{\pm 1}]_{\bar{\alpha} \otimes \rho_n}^n) \otimes_\chi \mathbb{C} \to H_1(\overline{M}, \chi \otimes \rho_n)$$

is an isomorphism, hence (6) is a monomorphism.

The second step is to show that, if we consider the algebraic subvariety $V_j \subset (\mathbb{C}^*)^r$ defined by $t^{\alpha(m_j)} = 1$, then $V_j \cap (\mathbb{S}^1)^r$ is Zariski dense in V_j. To prove this, by the action of $\mathrm{SL}(r, \mathbb{Z})$ we may assume $\alpha(m_j) = (k, 0, \ldots, 0)$ for some $k \in \mathbb{Z} \setminus \{0\}$. Then the subvariety $V_j \subset (\mathbb{C}^*)^r$ is defined by $t_1^k = 1$ and the assertion is clear.

After these preliminary claims, we prove (b): let

$$a \in \ker(H_1(\partial \overline{M}, \mathbb{C}[t^{\pm 1}]_{\bar{\alpha} \otimes \rho_n}^n) \to H_1(\overline{M}, \mathbb{C}[t^{\pm 1}]_{\bar{\alpha} \otimes \rho_n}^n))$$

be an element of the kernel, consider

$$a_j \in H_1(T_j^2, \mathbb{C}[t^{\pm 1}]_{\bar{\alpha} \otimes \rho_n}^n) \cong \mathbb{C}[t^{\pm 1}]/(t^{\alpha(m_j)} - 1)$$

its projection to the j-th component. Viewing $\mathbb{C}[t^{\pm 1}]/(t^{\alpha(m_j)}-1)$ as the function ring of $V_j \subset (\mathbb{C}^*)^r$, injectivity of (6) implies that a_j vanishes when evaluated on $V_j \cap (\mathbb{S}^1)^r$. By Zariski density of $V_j \cap (\mathbb{S}^1)^r$ in V_j, a_j vanishes on the whole V_j, hence $a_j = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Lemma 2.8. By Turaev [43] we have an equality (up to units in $\mathbb{C}[t^{\pm 1}]$):

$$\frac{1}{\mathrm{tor}(M, \bar{\alpha} \otimes \rho_n)} = \frac{\mathrm{order}(H_1(M, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \otimes \rho_n}))}{\mathrm{order}(H_0(M, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \otimes \rho_n}))\,\mathrm{order}(H_2(M, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \otimes \rho_n}))}$$

As the manifold M has the simple homotopy type of a 2-dimensional complex, we have that $\mathrm{order}(H_2(M, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \otimes \rho_n})) = 1$. Hence it suffices to prove that $\mathrm{order}(H_0(M, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \otimes \rho_n})) = 1$. So, looking for a contradiction, assume $\mathrm{order}(H_0(M, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \otimes \rho_n})) \neq 1$ and pick $\lambda = (\lambda_1, \ldots, \lambda_r) \in (\mathbb{C}^*)^r$ a root of the polynomial $\mathrm{order}(H_0(M, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \otimes \rho_n}))$ (this polynomial defines a hypersurface in $\mathbb{C}^r$, and since the order is defined up to factors $t_i^{\pm 1}$, it intersects $(\mathbb{C}^*)^r$ non trivially). Evaluation at $(t_1, \ldots, t_r) = (\lambda_1, \ldots, \lambda_r)$ defines a morphism $\Lambda \colon \pi_1(M) \to \mathbb{C}^*$, by $\Lambda(\gamma) = \lambda_1^{\alpha_1(\gamma)} \cdots \lambda_r^{\alpha_r(\gamma)}$. Since λ is a root of the polynomial $\mathrm{order}(H_0(M, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \otimes \rho_n}))$, $H_0(M, \Lambda \otimes \rho_n) \neq 0$. This means that $\mathbb{C}^n$ has non-zero coinvariants for the action of $\Lambda \otimes \rho_n$. By duality, $\mathbb{C}^n$ has non-zero invariants by the action of $\overline{\Lambda} \otimes \rho_n$, in particular $\mathbb{C}^n$ has a proper subspace preserved by ρ_n. By Zariski density of the holonomy representation, this contradicts irreducibility of Sym^{n-1}. This proves (a).

For (b) construct the twisted chain complex from a CW–complex K, with $|K| = M$:

$$C_*(K, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \rho_n}) = \mathbb{C}[t^{\pm 1}]^n \otimes_{\bar{\alpha} \otimes \rho_n} C_*(\widetilde{K}, \mathbb{Z}).$$

We may assume furthermore that there are 2-cells of K, denoted by e_i^2, representing T_i^2 and 1-cells denoted by e_i^1 representing l_i. Chose respective lifts $\tilde{e}_i^2$ of e_i^2, and $\tilde{e}_i^1$ of e_i^1, in the universal cover $\widetilde{K}$ that correspond to the same connected component of the lift of the peripheral torus T_i^2 in the universal covering. Moreover, chose $\tilde{e}_i^1$ to be adjacent to $\tilde{e}_i^2$, so that

$$(7) \qquad\qquad\qquad \partial \tilde{e}_i^2 = (m_i - 1)\tilde{e}_i^1 + (1 - l_i)\tilde{f}_i^1$$

for some other 1-cell f_i^1. Notice that $\langle m_i, l_i \rangle \cong \pi_1(T_i^2)$, with $\bar{\alpha}(l_i) = 1$ and $\bar{\alpha}(m_i) = t^{\alpha(m_i)}$. Chose also $h_i \in \mathbb{C}^n$ a non-zero element invariant by $\rho_n(\pi_1(T_i^2))$. Let $L_* \subset C_*(K, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \rho_n})$ be the $\mathbb{C}[t^{\pm 1}]$-subcomplex generated by the elements $h_i \otimes \tilde{e}_i^j$, $j = 1, 2$, $i = 1, \ldots, l$. By the choice of lifts:

$$(8) \qquad\qquad\qquad \partial(h_i \otimes \tilde{e}_i^2) = (t^{\alpha(m_i)} - 1)h_i \otimes \tilde{e}_i^1,$$

up to sign and up to powers of $t^{\alpha(m_i)}$, and $\partial(h_i \otimes \tilde{e}_i^1) = 0$ since e_i^1 represents l_i. Hence L_* is a subcomplex of $C_*(K, \bar{\alpha} \otimes \rho_n)$. Moreover it follows from (8) that this complex L_* is also acyclic as a complex of $\mathbb{C}(t)$-vector spaces. We have a short exact sequence of acyclic complexes of $\mathbb{C}(t)$-vector spaces:

$$(9) \qquad 0 \to L_* \to C_*(K, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \rho_n}) \to C_*(K, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \rho_n})/L_* \to 0.$$

Furthermore we can construct geometric bases à la Milnor for $C_*(K, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \rho_n})$ that include the elements $h_i \otimes \tilde{e}_i^j$, Definition B.2. Thus there are compatible

geometric bases in the sequence and the multiplicativity formula for the torsion [31, Theorem 3.2] provides the equality:

$$\mathrm{tor}(M, \bar{\alpha} \otimes \rho_n) = \mathrm{tor}(L_*)\,\mathrm{tor}(C_*(K, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \otimes \rho_n})/L_*).$$

From (8) the contribution of $\mathrm{tor}(L_*)$ is $(t^{m_1} - 1) \cdots (t^{m_l} - 1)$. Finally, we show that the zeroth and second homology groups of $C_*(K, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha}\rho_n})/L_*$ vanish (hence its torsion is the inverse of a Laurent polynomial). For that purpose, notice that from Lemma 2.10 (a) and (8) we have a natural isomorphism

$$H_1(L_*) \cong H_1(\partial M, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \otimes \rho_n}).$$

Hence by Lemma 2.10 (b) we have an injection induced by inclusion

$$H_1(L_*) \hookrightarrow H_1(M, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \otimes \rho_n}).$$

Using this monomorphism, the long exact sequence in homology corresponding to (9), and the vanishing of $H_i(M, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \otimes \rho_n})$ for $i = 0, 2$, it follows that zeroth and second homology groups of $C_*(K, \mathbb{C}[t^{\pm 1}]^n_{\bar{\alpha} \otimes \rho_n})/L_*$ also vanish. $\qquad\square$

Proposition 2.11. *For n even,*

$$|\Delta^{\alpha,n}_M(\zeta_1, \ldots, \zeta_r)| = \frac{1}{|\mathrm{tor}(M, \chi \otimes \rho_n)|}.$$

For n odd,

$$|\Delta^{\alpha,n}_M(\zeta_1, \ldots, \zeta_r)| = \frac{1}{|\mathrm{tor}(M, \chi \otimes \rho_n; b_1, b_2)|} \prod_{\zeta^{\alpha(m_i)} \neq 1} \frac{1}{|\zeta^{\alpha(m_i)} - 1|}.$$

In the proposition, b_2 and b_1 are the basis in homology of Lemma 2.4, according to the components where $\chi(\pi_1(T_i^2))$ is trivial, $\alpha(m_i) \in \mathbb{Z}^r$ is a generator of the image of $\alpha(\pi_1(T_i^2))$. We use the notation $\zeta^{\alpha(m_i)} = \zeta_1^{\alpha_1(m_i)} \cdots \zeta_r^{\alpha_r(m_i)}$. The product in the odd case runs on the components where $\chi(\pi_1(T_i^2))$ is non trivial.

Proof. In the acyclic case (when n is even or when χ is non-trivial on each peripheral subgroup) the proposition follows from naturality, cf. [31, § 6].

The proof of the non-acyclic case is very similar to the proof of Lemma 2.8 b), but the subcomplex L_* is only constructed from the peripheral tori for which the restriction of χ is trivial. Namely, assume that n is odd and that χ is trivial precisely on $\pi_1(T_1^2), \ldots, \pi_1(T_s^2)$. Then, choosing a CW-complex K as in the proof of Lemma 2.8, we take L_* to be the subcomplex of $C_*(K, \chi \otimes \rho_n)$ generated by elements $h_i \otimes \tilde{e}_i^j$, $j = 1, 2$, $i = 1, \ldots, s$. In this case the boundary operator is zero in L_*, and a geometric basis for L_* is precisely a lift of the basis b_1 and b_2 for $H_*(M, \chi \otimes \rho_n)$, by Lemma 2.4. From the defining short exact sequence and the previous consideration, it follows that $C_*(K, \chi \otimes \rho_n)/L_*$ is acyclic and its torsion equals $|\mathrm{tor}(M, \chi \otimes \rho_n; b_1, b_2)|$. Then the lemma follows from naturality applied to $C_*(K, \mathbb{C}[t^{\pm 1}]^n)/L'_*$, where L'_* is the subcomplex $\mathbb{C}[t^{\pm 1}]$-generated by the same elements as L_*, $h_i \otimes \tilde{e}_i^j$, $j = 1, 2$, $i = 1, \ldots, s$. $\qquad\square$

For $\rho_3 = \mathrm{Sym}^2 \circ \rho = \mathrm{Ad} \circ \rho$ and χ trivial, Proposition 2.11 has been proved by Yamaguchi in [48] for knot exteriors and by Dubois and Yamaguchi in [9] in a more general setting.

Proof of Theorem 1.11. Proposition 2.11 expresses the evaluation of the twisted Alexander polynomial at unitary complex numbers as a Reidemeister torsion, which is an element of $\mathbb{C}^*$, up to multiplication by a unit complex number. In particular it is not zero. □

3. Dehn fillings and rational twists

In this section we consider Dehn fillings on M and sequences of those fillings that converge geometrically to M. In the first subsection we discuss compatibility conditions with the twist and with the lift of the holonomy (equivalently the spin structure). In particular we restrict to rational Dehn fillings. In the second subsection we discuss surgery formulas for the torsion.

Definition 3.1. A unitary twist $\chi\colon \pi_1(M) \to \mathbb{S}^1 \subset \mathbb{C}$ is called *rational* if it takes values in $e^{2\pi i \mathbb{Q}}$.

Assumption 3.2. In this section we assume that:

(a) χ is rational,
(b) χ restricted to each peripheral subgroup $\pi_1(T_j^2) \subset \pi_1(M)$ is nontrivial.

Assumption 3.2 (a) implies that χ induces a twist of certain Dehn fillings, as we will explain in the next subsection. Assumption 3.2 (b) allows to simplify the formulas when n is odd. The results in this section could be generalized without Assumption (b) by considering bases in homology, but this simpler version is sufficient for applying them in Section 6.

3.1. **Compatible Dehn fillings.** For each peripheral torus T_i^2 we have an α-longitude, namely an element $l_i \in \pi_1(T_i^2)$ that generates the kernel of $\alpha|_{\pi_1(T_i^2)}$, by Assumption 1.2. We fix a basis for the fundamental group of the peripheral group that contains this element: $\langle m_i, l_i \rangle = \pi_1(T_i^2) \cong \mathbb{Z}^2$. As $\mathrm{trace}(\rho(l_i)) = -2$ (Assumption 1.3), we may chose $\mathrm{trace}(\rho(m_i)) = +2$, after replacing m_i by $m_i l_i$ if needed.

Once we have fixed the m_i and l_i, given pairs of coprime integers p_i, q_i, the Dehn filling with filling meridians $p_i m_i + q_i l_i$ is denoted by $M_{p_1/q_1,\ldots,p_l/q_l}$. To simplify notation we write

$$M_{p/q} := M_{p_1/q_1,\ldots,p_l/q_l}.$$

The inclusion map is denoted by $i\colon M \to M_{p/q}$, it induces an epimorphism $i_*\colon \pi_1(M) \to \pi_1(M_{p/q})$. Another convention is that $(p, q) \to \infty$ means that $p_i^2 + q_i^2 \to +\infty$ for $i = 1, \ldots, l$.

By Thurston's hyperbolic Dehn filling theorem, when $p_i^2 + q_i^2$ is sufficiently large for each $i = 1\ldots, l$, then $M_{p/q}$ is hyperbolic. The (conjugacy class of the) holonomy of $M_{p/q}$ composed with i_* converges to the (conjugacy class of the) holonomy of M in the set of conjugacy classes of such representations $\mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C})$.

As we work with representations in $\mathrm{SL}(2, \mathbb{C})$, we need to impose compatibility conditions on the Dehn filling to get the same conclusion for the lifts. We shall also impose conditions so that the rational twist χ factors through i_* to a twist of $M_{p/q}$.

Definition 3.3. The Dehn filling $M_{p/q} = M_{p_1/q_1,\dots,p_1/q_l}$ is called *compatible* with χ and ρ if, for each $i = 1, \dots, l$:

(1) $\chi(m_i^{p_i}) = 1$, and
(2) $q_i \equiv 1 \mod 2$.

Since $\chi(l_i) = 1$ by Assumption 1.2, Condition (1) in the definition amounts to say that the twist of M factors through $\pi_1(M_{p/q})$. As we assume χ rational, this is achieved by taking $p_i \in \text{order}(\chi(m_i))\mathbb{Z}$. Condition (2) in the definition is explained by the following lemma.

