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A fractional Michael–Simon Sobolev inequality on convex hypersurfaces

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Abstract. The classical Michael–Simon and Allard inequality is a Sobolev inequality for functions defined on a submanifold of Euclidean space. It is governed by a universal constant independent of the manifold, thanks to an additional L^p term on the right-hand side which is weighted by the mean curvature of the underlying manifold. We prove here a fractional version of this inequality on hypersurfaces of Euclidean space that are boundaries of convex sets. It involves the Gagliardo seminorm of the function, as well as its L^p norm weighted by the fractional mean curvature of the hypersurface.

As an application, we establish a new upper bound for the maximal time of existence in the smooth fractional mean curvature flow of a convex set. The bound depends on the perimeter of the initial set instead of on its diameter.

1. Introduction

The Michael–Simon and Allard inequality is a Sobolev inequality on submanifolds of Euclidean space which includes, on its right-hand side, an additional L^p integral weighted by a power of the submanifold's mean curvature norm. Remarkably, the presence of this extra geometric term enables the inequality to hold with a universal constant independent of the manifold. As a consequence, this classical result has important applications to the regularity of surfaces with prescribed mean curvature [6,20] and to the theory of geometric flows [26], among others.

In this article we establish a fractional version of the inequality on convex hypersurfaces of Euclidean space – that is, hypersurfaces which are the boundary of an open convex set. It involves the Gagliardo fractional seminorm of a function defined on the surface, as well as an additional L^p norm weighted now by a power of the nonlocal mean curvature. As for its classical counterpart, our inequality carries a universal constant. The validity of a similar inequality in nonconvex surfaces is still an open question.

Prior to this work, the only available fractional Michael–Simon and Allard inequality was established by the first two authors in [11] for functions defined on nonlocal minimal

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surfaces. It was conceived and used in [11] to derive a gradient estimate for nonlocal minimal graphs. Since nonlocal minimal surfaces are never convex (except for hyperplanes), the result of [11] and the one presented here complement each other.

As an application of the functional inequalities developed in the current paper, we obtain an upper bound on the maximal time of existence for the smooth fractional α-mean curvature flow of a convex set. The fractional mean curvature flow was introduced by Caffarelli and Souganidis [14] and by Imbert [31] in connection with diffusion phenomena with long range interactions. Similarly to the standard motion by mean curvature, bounded sets evolving according to this flow will become smaller after some time and ultimately disappear in finite time. A bound from above for the maximal time of existence of the smooth flow has been obtained in Sáez and Valdinoci [37, Corollary 7] by comparison with shrinking spheres. It reads

$$T^* \leqslant C \operatorname{diam}(\Omega_0)^{1+\alpha},$$

where $\operatorname{diam}(\Omega_0)$ is the diameter of the initial set Ω_0 and the constant C depends only on n and α.

Assuming the initial set to be convex, Chambolle, Novaga, and Ruffini [16] showed that convexity is preserved along the flow. By combining this fact with our fractional Michael–Simon-type inequalities, we are able to improve, in the case of smooth convex evolutions, the aforementioned result of [37] to an estimate involving the area of the initial surface. Specifically, we prove that if $\{\Omega_t\}_{t\geqslant 0}$ is a family of C^2 open subsets of $\mathbb{R}^{n+1}$ evolving by fractional α-mean curvature flow, with Ω_0 convex, then the maximal time of existence T^* satisfies

$$T^* \leqslant C|\partial\Omega_0|^{\frac{1+\alpha}{n}},$$

for some constant C depending only on n and α.

1.1. The classical Michael–Simon and Allard inequality

This inequality is an extension of the classical Sobolev inequality to m-dimensional submanifolds of $\mathbb{R}^{n+1}$. It was proved in the seventies independently by Allard [4] and by Michael and Simon [35] – the latter for a class of generalized submanifolds, the former in an even broader varifold setting. The following is the statement in the context of C^2 hypersurfaces $M \subset \mathbb{R}^{n+1}$. It makes no assumption on the topology of M, in particular whether it is compact or not. We denote the space of C^1 functions in M with compact support by $C^1_c(M)$, which agrees with $C^1(M)$ when M is compact.

Theorem 1.1 (Allard [4]; Michael and Simon [35]). *Let $n \geqslant 2$ be an integer, $p \in [1, n)$, and $M \subset \mathbb{R}^{n+1}$ a C^2 hypersurface. Then there exists a constant C depending only on n and p, such that*

$$\|u\|_{L^{p^*}(M)} \leqslant C\big(\|\nabla_M u\|_{L^p(M)} + \|Hu\|_{L^p(M)}\big) \quad \text{for all } u \in C^1_c(M), \tag{1.1}$$

where $p^* := np/(n-p)$, ∇_M is the tangential gradient on M, and H is the mean curvature of M.

We refer the reader to the recent paper [12] by Miraglio and the first author where, combining the ideas of [4,35], a quick and easy-to-read proof of Theorem 1.1 is provided.

Exactly as for the Euclidean Sobolev inequality, Theorem 1.1 can be deduced, using the coarea formula and Hölder's inequality, from the case when $p = 1$ and $u = \chi_E$ is the characteristic function of a sufficiently regular subset $E \subset M$. For these choices, inequality (1.1) is an isoperimetric one and reads

$$|E|^{\frac{n-1}{n}} \leqslant C\left(\mathrm{Per}_M(E) + \int_E |H(x)|\,dx\right), \tag{1.2}$$

where $|E|$ stands for the n-dimensional Hausdorff measure of E, dx indicates the restriction of such a measure to M, and $\mathrm{Per}_M(E)$ denotes the perimeter of E in M.

We emphasize that the constant C does not depend on M and that therefore all the information about the geometry of M is captured by its mean curvature H appearing on the right-hand side of (1.1). In particular, if M is a minimal surface, i.e., if $H = 0$, then estimate (1.1) holds true with only $\|\nabla_M u\|_{L^p(M)}$ appearing on its right-hand side, exactly as in the Euclidean case. Such a universal Sobolev inequality on minimal surfaces was first obtained by Bombieri, De Giorgi, and Miranda [5] – and consequently prior to that of Michael–Simon and Allard.

Determining the best constant in (1.2) remained an open question for many years, even when $H \equiv 0$. In a very recent paper, Brendle [8] has proved that, in every minimal surface, (1.2) holds true taking C to be the isoperimetric constant in $\mathbb{R}^n$. Moreover, equality is achieved only by flat n-dimensional balls. Brendle's argument is a far-reaching extension of the proof of the Euclidean isoperimetric inequality via the Aleksandrov–Bakelman–Pucci method found by the first author – see, e.g., [10].

Another interesting class of hypersurfaces are those that are compact (with no boundary). In this case, one can plug $u \equiv 1$ into (1.1). This leads to an estimate from below for the integral of the modulus of the mean curvature of M in terms of the measure of M:

$$|M|^{\frac{n-1}{n}} \leqslant C \int_M |H(x)|\,dx. \tag{1.3}$$

For a convex hypersurface M – that is, when $M = \partial\Omega$ is the boundary of a convex subset Ω of $\mathbb{R}^{n+1}$ – estimate (1.3) is a particular case of the Aleksandrov–Fenchel inequalities. In this convex case, it is known to hold with the optimal constant – which is achieved by all spheres $M = \partial B_R(x)$. However, it is still an open problem to determine the optimal constant for general compact hypersurfaces. See [1,2] and also Chang and Wang [17] for a recent survey on this topic.

1.2. A fractional Michael–Simon and Allard inequality on convex hypersurfaces

It is a well-known fact that an appropriate Sobolev embedding holds for Sobolev spaces of fractional order in Euclidean space. Indeed, for every $s \in (0,1)$, every integer $n \geqslant 1$, and every $p \in [1, n/s)$, there exists a constant C depending only on n, p, and s, such that

$$\|u\|_{L^{p_s^*}(\mathbb{R}^n)} \leqslant C[u]_{W^{s,p}(\mathbb{R}^n)} \quad \text{for all } u \in W^{s,p}(\mathbb{R}^n). \tag{1.4}$$

Here $W^{s,p}(\mathbb{R}^n)$ is the fractional Sobolev space of functions $u \in L^p(\mathbb{R}^n)$ for which the Gagliardo seminorm

$$[u]_{W^{s,p}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}} \tag{1.5}$$

is finite and

$$p_s^* := \frac{np}{n-sp}$$

is the relevant critical Sobolev exponent.

Historically, fractional Sobolev spaces were introduced to measure the smoothness of functions defined on *curved* hypersurfaces of Euclidean spaces, with special interest in boundaries of bounded open Lipschitz sets $\Omega \subset \mathbb{R}^{n+1}$. Indeed, Aronszajn [3], Slobodeckiĭ and Babič [40], and Gagliardo [28] showed that, for $p > 1$, the trace space of $W^{1,p}(\Omega)$ is $W^{(p-1)/p,p}(\partial\Omega)$. The fractional Sobolev space $W^{s,p}(M)$ on a hypersurface $M \subset \mathbb{R}^{n+1}$ can be defined, similarly to the Euclidean case, as the collection of $L^p(M)$ functions having finite seminorm $[\,\cdot\,]_{W^{s,p}(M)}$. This seminorm is defined as in (1.5) by replacing the domain of integration $\mathbb{R}^n$ with M, writing dx to mean integration with respect to the n-dimensional Hausdorff measure, and understanding $|x-y|$ to be the standard Euclidean distance in $\mathbb{R}^{n+1}$:

$$[u]_{W^{s,p}(M)} := \left(\int_M \int_M \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}}.$$

In the current paper we study the existence of a version of the Michael–Simon and Allard inequality for fractional Sobolev spaces on hypersurfaces of Euclidean space. Our interest originates from the theory of nonlocal minimal surfaces. Given $\alpha \in (0,1)$, nonlocal α-minimal surfaces are defined as being (the boundaries of) the critical points of the fractional α-perimeter functional

$$\mathbb{R}^{n+1} \supset \Omega \mapsto \operatorname{Per}_\alpha(\Omega) := \frac{1}{2}[\chi_\Omega]_{W^{\alpha,1}(\mathbb{R}^{n+1})} = \int_\Omega \int_{\mathbb{R}^{n+1}\setminus\Omega} \frac{dx\, dy}{|x-y|^{n+1+\alpha}}. \tag{1.6}$$