Lemma 3.4. *For an infinite family of compatible Dehn fillings $M_{p/q}$ such that $(p,q) \to \infty$, there exists a lift of the holonomy $\rho^{p/q}$ of $M_{p/q}$ in $\mathrm{SL}(2,\mathbb{C})$ such that*

$$\lim_{(p,q)\to\infty} [\rho^{p/q} \circ i_*] = [\rho]$$

in $\mathrm{Hom}(\pi_1(M), \mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C})$, where ρ is the lift of the holonomy of the complete structure on M.

Proof. As we chose m_i so that $\text{trace}(\rho(m_i)) = +2$, Condition (2) in Definition 3.3 means $\text{trace}(\rho(p_i m_i + q_i l_i)) = -2$. Using the natural bijection between spin structures and lifts of the holonomy, this is precisely the condition required for a spin structure on M to extend to $M^{p/q}$, see [29] for instance. In terms of lifts of representations, this is the compatibility condition for the lifts of the deformation for the holonomy in Thurston's hyperbolic Dehn filling theorem. $\square$

We shall use the following notation:

$$\rho_n^{p/q} := \text{Sym}^{n-1} \circ \rho^{p/q} \colon \pi_1(M_{p/q}) \to \mathrm{SL}_n(\mathbb{C}),$$
$$\varrho_n^{p/q} := \rho_n^{p/q} \circ i_* \colon \pi_1(M) \to \mathrm{SL}_n(\mathbb{C}).$$

Thus for large $p_i^2 + q_i^2$, $i = 1, \dots, l$, $[\varrho_n^{p/q}]$ lies in a neighborhood of $[\rho_n]$ in $\mathrm{Hom}(\pi_1(M), \mathrm{SL}_n(\mathbb{C}))/\mathrm{SL}_n(\mathbb{C})$.

3.2. **Dehn filling formula.** We shall only consider compatible Dehn fillings. In particular χ factors through $\pi_1(M_{p/q})$. Since $M_{p/q}$ is closed, Corollary C.7 yields that $H^*(M_{p/q}, \chi \otimes \rho_n^{p/q})$ vanishes. However $H^*(M, \chi \otimes \varrho_n^{p/q})$ does not need to vanish, we have:

Lemma 3.5. *If Assumption 3.2 holds for χ, then for sufficiently large $p_i^2 + q_i^2$, $i = 1, \dots, l$, we have:*

$$H^*(M, \chi \otimes \varrho_n^{p/q}) = 0.$$

Proof. We first claim that the inclusion $\partial \overline{M} \to \overline{M}$ induces a monomorphism

(10) $$0 \to H^*(M, \chi \otimes \varrho_n^{p/q}) \to H^*(\partial \overline{M}, \chi \otimes \varrho_n^{p/q}).$$

The proof of the claim is analogous to Theorem 2.3, but simpler, as we no not need a theorem in L^2-cohomology but Mayer-Vietoris exact sequence. More precisely, we apply Mayer-Vietoris to the pair $(\overline{M}, V)$, where V is the union of the attached solid tori in the Dehn filling, so that $V \cap \overline{M} = \partial \overline{M}$ and $V \cup \overline{M} =$

$M_{p/q}$. As $H^*(M_{p/q}, \chi \otimes \rho_n^{p/q}) = 0$, Mayer-Vietoris long exact sequence yields an isomorphism induced by inclusion maps:

$$(11) \qquad H^*(\overline{M}; \chi \otimes \varrho_n^{p/q}) \oplus H^*(V; \chi \otimes \varrho_n^{p/q}) \cong H^*(\partial \overline{M}; \chi \otimes \varrho_n^{p/q}),$$

which proves that (10) is a monomorphism. Using this, to prove the lemma we show the vanishing of the cohomology of each peripheral 2-tori T_j^2. For this purpose, we use that

$$H^0(T_j^2, \chi \otimes \varrho_n^{p/q}) \cong (\mathbb{C}^n)^{(\chi \otimes \varrho_n^{p/q})(\pi_1(T_j^2))}$$

where $(\mathbb{C}^n)^{(\chi \otimes \varrho_n^{p/q})(\pi_1(T_j^2))}$ denotes the subspace invariant by $(\chi \otimes \varrho_n^{p/q})(\pi_1(T_j^2))$. If γ_j denotes the soul of the j-th attached solid torus V_j, then, after conjugation,

$$(12) \qquad \rho^{p/q}(\gamma_j) = \begin{pmatrix} e^{\lambda(\gamma_j)/2} & 0 \\ 0 & e^{-\lambda(\gamma_j)/2} \end{pmatrix}$$

where $\lambda(\gamma_j)$ is the complex length, $\lambda(\gamma_j) = \ell(\gamma_j) + i\,\theta(\gamma_j)$, and $\ell(\gamma_j) > 0$ is the (real) length of the geodesic γ_j in $M_{p/q}$. Therefore, as we assume that χ is non trivial on every peripheral subgroup:

$$(\mathbb{C}^n)^{(\chi \otimes \varrho_n^{p/q})(\pi_1(T_j^2))} = (\mathbb{C}^n)^{\chi(\gamma_j)\rho_n^{p/q}(\gamma_j)} = 0.$$

As $H^0(T_j^2, \chi \otimes \varrho_n^{p/q}) = 0$, from Poincaré duality and $\chi(T_j^2) = 0$ we deduce the vanishing of the cohomology groups in every dimension, $H^*(T_j^2, \chi \otimes \varrho_n^{p/q}) = 0$, which concludes the lemma. $\qquad \square$

By applying Milnor's formula on the torsion of a Mayer-Vietoris sequence, we have (further details on the proof can be found in [29, Lemmas 5.7 and 5.11]):

Proposition 3.6. *Under Assumption 3.2 for χ:*

$$\mathrm{tor}(M_{p/q}, \chi \otimes \rho_n^{p/q}) = \mathrm{tor}(M, \chi \otimes \varrho_n^{p/q}) \prod_{j=1}^{l} \prod_{k=0}^{n-1} (e^{\frac{\lambda(\gamma_j)}{2}(n-1-2k)} \chi(m_j) - 1).$$

where the complex length $\lambda(\gamma_j)$ is defined in (12).

We want to consider sequences of admissible Dehn fillings $M_{p/q}$ such that $(p, q) \to \infty$, hence by Lemma 3.4 $\varrho^{p/q} \to \rho$. By naturality of the torsions:

Lemma 3.7. *Under Assumption 3.2 for χ:*

$$\lim_{(p,q) \to \infty} \mathrm{tor}(M, \chi \otimes \varrho_n^{p/q}) = \mathrm{tor}(M, \chi \otimes \rho_n).$$

As mentioned at the beginning of the section, if we did not assume non-triviality of χ on each peripheral torus, a generalization of Proposition 3.6 and Lemma 3.7 would also holds, but one would need to consider bases in homology.

4. Ruelle zeta functions and Fried's theorem

In this section we establish the analytic tools that will be used in Section 6 to prove our version (Theorem 6.1) of a result of Müller, relating the logarithm of the Reidemeister torsion of a closed hyperbolic 3-manifold, its volume and some geometric quantity expressed as a function on the lengths of the geodesics. The new point in our Theorem 6.1 is the appearance of a unitary twist χ.

In this section we fix a closed hyperbolic 3-manifold N with fundamental group $\pi_1(N) = \Gamma$ and a lift of its holonomy $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$. For any natural integer n, we denote by $\rho_n\colon \Gamma \to \mathrm{SL}_n(\mathbb{C})$ the $(n-1)$-th symmetric power of the holonomy ρ. Finally, we fix a group homomorphism $\chi\colon \Gamma \to \mathbb{S}^1$ (sometimes called the twist).

The two statements we need to prove in this section are the following, the definitions of the two different Ruelle zeta functions come right after in Subsection 4.1.

Proposition 4.1. *For any integer k, the unitary Ruelle zeta function $R_{\chi,k}$ extends meromorphically to the whole complex plane. Moreover, it satisfies the functional equation:*

$$|R_{\chi,k}(s)| = e^{4\,\mathrm{Vol}(N)s/\pi}|R_{\chi,-k}(-s)|.$$

Theorem 4.2 (Fried's Theorem)**.** *The Ruelle zeta function $\mathcal{R}_{\chi\otimes\rho_n}$ extends meromorphically to $\mathbb{C}$, it is holomorphic at $s = 0$ and*

$$|\mathcal{R}_{\chi\otimes\rho_n}(0)| = |\operatorname{tor}(N, \chi\otimes\rho_n)|^2.$$

4.1. **Ruelle zeta functions.** In this paragraph we introduce two Ruelle zeta functions $\mathcal{R}_{\chi\otimes\rho_n}$ and $R_{\chi,k}$ (we call the latter the *unitary* Ruelle zeta function). Those functions are defined as in [35], except for the appearance of the twist χ. Since the twist has modulus one, it does not modify the arguments from [35] showing the convergence of the defining series, hence we will not discuss this point. On the other hand, the unitary Ruelle zeta function, as well as Selberg zeta functions with a twist that we do not define here, are extensively studied in Bunke's and Olbrich's book [6]. The basics of the study of those functions are due to Fried in his seminal article [12].

Any closed geodesic ϕ in the manifold N corresponds uniquely to a conjugacy class $[\gamma]$ in the set $[\Gamma]$ of conjugacy classes in the fundamental group of N. A prime element in Γ is an element that cannot be written as a non-trivial power of another one. We will use the notation $\mathcal{PC}(N)$ for the set of conjugacy classes $[\gamma]$ of prime elements γ of Γ (equivalently, the set of prime closed geodesics in N). Any γ is mapped by the holonomy representation to a conjugate of the matrix $\begin{pmatrix} e^{\lambda(\gamma)/2} & 0 \\ 0 & e^{-\lambda(\gamma)/2} \end{pmatrix}$ where $\lambda(\gamma) = \ell(\gamma) + i\theta(\gamma)$ is the complex length of the geodesic ϕ represented by γ. In particular $\ell(\gamma) > 0$ denotes the length of γ and the parameter $\theta(\gamma)$ is determined modulo $4\pi i\mathbb{Z}$ (4π instead of 2π because we take care of the spin structure, equivalently, the lift to $\mathrm{SL}(2,\mathbb{C})$).

Definition 4.3. For $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 2$, the *Ruelle zeta function* is

$$\mathcal{R}_{\chi \otimes \rho_n}(s) = \prod_{[\gamma] \in \mathcal{PC}(N)} \det \left(\operatorname{Id} - \chi(\gamma) \rho_n(\gamma) e^{-s\ell(\gamma)} \right).$$

The *unitary Ruelle zeta function* is

$$R_{\chi,k}(s) = \prod_{[\gamma] \in \mathcal{PC}(N)} (1 - \chi(\gamma) e^{\frac{k}{2} i\theta(\gamma)} e^{-s\ell(\gamma)}).$$

The term unitary comes from the unitary representation $\gamma \mapsto e^{\frac{k}{2} i\theta(\gamma)}$.

The relation between both functions is the following, whose proof is a termwise direct computation, see [35, (3.14)] for a proof.

Lemma 4.4. *For $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 2$,*

$$\mathcal{R}_{\chi \otimes \rho_n}(s) = \prod_{k=0}^{n} R_{\chi, n-2k}(s - (\frac{n}{2} - k)).$$

In particular meromorphic continuation of the former follows from meromorphic continuation of the latter, which is the content of the first statement of Proposition 4.1.

The proof of Proposition 4.1 is exactly the same as [35, Proposition 3.2]: one uses a similar functional equation for a Selberg zeta function $Z_{\chi,k}$ proved in [6, Section 3.3.2] (see also [35, Section 4], though there is no twist χ in the latter reference). Then one expresses the unitary Ruelle zeta function $R_{\chi,k}$ as a product of Selberg zeta functions, and the proposition follows from the study of Selberg zeta functions carried out in [6].

4.2. **Fried's theorem.** In this subsection we prove Theorem 4.2. Again, it follows the lines of [35], which itself reproduces the proof contained in Wotzke's PhD thesis [46]. One needs some care to introduce our twist χ, and we will describe the global picture of the proof, stressing out the points where the twist plays a role.

4.2.1. *Analytic torsion.* The first step is to replace the Reidemeister torsion by the analytic torsion: the representation $\chi \otimes \rho_n$, despite non-unitary, allows to equip the flat bundle $E_{\chi \otimes \rho_n} = \Gamma \backslash (\mathbb{H}^3 \times \mathbb{C}^n)$ with a canonical hermitian metric, as we explain in Appendix C. Using the Hodge star on N and this hermitian metric on $E_{\chi \otimes \rho_n}$, one defines the Hodge-Laplace operator $\Delta_{\chi,n}^p$ acting on the space $\Omega^p(N, E_{\chi \otimes \rho_n})$ of p-forms on N with coefficients in $E_{\chi \otimes \rho_n}$.

Since N is compact, for any $t > 0$ the heat operator $e^{-t\Delta_{\chi,n}^p}$ is of trace class. The asymptotic behavior for small and large t of the heat trace $\operatorname{Tr} e^{-t\Delta_{\chi,n}^p}$ is local in N, in the sense that they are properties of the Laplacian acting on the universal cover $\mathbb{H}^3$ where the twist χ does not comes into play and it follows from the usual arguments that the expression

$$\log T(N, \chi \otimes \rho_n) = \frac{1}{2} \frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{p=1}^3 (-1)^p p \operatorname{Tr}(e^{-t\Delta_{\chi,n}^p}) dt \right) \Bigg|_{s=0}$$

defines an invariant of $(N, \chi \otimes \rho_n)$ called the *analytic torsion*.

In the rest of the section, the following alternating sum of heat traces will be denoted by

$$(13) \qquad K_{\chi,n}(t) = \sum_{p=1}^{3}(-1)^p p \operatorname{Tr}(e^{-t\Delta_{\chi,n}^p}).$$

The representation ρ_n has determinant one, and as χ has module one, $\chi \otimes \rho_n$ is unimodular, hence we can apply Cheeger-Müller theorem, in the version of [33] (see also [4]):

Theorem 4.5 (Cheeger-Müller). *For N closed, the combinatorial and analytic torsions coincide, that is*

$$|\mathrm{tor}(N, \chi \otimes \rho_n)| = T(N, E_{\chi \otimes \rho_n}).$$

Hence to prove our Theorem 4.2 we are led to relate the analytic torsion with the evaluation at 0 of the Ruelle zeta function.

4.2.2. *Selberg trace formula.* The second ingredient is the Selberg trace formula. Again $[\Gamma]$ denotes the set of conjugacy classes in $\Gamma = \pi_1(N)$.

From the identification $N = \Gamma \backslash \mathbb{H}^3$ it follows that forms in $\Omega^p(N, E_{\chi \otimes \rho_n})$ are exactly the equivariant p-forms on $\mathbb{H}^3$ with value in $\mathbb{C}^n$, and the Laplace operator $\Delta_{\chi,n}^p$ can be seen as the restriction of the Laplace operator $\widetilde{\Delta}_n^p$ acting on $\mathbb{C}^n$-valued forms on $\mathbb{H}^3$.

Using invariance properties of the Laplace operator, and denoting $G = \mathrm{SL}_2(\mathbb{C})$, it follows that the heat trace can be decomposed as a series

$$(14) \qquad \operatorname{Tr} e^{-t\Delta_{\chi,n}^p} = \sum_{\gamma \in \Gamma} \chi(\gamma) \int_{\Gamma \backslash G} h_n^p(t, g^{-1}\gamma g) dg$$

where $h_n^p(t,g)$ is the trace of the kernel of the heat operator $e^{-t\widetilde{\Delta}_n^p}$ that we compose with the regular representation $R(g)$ to get an endomorphism of the fiber of $E_{\chi \otimes \rho_n}$-valued p-forms at the class of the point g in $\mathbb{H}^3 = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$.

It turns out that one can compute the heat trace on the ground manifold N with the heat trace on its universal cover, summing up on all translated fundamental domains in a tiling of $\mathbb{H}^3$. A reference for a proof of (14) is [34, (5.7)].

Inserting (14) in (13), one obtains

$$(15) \qquad K_{\chi,n}(t) = \sum_{\gamma \in \Gamma} \chi(\gamma) \int_{\Gamma \backslash G} k_n(t, g^{-1}\gamma g) dg$$

where $k_n(t) = \sum_{p=1}^{3}(-1)^p p\, h_n^p(t)$.

In fact, one can regroup the summands by conjugacy classes and compute explicitly the integrals in the right hand side of (15). This is performed in [35, Section 7] and the result is [(7.10), loc. cit.], note that here the twist χ is just

added in each summand:

$$(16) \quad K_{\chi,n}(t) = \mathrm{Vol}(N) \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} P_k(i\lambda) \Theta_{k,\lambda}(k_n(t)) d\lambda$$

$$+ \sum_{[\gamma] \neq 1} \frac{\chi(\gamma)\ell(\gamma)}{2\pi D(\gamma) n_{\Gamma}(\gamma)} \sum_{k \in \mathbb{Z}} e^{-k\theta(\gamma)/2} \int_{\mathbb{R}} \Theta_{k,\lambda}(k_n(t)) e^{-\ell(\gamma)\lambda} d\lambda,$$

where

- $\ell(\gamma)$ and $\theta(\gamma)$ are the real and the imaginary part of the complex length $\lambda(\gamma)$ of γ.
- P_k is the Plancherel polynomial $P_k(z) = \frac{1}{4\pi^2}(\frac{k^2}{4} - z^2)$.
- $\Theta_{k,\lambda}$ is the character of the corresponding representation $\pi_{k,\lambda}$ of G called the unitary principal series, see [35, Section 2.4].
- The integer $n_{\Gamma}(\gamma)$ is defined as being maximal for the property that there exists $\gamma_0 \in \Gamma$ such that $\gamma = \gamma_0^{n_{\Gamma}(\gamma)}$.
- $D(\gamma)$ is the *Weyl denominator* of γ:

$$D(\gamma) = (1 - e^{-(\ell(\gamma)+i\theta(\gamma))})(1 - e^{-(\ell(\gamma)-i\theta(\gamma))}).$$

It turns out that the series indexed by k in (16) are indeed finite sums, since there are only finitely many indices k such that $\Theta_{k,\lambda}(k_n(t))$ is not zero. Precisely, there are exactly four indices $k_j, j = 1, \ldots, 4$, such that $\Theta_{k_j,\lambda}(k_n(t)) \neq 0$. They are computed in [35, Section 7]:

$$k_1 = n + 1, \quad k_2 = -k_1, \quad k_3 = -n - 3, \quad k_4 = -k_3,$$

and one obtains (see [35, (7.17)]):

$$(17) \quad K_n(t) = \sum_{j=1}^{4} (-1)^j e^{-t\lambda_j^2}$$

$$\times \left(\mathrm{Vol}(N) \int_{\mathbb{R}} e^{-t\lambda^2} P_{k_j}(i\lambda) d\lambda + \sum_{[\gamma] \neq 1} \frac{\chi(\gamma)\ell(\gamma)e^{k_j\theta(\gamma)/2}}{D(\gamma) n_{\Gamma}(\gamma)} \frac{e^{-\ell(\gamma)^2/(4t)}}{(4\pi t)^{1/2}} \right)$$

where the corresponding λ_j are computed in [35, (3.31)]:

$$\lambda_1 = \frac{n+3}{2}, \quad \lambda_2 = -\lambda_1, \quad \lambda_3 = -\frac{n+1}{2}, \quad \lambda_4 = -\lambda_3.$$

4.2.3. *Auxiliary operators.* At this stage it is not clear yet that the expansion (17) of the sum of heat traces $K_{\chi,n}(t)$ that defines the analytic torsion can be related with the evaluation at 0 of the Ruelle zeta function $\mathcal{R}_{\chi \otimes \rho_n}$. The idea of Müller and Wotzke is to use some auxiliary operators $\Delta_{\chi,n}(j), j = 1, \ldots, 4$, and to show that each of the four summands in (17) can indeed be written as the (super-)trace of one of this operator acting on sections of some graded vector bundles. Those operators $\Delta_{\chi,n}(j)$ are defined in [35, (6.15)] as a combination of some Casimir type operators studied in [6, Section 1.1.3]. Again, Müller does not include the case of a non-trivial unitary twist χ that we need, but all the basic work is done in [6] for those Casimir operators with a twist.

Applying the same strategy that in Subsection 4.2.2 and expressing the trace of those operators with the Selberg trace formula yields:

$$(18) \quad \mathrm{Tr}_s(e^{-t\Delta_{\chi,n}(j)}) = \frac{1}{2}e^{-t\lambda_j^2}$$

$$\times \left(\mathrm{Vol}(N) \int_{\mathbb{R}} e^{-t\lambda^2} P_{k_j}(i\lambda)d\lambda + \sum_{[\gamma]\neq 1} \frac{\chi(\gamma)\ell(\gamma)e^{k_i\theta(\gamma)/2}}{D(\gamma)n_{\Gamma}(\gamma)} \frac{e^{-\ell(\gamma)^2/(4t)}}{(4\pi t)^{1/2}} \right)$$

so that inserting (18) in (17) gives

$$(19) \qquad K_{\chi,n}(t) = \frac{1}{2}\sum_{j=1}^{4}(-1)^j \mathrm{Tr}_s(e^{-t\Delta_{\chi,n}(j)}).$$

We need to check that those operators have trivial kernels, it is done without a twist in [35, Lemma 7.2]:

Lemma 4.6. *For any $j = 1,\dots,4$, we have $\Delta_{\chi,n}(j) > 0$.*

Proof. We will show that it is a consequence of the vanishing of the kernel of the Hodge Laplacians $\Delta_{\chi,n}^p$, see Theorem C.1.

First, we can express the Hodge Laplacian as a direct sum of Bochner-Laplace operators on $\mathbb{H}^3$ (see [36, (5.7)]) that are denoted with a *tilde*. (recall that Sym^n is $(n+1)$-dimensional):

$$\widetilde{\Delta}_{n+1}^p = \bigoplus_{\substack{\nu\in\widehat{K} \\ [\nu_{n+1}^p:\nu]\neq 0}} \widetilde{\Delta}_\nu + (\mathrm{Sym}^n(\Omega) - \nu(\Omega_K))\,\mathrm{Id}\,.$$

Here $K = \mathrm{SU}(2)$, the representations ν of K in the sum are those that appear in the decomposition into irreducible representations of the restriction ν_{n+1}^p to K of the action of G on $\mathbb{C}^{n+1}$-valued p-forms through Sym^n. The scalars $\mathrm{Sym}^n(\Omega)$ and $\nu(\Omega_K)$ are corresponding eigenvalues of the Casimir operator, the first one is computed in [35, (6.16)].

From this equation and the fact that for any p, one has $\Delta_{\chi,n+1}^p > 0$, we deduce that the operator $A_{\chi,\nu}$ defined as $A_{\chi,\nu} = \Delta_{\chi,\nu} - \nu(\Omega_K)\,\mathrm{Id}$ (see [35, (4.7)] for the notations) is bounded below by $-\mathrm{Sym}^n(\Omega)$ for any ν such that $[\nu_{n+1}^p:\nu]\neq 0$. Inserting that in the definition of $\Delta_{\chi,n}(j)$ given in [35, (6.15)] it follows that $\Delta_{\chi,n}(j) > 0$, as claimed. $\qquad\square$

Lemma 4.6 ensures that we can apply the Mellin transform to (19), one gets:

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K_{\chi,n}(t)dt = \frac{1}{2\Gamma(s)}\int_0^\infty t^{s-1}\sum_{j=1}^{4}(-1)^j \mathrm{Tr}_s(e^{-t\Delta_{\chi,n}(j)})dt$$

and taking the derivative at $s = 0$ gives

$$(20) \qquad T(N, E_{\chi\otimes\varrho_n})^4 = \prod_{j=1}^{4} \mathrm{det}_{\mathrm{gr}}(\Delta_{\chi,n}(j))^{(-1)^{j+1}}.$$

4.2.4. *A determinant formula.* The relation between the right hand side of (20) and the Ruelle zeta function is given by the following proposition:

Proposition 4.7. *Denote by $\overline{\rho}_n \colon \Gamma \to \mathrm{SL}_n(\mathbb{C})$ the complex conjugate of ρ_n, we have*

$$\mathcal{R}_{\chi \otimes \rho_n}(s) \mathcal{R}_{\chi \otimes \overline{\rho}_n}(s) = e^{-\frac{4n \, \mathrm{Vol}(N)s}{\pi}} \prod_{j=1}^{4} \mathrm{det}_{\mathrm{gr}}(s^2 - 2\lambda_j s + \Delta_{\chi,n}(j))^{(-1)^{j+1}}.$$

This proposition is stated and proved as [35, Proposition 6.2] without the twist χ, nevertheless its proof relies on two facts that are known to be true with the twist inserted:

- A determinant formula for the Selberg zeta function ([6, Theorem 3.19]).
- The expression of the Ruelle zeta function as a product of Selberg zeta functions, whose proof [35, Proposition 3.5] works identically in our situation.

With those two points, Müller's proof generalizes *verbatim* to our statement.

Proof of Theorem 4.2. It follows from Lemma 4.6 that we can take the limit as s goes to 0 in Proposition 4.7, we obtain

$$(21) \qquad \mathcal{R}_{\chi \otimes \rho_n}(0) \mathcal{R}_{\chi \otimes \overline{\rho_n}}(0) = \prod_{j=1}^{4} \mathrm{det}_{\mathrm{gr}}(\Delta_{\chi,n}(j))^{(-1)^{j+1}}$$

Note that in the product on $[\gamma]$, the conjugacy classes of both γ and of γ^{-1} are taken into account. Since $\rho(\gamma)$ is conjugated in $\mathrm{SL}_2(\mathbb{C})$ to $\rho(\gamma^{-1})$ and using $\chi(\gamma^{-1}) = \overline{\chi(\gamma)}$, one has $\det(1 - \overline{\chi}(\gamma)\overline{\rho_n(\gamma)}) = \det(1 - \chi(\gamma^{-1})\overline{\rho_n(\gamma^{-1})})$. Exchanging the role of γ and γ^{-1} in the product defining $\mathcal{R}_{\chi \otimes \rho_n}$ we obtain:

$$\mathcal{R}_{\chi \otimes \overline{\rho_n}}(0) = \overline{\mathcal{R}_{\chi \otimes \rho_n}(0)}$$

hence the left hand side of (21) is equal to $|\mathcal{R}_{\chi \otimes \rho_n}(0)|^2$. Inserting this in (20) yields

$$T(N, E_{\chi \otimes \varrho_n})^2 = |\mathcal{R}_{\chi \otimes \rho_n}(0)|$$

and Theorem 4.2 is proved. $\qquad \square$

5. Approximation by Dehn fillings

In this section we describe the geometric convergence of Dehn fillings $M_{p/q}$ to M, focusing on the behavior of geodesics and their role in Ruelle functions.

5.1. **Geometric convergence.** For a sequence of compatible Dehn fillings $M_{p/q}$ such that $(p, q) \to \infty$, not only we have convergence of representations $[\varrho_n^{p/q}] \to [\rho_n]$, but we have also pointed bi-Lipschitz convergence, see Thurston's notes [42] or [15] for an overview, and [2, Theorem E.5.1] for this precise statement:

Theorem 5.1 (Thurston)**.** *Given $\epsilon > 0$ and $\delta > 0$, there exists $C > 0$ such that, if $p_j^2 + q_j^2 > C(\epsilon, \delta)$ for $i = 1, \ldots, l$, then there is a $(1 + \epsilon)$-bi-Lipschitz homeomorphism of the δ-thick parts $M^{[\delta, +\infty)} \to M_{p/q}^{[\delta, +\infty)}$.*

The δ-thick part of N is defined as

$$N^{[\delta,+\infty)} = \{x \in N \mid \mathrm{inj}(x) \geq \delta\},$$

where $\mathrm{inj}(x)$ denotes the injectivity radius of x.

Let $\gamma_1^{p_1/q_1}, \ldots, \gamma_l^{p_l/q_l}$ denote the souls of the filling solid tori of $M_{p/q}$, whose length converges to zero as $(p,q) \to \infty$.

Proposition 5.2. *Except for $\gamma_1^{p_1/q_1}, \ldots, \gamma_l^{p_l/q_l}$, all primitive closed geodesics of $M_{p/q}$ must intersect the δ-thick part, provided that $0 < \delta < \delta_0$ for a $\delta_0 > 0$ depending only on M.*

Proof. By Margulis lemma, using the thin-thick decomposition (see [42] again) and taking $\delta_0 > 0$ less than half the length of the shortest geodesic of M, $M \setminus M^{[\delta,+\infty)}$ is the union of cusp neighborhoods. Therefore, by choosing δ_0 even less, by Theorem 5.1 $M_{p/q} \setminus M_{p/q}^{[\delta,+\infty)}$ is the union of Margulis tubes around the geodesics $\gamma_1^{p_1/q_1}, \ldots, \gamma_l^{p_l/q_l}$. Then the proposition holds true because Margulis tubes contain no closed geodesics other than their souls. $\square$

As the diameter of $M^{[\delta,+\infty)}$ goes to infinity when $\delta \to 0$, by Proposition 5.2 and geometric convergence we have:

Proposition 5.3. *For given $L > 0$ generic (so that L is not the length of any geodesic in M) there exists a constant $C(L) > 0$ such that if $p_j^2 + q_j^2 > C(L)$, for $i = 1 \ldots, l$, then the inclusion induces a bijection:*

$$\{[\gamma] \in \mathcal{PC}(M) \mid \ell(\gamma) \leq L\} \longleftrightarrow \{[\gamma] \in \mathcal{PC}(M_{p/q}) \mid \ell(\gamma) \leq L,\ \gamma \neq (\gamma_i^{p_i/q_i})^{\pm 1}\}.$$

As the length of the γ_i converges to zero, the inclusion induces a bijection:

$$\{[\gamma] \in \mathcal{PC}(M) \mid \ell(\gamma) \leq L\} \longleftrightarrow \{[\gamma] \in \mathcal{PC}(M_{p/q}) \mid \frac{1}{L} \leq \ell(\gamma) \leq L\}.$$

Furthermore, the length and the holonomy of each geodesic in $\mathcal{PC}(M)$ is the limit of length and holonomy of the corresponding geodesic in $M_{p/q}$ as $(p,q) \to \infty$.