They were introduced by Caffarelli, Roquejoffre, and Savin in [13], and are related to phase-transition models with strongly nonlocal interactions. Such critical points are characterized by the equation

$$H_\alpha[\Omega] = 0 \quad \text{on } \partial\Omega,$$

where

$$H_\alpha(x) = H_\alpha[\Omega](x) := \frac{\alpha}{2} \text{P.V.} \int_{\mathbb{R}^{n+1}} \frac{\chi_{\mathbb{R}^{n+1}\setminus\Omega}(y) - \chi_\Omega(y)}{|y-x|^{n+1+\alpha}} \, dy \tag{1.7}$$

$$= \text{P.V.} \int_{\partial\Omega} \frac{(y-x)\cdot\nu(y)}{|y-x|^{n+1+\alpha}} \, dy \quad \text{for } x \in \partial\Omega \tag{1.8}$$

is the so-called nonlocal (or fractional) α-mean curvature of Ω at the point $x \in \partial\Omega$ and ν denotes the exterior unit normal vector to $\partial\Omega$. Note that the last equality follows from the divergence theorem. It is known that these surfaces satisfy a density estimate

$$|M \cap B_R(x_0)| \geqslant c_* R^n \quad \text{for all } x_0 \in M := \partial\Omega \text{ and } R > 0, \tag{1.9}$$

as in the case of standard minimal surfaces. Here, the positive constant c_* depends only on n and α.

It was the study of nonlocal α-minimal surfaces that led to the first result on a fractional Michael–Simon inequality, obtained by the first two authors. In [11] we obtained the following new universal fractional Sobolev inequality on nonlocal α-minimal surfaces, as well as on classical minimal surfaces. We established it by extending a beautiful proof of the fractional Sobolev inequality in Euclidean space due to Brezis [9]. In [11], this result played a central role in the proof of a gradient estimate for nonlocal minimal graphs.

Theorem 1.2 (Cabré and Cozzi [11]). *Let $n \geqslant 1$ be an integer, $s \in (0,1)$, and $p \geqslant 1$ be such that $n > sp$. Let $M \subset \mathbb{R}^{n+1}$ be either a nonlocal α-minimal surface or a classical minimal surface – more generally, it suffices to assume that $M \subset \mathbb{R}^{n+1}$ is a set with locally finite n-dimensional Hausdorff measure that satisfies* (1.9) *for some positive constant c_*.*

Then there exists a constant C depending only on n, s, p, and c_*, such that

$$\|u\|_{L^{p_s^*}(M)} \leqslant C[u]_{W^{s,p}(M)} \quad \textit{for all } u \in W^{s,p}(M). \tag{1.10}$$

Also recently, and independently from [11], inequality (1.10) has been obtained by Dyda et al. [21] as part of a more general family of Hardy–Sobolev-type inequalities for weighted fractional Sobolev spaces defined on metric measure spaces – see [21, Theorem 5.3]. However, when restricted to a hypersurface M of Euclidean space, their inequalities hold under stronger assumptions than the density estimate (1.9) – namely, a connectivity-type hypothesis on M and the validity of quantitative doubling and reverse doubling conditions on the n-dimensional Hausdorff measure restricted to M, in addition to (1.9).

In light of Theorem 1.2 and the classical Michael–Simon inequality, it is conceivable that (1.10) could be extended to general hypersurfaces by including an additional remainder L^p-term involving the nonlocal mean curvature. The following result – which is the main contribution of our paper – shows that this is indeed the case for convex hypersurfaces. The question remains open in the nonconvex case.

In order to state our theorem, note first that if $\Omega \subset \mathbb{R}^{n+1}$ is an open convex set, then $\partial\Omega$ is a Lipschitz hypersurface and thus, by Rademacher's theorem, differentiable at almost every point $x \in \partial\Omega$. On the other hand, by either expression (1.7) or (1.8), we see that $H_\alpha(x)$ is a well-defined quantity in $[0, +\infty]$, since Ω is convex. Furthermore, by Aleksandrov's theorem, $\partial\Omega$ is (pointwise) twice differentiable at almost every $x \in \partial\Omega$. At these points, the nonlocal mean curvature $H_\alpha(x)$ is finite.

Note also that every bounded open convex set $\Omega \subset \mathbb{R}^{n+1}$ has finite perimeter, that is, $|\partial\Omega| < +\infty$. This follows from the classical isodiametric inequality for the perimeter of convex sets, stated in Proposition 3.6 and proved in Appendix A. This fact will be important within some of our proofs, to avoid the indetermination $0 \cdot \infty$.

We also need to define the nonlocal α-perimeter functional on hypersurfaces, as the natural generalization of the Euclidean nonlocal perimeter functional (1.6). Given a hypersurface $M \subset \mathbb{R}^{n+1}$ and a subset $F \subset M$, for $s \in (0,1)$ we define

$$\operatorname{Per}_{M,s}(F) := \int_F \int_{M \setminus F} \frac{dx\, dy}{|x-y|^{n+s}}.$$

Theorem 1.3. *Let $n \geqslant 1$ be an integer, $\alpha \in (0,1)$, $s \in (0,1)$, and $p \geqslant 1$ be such that $n > sp$. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open convex set. Then there exists a constant C depending only on n, α, s, and p, such that*

$$\|u\|_{L^{p_s^*}(\partial\Omega)} \leqslant C\left(\frac{1}{2}\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy + \int_{\partial\Omega} H_\alpha(x)^{\frac{sp}{\alpha}} |u(x)|^p\, dx\right)^{\frac{1}{p}} \tag{1.11}$$

holds true for every $u \in W^{s,p}(\partial\Omega)$, where $p_s^* = np/(n-sp)$.

As a consequence, taking $p = 1$ and u to be the characteristic function of a set E, we have

$$|E|^{\frac{n-s}{n}} \leqslant C\left(\operatorname{Per}_{\partial\Omega,s}(E) + \int_E H_\alpha(x)^{\frac{s}{\alpha}}\, dx\right) \tag{1.12}$$

for every measurable subset $E \subset \partial\Omega$ with finite measure, where C is a constant depending only on n, α, and s. In particular, if $\partial\Omega$ has finite measure, the choice $E = \partial\Omega$ leads to

$$|\partial\Omega|^{\frac{n-s}{n}} \leqslant C \int_{\partial\Omega} H_\alpha(x)^{\frac{s}{\alpha}}\, dx. \tag{1.13}$$

Note that no relation between the parameters s and α is assumed within the theorem.

Inequality (1.13) is a fractional extension of the classical Aleksandrov–Fenchel-type inequality (1.3). In (1.13), it is still unknown what the best constant is. In addition, its validity in nonconvex surfaces – after replacing H_α by $|H_\alpha|$ – remains an open question.

Observe that neither is Theorem 1.2 a particular case of Theorem 1.3 (since the former makes no convexity assumption, and thus includes classical and nonlocal minimal surfaces), nor can one deduce the latter from the former. Indeed, Theorem 1.3 holds not only for unbounded convex sets but also for bounded ones, and in this last case, the density estimate (1.9) cannot hold because $|\partial\Omega \cap B_R(x_0)| = |\partial\Omega|$ for all sufficiently large balls $B_R(x_0)$. Note also that, by convexity, the fractional mean curvature H_α is positive at all points of $\partial\Omega$ except when $\partial\Omega$ is a hyperplane.

1.3. An application to the fractional mean curvature flow of convex sets

In Section 5 we give an application of our results to get a new bound on the maximal time of existence for the smooth fractional mean curvature flow of convex hypersurfaces.

For this, we will use the pointwise inequality (1.17) reported below – which is a key tool within the proof of our main result, Theorem 1.3 – as well as the classical Michael–Simon inequality.

Without entering into regularity issues, a family of open sets $\{\Omega_t\}_{t\geqslant 0}$ evolves by fractional α-mean curvature if the inner normal velocity at a point $x \in \partial\Omega_t$ is equal to the fractional α-mean curvature of Ω_t at x. This flow has been investigated recently in several works. The existence and uniqueness of viscosity solutions to the generalized level set flow was obtained by Imbert [31]. Julin and La Manna [32] established that, if the initial set Ω_0 is bounded and of class C^2, then Ω_t is smooth for sufficiently small times t. Thus, the time

$$T^* := \sup\big\{t > 0 : \Omega_\tau \text{ is nonempty and has } C^2 \text{ boundary for all } \tau \in [0, t)\big\} \tag{1.14}$$

is positive. The sets Ω_t will become empty in finite time, possibly developing singularities prior to extinction. Indeed, Sáez and Valdinoci [37, Corollary 7] have shown that

$$T^* \leqslant C \operatorname{diam}(\Omega_0)^{1+\alpha} \tag{1.15}$$

for some constant C depending only on n and α, whereas an example in which singularities arise before extinction for a nonconvex initial datum has been produced by Cinti, Sinestrari, and Valdinoci [18].[1]

When Ω_0 is convex, then each Ω_t is convex as well, as shown by Chambolle, Novaga, and Ruffini [16]. Thanks to this observation, by combining the pointwise nonlocal estimate (1.17) with the classical Michael–Simon inequality of Theorem 1.1, we establish the following result.

Theorem 1.4. *Let $n \geqslant 1$, $\alpha \in (0,1)$, and $\Omega_0 \subset \mathbb{R}^{n+1}$ be a bounded open convex set with C^2 boundary. Let $\{\partial\Omega_t\}$ be the flow of hypersurfaces moving by fractional mean curvature H_α. Then the maximal time T^* defined by (1.14) satisfies*

$$T^* \leqslant C|\partial\Omega_0|^{\frac{1+\alpha}{n}}, \tag{1.16}$$

for some constant C depending only on n and α.

The corresponding estimate for the classical mean curvature flow (where $\alpha = 1$) was established by Evans and Spruck [26] – see Evans [24, Section F.2] for a simpler proof

[1] We stress that T^* is the maximal time for which the fractional α-mean curvature flow originating from a C^2 convex set Ω_0 remains C^2. Due to the possible formation of singularities, this time might be in principle smaller than the extinction time $T_e := \sup\{t > 0 : \Omega_t \text{ is nonempty}\}$ of the generalized level set flow considered, e.g., in [16, 31]. Since, for Ω_0 convex, this possibility has not been ruled out at the current time (in contrast with the case of convex sets evolving by classical mean curvature flow [26, 30]), we keep this distinction. We also point out that, via results and ideas from [15, 31], the upper bound on T^* provided by [37] can actually be improved to an estimate on the extinction time T_e, regardless of the convexity of the initial datum.

in the case of a smooth flow. Both arguments make crucial use of the Michael–Simon inequality.

We stress that Theorem 1.4 assumes $\partial\Omega_t$ to be a C^2 hypersurface for all $t \in [0, T^*)$. Hence, our result must be understood as an estimate for the maximal time of existence of the C^2 flow, and not as a bound on the true extinction time T_e. As commented in footnote 1, it is still not known whether for a convex C^2 initial surface $\partial\Omega_0$ the flow remains C^2 for all times prior to extinction and there is no formation of singularities, such as, for instance, corners or edges of a polytope.

Note that Theorem 1.4 improves estimate (1.15) from [37] (when restricted to convex evolutions) in the dependence on Ω_0. Indeed, any bounded convex set Ω_0 satisfies the nontrivial inequality $|\partial\Omega_0| \leqslant C(n)\,\mathrm{diam}(\Omega_0)^n$ – see Proposition 3.6 below for its sharp version, in which $C(n) = 2^{-n}|\partial B_1| = 2^{-n}|\mathbb{S}^n|$. On the other hand, for $n \geqslant 2$ one can produce examples of convex sets with diameter equal to 1 and arbitrarily small surface area – e.g., shrinking tubular neighborhoods of a segment.

1.4. Sketch of the proof of Theorem 1.3

Our analysis stems from the following observation. If $\Omega \subset \mathbb{R}^{n+1}$ is an open convex set, then

$$|\partial\Omega|^{-\frac{\alpha}{n}} \leqslant C H_\alpha(x) \quad \text{for a.e. } x \in \partial\Omega, \tag{1.17}$$

for some universal constant C depending only on n and α. This pointwise inequality for convex sets cannot hold for general domains – after replacing H_α by $|H_\alpha|$. Indeed, one can easily construct a smooth bounded domain Ω with $0 \in \partial\Omega$ and $H_\alpha(0) = 0$; we will then have $|H_\alpha| \leqslant \varepsilon$ in a set of positive measure (a small neighborhood of 0 on $\partial\Omega$), for every $\varepsilon > 0$. It also has no counterpart in the local setting, since $\partial\Omega$ may have flat parts where the standard mean curvature vanishes. The proof of (1.17) will be rather simple but, in any case, at the end of this section we will discuss how we originally found it in the plane, that is, when $\Omega \subset \mathbb{R}^2$.

The next step in proving the main theorem is to consider subsets $E \subset \partial\Omega$ and a dichotomy argument. We will distinguish, vaguely speaking, between two situations: either $\partial\Omega$ has, at some well-chosen scales depending on $|E|$, small density around x, or it does not. In the former case of points x of low density – occurring, say, where $\partial\Omega$ has a tentacle-like shape – the proof of (1.17) still carries through and one obtains

$$|E|^{-\frac{\alpha}{n}} \leqslant C H_\alpha(x)$$

at such points x. In the latter case when x has high density, we take advantage of the other term in the right-hand side of our fractional Michael–Simon inequality (in this exposition we take $s = \alpha$ to simplify) and prove that

$$|E|^{-\frac{\alpha}{n}} \leqslant C \int_{\partial\Omega\setminus E} \frac{dy}{|x-y|^{n+\alpha}}.$$

This second case is what happens for nonlocal minimal surfaces (Theorem 1.2), where every point has high density. Either way, the following pointwise inequality will hold true:

$$|E|^{-\frac{\alpha}{n}} \leqslant C\left(\int_{\partial\Omega\setminus E} \frac{dy}{|x-y|^{n+\alpha}} + H_\alpha(x)\right) \quad \text{for every } E \subset \partial\Omega \text{ and a.e. } x \in E. \tag{1.18}$$

Inequality (1.18) – see Proposition 3.1 – is the key step towards the main theorem. It turns out to be the nonflat version of the pointwise estimate established in $\mathbb{R}^n$ by Savin and Valdinoci [39, Appendix A]. Their estimate states that

$$|E|^{-\frac{\alpha}{n}} \leqslant C \int_{\mathbb{R}^n\setminus E} \frac{dy}{|x-y|^{n+\alpha}} \quad \text{for all } x \in \mathbb{R}^n \text{ and } E \subset \mathbb{R}^n, \tag{1.19}$$

for some constant C depending only on n and α. This is a rearrangement inequality that follows immediately from the observation that integrating over the complement of the ball $B_\rho(x)$, with $|B_\rho| = |E|$, instead of $\mathbb{R}^n \setminus E$ does not increase the right-hand side of (1.19).

Integrating (1.18) over E, we are led to the fractional isoperimetric inequality

$$|E|^{\frac{n-\alpha}{n}} \leqslant C\left(\mathrm{Per}_{\partial\Omega,\alpha}(E) + \int_E H_\alpha(x)\,dx\right).$$

This is our fractional Sobolev inequality (1.11) for $s=\alpha$, $p=1$, and characteristic functions – i.e., inequality (1.12). To extend it to any $p \geqslant 1$ and arbitrary functions we follow the strategy devised by Di Nezza, Palatucci, and Valdinoci in [19, Section 6] to deduce the Euclidean fractional Sobolev inequality (1.4) from the pointwise inequality (1.19). As we will see later, when $p=1$ a fractional Sobolev inequality can be established more easily using the corresponding fractional isoperimetric inequality in combination with the fractional coarea formula of Visintin [41] – see Lemma 4.1 below. This is true both in the Euclidean framework and in the context of hypersurfaces. However, to the best of our knowledge, it is not known whether one can then derive the fractional Sobolev inequality for $p>1$ from the case $p=1$ (even in the Euclidean case), in contrast with the case of Sobolev inequalities of integer order.

Finally, the following is an elementary proof of the pointwise lower bound (1.17) on the nonlocal mean curvature for bounded and strictly convex sets of $\mathbb{R}^2$. This was the starting point of our work. Up to a rigid movement, we may assume that $x=0\in\partial\Omega$ and that $\Omega\subset\mathbb{R}^2_+ = \{y\in\mathbb{R}^2 : y_2>0\}$. As Ω is strictly convex, it can be parametrized by a function $y:[0,\pi]\to\partial\Omega\subset\mathbb{R}^2$ of the form $y(\theta)=r(\theta)(\cos\theta,\sin\theta)$, with $r>0$ in $(0,\pi)$ and $r(0)=r(\pi)=0$. In this parametrization, for $y=y(\theta)$ and $r=r(\theta)$, it holds that

$$y\cdot\nu(y) = \frac{r^2}{\sqrt{r^2+\dot r^2}} \quad \text{and} \quad \frac{y\cdot\nu(y)}{|y|^{2+\alpha}} = \frac{1}{r^\alpha\sqrt{r^2+\dot r^2}}. \tag{1.20}$$

From the fact that $r/\sqrt{r^2+\dot r^2}\leqslant 1$ one obtains

$$\pi = \int_0^\pi d\theta \leqslant \int_0^\pi \left(\frac{r}{\sqrt{r^2+\dot r^2}}\right)^{-\frac{\alpha}{1+\alpha}} d\theta$$

$$= \int_0^\pi r^{-\frac{\alpha}{1+\alpha}} \left(\sqrt{r^2+\dot{r}^2}\right)^{\frac{\alpha}{1+\alpha}} d\theta$$
$$\leqslant \left(\int_0^\pi \frac{d\theta}{r^\alpha}\right)^{\frac{1}{1+\alpha}} \left(\int_0^\pi \sqrt{r^2+\dot{r}^2}\, d\theta\right)^{\frac{\alpha}{1+\alpha}}. \tag{1.21}$$

From representation (1.8) for the fractional α-mean curvature and the second identity in (1.20) it follows that $\int_0^\pi r^{-\alpha}\, d\theta = H_\alpha(0)$. Hence we proved that $\pi^{1+\alpha} \leqslant H_\alpha(0)|\partial\Omega|^\alpha$, which is precisely (1.17) for $n = 1$.

1.5. Plan of the paper

We shall prove Theorem 1.3 in increasing order of generality, using in each section the previous less general results or the main ingredients of their proofs.

In Section 2 we prove the pointwise lower bound (1.17) for H_α, as well as its integral consequence (1.13). This is the last statement of Theorem 1.3.

In Section 3 we extend the pointwise inequality (1.17) to proper subsets E of $\partial\Omega$ – i.e., we prove the pointwise lower bound (1.18).

In Section 4 we deduce Theorem 1.3 in its full generality from the pointwise lower bound (1.18).

In Section 5 we apply the pointwise inequality (1.17) to the fractional mean curvature flow and establish Theorem 1.4.

In Appendix A we provide a simple proof of a known isodiametric inequality for the perimeter of convex sets.

1.6. Notation

Throughout the paper, the word *measurable* refers to the n-dimensional Hausdorff measure $\mathcal{H}^n$ on a hypersurface M of $\mathbb{R}^{n+1}$, if not stated explicitly otherwise. The measure of a set $E \subset M$ will be denoted by $|E|$ and the integration element simply by dx, instead of $d\mathcal{H}^n(x)$. Open balls are understood as balls in the ambient space $\mathbb{R}^{n+1}$, i.e., $B_R(x) = \{y \in \mathbb{R}^{n+1} : |y - x| < R\}$ and $|y - x|$ is the Euclidean distance in $\mathbb{R}^{n+1}$. If $x = 0$ we write $B_R = B_R(0)$, while $\mathbb{S}^n = \partial B_1$ is the n-dimensional unit sphere in $\mathbb{R}^{n+1}$.