See [29, Section 6.3 and 6.4] for a detailed proof, for instance. Another consequence of bi-Lipschitz convergence is a uniform estimate on the growth of geodesics. We next quote Lemma 6.3 from [29], based on [7]:

Lemma 5.4. *Let X be a complete hyperbolic 3-manifold. For a compact domain $K \subset X$,*

$$\#\{[\gamma] \in \mathcal{PC}(X) \mid \gamma \cap K \neq \emptyset,\ \ell(\gamma) \leq L\} \leq Ce^{2L},$$

with $C = \pi\, e^{8\,\mathrm{diam}(K)}/\mathrm{vol}(K)$.

This is not the best estimate, for instance the estimate

$$\#\{[\gamma] \in \mathcal{PC}(N) \mid \ell(\gamma) \leq L\} < Ce^{2L}/2L.$$

due to Margulis [24] is better, see also [7], but Lemma 5.4 provides a uniform bound for the family of Dehn fillings. From Proposition 5.2 and Theorem 5.1, by taking $K = M^{[\delta,+\infty)}$ or $K = M_{p/q}^{[\delta,+\infty)}$, Lemma 5.4 yields:

Lemma 5.5. *There is a uniform C such that*

$$\#\{[\gamma] \in \mathcal{PC}(X) \mid \ell(\gamma) \leq t\} \leq Ce^{2t},$$

for $X = M$ and $X = M_{p/q}$.

5.2. **Estimates for Ruelle functions.** We want to apply the results of the previous subsection to find uniform estimates on Ruelle functions for the Dehn fillings. We start with two elementary inequalities:

(22) $\qquad$ For $z \in \mathbb{C}$, $|z| < 1$, $\qquad \big| \log |1 - z| \big| \leq \big| \log(1 - |z|) \big|$.

(23) $\qquad$ For $z \in \mathbb{C}$, $|z| < 1/2$, $\qquad \big| \log |1 - z| \big| \leq 4|z|$.

To prove (22) apply logarithms to

$$1 - |z| \leq |1 - z| \leq 1 + |z| \leq \frac{1}{1 - |z|}$$

and take into account that $\log(1 - |z|) < 0$. Inequality (23) is then straightforward.

The next lemma reformulates the key calculus required for analysis of Ruelle functions, without using the formalism of measures of [29].

Lemma 5.6. *For $\epsilon > 0$ there exists $C'(\epsilon)$ such that, if $s > 2 + \epsilon$ and $L \geq 1$, then*

$$\sum_{\substack{[\gamma] \in \mathcal{PC}(X) \\ \ell(\gamma) > L}} \big| \log |1 - \chi(\gamma)e^{-s\,\ell(\gamma)}| \big| \leq C'(\epsilon)\, e^{L(2+\epsilon-s)}$$

for $X = M_{p/q}$ or $X = M$, where $C'(\epsilon)$ is uniform on X and the unitary twist χ.

Proof. We omit the subscript $[\gamma] \in \mathcal{PC}(X)$ from the sums, which is always understood in the summations along the proof, and is combined with restrictions on the length of the geodesics. First, by (23)

$$\sum_{\ell(\gamma) > L} \big| \log |1 - \chi(\gamma)e^{-s\,\ell(\gamma)}| \big| \leq 4 \sum_{\ell(\gamma) > L} e^{-s\,\ell(\gamma)}.$$

We divide the set $\mathcal{PC}(X)$ according to lengths. Set

$$l_j = (1 + \tfrac{j}{2}\epsilon)L.$$

Then by using Lemma 5.5:

(24) $\qquad \displaystyle\sum_{\ell(\gamma) > L} e^{-s\,\ell(\gamma)} \leq \sum_{j=0}^{\infty} \sum_{l_j < \ell(\gamma) \leq l_{j+1}} e^{-s\,\ell(\gamma)} \leq \sum_{j=0}^{\infty} C\, e^{2l_{j+1}} e^{-s\,l_j}$

Since

$$2l_{j+1} - s\,l_j = (2 + \epsilon - s)L + j(-\tfrac{s}{2} + 1)\epsilon\,L,$$

the bound in (24) can be explicitly computed:

$$\sum_{j=0}^{\infty} C\, e^{2l_{j+1}} e^{-s\,l_j} = C\frac{e^{L(2+\epsilon-s)}}{1 - e^{(-s/2+1)\epsilon\,L}} \leq \frac{C}{1 - e^{-\epsilon^2/2}}$$

and we are done. $\qquad\qquad\square$

The following bound is used in the proof of the theorem on the asymptotic behavior.

Lemma 5.7. *For a closed hyperbolic three-manifold N there exists a constant $C(N)$ depending only on N such that*

$$(25) \qquad \sum_{k=5}^{\infty} \big| \log |R_{\chi,-k}(k/2)| \big| \leq C(N).$$

Proof. For each $k \geq 5$ split $\log |R_{\chi,-k}(k/2)|$ into two summations:

$$\big| \log |R_{\chi,-k}(k/2)| \big| \leq \sum_{\substack{[\gamma] \in \mathcal{PC}(N) \\ \ell(\gamma) \leq 1}} \big| \log |1 - \chi(\gamma) e^{-k\lambda(\gamma)/2}| \big|$$

$$+ \sum_{\substack{[\gamma] \in \mathcal{PC}(N) \\ \ell(\gamma) > 1}} \big| \log |1 - \chi(\gamma) e^{-k\lambda(\gamma)/2}| \big|.$$

We bound the contribution of the first summation. There exists k_0 (depending on the length of the shortest geodesic of N) such that for each $k > k_0$ we have for all $[\gamma] \in \mathcal{PC}(N)$:

$$|\chi(\gamma) e^{-k\lambda(\gamma)/2}| = |e^{-k\ell(\gamma)/2}| < \frac{1}{2}.$$

By (23) we obtain for all $[\gamma] \in \mathcal{PC}(N)$:

$$\big| \log |1 - \chi(\gamma) e^{-k\lambda(\gamma)/2}| \big| \leq 4 e^{-k\ell(\gamma)/2}.$$

As the number of geodesics of length ≤ 1 is finite, the contribution of the summation indexed by $\ell(\gamma) \leq 1$ in the left-hand side of (25) is bounded (by finitely many geometric series, starting from k_0).

For the summation of geodesics $[\gamma]$ with $\ell(\gamma) > 1$, we use Lemma 5.6 (that we stated for Dehn fillings but applies to any closed hyperbolic manifold if we do not require uniformity on the manifold). As $k \geq 5$, this yields again a bound by a geometric series. $\qquad\square$

Remark 5.8. In Lemma 5.7 we do not have uniformity on the Dehn fillings because of short geodesics (k_0 depends on the length of the shortest geodesic in N). We will get rid of short geodesics by Dehn filling formulas in the next section (see Lemma 6.3). Notice that we do have uniformity on the twist χ.

6. Asymptotic behavior of torsions

I this section we prove Theorem 1.6 and Theorem 1.8 from the introduction.

6.1. **Müller's theorem for closed Dehn filling.** We give first the proof of Müller's theorem for the Dehn fillings $M_{p/q}$. We follow [35], just with the minor change of the rational unitary twist χ:

Theorem 6.1 (Müller)**.** *For χ rational, $M_{p/q}$ a compatible Dehn filling and $m \geq 3$:*

$$\log \left| \frac{\operatorname{tor}(M_{p/q}, \chi \otimes \rho_{2m})}{\operatorname{tor}(M_{p/q}, \chi \otimes \rho_4)} \right| = \sum_{k=2}^{m-1} \log |R_{\chi, -2k-1}(k + \tfrac{1}{2})| - \frac{1}{\pi} \operatorname{Vol}(M_{p/q})(m^2 - 4),$$

$$\log \left| \frac{\operatorname{tor}(M_{p/q}, \chi \otimes \rho_{2m+1})}{\operatorname{tor}(M_{p/q}, \chi \otimes \rho_5)} \right| = \sum_{k=3}^{m} \log |R_{\chi, -2k}(k)| - \frac{1}{\pi} \operatorname{Vol}(M_{p/q})(m - 2)(m + 3).$$

Proof. We prove the odd-dimensional case, the even dimensional case is similar. Observe first that, by Lemma 4.4,

$$\mathcal{R}_{\chi \otimes \rho_{2m+1}}(s) = \prod_{k=0}^{2m} R_{\chi, 2m-2k}(s - (m - k))$$

$$= R_{\chi, 0}(s) \prod_{k=1}^{m} R_{\chi, 2k}(s - k) R_{\chi, -2k}(s + k)$$

$$(26) \qquad = \mathcal{R}_{\chi \otimes \rho_5}(s) \prod_{k=3}^{m} R_{\chi, 2k}(s - k) R_{\chi, -2k}(s + k)$$

Then, taking $s = 0$ in (26) and by Theorem 4.2:

$$(27) \qquad T(M_{p/q}, E_{\chi \otimes \rho_{2m+1}})^2 = T(M_{p/q}, E_{\chi \otimes \rho_5})^2 \prod_{k=3}^{m} |R_{\chi, 2k}(-k)| |R_{\chi, -2k}(k)|.$$

Next recall from Proposition 4.1 that

$$|R_{\chi, 2k}(-k)| = |R_{\chi, -2k}(k)| e^{-4k \operatorname{Vol}(M_{p/q})/\pi},$$

then (27) becomes

$$\log \frac{T(M_{p/q}, E_{\chi \otimes \rho_{2m+1}})}{T(M_{p/q}, E_{\chi \otimes \rho_5})} = -\frac{2}{\pi} \operatorname{Vol}(N) \sum_{k=3}^{m} k + \sum_{k=3}^{m} \log |R_{\chi, -2k}(k)|$$

and the statement follows from Cheeger–Müller Theorem, Thm. 4.5. □

This theorem holds for any closed oriented hyperbolic 3-manifold, not only for Dehn fillings. Combined with Lemma 5.7 it yields a twisted version of Müller's theorem:

Corollary 6.2 ([35])**.** *Let N be a closed hyperbolic, oriented 3-manifold. Let $\chi \colon \pi_1(N) \to \mathbb{S}^1$ be a homomorphism. Then:*

$$\lim_{n \to \infty} \frac{\log |\operatorname{tor}(N, \chi \otimes \rho_n)|}{n^2} = -\frac{\operatorname{Vol}(N)}{4\pi}.$$

Note that the corollary above is also a consequence of the more general work of Bismut–Ma–Zhang in [3].

6.2. **Proof of the main theorem.** Assume that the twist χ is rational and that χ restricted to every peripheral subgroup of M is nontrivial, until Lemma 6.7. For an admissible Dehn filling $M_{p/q}$, let

$$A = \{(\gamma_{p_1/q_1})^{\pm 1}, \ldots, (\gamma_{p_l/q_l})^{\pm 1}\}$$

denote the set of oriented souls of the filling tori, namely the l short geodesics added for the Dehn filling, with both orientations (hence A has cardinality $2l$). Define, for $k \geq 5$,

$$B_{\chi,k}^{p/q} = \sum_{[\gamma] \in \mathcal{PC}(M_{p/q}) - A} \log|1 - \chi(\gamma)e^{-k\lambda_{p/q}(\gamma)/2}|.$$

The convergence of this series follows from Margulis' estimate ([24]), because $k/2 \geq 5/2 > 2$. We discuss below in Lemma 6.4 further properties of this series.

Recall that $\varrho_n^{p/q} = \rho_n^{p/q} \circ i_*$, where $\rho_n^{p/q}$ is the symmetric power of the lift of the holonomy of $M_{p/q}$ and $i_* \colon \pi_1(M) \to \pi_1(M_{p/q})$ is induced by inclusion.

Lemma 6.3. *Given a rational twist χ of M that is nontrivial on each peripheral torus, for any integer $m \geq 3$:*

$$\log\left|\frac{\mathrm{tor}(M, \chi \otimes \varrho_{2m}^{p/q})}{\mathrm{tor}(M, \chi \otimes \varrho_4^{p/q})}\right| = -\frac{m^2 - 4}{2}\left(\sum_{i=1}^{l} \ell(\gamma_{p_i/q_i}) + \frac{2}{\pi}\mathrm{Vol}(M_{p/q})\right) + \sum_{k=2}^{m-1} B_{\chi,2k+1}^{p/q}$$

and

$$\log\left|\frac{\mathrm{tor}(M, \chi \otimes \varrho_{2m+1}^{p/q})}{\mathrm{tor}(M, \chi \otimes \varrho_5^{p/q})}\right| = -\frac{(m-2)(m-3)}{2}\left(\sum_{i=1}^{l} \ell(\gamma_{p_i/q_i}) + \frac{2}{\pi}\mathrm{Vol}(M_{p/q})\right) + \sum_{k=3}^{m} B_{\chi,2k}^{p/q}$$

Proof. We discuss the even case, $2m$, and assume for simplicity that there is only one cusp, $l = 1$. Set $\lambda = \lambda(\gamma_{p_1/q_1})$ and $\zeta = \chi(m_1)$ the image by the twist of any meridian of the boundary torus. By Proposition 3.6:

$$\log\left|\frac{\mathrm{tor}(M_{p/q}, \chi \otimes \rho_{2m}^{p/q})}{\mathrm{tor}(M, \chi \otimes \varrho_{2m}^{p/q})}\right| = \sum_{k=0}^{2m-1} \log|e^{(2m-1-2k)\lambda/2}\zeta - 1|$$

$$= \sum_{k=0}^{m-1} \log|(e^{(k+\frac{1}{2})\lambda}\zeta - 1)(e^{-(k+\frac{1}{2})\lambda}\zeta - 1)|.$$

Thus

$$\log\left|\frac{\mathrm{tor}(M_{p/q}, \chi \otimes \rho_{2m}^{p/q})}{\mathrm{tor}(M_{p/q}, \chi \otimes \rho_4^{p/q})}\frac{\mathrm{tor}(M, \chi \otimes \varrho_4^{p/q})}{\mathrm{tor}(M, \chi \otimes \varrho_{2m}^{p/q})}\right| = \sum_{k=2}^{m-1} \log|(e^{(k+\frac{1}{2})\lambda}\zeta - 1)(e^{-(k+\frac{1}{2})\lambda}\zeta - 1)|.$$