2. A lower bound on the nonlocal mean curvature

We start by proving the last bound (1.13) of Theorem 1.3, which is the simplest statement within the theorem. It will follow from the pointwise inequality

$$|\partial\Omega|^{-\frac{\alpha}{n}} \leqslant C H_\alpha(x) \quad \text{for a.e. } x \in \partial\Omega, \text{ where } C = \left(\frac{2}{|\mathbb{S}^n|}\right)^{\frac{n+\alpha}{n}}. \tag{2.1}$$

Here, $\alpha \in (0, 1)$ and Ω is any bounded open convex set of $\mathbb{R}^{n+1}$.

The proof of (2.1) relies on the following two simple lemmas. The first one extends the first identity in (1.21) to higher dimensions. It is a well-known result in the theory of double layer potentials (it is sometimes called Gauss's law) and does not require convexity; see, e.g., [27, Proposition 3.19]. Later we will use the lemma with $\Omega_b = \Omega \cap B_R(x)$ for some radius R, where Ω is our convex set and $x \in \partial\Omega$.

Lemma 2.1. *Let $\Omega_b \subset \mathbb{R}^{n+1}$ be a bounded domain with Lipschitz boundary. Then*

$$\text{P.V.} \int_{\partial\Omega_b} \frac{(y-x)\cdot\nu(y)}{|y-x|^{n+1}}\,dy = \frac{|\mathbb{S}^n|}{2}$$

holds true in the principal value sense at every point $x \in \partial\Omega_b$ at which $\partial\Omega_b$ is differentiable.

Proof. Let Φ be the fundamental solution of the Laplacian centered at x, i.e., $-\Delta\Phi = \delta_x$ in $\mathbb{R}^{n+1}$. We have that $\nabla\Phi(y) = -|\mathbb{S}^n|^{-1}|y-x|^{-n-1}(y-x)$ for every $y \neq x$. Let $\varepsilon > 0$ be sufficiently small. By applying the divergence theorem in $\Omega_b \setminus \bar{B}_\varepsilon(x)$, a Lipschitz domain, we get

$$\begin{aligned} 0 &= \int_{\Omega_b\setminus\bar{B}_\varepsilon(x)} \Delta\Phi(y)\,dy = \int_{\partial(\Omega_b\setminus\bar{B}_\varepsilon(x))} \nabla\Phi(y)\cdot\nu(y)\,dy \\ &= \frac{1}{|\mathbb{S}^n|}\left\{-\int_{\partial\Omega_b\setminus\bar{B}_\varepsilon(x)} \frac{(y-x)\cdot\nu(y)}{|y-x|^{n+1}}\,dy + \frac{|\Omega_b\cap\partial B_\varepsilon(x)|}{\varepsilon^n}\right\}. \end{aligned}$$

The claim follows by letting $\varepsilon \to 0^+$ and noticing that $\varepsilon^{-n}|\Omega_b \cap \partial B_\varepsilon(x)| \to |\mathbb{S}^n|/2$, since $\partial\Omega$ is differentiable at x. This shows in particular that the principal value in the statement exists. ∎

The second lemma is an extension of the inequalities in (1.21) to any dimension $n \geqslant 1$. For the proof of (2.1) we will only need the next lemma for $E = \partial\Omega$, but in Section 3 we will require the estimate for general subsets $E \subset \partial\Omega$. Here it is useful to recall the comments made before Theorem 1.3 on the differentiability properties of open convex sets Ω and the definition (1.8) of $H_\alpha(x)$ for $x \in \partial\Omega$.

Lemma 2.2. *Let $\alpha \in (0,1)$ and $\Omega \subset \mathbb{R}^{n+1}$ be an open convex set. Then*

$$\int_E \frac{(y-x)\cdot\nu(y)}{|y-x|^{n+1}}\,dy \leqslant |E|^{\frac{\alpha}{n+\alpha}} H_\alpha(x)^{\frac{n}{n+\alpha}}$$

for every measurable subset $E \subset \partial\Omega$ and almost every point $x \in \partial\Omega$. Here, the integral is well defined in $[0, +\infty]$ since its integrand is a nonnegative function.

Proof. Using that $0 \leqslant (y-x)\cdot\nu(y)/|y-x| \leqslant 1$ for almost every x and y on $\partial\Omega$, together with Hölder's inequality, we see that

$$\begin{aligned} \int_E \frac{(y-x)\cdot\nu(y)}{|y-x|^{n+1}}\,dy &= \int_E \frac{(y-x)\cdot\nu(y)}{|y-x|}\,\frac{dy}{|y-x|^n} \\ &\leqslant \int_E \left(\frac{(y-x)\cdot\nu(y)}{|y-x|}\right)^{\frac{n}{n+\alpha}} \frac{dy}{|y-x|^n} \end{aligned}$$

$$
\begin{aligned}
&= \int_E \left(\frac{(y-x)\cdot \nu(y)}{|y-x|^{n+1+\alpha}} \right)^{\frac{n}{n+\alpha}} dy \\
&\leqslant |E|^{\frac{\alpha}{n+\alpha}} \left(\int_E \frac{(y-x)\cdot \nu(y)}{|y-x|^{n+1+\alpha}}\, dy \right)^{\frac{n}{n+\alpha}}.
\end{aligned}
$$

The claim follows from this inequality and expression (1.8) for $H_\alpha(x)$, combined with the fact that $(y-x)\cdot\nu(y) \geqslant 0$ for almost every x and y on $\partial\Omega$ since Ω is convex. ■

Proofs of inequalities (2.1) *and* (1.13). Since Lemma 2.1 requires the domain to be bounded, while (1.13) is claimed for convex sets with finite perimeter, we first point out that an open convex set is bounded if and only if it has finite perimeter.[2]

Now, from Lemmas 2.1 and 2.2 applied with $\Omega_b = \Omega$ and $E = \partial\Omega$, respectively, we obtain (2.1). By raising (2.1) to the power s/α and integrating over $\partial\Omega$, we infer the validity of (1.13). ■

3. A fractional Michael–Simon-type isoperimetric inequality

In this section we shall prove the key pointwise inequality involved in the proof of Theorem 1.3. This is the content of the following proposition.

Proposition 3.1. *Let $\alpha \in (0,1)$, $s \in (0,1)$, $p > 0$, and $\Omega \subset \mathbb{R}^{n+1}$ be an open convex set. Then, for every $E \subset \partial\Omega$ with finite positive measure and a.e. $x \in E$, it holds that*

$$
|E|^{-\frac{sp}{n}} \leqslant C \left(\int_{\partial\Omega\setminus E} \frac{dy}{|x-y|^{n+sp}} + H_\alpha(x)^{\frac{sp}{\alpha}} \right) \tag{3.1}
$$

for some constant C depending only on n, α, s, and p.

The proof of this result relies on the following ingredients:

(i) A classical rearrangement result, Lemma 3.2, which reduces the proof of (3.1) for a general set E to the case $E = \partial\Omega \cap B_R(x)$.

(ii) The double layer potential identity of Lemma 2.1, and the localized pointwise inequality that will follow from it, Lemma 3.4.

(iii) A dichotomy argument. We will essentially distinguish between two cases, depending on whether $R^{-n}|\partial\Omega \cap B_R(x)|$ is smaller or larger than a certain constant for some appropriate radii R which depend on $|E|$.

[2]This follows from the monotonicity of the perimeter of open convex sets with respect to inclusion (see Appendix A for the proof of this classical result). Using this fact, one part of the claim is obvious, while the other is checked as follows. Any unbounded open convex set contains a ball, and hence also the convex cones generated by a vertex going to infinity and the ball. Note finally that such convex cones have arbitrarily large perimeter.

(iv) A kind of reverse perimeter-energy estimate for convex sets, Lemma 3.5 (*reverse* here is meant in comparison with the natural upper bound on the perimeter that holds for minimizing minimal surfaces).

(v) A known isodiametric inequality for the perimeter of convex sets, Proposition 3.6.

We start with (i) – the rearrangement result – which is the content of the next lemma. From it, it will be enough to prove Proposition 3.1 for sets of the form $E = \partial\Omega \cap B_R(x)$. Indeed, the lemma states that replacing E by $\partial\Omega \cap B_R(x)$, with $|E| = |\partial\Omega \cap B_R(x)|$, does not increase the right-hand side of (3.1). This elementary observation does not require any convexity assumption on Ω, nor that the hypersurface $\partial\Omega$ is a boundary in the first place. In addition, all that is needed for the exponent $n + sp$ appearing in the statement is to be larger than n. This is why we allow $p > 0$ in the lemma – though we will use it always with $p \geqslant 1$.

Lemma 3.2. *Let $s \in (0, 1)$, $p \in (0, +\infty)$, and $M \subset \mathbb{R}^{n+1}$ be a set of locally finite n-dimensional Hausdorff measure. Let $E \subset M$ be a set with positive measure and $x \in E$. Assume that*

$$|E| = |M \cap B_R(x)| \tag{3.2}$$

for some $R > 0$.