With Theorem 6.1 it yields

(28)

$$\log\left|\frac{\mathrm{tor}(M, \chi \otimes \varrho_{2m}^{p/q})}{\mathrm{tor}(M, \chi \otimes \varrho_4^{p/q})}\right| = \sum_{k=2}^{m-1} \log|R_{\chi,-2k-1}^{p/q}(k+\frac{1}{2})| - \frac{1}{\pi}\mathrm{Vol}(M_{p/q})(m^2 - 4)$$

$$- \sum_{k=2}^{m-1} \log|(e^{(k+\frac{1}{2})\lambda}\zeta - 1)(e^{-(k+\frac{1}{2})\lambda}\zeta - 1)|,$$

where $R^{p/q}_{\chi,-2k-1}$ denotes the twisted Ruelle zeta function of $M_{p/q}$. By definition of $B^{p/q}_k$:

$$(29) \quad \log|R^{p/q}_{\chi,-2k-1}(k+\tfrac{1}{2})| = \log|B^{p/q}_{\chi,2k+1}| + \log|(1-\zeta e^{-(k+\frac{1}{2})\lambda})(1-\overline{\zeta}e^{-(k+\frac{1}{2})\lambda})|.$$

To combine (28) and (29), we use:

$$(30) \quad \left|\frac{(1-\zeta e^{-(k+\frac{1}{2})\lambda})(1-\overline{\zeta}e^{-(k+\frac{1}{2})\lambda})}{(1-\zeta e^{-(k+\frac{1}{2})\lambda})(1-\zeta e^{(k+\frac{1}{2})\lambda})}\right| = \left|\frac{1-\overline{\zeta}e^{-(k+\frac{1}{2})\lambda}}{1-\zeta e^{(k+\frac{1}{2})\lambda}}\right|$$

$$= \left|e^{-\lambda(\frac{1}{2}+k)}\right|\left|\frac{1-\overline{\zeta}e^{-(k+\frac{1}{2})\lambda}}{\overline{\zeta}e^{-(k+\frac{1}{2})\lambda}-1}\right| = e^{-\ell(\lambda)(k+\frac{1}{2})}.$$

The lemma follows from (28), (29) and (30). □

Define, for $k \geq 5$, a unitary Ruelle function on the cusped manifold M:

$$R_{\chi,-k}(k/2) = \prod_{\gamma \in \mathcal{PC}(M)}(1-\chi(\gamma)e^{-k\lambda(\gamma)/2}).$$

Lemma 6.4. *For $k \geq 5$:*

(a) The series

$$\sum_{\gamma \in \mathcal{PC}(M)} \log|1-\chi(\gamma)e^{-k\lambda(\gamma)/2}| \quad \text{and} \quad \sum_{\gamma \in \mathcal{PC}(M)} \log|1-e^{-k\ell(\gamma)/2}|$$

converge uniformly.

(b) There exists a constant $C > 0$, uniform in χ, such that

$$\sum_{k=3}^{\infty}\big|\log|R_{\chi,-2k}(k)|\big| \leq C, \quad \text{and} \quad \sum_{k=3}^{\infty}\big|\log|R_{\chi,-2k-1}(k+\tfrac{1}{2})|\big| \leq C.$$

(c) The series $B^{p/q}_{\chi,k}$ also converges uniformly, uniformly on (p,q) and the twist χ. In addition,

$$\lim_{(p,q)\to\infty} B^{p/q}_{\chi,k} = R_{\chi,-k}(k/2)$$

uniformly on the twist χ.

In the lemma, uniformity on (p,q) or on χ means that the series can be bounded term-wise in absolute value by a convergent series, independently on (p,q) or/and on χ.

Proof. Assertion (a) follows from Margulis bound on geodesic length growth, using inequalities (22) and (23). Assertion (b) has the very same proof as Lemma 5.7. For (c), we get uniformity on (p,q) from Lemma 5.5 and the fact that the sum does not include any of the short geodesics of the set $A = \{(\gamma^{p/q}_1)^{\pm 1}, \ldots, (\gamma^{p/q}_n)^{\pm 1}\}$; hence there is a uniform lower bound away from zero on the length of the geodesics that appear in the sum of $B^{p/q}_{\chi,k}$, and from this, with Proposition 5.3 and Lemma 5.5, we get uniformity. Finally, the limit follows also from Proposition 5.3. □

Proposition 6.5. *For a rational twist that is nontrivial on each peripheral torus, and $m \geq 3$,*

$$\log \left| \frac{\mathrm{tor}(M, \chi \otimes \rho_{2m+1})}{\mathrm{tor}(M, \chi \otimes \rho_5)} \right| = \sum_{k=3}^{m} \log |R_{\chi,-2k}(k)| - \frac{1}{\pi} \mathrm{Vol}(M)(m-2)(m+3)$$

and

$$\log \left| \frac{\mathrm{tor}(M, \chi \otimes \rho_{2m})}{\mathrm{tor}(M, \chi \otimes \rho_4)} \right| = \sum_{k=2}^{m} \log |R_{\chi,-2k-1}(k+\tfrac{1}{2})| - \frac{1}{\pi} \mathrm{Vol}(M)(m-2)(m+2)$$

Proof. We take limits on Lemma 6.3 when $(p,q) \to \infty$. On the left hand side of the formula in Lemma 6.3, we apply Lemma 3.7. On the right hand side, we apply that $\mathrm{Vol}(M_{p/q}) \to \mathrm{Vol}(M)$, that $\ell(\gamma_{p_i/q_i}) \to 0$ [42, 15], and Lemma 6.4. $\qquad\square$

Using Propositions 2.11 and 6.5, we get:

Corollary 6.6. *Assume that $\zeta = (\zeta_1, \ldots, \zeta_r) \in (\mathbb{S}^1)^r$ satisfies that $\zeta_j \in e^{\pi i \mathbb{Q}}$, for $j = 1, \ldots, r$, and $\zeta^{\alpha(m_k)} \neq 1$, for $k = 1, \ldots, l$. Then:*

(a) for $2m$ even:

$$\log \left| \frac{\Delta_M^{\alpha,2m}(\zeta)}{\Delta_M^{\alpha,4}(\zeta)} \right| = \frac{1}{\pi} \mathrm{Vol}(M)(m-2)(m+2) - \sum_{k=2}^{m} \log |R_{\chi,-2k-1}(k+\tfrac{1}{2})|,$$

(b) for $2m+1$ odd:

$$\log \left| \frac{\Delta_M^{\alpha,2m+1}(\zeta)}{\Delta_M^{\alpha,5}(\zeta)} \right| = \frac{1}{\pi} \mathrm{Vol}(M)(m-2)(m+3) - \sum_{k=3}^{m} \log |R_{\chi,-2k}(k)|.$$

The proof of Theorem 1.6 for χ rational and non trivial on peripheral subgroups follows from Corollary 6.6 and Lemma 6.4 (b). Next we remove the hypothesis on the twist χ.

Lemma 6.7. *Corollary 6.6 holds for any $\zeta = (\zeta_1, \ldots, \zeta_r) \in (\mathbb{S}^1)^r$, without any assumption on rationality of $\zeta_1, \ldots, \zeta_r$.*

Proof. The proof is a density argument, using continuity of the terms that appear in Corollary 6.6, that we need to justify.

By Theorem 1.11, we know that $\Delta_M^{\alpha,n}(\zeta)$ does not vanish, hence $\log |\Delta_M^{\alpha,n}(\zeta)|$ is continuous on $\zeta = (\zeta_1, \ldots, \zeta_r) \in \mathbb{S}^1$, for $n \geq 2$.

For the continuity of $R_{\chi,-k}(k/2)$, $k \geq 5$, as in the proof of Lemma 5.7 we split again the series $\log |R_{\chi,-k}(k/2)|$ in two: a finite sum indexed by geodesics of length $< L$ and a series indexed by geodesics of length $> L$. The finite sum is continuous on ζ, so we need to chose L so that the series indexed by geodesics of length $> L$ is arbitrarily small, uniformly on ζ. More precisely, by (22), for each $[\gamma] \in \mathcal{PC}(M)$

$$|1 - \chi(\gamma)e^{-k\lambda(\gamma)/2}| \leq |1 - e^{-k\ell(\gamma)/2}|.$$

As the series $\sum_{[\gamma]\in\mathcal{PC}(M)} \big| \log|1 - e^{-k\ell(\gamma)/2}| \big|$ converges (Lemma 6.4), for every $\varepsilon > 0$ there exists $L = L(\varepsilon) > 0$ such that

$$\sum_{\substack{[\gamma]\in\mathcal{PC}(M) \\ \ell(\gamma)>L}} \big| \log|1 - \chi(\gamma)e^{-k\lambda(\gamma)/2}| \big| < \varepsilon,$$

uniformly on χ. As $\mathcal{PC}(M)$ has finitely many elements of length $\leq L$, continuity of $R_{\chi,-k}(k/2)$ on χ is clear. $\qquad\square$

Proof of Theorems 1.6 and 1.8. By Lemma 6.7, equations of Corollary 6.6 hold true for any ζ. Moreover, the bound on the series in the right-hand side given by Lemma 6.4 (b) is uniform in χ, so that we obtain Theorem 1.6 by dividing by $(2m)^2$ in case (a), or by $(2m + 1)^2$ in case (b), and by letting m tend to infinity. This proves Theorem 1.6, and Theorem 1.8 follows in the same way, again using uniformity on ζ of the bound from Lemma 6.4. $\qquad\square$

APPENDIX A. SOME TWISTED ALEXANDER POLYNOMIALS FOR THE FIGURE-EIGHT KNOT

We computed the twisted Alexander polynomials $\Delta_{4_1}^{\rho_n}$ for the figure-eight knot for n up to 10.

n	$\Delta_{4_1}^{\rho_n}(t)$
2	$t^2 - 4t + 1$
3	$t^2 - 5t + 1$
4	$\left(t^2 - 4t + 1\right)^2$
5	$t^4 - 9t^3 + 44t^2 - 9t + 1$
6	$t^6 - 12t^5 + 156t^4 - 388t^3 + 156t^2 - 12t + 1$
7	$\left(t^2 - 6t + 1\right)\left(t^4 - 7t^3 + 372t^2 - 7t + 1\right)$
8	$t^8 - 16t^7 + 1045t^6 - 15808t^5 + 39160t^4 - 15808t^3 + 1045t^2 - 16t + 1$
9	$\left(t^6 + 9t^5 + 2403t^4 - 15690t^3 + 2403t^2 + 9t + 1\right)\left(t^2 - 26t + 1\right)$
10	$t^{10} - 20t^9 + 3630t^8 - 350916t^7 + 5322645t^6 - 13176480t^5$ $\qquad\qquad + 5322645t^4 - 350916t^3 + 3630t^2 - 20t + 1$

APPENDIX B. REVIEW ON COMBINATORIAL TORSION

The goal of this appendix is to review basic properties of combinatorial torsion.

We restrict to compact orientable three-dimensional manifolds N, possibly with boundary. To simplify notation, we write $\Gamma = \pi_1(N)$. We also fix a field $\mathbb{F}$ of characteristic 0, and a representation $\rho\colon \Gamma \to \mathrm{GL}_n(\mathbb{F})$.

B.1. **Twisted chain complexes.** Fix a CW-complex structure K on N. The complex of chains on the universal covering $\widetilde{K}$ is the free $\mathbb{Z}$-module on the cells of $\widetilde{K}$, equipped with the usual boundary operator, and it is denoted by $C_*(\widetilde{K}, \mathbb{Z})$. It has an action of $\Gamma = \pi_1(N)$ that turns it into a left $\mathbb{Z}[\Gamma]$-module. The group Γ acts on $\mathbb{F}^n$ via ρ on the left, and for the tensor product Γ acts on $\mathbb{F}^n$ on the right using inverses: any $\gamma \in \Gamma$ maps $v \in \mathbb{F}^n$ to $\rho(\gamma^{-1})(v)$. We write $_\rho\mathbb{F}^n$ and $\mathbb{F}^n_\rho$ to emphazise the left and right $\mathbb{Z}[\Gamma]$-module structures, respectively. The *twisted* chain and cochain complexes are defined as:

$$(31) \qquad C_*(K, \rho) = \mathbb{F}^n_\rho \otimes_\Gamma C_*(\widetilde{K}, \mathbb{Z}),$$

$$(32) \qquad C^*(K, \rho) = \mathrm{Hom}_\Gamma(C_*(\widetilde{K}, \mathbb{Z}), {}_\rho\mathbb{F}^n).$$

Those are complexes and co-complexes of finite-dimensional vector spaces, and the corresponding homology and cohomology groups are denoted by $H_*(K, \rho)$ and $H^*(K, \rho)$.

B.2. **Geometric bases.** For a cell $\widetilde{e} \in \widetilde{K}$, $\mathbb{Z}[\Gamma]\widetilde{e}$ denotes the free $\mathbb{Z}[\Gamma]$-module of rank one on its Γ-orbit (i.e. the free module on all lifts of a given cell e in K).

Lemma B.1. *We have natural isomorphisms of $\mathbb{F}$-vector spaces:*

$$\mathrm{Hom}_\Gamma(\mathbb{Z}[\Gamma]\widetilde{e}, \mathbb{F}^n) \;\;\to\;\; \mathbb{F}^n \qquad\qquad \mathbb{F}^n \otimes_\Gamma \mathbb{Z}[\Gamma]\widetilde{e} \;\;\to\;\; \mathbb{F}^n$$
$$\theta \;\;\mapsto\;\; \theta(\widetilde{e}) \qquad\qquad\qquad v \otimes \widetilde{e} \;\;\mapsto\;\; v$$

The proof is straightforward.

Chose $\{v_1, \ldots, v_n\}$ a basis for $\mathbb{F}^n$ and let $\{e_1^i, \ldots e_{j_i}^i\}$ be the set of i-dimensional cells of K. For each cell e_j^i chose a lift $\widetilde{e}_j^i$ to $\widetilde{K}$. Then $\{v_k \otimes \widetilde{e}_j^i\}_{i,j,k}$ is an $\mathbb{F}$-basis for $C_i(K, \rho)$. Similarly $\{(\widetilde{e}_j^i)^* \otimes v_l\}_{i,j,l}$ is an $\mathbb{F}$-basis for $C^i(K, \rho)$ where $\big((\widetilde{e}_j^i)^* \otimes v_l\big)(\gamma\widetilde{e}_k^i) = \rho(\gamma)v_l\delta_{jk}$.

Definition B.2. We call this basis a *geometric basis* for $C_*(K, \rho)$, respectively for $C^*(K, \rho)$.

B.3. **Combinatorial torsion.** Recall the definition of torsion of a complex of finite dimensional $\mathbb{F}$-vector spaces C_* with bases $\{c^i\}_i$ for the chain complexes and bases $\{h^i\}_i$ for the homology groups, following for instance [31]. For that purpose we consider the space of boundaries $B_i = \mathrm{im}(\partial\colon C_{i+1} \to C_i)$, the space of cycles $Z_i = \ker(\partial\colon C_i \to C_{i-1})$ and the homology $H_i = Z_i/B_i$. We chose b^i an $\mathbb{F}$-basis for B_i. Using the exact sequences

$$0 \to B_i \to Z_i \to H_i \to 0, \quad 0 \to Z_i \to C_i \to B_{i-1} \to 0$$

we lift b^i it to a subset $\widetilde{b}^i$ of C_{i+1}, and h^i to a subset $\widetilde{h}^i$ of C_i, so that $\widetilde{b}^{i-1} \cup \widetilde{h}^i \cup b^i$ is an $\mathbb{F}$-basis for C_i. We denote $[\widetilde{b}^{i-1} \cup \widetilde{h}^i \cup b^i : c^i]$ the determinant of the matrix which takes c^i to $\widetilde{b}^{i-1} \cup \widetilde{h}^i \cup b^i$ (in the colomns of the matrix are the coordinates of $\widetilde{b}^{i-1} \cup \widetilde{h}^i \cup b^i$ with respect to c^i). Then we define

$$\mathrm{tor}(C_*, \{c^i\}_i, \{h^i\}_i) = \prod_{i=0}^{3} [\widetilde{b}^{i-1} \cup \widetilde{h}^i \cup b^i : c^i]^{(-1)^i} \in \mathbb{F}^*$$

If we have defined a geometric basis $\{c^i\}_i$ as in Definition B.2, then the torsion is:
$$\operatorname{tor}(N, \rho, \{h^i\}_i) = \operatorname{tor}(C_*(\widetilde{K}, \rho), \{c^i\}_i, \{h^i\}_i) \in \mathbb{F}^* / \pm \det\big(\rho(\Gamma)\big).$$
It is straightforward to check that it is well defined (see [31] and [38]). Topological invariance follows from uniqueness of triangulations on three-manifolds.

B.4. **Duality homology-cohomology.** We aim to define the torsion from the cohomological point of view. Let V be a finite dimensional $\mathbb{F}$-vector space, and let $\rho \colon \Gamma \to \operatorname{GL}(V)$ be a representation. The *contravariant representation* or *dual* representation $\rho^* \colon \Gamma \to \operatorname{GL}(V^*)$ is defined by $\rho^*(\gamma)(f) = f \circ \rho(\gamma^{-1})$.

Lemma B.3. *The representations ρ and ρ^* are equivalent if and only if there exists a non-degenerate bilinear form $B \colon V \otimes V \to \mathbb{F}$ which is Γ-invariant.*

If we choose a basis in V and its dual basis in V^*, we obtain matrix representations ρ, $\rho^* \colon \Gamma \to \operatorname{GL}_n(\mathbb{F})$, and they are related by $\rho^*(\gamma) = \rho(\gamma^{-1})^t$. Notice that $(\rho^*)^* = \rho$.

Example B.4. For any representation $\rho \colon \Gamma \to \operatorname{SL}_2(\mathbb{F})$, the module $V = \mathbb{F}^2$ has a skew-symmetric non-degenerate bilinear form defined by the determinant. Namely, the vectors (x_1, x_2) and $(y_1, y_2) \in \mathbb{F}^2$ are mapped to
$$\det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}.$$
In view of Lemma B.3, ρ^* and ρ are equivalent. More concretely, for any matrix $A \in \operatorname{SL}_2(\mathbb{F})$ we have
$$\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) A \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = (A^{-1})^t.$$

The pairing $V^* \otimes V \to \mathbb{F}$ induces a perfect pairing of complexes
$$\langle\,,\,\rangle \colon C_i(K, \rho) \otimes C^i(K, \rho^*) \to \mathbb{F},$$
defined by:
$$\langle v \otimes \tilde{e}, \theta \rangle = \theta(\tilde{e})(v),$$
where $\tilde{e}$ is a cell of $\widetilde{K}$, $v \in V$ and $\theta \in \hom_\Gamma(C_*(\widetilde{K}), V^*)$. It is easy to check that it is well defined, non-degenerate and that it is compatible with the boundaries and coboundaries:
$$\langle \partial \cdot, \cdot \rangle = \pm \langle \cdot, \delta \cdot \rangle$$
where the sign depends only on the dimension. Hence in its turn it induces a non-degenerate Kronecker pairing between homology and cohomology
$$H_i(K, \rho) \times H^i(K, \rho^*) \to \mathbb{F}.$$
Now we can relate the torsion in homology with the torsion in cohomology. We denote by B^i, Z^i and H^i the coboundary, cocycle and cohomology spaces, respectively. In addition, we take $\bar{b}^i$ basis for B^i that we lift to $\widetilde{b}^i$ in C^{i-1}. We define the torsion of a cocomplex with bases in cohomology h^i as:
$$\operatorname{tor}(C^*, \{\bar{c}^i\}_i, \{\bar{h}^i\}_i) = \prod_{i=0}^3 [\widetilde{b}^{i+1} \cup \widetilde{h}^i \cup \bar{b}^i \colon \bar{c}^i]^{(-1)^{i+1}} \in \mathbb{F}^*$$

To relate torsion in homology and cohomology, notice that the geometric basis c^i of $C_i(K, \rho)$ and $\bar{c}^i$ of $C^i(K, \rho^*)$ can be chosen to be dual. Then the matrices of the boundary operators with respect to those basis are transpose to the matrices of the respective coboundary operators. From this, we have:

Proposition B.5. *If the basis h^i for $H_i(K, \rho)$ and the basis $\bar{h}^i$ for $H^i(K, \rho^*)$ are dual for each i, then*

$$\mathrm{tor}(C_i(K, \rho), \{c^i\}, \{h_i\}) = \mathrm{tor}(C^i(K, \rho^*), \{\bar{c}^i\}, \{\bar{h}^i\}).$$

Remark B.6. We shall also use Poincaré duality with twisted coefficients, see for instance [18, 38]. For N a compact orientable manifold:

$$H^i(N; \rho)^* \cong H^{\dim N - i}(N, \partial N; \rho^*)$$

B.5. **The representations we are interested in the paper.** Here we list the representations we use in the paper. We describe in which space they are defined. Since the torsion lies by definition in $\mathbb{F}/\pm \det \rho(\Gamma)$, we need to understand $\det \rho(\Gamma)$. We start with a representation $\rho \colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$, and we put $\rho_n := \mathrm{Sym}^{n-1} \circ \rho$.

(1) For the representation $\rho_n \colon \Gamma \to \mathrm{SL}_n(\mathbb{C})$, the torsion is well defined up to sign, as $\det \rho(\Gamma) = \{1\}$. Recall that $\mathbb{C}^n$ has a non-degenerate Sym^{n-1}-invariant bilinear form which is symmetric for n odd and antisymmetric for n even. By irreducibility, the form is unique up to scalar. For $n = 2$ this form is the determinant (see Example B.4). For general n, it is the symmetrization of this bilinear form on $\mathbb{C}^2$, an explicit formula is given in Lemma 3.1.4 in [41]. Thus

$$\rho_n^* \cong \rho_n.$$

(2) For the representation $\alpha \otimes \rho_n \colon \Gamma \to \mathrm{GL}_n(\mathbb{C}(t_1, \ldots, t_r))$, the torsion is well defined up to sign and multiplication by monomials $t^m = t_1^{m_1} \cdots t_r^{m_r}$. There is no Γ-invariant bilinear form on $\mathbb{C}(t_1, \ldots, t_r)^n$, and hence $(\alpha \otimes \rho_n)^*$ and $\alpha \otimes \rho_n$ are not equivalent (for non-trivial α), nevertheless

$$(\alpha \otimes \rho_n)^* = \alpha^{-1} \otimes \rho_n^* \cong \alpha^{-1} \otimes \rho_n.$$

(3) In the case of the representation $\chi \otimes \rho_n \colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$ for a character $\chi \colon \Gamma \to \mathbb{S}^1 \subset \mathbb{C}$, only the modulus of the torsion is well defined. There is no Γ-invariant bilinear form on $\mathbb{C}^n$, and

$$(\chi \otimes \rho_n)^* = \overline{\chi} \otimes \rho_n^* \cong \overline{\chi} \otimes \rho_n.$$

By the classical duality theorems of Franz [11] and Milnor [30] we have

$$\mathrm{tor}(N, \alpha \otimes \rho_n) = \pm t^m \, \mathrm{tor}(N, \alpha^{-1} \otimes \rho_n)$$

for some multiplicative factor $\pm t^m$, and

$$|\mathrm{tor}(N, \chi \otimes \rho_n)| = |\mathrm{tor}(N, \bar{\chi} \otimes \rho_n)|.$$

Remark B.7. Let us recall some basic facts about the irreducible representation $\mathrm{Sym}^{n-1} \colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$. For details we refer to Springer's book [41, Section 3.1].

- The representation Sym^{n-1} factors through $\mathrm{PSL}_2(\mathbb{C})$ for n odd.

- The space $\mathbb{C}^n$ has a non-degenerate Sym^{n-1}-invariant bilinear form, that is symmetric for n odd and antisymmetric for n even. By irreducibility, this form is unique up to scalar. An explicit formula is given in Lemma 3.1.4 in [41] (for $n = 2$ see Example B.4). In higher dimensions the Sym^{n-1}-invariant bilinear form is the symmetrization of the determinant.
- The image of a non-trivial unipotent element in $\mathrm{SL}_2(\mathbb{C})$ is a *regular* unipotent element in $\mathrm{SL}_n(\mathbb{C})$, i.e. it is conjugate to an upper-triangular matrix which has only ones on the diagonal and a *single block* in the Jordan-Hölder form.

  It follows that the image of a parabolic element $g \in \mathrm{SL}_2(\mathbb{C})$, with trace $\epsilon_g 2$, for some $\epsilon_g = \pm 1$ has a unique eigenspace, of dimension one and with eigenvalue ϵ_g^{n-1}. Moreover, this 1-dimensional eigenspace is an *isotropic* subspace of $\mathbb{C}^n$.
- We also use Clebsch-Gordan formula:

$$(33) \qquad \mathrm{Ad} \circ \mathrm{Sym}^{n-1} \cong \mathrm{Sym}^{2(n-1)} \oplus \mathrm{Sym}^{2(n-2)} \oplus \cdots \oplus \mathrm{Sym}^4 \oplus \mathrm{Sym}^2$$

(see for instance [41, Exercise 3.2.4])

Appendix C. Vanishing of L^2-cohomology

The goal of this appendix is to show that the classical vanishing theorems in cohomology à la Matsushima-Murakami [25] apply to our situation with a twist χ.

C.1. **Review on L^2-forms on hyperbolic manifolds.** In this appendix M is an oriented hyperbolic three-manifold (possibly of infinite volume) and $\chi \colon \pi_1(M) \to \mathbb{S}^1$ a unitary character, possibly trivial. In the rest of the paper we assume that M has finite volume, but not in this appendix. Let

$$\rho \colon \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$$

be a lift of the holonomy, and

$$\mathrm{Sym}^{n-1} \colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$$

be the n-dimensional holomorphic irreducible representation. The composition is denoted by

$$\rho_n = \mathrm{Sym}^{n-1} \circ \rho \colon \pi_1(M) \to \mathrm{SL}_n(\mathbb{C}).$$

We consider the flat vector bundle $\mathbb{C}^n \to E_{\chi \otimes \rho_n} \to M$ with total space:

$$E_{\chi \otimes \rho_n} = \mathbb{C}^n \times_{\chi \otimes \rho_n} \widetilde{M}.$$

We describe the hermitian metric on the bundle (i.e. on each fibre). View the universal covering $\widetilde{M}$ as the quotient $\mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$, start with a hermitian product on $\mathbb{C}^n$ invariant by the action of the compact group $\mathrm{SU}(2)$, and translate it along $\mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2) \cong \mathbb{H}^3$, via Sym^{n-1}. This hermitian product is compatible with the action of ρ_n by construction, but also with the action of χ, because hermitian products are invariant by multiplication by unit complex numbers. Thus it induces a non-flat hermitian metric on the bundle $E_{\chi \otimes \rho_n}$.

We consider $\Omega^p(M, E_{\chi \otimes \rho_n})$ the space of p-forms valued in $E_{\chi \otimes \rho_n}$, namely smooth sections of the bundle $E_{\chi \otimes \rho_n} \otimes \Omega^p(M)$. The Riemannian metric on TM and the hermitian metric on the fibres yield a Hodge star operator $*$, a codifferential δ, and a Laplacian $\Delta^p_{\chi,n}$ on $\Omega^p(M, E_{\chi \otimes \rho_n})$. They also provide a hermitian product on p-forms:

$$(\phi, \psi) = \int_M \phi \wedge *\psi \qquad \forall \phi, \psi \in \Omega^p_c(M, E_{\chi \otimes \rho_n}),$$

where $\wedge$ denotes the exterior product on forms in $\Omega^*(M)$ combined with the hermitian product on $E_{\chi \otimes \rho_n}$, and $\Omega^p_c(M, E_{\chi \otimes \rho_n})$ the space of compactly supported forms in $\Omega^p(M, E_{\chi \otimes \rho_n})$

Pointwise we use the Riemannian metric on M and the hermitian product on the bundle to define a hermitian product on forms $\langle \phi, \psi \rangle_x$ at any $x \in M$, so that

$$(\phi \wedge *\psi)_x = \langle \phi, \psi \rangle_x \, d\text{vol} \qquad \text{for every } x \in M.$$

De Rham cohomology of the cocomplex $(\Omega^*(M, E_{\chi \otimes \rho_n}), d)$ is denoted by $H^*(M, E_{\chi \otimes \rho_n})$; it is isomorphic to the simplicial cohomology $H^*(M; \chi \otimes \rho_n)$. The aim of this appendix is to prove that every closed form with finite norm is exact:

Theorem C.1. *For any form $\omega \in \Omega^1(M, E_{\chi \otimes \rho_n})$ satisfying $(\omega, \omega) < \infty$, if $d\omega = 0$ then there exists $\eta \in \Omega^0(M, E_{\chi \otimes \rho_n})$ such that $\omega = d\eta$.*

Theorem C.1 is in fact a theorem on vanishing of L^2-cohomology and it is a version of a theorem of Garland [13]. This theorem is proved in Subsection C.2.

C.2. **Proof of the theorem.** The proof is based on the following theorem of Andreotti-Vesentini [1] and Garland [13]: uniform ellipticity implies that closed forms of finite norm are exact.

Theorem C.2 (Thm 3.22 in [13]). *If there exists a constant $c > 0$ such that for every form $\omega \in \Omega^1(M, E_{\chi \otimes \rho_n})$ with compact support*

$$(34) \qquad (d\,\omega, d\,\omega) + (\delta\,\omega, \delta\,\omega) \geq c(\omega, \omega),$$

then Theorem C.1 holds.