Then

$$\int_{M\setminus E} \frac{dy}{|x-y|^{n+sp}} \geqslant \int_{M\setminus B_R(x)} \frac{dy}{|x-y|^{n+sp}}.$$

Proof. By assumption (3.2) we have

$$|E \cap B_R(x)| + |(M \setminus E) \cap B_R(x)| = |E \cap B_R(x)| + |E \setminus B_R(x)|,$$

and thus

$$|(M \setminus E) \cap B_R(x)| = |E \setminus B_R(x)|.$$

By using this identity we see that

$$\begin{aligned}
\int_{M\setminus E} \frac{dy}{|y-x|^{n+sp}} &= \int_{(M\setminus E)\cap B_R(x)} \frac{dy}{|y-x|^{n+sp}} + \int_{(M\setminus E)\setminus B_R(x)} \frac{dy}{|y-x|^{n+sp}} \\
&\geqslant R^{-n-sp}|(M \setminus E) \cap B_R(x)| + \int_{(M\setminus E)\setminus B_R(x)} \frac{dy}{|y-x|^{n+sp}} \\
&= R^{-n-sp}|E \setminus B_R(x)| + \int_{(M\setminus E)\setminus B_R(x)} \frac{dy}{|y-x|^{n+sp}} \\
&\geqslant \int_{E\setminus B_R(x)} \frac{dy}{|y-x|^{n+sp}} + \int_{(M\setminus E)\setminus B_R(x)} \frac{dy}{|y-x|^{n+sp}} \\
&= \int_{M\setminus B_R(x)} \frac{dy}{|y-x|^{n+sp}}.
\end{aligned}$$

This proves the lemma. ∎

The next lemma is of a technical nature and we will use it twice. Within the proof of Proposition 3.1 it will guarantee that, under appropriate assumptions on M and for almost every $x \in E \subset M$, hypothesis (3.2) is actually satisfied for some radius R depending on x – a property that may not be satisfied by all $x \in E$, as we will see.

Lemma 3.3. *Let $M \subset \mathbb{R}^{n+1}$ be a set of locally finite n-dimensional Hausdorff measure. Then the following statements hold true.*

(a) *The set*

$$D := \left\{ x \in M : \text{there exists } R_x > 0 \text{ such that } |M \cap \partial B_{R_x}(x)| > 0 \right\}$$

is at most countable.

(b) *For every $x \in M$, the function*

$$\mathcal{A}_x \colon (0, +\infty) \to [0, +\infty), \text{ defined by } \mathcal{A}_x(R) := |M \cap B_R(x)| \text{ for } R > 0,$$

is nondecreasing and continuous from the left. Furthermore, it is continuous if and only if $x \in M \setminus D$.

Proof. Consider, for $j, k \in \mathbb{N}$, the sets $M_j := M \cap B_j$ and

$$D_{j,k} := \left\{ x \in M_j : \text{there exists } R_x > 0 \text{ such that } |M_j \cap \partial B_{R_x}(x)| \geqslant \tfrac{1}{k} \right\}.$$

It is clear that $D = \bigcup_{j,k \in \mathbb{N}} D_{j,k}$. Note that, if x and y are two distinct points in $D_{j,k}$, then $|(M_j \cap \partial B_{R_x}(x)) \cup (M_j \cap \partial B_{R_y}(y))| = 0$. Moreover, as M has locally finite n-dimensional measure, we have that $|M_j| < +\infty$. From the last two facts we deduce that each $D_{j,k}$ contains no more than $k|M_j|$ points. Hence, D is at most countable and (a) is proved.

We now address point (b). The monotonicity of the function $\mathcal{A}_x$ is obvious, while its left-continuity follows from $B_R(x)$ being open. The last statement is a consequence of the fact that $|M \cap \partial B_R(x)| = \lim_{\rho \to R^+} \mathcal{A}_x(\rho) - \mathcal{A}_x(R)$ for every $x \in M$ and $R > 0$. We stress that for the last two claims we took advantage of the $\mathcal{H}^n$-measurability of M and of standard formulas for the measure of increasing unions and decreasing intersections of sets. ∎

In the following result we apply the double layer potential identity of Lemma 2.1 with $\Omega_b = \Omega \cap B_R(x)$. This allows us to obtain a localized version of Lemma 2.2.

Lemma 3.4. *Let $\alpha \in (0, 1)$ and $\Omega \subset \mathbb{R}^{n+1}$ be an open convex set. Then*

$$\begin{aligned} \frac{|\mathbb{S}^n|}{2} &= \int_{\partial\Omega \cap B_R(x)} \frac{(y - x) \cdot \nu(y)}{|y - x|^{n+1}} \, dy + \frac{|\Omega \cap \partial B_R(x)|}{R^n} \\ &\leqslant |\partial\Omega \cap B_R(x)|^{\frac{\alpha}{n+\alpha}} H_\alpha(x)^{\frac{n}{n+\alpha}} + \frac{|\Omega \cap \partial B_R(x)|}{R^n} \end{aligned}$$

for every $R > 0$ and almost every $x \in \partial\Omega$.

Proof. First, recall that, as Ω is convex, its boundary is Lipschitz and has therefore locally finite n-dimensional Hausdorff measure. Hence, we may apply Lemma 3.3(a) and deduce that, for all but a countable number of points $x \in \partial\Omega$, it holds that $|\partial\Omega \cap \partial B_R(x)| = 0$ for every $R > 0$. Moreover, $\partial\Omega$ is differentiable at almost all such points.

Consider the convex set $\Omega_b = \Omega \cap B_R(x)$. Its boundary $\partial\Omega_b$ is therefore Lipschitz and, in addition, it is equal, up to a set of measure zero, to the disjoint union of the two sets $\partial\Omega \cap B_R(x)$ and $\Omega \cap \partial B_R(x)$ – we used here the fact, noted earlier, that $|\partial\Omega \cap \partial B_R(x)| = 0$. Applying the double layer potential identity of Lemma 2.1, we get

$$\begin{aligned}\frac{|\mathbb{S}^n|}{2} &= \int_{\partial\Omega_b} \frac{(y-x)\cdot\nu(y)}{|y-x|^{n+1}}\,dy \\ &= \int_{\partial\Omega\cap B_R(x)} \frac{(y-x)\cdot\nu(y)}{|y-x|^{n+1}}\,dy + \int_{\Omega\cap\partial B_R(x)} \frac{(y-x)\cdot\nu(y)}{|y-x|^{n+1}}\,dy.\end{aligned}$$

As $(y-x)\cdot\nu(y) = |y-x| = R$ for all y on $\partial B_R(x)$, the first identity in the lemma is proved. The second inequality follows from Lemma 2.2, applied to the set $E = \partial\Omega \cap B_R(x)$. ■

When $\partial\Omega$ has low density around a point x at a certain scale, we will absorb the last term in the inequality of Lemma 3.4 within its left-hand side. For this we will need the following reverse perimeter-energy estimate.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open convex set, $x \in \partial\Omega$, and $R > 0$. Then*

$$|\Omega \cap \partial B_R(x)| \leqslant \frac{C_n}{R} |\partial\Omega \cap B_R(x)|^{\frac{n+1}{n}}$$

for some constant $C_n \geqslant 1$ depending only on n.

Proof. Of course, we can assume that $\Omega \not\subset B_R(x)$, since otherwise there is nothing to prove. Let $\mathcal{C}$ be the open cone of vertex x spanned by $\Omega \cap \partial B_R(x)$. By the coarea formula and the homogeneity of cones (here $\mathcal{H}^{n+1}$ denotes the $(n+1)$-dimensional Lebesgue measure in $\mathbb{R}^{n+1}$),

$$\begin{aligned}\mathcal{H}^{n+1}(\mathcal{C}\cap B_R(x)) = \int_0^R |\mathcal{C}\cap\partial B_\rho(x)|\,d\rho &= \frac{|\mathcal{C}\cap\partial B_R(x)|}{R^n}\int_0^R \rho^n\,d\rho \\ &= \frac{|\mathcal{C}\cap\partial B_R(x)|R}{n+1}.\end{aligned}$$

Moreover, as $\Omega \cap B_R(x)$ is convex, we have that $\mathcal{C} \cap B_R(x) \subset \Omega \cap B_R(x)$. Consequently,

$$\begin{aligned}|\Omega\cap\partial B_R(x)| = |\mathcal{C}\cap\partial B_R(x)| &= (n+1)\frac{\mathcal{H}^{n+1}(\mathcal{C}\cap B_R(x))}{R} \\ &\leqslant (n+1)\frac{\mathcal{H}^{n+1}(\Omega\cap B_R(x))}{R}.\end{aligned}$$

Now, by the relative isoperimetric inequality in Euclidean balls (see, e.g., [34, Proposition 12.37 and Remark 12.38]),

$$\min\{\mathcal{H}^{n+1}(\Omega \cap B_R(x)), \mathcal{H}^{n+1}(B_R(x) \setminus \Omega)\} \leqslant C_I |\partial\Omega \cap B_R(x)|^{\frac{n+1}{n}}$$

for some constant C_I depending only on n. Using again the convexity of Ω to ensure that the minimum on the left-hand side is $\mathcal{H}^{n+1}(\Omega \cap B_R(x))$, we conclude that

$$|\Omega \cap \partial B_R(x)| \leqslant \frac{C_n}{R} |\partial\Omega \cap B_R(x)|^{\frac{n+1}{n}},$$

where $C_n = (n+1)C_I$. ■

To deal with the second case in the dichotomy (iii), where the point $x \in \partial\Omega$ has large density for some radii R, we will need the following isodiametric inequality for the perimeter of convex sets.

Proposition 3.6 (Rosenthal–Szász-type inequality; see, e.g., [7]). *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded open convex set. Then*

$$\frac{|\partial\Omega|}{\operatorname{diam}(\Omega)^n} \leqslant \frac{|\partial B_1|}{\operatorname{diam}(B_1)^n} = \frac{|\mathbb{S}^n|}{2^n}. \tag{3.3}$$

As a consequence,

$$|\partial\Omega \cap B_R(x)| \leqslant |\mathbb{S}^n| R^n \tag{3.4}$$

for every open convex set $\Omega \subset \mathbb{R}^{n+1}$, $x \in \partial\Omega$, and $R > 0$.

The Rosenthal–Szász inequality (3.3) is classical and probably well known to expert readers. It is stated in [7, Section 44] as inequality (6) and proved in that monograph over several sections. Since we could not find a reference with a short proof of the inequality, we will include it in Appendix A. Estimate (3.4) is immediately deduced by applying (3.3) to the bounded open convex set $\Omega \cap B_R(x)$ and using that $\partial\Omega \cap B_R(x) \subset \partial(\Omega \cap B_R(x))$.