Inequality (34) is called *uniform ellipticity* because, for forms ω with compact support, it is equivalent to

$$(\Delta^1_{\chi,n}\omega, \omega) \geq c(\omega, \omega).$$

In order to prove uniform ellipticity, we use the formalism of Matsushima and Murakami [25], as in [13, 39, 17, 28, 40]. Since $\mathbb{H}^3 \cong \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$, from the decomposition $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$, orthogonal for the real Killing form, there is a natural identification ϕ_x of the tangent space at each point $T_x\mathbb{H}^3$ with $i\mathfrak{su}(2)$.

Let $E = \mathbb{C}^n \times \mathbb{H}^3$ be the trivial bundle, equipped with the natural flat connection, with covariant derivative ∇_v and the standard differential $d \colon \Omega^p(\mathbb{H}^3, E) \to \Omega^{p+1}(\mathbb{H}^3, E)$. Following [25] we define a new covariant derivative

$$\widetilde{\nabla}_v = \nabla_v - \mathrm{sym}^{n-1}(\phi_x(v)), \qquad \forall v \in T_x\mathbb{H}^3,$$

where $\operatorname{sym}^{n-1}\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C})$ is the representation of the Lie algebra associated to Sym^{n-1}. The corresponding connection $D\colon \Omega^0(\mathbb{H}^3, E) \to \Omega^1(\mathbb{H}^3, E)$ is given by

$$Ds(v) = d\,s(v) - \operatorname{sym}^{n-1}(\phi_x(v))(s)$$

for every section $s \in \Omega^0(\mathbb{H}^3, E)$ and every tangent vector $v \in T_x\mathbb{H}^3$. By construction, D is a connection: $D(f\,s) = fDs + sd\,f$ for any function f and any section s, and it can be checked that it is metric:

$$d\langle s_1, s_2 \rangle_x = Ds_1 \wedge s_2 + s_1 \wedge Ds_2,$$

where $x \mapsto \langle s_1, s_2 \rangle_x$ is a function on $\mathbb{H}^3$. The connection D is introduced in [25] as induced from the natural connection associated to the principal bundle on $\operatorname{SL}_n(\mathbb{C}) \to \operatorname{SL}_2(\mathbb{C})$, corresponding to the representation Sym^{n-1}.

Now, for any frame $\{e_1, e_2, e_3\}$ of $\mathbb{H}^3$, let $\{\omega^1, \omega^2, \omega^3\}$ denote its dual coframe. As $d = \sum_{j=1}^3 \omega^j \wedge \nabla_{e_j}$ [47, (6.19)] we have:

Proposition C.3. [25] *On* $\Omega^*(\mathbb{H}^3, E)$ *we have*

$$d = D + T \quad and \quad \delta = D^* + T^*,$$

where

(35) $$D = \sum_j \omega^j \wedge \widetilde{\nabla}_{e_j}, \qquad\qquad T = \sum_j \omega^j \wedge \operatorname{sym}^{n-1}(\phi_x(e_j)),$$

(36) $$D^* = -\sum_j i(e_j)\widetilde{\nabla}_{e_j}, \qquad\qquad T^* = \sum_j i(e_j)\operatorname{sym}^{n-1}(\phi_x(e_j)).$$

Up to now these operators are defined on the bundle E on $\mathbb{H}^3$, and we want to descend them to the bundle on M twisted by $\chi \otimes \rho_n$. Notice that the operators and Proposition C.3 are found in the literature without the twist, so we need to justify why it works in our situation. We may view the definitions and Proposition C.3 on the trivial bundle $E = \mathbb{C}^n \times \mathbb{H}^3 \to \mathbb{H}^3$ as being equivariant for the action of $\pi_1(M)$ via the representation ρ_n. On the other hand, these formulas are $\mathbb{C}$-linear, so they are equivariant for the action via $\chi \otimes \rho_n$:

Remark C.4. Proposition C.3 holds true on $\Omega^*(M, E_{\chi \otimes \rho})$.

Proposition C.5. [25] *There is a Weitzenböck formula:*

$$\Delta = d\delta + \delta d = D\,D^* + D^*D + T\,T^* + T^*T = \Delta_D + H$$

where $\Delta_D = D\,D^* + D^*D$ *and* $H = T\,T^* + T^*T$.

Moreover, for any form $\omega \in \Omega^1(M, E_{\chi \otimes \rho_n})$ with compact support:

(37) $$(d\tilde{\omega}, d\tilde{\omega}) + (\delta\tilde{\omega}, \delta\tilde{\omega}) = (D\tilde{\omega}, D\tilde{\omega}) + (D^*\tilde{\omega}, D^*\tilde{\omega}) + (H\tilde{\omega}, \tilde{\omega}).$$

A proof of both propositions can be found in [25] and also in [17, 28, 40]. The Weitzenböck formula requires the identity

$$D\,T^* + T\,D^* + D^*\,T + T^*\,D = 0$$

on forms with compact support. Identity (37) is proved from Stokes theorem, cf. [17, Equation (5)]. On the other hand:

Proposition C.6. [25] *For any $\omega \in \Omega^1(M, E_{\chi \otimes \rho_n})$, pointwise,*

$$\langle H\omega, \omega \rangle_x \geq c_n \langle \omega, \omega \rangle_x, \qquad \text{at every point } x \in M,$$

for a uniform constant c_n that depends only on Sym^{n-1}.

See also [39, 28] for a proof. Thus uniform ellipticity holds for any form in $\omega \in \Omega^1(M, E_{\chi \otimes \rho_n})$ with compact support; so Theorem C.2 applies.

Next we assume that the orientable hyperbolic manifold M has finite topology, that it has a compactification $\overline{M}$ that consists in adding surfaces, as it has finitely many ends that are topologically products. Every element in the kernel of $H^1(M; \chi \otimes \rho_n) \to H^1(\partial \overline{M}; \chi \otimes \rho_n)$ is represented by a differential form with compact support in M, in particular it has finite norm. Thus, by Theorem C.1:

Corollary C.7. *We have an injection induced by inclusion:*

$$H^1(M; \chi \otimes \rho_n) \hookrightarrow H^1(\partial \overline{M}; \chi \otimes \rho_n).$$

APPENDIX D. DYNAMICS OF PSEUDO-ANOSOV DIFFEOMORPHISMS ON THE
VARIETY OF CHARACTERS

Let Σ be a compact orientable surface, possibly with boundary, connected and with negative Euler characteristic, and let

$$\phi \colon \Sigma \to \Sigma$$

be a pseudo-Anosov diffeomorphism. Note that ϕ does not necessarily act by the identity on $\partial \Sigma$, but it may permute the boundary components.

Consider the mapping torus of ϕ:

$$M(\phi) = \Sigma \times [0, 1]/(x, 1) \sim (\phi(x), 0).$$

Its fundamental group is a semi-direct product

$$(38) \quad \pi_1(M(\phi)) \cong \pi_1(\Sigma) \rtimes \mathbb{Z} \cong \langle \pi_1(\Sigma), \tau \mid \tau \gamma \tau^{-1} = \phi_\#(\gamma) \text{ for all } \gamma \in \pi(\Sigma) \rangle.$$

Here $\phi_\# \colon \pi_1(\Sigma) \to \pi_1(\Sigma)$ denotes the isomorphism induced by ϕ. Notice that a different choice of τ would yield the composition of $\phi_\#$ with an inner automorphism.

According to Thurston's hyperbolization theorem, $M(\phi)$ admits a finite volume and complete hyperbolic metric [37]. The holonomy of this hyperbolic structure lifts to a representation $\pi_1(M(\phi)) \to \mathrm{SL}_2(\mathbb{C})$ and its restriction to $\pi_1(\Sigma)$ yields a representation $\rho_2 \colon \pi_1(\Sigma) \to \mathrm{SL}_2(\mathbb{C})$. The composition of ρ_2 with Sym^{n-1} will be denoted by $\rho_n := \mathrm{Sym}^{n-1} \circ \rho_2$. Notice that, by (38), ρ_n and

$$\phi^*(\rho_n) := \rho_n \circ \phi_\#$$

are conjugate by $\rho_n(\tau)$. This implies that the equivalence class $[\rho_n]$ of ρ_n is a fixed point of the action of ϕ^* on the character variety

$$\mathcal{R}\big(\Sigma, \mathrm{SL}_n(\mathbb{C})\big) := \mathrm{Hom}\big(\pi_1(\Sigma), \mathrm{SL}_n(\mathbb{C})\big) /\!\!/ \, \mathrm{SL}_n(\mathbb{C}).$$

For definitions and more details see [20, Section 4.3] or [16].

In the case of a closed surface ($\partial\Sigma = \emptyset$) and $n = 2$, M. Kapovich proved in [19] that $[\rho]$ is a hyperbolic fixed point of $\phi^* \colon \mathcal{R}(\Sigma, \mathrm{SL}_2(\mathbb{C})) \to \mathcal{R}(\Sigma, \mathrm{SL}_2(\mathbb{C}))$, namely the tangent map

$$(d\phi^*)_{[\rho_2]} \colon T_{[\rho_2]}\mathcal{R}(\Sigma, \mathrm{SL}_2(\mathbb{C})) \to T_{[\rho_2]}\mathcal{R}(\Sigma, \mathrm{SL}_2(\mathbb{C}))$$

has no eigenvalues of modulus one.

In the case of a surface with boundary this assertion is no longer true, as the trace functions of the peripheral elements are invariant under a power of ϕ^*. This causes $(d\phi^*)_{[\rho_2]}$ to have 1 as an eigenvalue (see [38, Section 4.5] for $n = 2$ and a punctured torus).

In order to generalize Kapovich's result, for surfaces with boundary we consider the relative character variety. Let

$$\partial\Sigma = \partial_1 \sqcup \cdots \sqcup \partial_s$$

be the decomposition in connected components. The relative character variety is

$$\mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C})) := \{[\rho] \in \mathcal{R}(\Sigma, \mathrm{SL}_n(\mathbb{C})) \mid \rho(\partial_i) \text{ and } \rho_n(\partial_i) \text{ are similar}\},$$

with the convention that $\mathcal{R}(\Sigma, \emptyset, \mathrm{SL}_n(\mathbb{C})) = \mathcal{R}(\Sigma, \mathrm{SL}_n(\mathbb{C}))$ for closed surfaces.

The main result of this appendix is the following:

Theorem D.1. *Let Σ be a compact orientable surface, possibly with boundary, connected and with negative Euler characteristic, and let $\phi \colon \Sigma \to \Sigma$ be a pseudo-Anosov diffeomorphism.*

Then the character $[\rho_n]$ is a hyperbolic fixed point of ϕ^*, i.e. the tangent map

$$(d\phi^*)_{[\rho_n]} \colon T_{[\rho_n]}\mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C})) \to T_{[\rho_n]}\mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C}))$$

has no eigenvalues of modulus one.

The proof is based on the cohomological interpretation of the tangent spaces to varieties of characters. Recall that by a result of A. Weil [45] there is a natural isomorphism

$$T_{[\rho_n]}\mathcal{R}(\Sigma, \mathrm{SL}_n(\mathbb{C})) \cong H^1(\Sigma, \mathrm{Ad} \circ \rho_n)$$

and the tangent space of the relative character variety can be interpreted as a kernel (see for instance [16, Proposition 18]):

$$T_{[\rho_n]}\mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C})) \cong \ker(H^1(\Sigma, \mathrm{Ad} \circ \rho_n) \to H^1(\partial\Sigma, \mathrm{Ad} \circ \rho_n)).$$

Moreover, the tangent map $d\phi^*_{[\rho_n]}$ corresponds to the induced map ϕ^* in cohomology.

There is a natural surjection

$$\alpha \colon \pi_1(M(\phi)) \twoheadrightarrow \mathbb{Z}, \quad \alpha(\tau) = 1, \text{ and } \alpha(\gamma) = 0 \text{ for } \gamma \in \pi_1(\Sigma),$$

induced by the fibration $\Sigma \to M(\phi) \to S^1$.

Remark D.2. In Section 2.2 we have defined the twisted Alexander polynomial for manifolds with cusps, but the same definition applies to closed manifolds, without requiring any assumption on α (just non-triviality). Thus we can define a one variable twisted Alexander polynomial $\Delta^{\alpha,n}_{M(\phi)}(t)$ even for $M(\phi)$ closed. The main results of the paper hold true and are simpler to prove in the closed

case, without needing the approximation results of Sections 3 and 6, for instance.

Proposition D.3. *Up to multiplication by a factor $\pm t^m$:*

$$\det((d\phi^*)_{[\rho_n]} - t\,\mathrm{Id}) = \prod_{k=1}^{n-1} \Delta_{M(\phi)}^{\alpha, 2k+1}(t).$$

Proof of Theorem D.1. Apply Proposition D.3 and Theorem 1.11.   □

Remark D.4. There is a natural $\mathbb{C}$-valued symplectic form on $\mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C}))$ [22, 16]. By naturality, this symplectic form is ϕ^*-invariant, therefore:

$$\det((d\phi^*)_{[\rho_n]}^{-1} - t\,\mathrm{Id}) = \det((d\phi^*)_{[\rho_n]} - t\,\mathrm{Id}).$$

Proposition D.3 is based on the following lemma:

Lemma D.5. *(a) When Σ is closed,*

$$\mathrm{tor}(M(\phi), \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n) = \det((d\phi^*)_{[\rho_n]} - t\,\mathrm{Id})^{-1}.$$

(b) When $\partial\Sigma \neq \emptyset$, if σ_ϕ denotes the permutation matrix on the components of $\partial\Sigma$, then

$$\mathrm{tor}(M(\phi), \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n) = \det((d\phi^*)_{[\rho_n]} - t\,\mathrm{Id})^{-1} \det(\sigma_\phi - t\,\mathrm{Id})^{1-n}.$$

Remark D.6. As $\partial\Sigma$ has s components, σ_ϕ is a permutation matrix of size $s \times s$, that decomposes into l cycles, where l is the number of components of $\partial M(\phi) = T_1^2 \sqcup \cdots \sqcup T_l^2$. Furthermore, if c_i is the order of the cycle corresponding to T_i^2, the i-th component of $\partial M(\phi)$, then $\alpha(\pi_1(T_i^2)) = c_i\mathbb{Z}$, $s = c_1 + \cdots + c_l$, and

$$\det(\sigma_\phi - t\,\mathrm{Id}) = (-1)^{s-l} \prod_{i=1}^{l} (1 - t^{c_i}).$$

Proof of Lemma D.5. Let K be CW-complex with underlying space $|K| = M(\phi)$. Consider its lift $\overline{K}$ to $\Sigma \times \mathbb{R}$ and its lift $\widetilde{K}$ to the universal covering. We work with the following chain complexes:

$$C^*(K, \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n) = \hom_{\pi_1(M(\phi))}(C_*(\widetilde{K}, \mathbb{Z}), \mathbb{C}(t) \otimes \mathfrak{sl}_n(\mathbb{C})),$$

$$C^*(\overline{K}, \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n) = \hom_{\pi_1(\Sigma)}(C_*(\widetilde{K}, \mathbb{Z}), \mathbb{C}(t) \otimes \mathfrak{sl}_n(\mathbb{C})).$$