Observe that (3.3) carries the optimal constant. For our purposes, we only need (3.4), and we do not need it with its best constant. That is, we will only use that $|\partial\Omega \cap B_R(x)| \leqslant CR^n$, for some dimensional constant C, for every open convex set Ω. We also include in Appendix A a simple proof of this nonoptimal inequality.

We have now all the preliminary results to prove Proposition 3.1.

Proof of Proposition 3.1. Without loss of generality we can take E to be bounded, by proving the proposition first for $E_k := E \cap B_k(x)$, $k \in \mathbb{N}$, and then letting $k \to +\infty$. To this aim, notice that

$$\left| \int_{\partial\Omega \setminus E_k} \frac{dy}{|x-y|^{n+sp}} - \int_{\partial\Omega \setminus E} \frac{dy}{|x-y|^{n+sp}} \right| = \int_{E \setminus B_k(x)} \frac{dy}{|x-y|^{n+sp}} \leqslant \frac{|E|}{k^{n+sp}} \to 0$$

as $k \to +\infty$, since $|E| < +\infty$.

Since now E is bounded, we can assume, using Lemmas 3.2 and 3.3, that E is of the form

$$E = \partial\Omega \cap B_R(x), \tag{3.5}$$

for some $R > 0$. Indeed, by Lemma 3.3(b), the function $R \mapsto |\partial\Omega \cap B_R(x)|$ is continuous for almost every $x \in \partial\Omega$. Thus, we can clearly choose a radius $R > 0$, depending on x, such that $|\partial\Omega \cap B_R(x)| = |E|$. Now, Lemma 3.2 says that, replacing E by $\partial\Omega \cap B_R(x)$, the right-hand side of (3.1) does not increase, while its left-hand side remains unaltered. We can therefore take E to be given by (3.5).

We now distinguish between three cases, involving different assumptions on the density of $\partial\Omega$ around x. We will compare the density $\rho^{-n}|\partial\Omega \cap B_\rho(x)|$ with the dimensional constant

$$\delta := \min\Big\{|\mathbb{S}^n|, \Big(\frac{|\mathbb{S}^n|}{4C_n}\Big)^{\frac{n}{n+1}}\Big\}$$

at the two different scales $\rho = R$ and $\rho = TR$, where $C_n \geqslant 1$ is the constant from Lemma 3.5 and

$$T := \Big(\frac{2|\mathbb{S}^n|}{\delta}\Big)^{\frac{1}{n}} > 1.$$

Case 1. Assume that

$$|\partial\Omega \cap B_R(x)| \leqslant \delta R^n. \tag{3.6}$$

Using Lemma 3.4 we deduce that

$$\frac{|\mathbb{S}^n|}{2} \leqslant |\partial\Omega \cap B_R(x)|^{\frac{\alpha}{n+\alpha}} H_\alpha(x)^{\frac{n}{n+\alpha}} + \frac{|\Omega \cap \partial B_R(x)|}{R^n}.$$

We estimate the second term on the right with the aid of Lemma 3.5, assumption (3.6), and the definition of δ, getting that

$$\frac{|\Omega \cap \partial B_R(x)|}{R^n} \leqslant C_n \delta^{\frac{n+1}{n}} \leqslant \frac{|\mathbb{S}^n|}{4}.$$

The combination of the previous two inequalities leads us to

$$\frac{|\mathbb{S}^n|}{4} \leqslant |\partial\Omega \cap B_R(x)|^{\frac{\alpha}{n+\alpha}} H_\alpha(x)^{\frac{n}{n+\alpha}}, \tag{3.7}$$

which, recalling (3.5), establishes (3.1) in this first case.

Case 2. We assume now that

$$|\partial\Omega \cap B_R(x)| > \delta R^n \quad \text{and} \quad |\partial\Omega \cap B_{TR}(x)| \leqslant \delta (TR)^n. \tag{3.8}$$

Arguing exactly as for (3.7), but now with R replaced by TR, we obtain that

$$\frac{|\mathbb{S}^n|}{4} \leqslant |\partial\Omega \cap B_{TR}(x)|^{\frac{\alpha}{n+\alpha}} H_\alpha(x)^{\frac{n}{n+\alpha}}.$$

The two inequalities in (3.8) give that $|\partial\Omega \cap B_{TR}(x)| \leqslant T^n|\partial\Omega \cap B_R(x)| = T^n|E|$. Hence,

$$\frac{|\mathbb{S}^n|}{4} \leqslant T^{\frac{n\alpha}{n+\alpha}}|E|^{\frac{\alpha}{n+\alpha}} H_\alpha(x)^{\frac{n}{n+\alpha}},$$

which yields (3.1) with a new constant C.

Case 3. Finally, we assume that

$$|\partial\Omega \cap B_R(x)| > \delta R^n \quad \text{and} \quad |\partial\Omega \cap B_{TR}(x)| > \delta (TR)^n.$$

Taking advantage of the perimeter bound (3.4), we see that

$$\begin{aligned} |\partial\Omega \cap (B_{TR}(x) \setminus B_R(x))| &= |\partial\Omega \cap B_{TR}(x)| - |\partial\Omega \cap B_R(x)| \\ &\geqslant \delta (TR)^n - |\mathbb{S}^n| R^n = |\mathbb{S}^n| R^n. \end{aligned}$$

Consequently, we find that

$$\begin{aligned} \int_{\partial\Omega \setminus E} \frac{dy}{|y-x|^{n+sp}} &\geqslant \int_{\partial\Omega \cap (B_{TR}(x) \setminus B_R(x))} \frac{dy}{|y-x|^{n+sp}} \geqslant \frac{|\partial\Omega \cap (B_{TR}(x) \setminus B_R(x))|}{(TR)^{n+sp}} \\ &\geqslant \frac{|\mathbb{S}^n|}{T^{n+sp}} R^{-sp} \geqslant \frac{|\mathbb{S}^n| \delta^{\frac{sp}{n}}}{T^{n+sp}} |\partial\Omega \cap B_R(x)|^{-\frac{sp}{n}} = \frac{|\mathbb{S}^n| \delta^{\frac{sp}{n}}}{T^{n+sp}} |E|^{-\frac{sp}{n}}, \end{aligned}$$

which yields (3.1) once again for some constant C.

As this was the last case, the proof of Proposition 3.1 is finished. ∎

4. Fractional Michael–Simon inequality for functions

In this section we establish our main result, inequality (1.11) of Theorem 1.3. Namely, for every measurable function $u \in W^{s,p}(\partial\Omega)$ it holds that

$$\begin{aligned} \|u\|_{L^{p_s^*}(\partial\Omega)} \leqslant C \bigg(\frac{1}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \\ + \int_{\partial\Omega} H_\alpha(x)^{\frac{sp}{\alpha}} |u(x)|^p \, dx \bigg)^{\frac{1}{p}}, \end{aligned} \tag{4.1}$$

where C is a constant depending only on n, α, s, and p.

We first give a proof when $p = 1$. This is simple and based on the fractional coarea formula of Visintin [41]. This first proof gives the same constant in (4.1) as the one in the isoperimetric inequality (1.12) of Theorem 1.3, which also agrees with the constant in the pointwise inequality of Proposition 3.1.

It is important to point out that in contrast with the local case, for $p > 1$ it is not known how to derive a fractional Sobolev inequality from a corresponding fractional isoperimetric inequality, even in Euclidean space. Thus we give a second proof of our fractional

Sobolev inequality that is valid for all $p \geqslant 1$. For $p = 1$, it gives a worse constant than the one found via the coarea formula. This second argument follows very closely the slicing procedure of Savin and Valdinoci [38] and Di Nezza, Palatucci, and Valdinoci [19, Section 6], with the necessary modifications to cope with the term involving H_α.

For the first proof, we will need the following version of the fractional coarea formula.

Lemma 4.1 (Fractional coarea formula on manifolds). *Let $M \subset \mathbb{R}^{n+1}$ be a Lipschitz hypersurface, $s \in (0,1)$, and $u\colon M \to \mathbb{R}$ be a measurable function. Then*

$$\frac{1}{2}\int_M \int_M \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy = \int_{-\infty}^{+\infty} \mathrm{Per}_{M,s}(\{u>t\})\,dt.$$

Proof. Using the layer cake representation, one writes

$$u(x)-u(y) = \int_{-\infty}^{+\infty} \chi_{\{u>t\}}(x)\chi_{\{u\leqslant t\}}(y)\,dt \quad \text{if } u(x)>u(y). \tag{4.2}$$

If $u(x) \leqslant u(y)$ then the right-hand side of (4.2) vanishes. Therefore

$$|u(x)-u(y)| = \int_{-\infty}^{+\infty} \big(\chi_{\{u>t\}}(x)\chi_{\{u\leqslant t\}}(y) + \chi_{\{u\leqslant t\}}(x)\chi_{\{u>t\}}(y)\big)dt.$$

Now, an application of Fubini's theorem on $\mathbb{R}\times M$ gives

$$\int_M\int_M \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy = 2\int_{-\infty}^{+\infty}\bigg(\int_M\int_M \frac{\chi_{\{u>t\}}(x)\chi_{\{u\leqslant t\}}(y)}{|x-y|^{n+s}}\,dx\,dy\bigg)dt.$$

Since

$$\int_M\int_M \frac{\chi_{\{u>t\}}(x)\chi_{\{u\leqslant t\}}(y)}{|x-y|^{n+s}}\,dx\,dy = \int_{\{u>t\}}\int_{M\setminus\{u>t\}} \frac{dy\,dx}{|x-y|^{n+s}} = \mathrm{Per}_{M,s}(\{u>t\}),$$

we conclude the claim of the lemma. ∎

We can now give the following proof:

First proof of Theorem 1.3 for $p = 1$. Without loss of generality we may assume u to be nonnegative. Indeed, the general case will then follow from this, by applying (4.1) to $|u|$ and noticing that $\big|\,|u(x)|-|u(y)|\,\big| \leqslant |u(x)-u(y)|$. We may also suppose that u has compact support – see the final argument in the proof of Theorem 1.3 for $p \geqslant 1$, presented later in this section, for details of how to remove this assumption.