Choosing τ, a representative in $\pi_1(M(\phi))$ of a generator of $\mathbb{Z}$, it acts on $C^*(\overline{K}, \bar{\alpha} \otimes \mathrm{Ad} \circ \rho)$ by

$$\theta \mapsto \alpha(\tau)\mathrm{Ad}_{\rho_n(\tau)} \circ \theta \circ \tau^{-1}, \qquad \forall \theta \in C^*(\overline{K}, \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n).$$

The action does not depend on the choice of the representative $\tau \in \pi_1(M(\phi))$ of the generator of $\mathbb{Z}$. We have then a short exact sequence of complexes:

$$0 \to C^*(K, \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n) \to C^*(\overline{K}, \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n) \xrightarrow{(\tau-1)} C^*(\overline{K}, \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n) \to 0.$$

It induces Wang's exact sequence in cohomology. Hence, as $H^i(\overline{K}, \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n) \cong H^i(\Sigma, \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n)$ vanishes for every $i \neq 1$, by Milnor's theorem on torsion of exact sequences [31, Theorem 3.2]:

(39) $$\mathrm{tor}(M(\phi), \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n) = \det(\tau^{*,1} - \mathrm{Id})^{-1},$$

where $\tau^{*,1}$ is the morphism on $H^1(\Sigma, \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n)$ induced by τ. As α is trivial on $\pi_1(\Sigma)$:

$$H^1(\Sigma, \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n) \cong \mathbb{C}(t) \otimes H^1(\Sigma, \mathrm{Ad} \circ \rho_n)$$

and $\tau^{*,1}$ corresponds via this isomorphism to $t \otimes \phi^{*,1}$, where $\phi^{*,1}$ is the action that ϕ induces on $H^1(\Sigma, \mathrm{Ad} \circ \rho_n)$. Hence

(40) $$\det(\tau^{*,1} - \mathrm{Id}) = \det(t\,\phi^{*,1} - \mathrm{Id}) = t^n \det(\phi^{*,1} - t^{-1}\,\mathrm{Id}).$$

From (39) and (40) we get

$$\mathrm{tor}(M(\phi), \bar{\alpha} \otimes \mathrm{Ad} \circ \rho_n) = \det(\phi^{*,1} - t^{-1}\,\mathrm{Id})^{-1},$$

up to a factor $\pm t^m$. Next we look at the action of $(\phi^{*,1})^{\pm 1}$ to the exact sequence coming from the cohomology of the pair $(\Sigma, \partial\Sigma)$

(41) $$0 \to \ker(i^*) \to H^1(\Sigma, \mathrm{Ad} \circ \rho_n) \xrightarrow{i^*} H^1(\partial\Sigma, \mathrm{Ad} \circ \rho_n)) \to 0.$$

We claim that:

(1) The action of $(\phi^{*,1})^{-1}$ on $\ker(i^*)$ corresponds to the action of $d\phi^*$ by the isomorphism $\ker(i^*) \cong T_{[\rho_n]}\mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C}))$.

(2) The action of $(\phi^{*,1})$ on $H^1(\partial\Sigma, \mathrm{Ad} \circ \rho_n))$ is equivalent to $\sigma_\phi \times \overset{(n-1)}{\cdots} \times \sigma_\phi$.

The proof of these claims and the product formula of torsions applied to the exact sequence (41) will complete the proof of Lemma D.5.

Proof of Claim (1). We consider the action on the variety of representations

$$\rho \mapsto \mathrm{Ad}_{\rho_n(\tau^{-1})} \circ \rho \circ \phi_\#$$

so that ρ_n is fixed and induces the previous action on the space of conjugacy classes of representations. Next we recall Weil's isomorphism between the tangent space and group cohomology:

(42) $$\begin{aligned} T_{[\rho_n]}\mathcal{R}(\Sigma, \mathrm{SL}_n(\mathbb{C})) &\cong& H^1(\pi_1(\Sigma), \mathrm{Ad} \circ \rho_n) \\ \tfrac{d}{dt}\rho_t|_{t=0} &\mapsto& \left[\gamma \mapsto \tfrac{d}{dt}\rho_t(\gamma)\rho_0(\gamma^{-1})|_{t=0}\right] \end{aligned}$$

where ρ_t is a path of representations, smooth on $t \in (-\varepsilon, \varepsilon)$ and with $\rho_0 = \rho_n$. Here $H^1(\pi_1(\Sigma), \mathrm{Ad} \circ \rho_n) = Z^1/B^1$ where Z^1 is the space of cocycles or crossed morphisms, i.e. maps $\theta \colon \pi_1(\Sigma) \to \mathfrak{sl}_n(\mathbb{C})$ that satisfy $\theta(\gamma_1 \gamma_2) = \theta(\gamma_1) + \mathrm{Ad}_{\rho_n(\gamma_1)}(\theta(\gamma_2))\ \forall \gamma_1, \gamma_2 \in \pi_1(\Sigma)$, and B^1 is the space of inner cocycles, i.e. maps $\gamma \mapsto a - \mathrm{Ad}_{\rho_n(\gamma)}(a)$ for some $a \in \mathfrak{sl}_n(\mathbb{C})$. Using Weil's isomorphism (42), the action of $d\phi^*$ on $H^1(\pi_1(\Sigma), \mathrm{Ad} \circ \rho_n)$ using classes of cocycles is

$$[\theta] \mapsto [\mathrm{Ad}_{\rho_n(\tau^{-1})} \circ \theta \circ \phi_\#] \qquad \forall \theta \in Z^1.$$

The claim follows using the isomorphism between group cohomology of $\pi_1(\Sigma)$ and cohomology of the surface Σ, which is aspherical, and the naturality of the actions induced by ϕ on each cohomology group. This establishes the first claim.

Proof of Claim (2). We use the isomorphism:

(43) $$H^1(\partial\Sigma, \mathrm{Ad} \circ \rho_n) = \bigoplus_{i=1}^{r} H^1(\partial_i, \mathrm{Ad} \circ \rho_n) \cong \bigoplus_{i=1}^{r} H_0(\partial_i, \mathrm{Ad} \circ \rho_n).$$

Namely, we decompose $\partial\Sigma$ along connected components and use Poincaré duality on each of the circles. Next, we use the canonical isomorphism

$$(44) \qquad H_0(\partial_i, \mathrm{Ad} \circ \rho_n) \cong \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\partial_i))} \cong \mathbb{C}^{n-1}$$

where $\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\partial_i))} = \mathfrak{sl}_n(\mathbb{C}) \otimes_{\pi_1(\partial_i)} \mathbb{Z}$ denotes the space of coinvariants. It is isomorphic to the quotient of $\mathfrak{sl}_n(\mathbb{C})$ by the image of $(\mathrm{Ad}(\rho_n(\partial_i)) - \mathrm{Id})$ [5], and its dimension $n-1$ has been computed for instance in [28].

Next we aim to understand the action of τ on these spaces. We view the chain complex for computing the homology of each ∂_i as a subcomplex of

$$C_*(\overline{K}, \mathrm{Ad}\,\rho_n) = \mathfrak{sl}_n(\mathbb{C}) \otimes_{\pi_1(\Sigma)} C_*(\widetilde{K}, \mathbb{Z}),$$

the complex that computes the homology of $\Sigma \times \mathbb{R}$, where K, $\overline{K}$ and $\widetilde{K}$ are as in the beginning of the proof of the lemma. Here every $\gamma \in \pi_1(\Sigma)$ acts on $C_*(\widetilde{K}, \mathbb{Z})$ by deck transformations and on $\mathfrak{sl}_n(\mathbb{C})$ by $\mathrm{Ad}(\rho_n(\gamma^{-1}))$. The action of τ maps the chain $m \otimes c$ (for $m \in \mathfrak{sl}_n(\mathbb{C})$ and $c \in C_*(\widetilde{K}, \mathbb{Z})$) to

$$m \otimes c \mapsto m \cdot \tau^{-1} \otimes \tau\, c = \mathrm{Ad}(\tau)(m) \otimes \tau\, c$$

(see [38]).

Each component ∂_i lifts to a union of lines in the universal covering $\widetilde{\Sigma}$ whose stabilizer by the action of $\pi_1(\Sigma)$ is precisely a representative of $\pi_1(\partial_i)$ in the conjugacy class. Then choosing $\tilde{e}_0$ a 0-cell of $\widetilde{K}$ that projects to ∂_i, the canonical isomorphism (44) is induced by the projection

$$\mathfrak{sl}_n(\mathbb{C}) \otimes \tilde{e}_0 \to \mathfrak{sl}_n(\mathbb{C}) \otimes_{\pi_1(\partial_i)} \mathbb{Z} = \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\partial_i))},$$

where the choice of $\pi_1(\partial_i)$ corresponds to the stabilizer of the line in $\widetilde{K}$ that contains $\tilde{e}_0$. A different choice of $\tilde{e}_0$ would be $\gamma\tilde{e}_0$ for $\gamma \in \pi_1(\Sigma)$, then the subgroup $\pi_1(\partial_i)$ should be replaced by $\gamma\pi_1(\partial_i)\gamma^{-1}$. This leads to the natural isomorphism between coinvariant subspaces

$$\begin{aligned} \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\partial_i))} &\cong \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\gamma\partial_i\gamma^{-1}))} \\ m &\mapsto \mathrm{Ad}_{\rho_n(\gamma)}(m) \end{aligned}$$

Furthermore, the action of $\phi^{*,1}$ on $H^1(\Sigma, \mathrm{Ad} \circ \rho_n)$ corresponds via (43) and (44) to

$$\begin{aligned} \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\partial_i))} &\to \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\phi_\#(\partial_i)))} \\ m &\mapsto \mathrm{Ad}_{\rho_n(\tau)}(m) \end{aligned}$$

If ϕ defines a cycle of order $c_i > 0$ on the component ∂_i (in particular $\phi^{c_i}(\partial_i) = \partial_i$), then, viewing ∂_i as an element of the fundamental group, there exists $\gamma \in \pi_1(\Sigma)$ that conjugates $\phi_\#^{c_i}(\partial_i) = \tau^{c_i}\partial_i\tau^{-c_i}$ and ∂_i. Namely $\gamma\tau^{c_i}$ commutes with ∂_i in $\pi_1(M(\phi))$, in fact both $\gamma\tau^{c_i}$ and ∂_i belong to the same peripheral subgroup $\pi_1(T_i^2)$ of $\pi_1(M(\phi))$. Therefore

$$\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\partial_i))} = \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\langle\partial_i,\gamma\tau^{c_i}\rangle))} = \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\pi_1(T_i^2)))}$$

and $\mathrm{Ad}(\rho_n(\gamma\tau^{c_i}))$ acts as the identity on $\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\partial_i))}$, by Lemma D.7 below. In other words, ϕ^{c_i} acts trivially on $H^1(\partial_i, \mathrm{Ad} \circ \rho_n) \cong \mathbb{C}^{n-1}$, and the second claim follows. $\qquad\square$

Lemma D.7. *For any peripheral torus T_i^2 and for any nontrivial $\gamma \in \pi_1(T_i^2)$, $\mathrm{Ad}(\rho_n(\gamma))$ acts trivially on the space of coinvariants $\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\pi_1(T_i^2)))}$.*

Proof. By construction, the action of $\mathrm{Ad}(\rho_n(\gamma))$ on the space of invariants

$$\mathfrak{sl}_n(\mathbb{C})^{\mathrm{Ad}(\rho_n(\pi_1(T_i^2)))} = \{m \in \mathfrak{sl}_n(\mathbb{C}) \mid \mathrm{Ad}_{\rho_n(g)}(m) = m, \ \forall g \in \pi_1(T_i^2)\}$$

is trivial. As the $\mathbb{C}$-valued Killing form on $\mathfrak{sl}_n(\mathbb{C})$ is Ad-invariant, the space of coinvariants is the quotient of $\mathfrak{sl}_n(\mathbb{C})$ by the orthogonal of the invariants. Namely

$$\left(\mathfrak{sl}_n(\mathbb{C})^{\mathrm{Ad}(\rho_n(\pi_1(T_i^2)))}\right)^\perp = \{m - \mathrm{Ad}_{\rho_n(g)}(m) \mid m \in \ \mathfrak{sl}_n(\mathbb{C}), \ \forall g \in \pi_1(T_i^2)\}$$

and

$$\mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\pi_1(T_i^2)))} = \mathfrak{sl}_n(\mathbb{C})/\left(\mathfrak{sl}_n(\mathbb{C})^{\mathrm{Ad}(\rho_n(\pi_1(T_i^2)))}\right)^\perp.$$

Therefore, the Killing form induces a pairing

$$\mathfrak{sl}_n(\mathbb{C})^{\mathrm{Ad}(\rho_n(\pi_1(T_i^2)))} \times \mathfrak{sl}_n(\mathbb{C})_{\mathrm{Ad}(\rho_n(\pi_1(T_i^2)))} \to \mathbb{C}.$$

As the Killing form is non-degenrate and Ad-invariant, this pairing is perfect and $\mathrm{Ad}(\rho_n(\gamma))$-invariant. In particular, as the action of $\mathrm{Ad}(\rho_n(\gamma))$ is trivial on the space of invariants, it is also trivial on the space of coinvariants. $\square$

Proof of Proposition D.3. When $\partial\Sigma = \emptyset$ this is a direct consequence of the Clebsch-Gordan formula (33) and of Lemma D.5.

When $\partial\Sigma \neq \emptyset$, first notice that α always satisfies Assumption 1.2. Furthermore, we do not need to care about Assumption 1.3 because $\mathrm{Ad} \circ \rho_n$ decomposes by (33) in a sum of odd dimensional representations and no spin structure is involved in the computation of the torsion in this case (however we could chose a lift of the holonomy satisfying this assumption).

Finally, we need to discuss the term $\det(\sigma_\phi - t\,\mathrm{Id})^{n-1}$. Following Remark D.6, we decompose σ_ϕ into l disjoint cycles of order $c_1, \ldots, c_l$, respectively, with $c_1 + \cdots + c_l = s$. Therefore

$$\det(\sigma_\phi - t\,\mathrm{Id}) = \pm \prod_{i=1}^l (t^{c_i} - 1)\,.$$

On the other hand, each cycle corresponds to a peripheral torus T_i^2 of $M(\phi)$ and $\alpha(\pi_1(T_i^2)) = c_i\mathbb{Z}$, for $i = 1, \ldots, l$. Thus the factor that appears in the definition of $\Delta_{M(\phi)}^{\alpha,2k+1}(t)$ (Definition 2.6) is also $\prod_{i=1}^l (t^{c_i} - 1)$. Finally the exponent $n-1$ is the number of factors in the Clebsch-Gordan formula. $\square$

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