From the expression

$$u(x) = \int_0^{+\infty} \chi_{\{u>t\}}(x)\,dt,$$

we use Minkowski's integral inequality to obtain

$$\begin{aligned}\|u\|_{L^{\frac{n}{n-s}}(\partial\Omega)} &= \bigg\|\int_0^{+\infty}\chi_{\{u>t\}}\,dt\bigg\|_{L^{\frac{n}{n-s}}(\partial\Omega)}\\ &\leqslant \int_0^{+\infty}\|\chi_{\{u>t\}}\|_{L^{\frac{n}{n-s}}(\partial\Omega)}\,dt = \int_0^{+\infty}|\{u>t\}|^{\frac{n-s}{n}}\,dt.\end{aligned}$$

We now apply the inequality of Proposition 3.1 with $E := \{u{>}t\}$ – observe that $|\{u{>}t\}| < +\infty$ as u has compact support. Integrating it over E we see that

$$|\{u > t\}|^{\frac{n-s}{n}} \leqslant C\left(\mathrm{Per}_{\partial\Omega,s}(\{u > t\}) + \int_{\{u>t\}} H_\alpha(x)^{\frac{s}{\alpha}}\, dx\right).$$

By combining the last two estimates we get

$$\|u\|_{L^{\frac{n}{n-s}}(\partial\Omega)} \leqslant C \int_0^{+\infty} \left(\mathrm{Per}_{\partial\Omega,s}(\{u > t\}) + \int_{\{u>t\}} H_\alpha(x)^{\frac{s}{\alpha}}\, dx\right) dt. \tag{4.3}$$

Finally, by Fubini's theorem we have

$$\int_0^{+\infty} \left(\int_{\{u>t\}} H_\alpha(x)^{\frac{s}{\alpha}}\, dx\right) dt = \int_{\partial\Omega\cap\{u>0\}} H_\alpha(x)^{\frac{s}{\alpha}} |u(x)|\, dx.$$

Plugging this into (4.3) and using Lemma 4.1 we deduce

$$\|u\|_{L^{\frac{n}{n-s}}(\partial\Omega)} \leqslant C\left(\frac{1}{2}\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\, dx\, dy + \int_{\partial\Omega} H_\alpha(x)^{\frac{s}{\alpha}} |u(x)|\, dx\right).$$

This settles the theorem for $p = 1$. ∎

We now present an adaptation of the slicing procedure of [38]. It will lead to the proof of Theorem 1.3 in the general case $p \geqslant 1$.

We first introduce some notation. Let $u\colon \partial\Omega \to \mathbb{R}$ be a bounded and nonnegative measurable function with compact support. For $i \in \mathbb{Z}$ we write

$$\begin{aligned} &A_i := \{u > 2^i\}, \quad a_i := |A_i|, \\ &D_i := A_i \setminus A_{i+1} = \{2^i < u \leqslant 2^{i+1}\}, \quad \text{and} \quad d_i := |D_i|. \end{aligned}$$

We have that the sets D_i are pairwise disjoint:

$$\{u = 0\} \cup \bigcup_{\substack{j\in\mathbb{Z}\\ j\leqslant i}} D_j = \partial\Omega \setminus A_{i+1}, \quad \bigcup_{\substack{j\in\mathbb{Z}\\ j\geqslant i}} D_j = A_i, \quad \text{and} \quad a_i = \sum_{\substack{j\in\mathbb{Z}\\ j\geqslant i}} d_j.$$

We will need the following auxiliary lemma – see [19, Lemma 6.2] for its proof, which is very short and only uses Hölder's inequality. Note that, as u is bounded, nonnegative, and has compact support, our sequence $\{a_i\}$ satisfies the hypotheses of the lemma for some $N \in \mathbb{Z}$.

Lemma 4.2. *Let $s \in (0,1)$, $p \geqslant 1$ such that $sp < n$, and $N \in \mathbb{Z}$. Suppose $\{a_i\}_{i\in\mathbb{Z}}$ is a bounded, nonnegative, and nonincreasing sequence with*

$$a_i = 0 \quad \textit{for all } i \geqslant N.$$

Then

$$\sum_{i\in\mathbb{Z}} 2^{pi} a_i^{(n-sp)/n} \leqslant 2^{p_s^*} \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_i^{-sp/n} a_{i+1}.$$

Note that, from the hypotheses made on the sequence $\{a_i\}$ in the lemma, clearly both series are convergent. The same happens for the series in the following inequality, which is taken from the proof of [19, Lemma 6.3] and that we will use later:

$$\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} 2^{pi} a_{i-1}^{-sp/n} a_{i+1} \leqslant \frac{1}{2} \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} 2^{pi} a_{i-1}^{-sp/n} a_i. \tag{4.4}$$

Its proof is simple:

$$\begin{aligned}
\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} 2^{pi} a_{i-1}^{-sp/n} a_{i+1} &= \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0, a_{i+1}\neq 0}} 2^{pi} a_{i-1}^{-sp/n} a_{i+1} = \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_{i-1}^{-sp/n} a_{i+1} \\
&\leqslant \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_i^{-sp/n} a_{i+1} = \frac{1}{2^p} \sum_{\substack{j\in\mathbb{Z}\\ a_{j-1}\neq 0}} 2^{pj} a_{j-1}^{-sp/n} a_j \\
&\leqslant \frac{1}{2} \sum_{\substack{j\in\mathbb{Z}\\ a_{j-1}\neq 0}} 2^{pj} a_{j-1}^{-sp/n} a_j.
\end{aligned}$$

The next lemma is the core of the proof and the analogue of [19, Lemma 6.3].

Lemma 4.3. *Let $s \in (0,1)$, $p \geqslant 1$ such that $n > sp$, and $\Omega \subset \mathbb{R}^{n+1}$ be an open convex set. Let $u \in L^\infty(\partial\Omega)$ be a nonnegative function with compact support. Then*

$$\frac{1}{2}\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy + \int_{\partial\Omega} H_\alpha(x)^{\frac{sp}{\alpha}} u(x)^p\,dx \geqslant c \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} 2^{pi} a_{i-1}^{-sp/n} a_i,$$

for some constant $c > 0$ depending only on n, α, s, and p.

Proof. Throughout the proof, we will use the notation $D_{-\infty} := \{u = 0\}$. Moreover, for any $k \in \mathbb{Z}$, we write $j \leqslant k$ to indicate that j is either an integer smaller than or equal to k or that $j = -\infty$.

Let $i \in \mathbb{Z}$ and $x \in D_i$. For every $j \leqslant i-2$ and $y \in D_j$ we have that $u(x)-u(y) \geqslant 2^{i-1}$ and therefore

$$\begin{aligned}
\sum_{j\leqslant i-2}\int_{D_j} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy &\geqslant 2^{p(i-1)} \sum_{j\leqslant i-2}\int_{D_j} \frac{dy}{|x-y|^{n+sp}} \\
&= 2^{p(i-1)} \int_{\partial\Omega\setminus A_{i-1}} \frac{dy}{|x-y|^{n+sp}}.
\end{aligned} \tag{4.5}$$

Suppose now that A_{i-1} has positive measure. From Proposition 3.1 we have that

$$a_{i-1}^{-sp/n} \leqslant C\left(\int_{\partial\Omega\setminus A_{i-1}} \frac{dy}{|x-y|^{n+sp}} + H_\alpha(x)^{\frac{sp}{\alpha}}\right)$$

for a.e. $x \in A_{i-1}$. As a consequence, using (4.5),

$$\sum_{j \leqslant i-2} \int_{D_j} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy \geqslant 2^{p(i-1)} \Big(\frac{a_{i-1}^{-sp/n}}{C} - H_\alpha(x)^{\frac{sp}{\alpha}} \Big)$$

for a.e. $x \in D_i \subset A_{i-1}$. Integrating over D_i, this gives

$$\begin{aligned}\sum_{j \leqslant i-2} \int_{D_i} \int_{D_j} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy\, dx &+ 2^{p(i-1)} \int_{D_i} H_\alpha(x)^{\frac{sp}{\alpha}}\, dx \\ &\geqslant \frac{2^{pi} a_{i-1}^{-sp/n} d_i}{2^p C} = \frac{2^{pi} a_{i-1}^{-sp/n} (a_i - a_{i+1})}{2^p C}.\end{aligned}$$

We now take the sum over all $i \in \mathbb{Z}$ such that $a_{i-1} \neq 0$ and use (4.4) to deduce

$$\begin{aligned}\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} \sum_{j \leqslant i-2} \int_{D_i} \int_{D_j} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy\, dx &+ \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} 2^{p(i-1)} \int_{D_i} H_\alpha(x)^{\frac{sp}{\alpha}}\, dx \\ &\geqslant \frac{1}{2^p C} \frac{1}{2} \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} 2^{pi} a_{i-1}^{-sp/n} a_i. \end{aligned} \tag{4.6}$$

Now, by symmetry,

$$\begin{aligned}\frac{1}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy &\geqslant \sum_{i\in\mathbb{Z}} \sum_{j \leqslant i-1} \int_{D_i} \int_{D_j} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy\, dx \\ &\geqslant \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} \sum_{j \leqslant i-2} \int_{D_i} \int_{D_j} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy\, dx. \end{aligned} \tag{4.7}$$

Since $u(x)^p > 2^{pi} > 2^{p(i-1)}$ for $x \in D_i$, we have

$$\int_{D_i} H_\alpha(x)^{\frac{sp}{\alpha}} u(x)^p\, dx \geqslant 2^{p(i-1)} \int_{D_i} H_\alpha(x)^{\frac{sp}{\alpha}}\, dx.$$

Thus,

$$\begin{aligned}\int_{\partial\Omega} H_\alpha(x)^{\frac{sp}{\alpha}} u(x)^p\, dx &\geqslant \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} \int_{D_i} H_\alpha(x)^{\frac{sp}{\alpha}} u(x)^p\, dx \\ &\geqslant \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} 2^{p(i-1)} \int_{D_i} H_\alpha(x)^{\frac{sp}{\alpha}}\, dx.\end{aligned}$$

It now follows from this, (4.7), and (4.6) that

$$\frac{1}{2} \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy + \int_{\partial\Omega} H_\alpha(x)^{\frac{sp}{\alpha}} u(x)^p\, dx \geqslant \frac{1}{2^{p+1} C} \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} 2^{pi} a_{i-1}^{-sp/n} a_i,$$

which concludes the proof of the lemma. ∎

Proof of Theorem 1.3. As for the proof in the case $p = 1$ presented previously, we may assume u to be nonnegative. Using truncations, we can also take u to be bounded. In addition, we suppose for the moment that u has compact support. We will show at the end of the proof that this hypothesis can be removed.

Under these assumptions we have

$$\|u\|_{L^{p_s^*}(\partial\Omega)}^{p_s^*} = \sum_{i\in\mathbb{Z}} \int_{D_i} u(x)^{p_s^*}\,dx \leqslant \sum_{i\in\mathbb{Z}} \int_{D_i} (2^{i+1})^{p_s^*}\,dx \leqslant \sum_{i\in\mathbb{Z}} 2^{p_s^*(i+1)} a_i.$$

From this and the elementary inequality $(\sum_i m_i)^\lambda \leqslant \sum_i m_i^\lambda$ for every sequence $m_i \geqslant 0$ and $\lambda \in [0,1]$, taking here $\lambda := p/p_s^* = (n-sp)/n \in (0,1)$, one concludes that

$$\|u\|_{L^{p_s^*}(\partial\Omega)}^{p} \leqslant 2^p \Big(\sum_{i\in\mathbb{Z}} 2^{p_s^* i} a_i\Big)^{p/p_s^*} \leqslant 2^p \sum_{i\in\mathbb{Z}} 2^{pi} a_i^{(n-sp)/n}.$$

Now using Lemmas 4.2 and 4.3 we get

$$\begin{aligned}\|u\|_{L^{p_s^*}(\partial\Omega)}^{p} &\leqslant 2^{p+p_s^*} \sum_{\substack{i\in\mathbb{Z}\\ a_i\neq 0}} 2^{pi} a_i^{-sp/n} a_{i+1} = 2^{p_s^*} \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} 2^{pi} a_{i-1}^{-sp/n} a_i \\ &\leqslant C\Big(\frac{1}{2}\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy + \int_{\partial\Omega} H_\alpha(x)^{\frac{sp}{\alpha}} u(x)^p\,dx\Big),\end{aligned}$$

which proves the theorem under the assumption that u has compact support.

We now show that the compactness of $\operatorname{supp}(u)$ is not needed. Let $R \geqslant 1$ and consider a cutoff function $\eta \in C_c^\infty(\mathbb{R}^{n+1})$ satisfying $0 \leqslant \eta \leqslant 1$ in $\mathbb{R}^{n+1}$, $\eta = 1$ in B_R, $\operatorname{supp}(\eta) \subset B_{2R}$, and $|\nabla\eta| \leqslant 2/R$. Given $u \in W^{s,p}(\partial\Omega)$, we define $v := \eta u$. By the inequality that we have just proved and since v has compact support, we have

$$\begin{aligned}\|u\|_{L^{p_s^*}(\partial\Omega\cap B_R)}^{p} &\leqslant \|v\|_{L^{p_s^*}(\partial\Omega)}^{p} \\ &\leqslant C\Big(\frac{1}{2}\int_{\partial\Omega}\int_{\partial\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\,dx\,dy + \int_{\partial\Omega} H_\alpha(x)^{\frac{sp}{\alpha}} |v(x)|^p\,dx\Big) \\ &\leqslant C\Big(\frac{1}{2}\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy + \int_{\partial\Omega} H_\alpha(x)^{\frac{sp}{\alpha}} |u(x)|^p\,dx \\ &\qquad + \int_{\partial\Omega} |u(x)|^p \Big(\int_{\partial\Omega} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{n+sp}}\,dy\Big)\,dx\Big),\end{aligned} \tag{4.8}$$

where for the last inequality we used that

$$|v(x)-v(y)|^p \leqslant 2^{p-1}\big(|u(x)-u(y)|^p + |u(x)|^p|\eta(x)-\eta(y)|^p\big) \quad \text{for a.e. } x,y \in \partial\Omega.$$

To control the last term in (4.8) we adapt some techniques from [11, Section 3.2]. First, using the Lipschitz property of η we have

$$\begin{aligned}\int_{\partial\Omega} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{n+sp}}\,dy &\leqslant \frac{2^p}{R^p}\int_{\partial\Omega\cap B_R(x)} \frac{dy}{|x-y|^{n-(1-s)p}} \\ &\qquad + \int_{\partial\Omega\setminus B_R(x)} \frac{dy}{|x-y|^{n+sp}}.\end{aligned} \tag{4.9}$$

To estimate the second term on the right, we argue similarly to [11, Lemma 3.3]. Taking advantage of the perimeter estimate (3.4), we deduce

$$\begin{aligned}\int_{\partial\Omega\setminus B_R(x)} \frac{dy}{|x-y|^{n+sp}} &= \sum_{j=1}^{+\infty} \int_{\partial\Omega\cap(B_{2^jR}(x)\setminus B_{2^{j-1}R}(x))} \frac{dy}{|x-y|^{n+sp}} \\ &\leqslant \sum_{j=1}^{+\infty} \frac{|\partial\Omega\cap B_{2^jR}(x)|}{(2^{j-1}R)^{n+sp}} \\ &\leqslant \frac{2^{n+sp}|\mathbb{S}^n|}{R^{sp}} \sum_{j=1}^{+\infty} 2^{-spj} \leqslant \frac{C}{R^{sp}}.\end{aligned}$$

As the first term on the right-hand side of (4.9) can be dealt with using [11, Lemma 3.4] – observe that [11, hypothesis (3.3)] is fulfilled thanks to our (3.4) – we infer that

$$\int_{\partial\Omega} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{n+sp}}\, dy \leqslant \frac{C}{R^{sp}}.$$

By plugging this into (4.8) and letting $R \to +\infty$, we conclude that u satisfies (1.11). The proof is thus complete. ■

5. Application to the fractional mean curvature flow

In this section we study the evolution of convex sets under fractional mean curvature flow. Using the pointwise inequality (2.1) in conjunction with the classical Michael–Simon inequality, we provide an upper bound for the maximal time of existence for the smooth fractional mean curvature flow of convex hypersurfaces. Namely, we prove Theorem 1.4. As in the classical local case, the argument is simple, once the appropriate Michael–Simon-type inequality is known.

We denote by $\Omega_0 \subset \mathbb{R}^{n+1}$ a bounded open convex set with C^2 boundary, and by Ω_t its evolution by fractional α-mean curvature flow. That is, the inner normal velocity is, at every point, the fractional α-mean curvature. The unit outer normal to Ω_t is denoted by ν_t and we take the mean curvature H of Ω (i.e., the sum of its principal curvatures) with the sign convention to be nonnegative for convex sets.

As in (1.14), we consider

$$T^* := \sup\bigl\{t > 0 : \Omega_\tau \text{ is nonempty and has } C^2 \text{ boundary for all } \tau \in [0,t)\bigr\}.$$

In view of the results of [32], Ω_τ has boundary of class C^2 – actually, C^∞ – for every small τ. Hence, $T^* > 0$. On the other hand, through comparison with shrinking balls in [37] it is proved that $T^* \leqslant C\,\mathrm{diam}(\Omega_0)^{1+\alpha}$ for some constant C depending only on n and α. These two results hold regardless of the convexity of Ω_0. Here, we show that, when Ω_0 is convex, the bound on T^* can be improved to (1.16).

First, we recall a general first variation formula. In our situation, we will apply it with $\varphi_t = -H_\alpha[\Omega_t]$. Note that, throughout this section, we emphasize the dependence of the classical and fractional mean curvatures on the set Ω by writing $H(x) = H[\Omega](x)$ and $H_\alpha(x) = H_\alpha[\Omega](x)$ for $x \in \partial\Omega$.

Lemma 5.1 (See, e.g., [22, Remark 4.2] or [29, Proposition 4]). *Let $\Omega_t \subset \mathbb{R}^{n+1}$ be a one-parameter family of open sets with C^2 boundary and with $|\partial\Omega_t| < +\infty$ for all $t \in (-a, a)$ and some $a > 0$. Assume that, corresponding to each point $p_0 \in \partial\Omega_0$, there is a differentiable curve $t \mapsto p(t)$ with $p(0) = p_0$, $p(t) \in \partial\Omega_t$ for all $t \in (-a, a)$, and satisfying*

$$\frac{d}{dt} p(t) = \varphi_t(p(t))\nu_t(p(t)) \quad \text{for all } t \in (-a, a),$$

for some continuous function $\varphi_t \colon \partial\Omega_t \to \mathbb{R}$.

Then

$$\frac{d}{dt}|\partial\Omega_t| = \int_{\partial\Omega_t} \varphi_t H[\Omega_t]\, dx.$$

We can now give the following proof:

Proof of Theorem 1.4. Recall that Ω_t remains convex, thanks to [16]. Using Lemma 5.1 we see that

$$\frac{d}{dt}|\partial\Omega_t| = -\int_{\partial\Omega_t} H_\alpha[\Omega_t]H[\Omega_t]\, dx. \tag{5.1}$$

By inequality (2.1) proved in Section 2, we know that

$$|\partial\Omega_t|^{-\frac{\alpha}{n}} \leqslant C_1 H_\alpha[\Omega_t](x) \quad \text{for all } x \in \partial\Omega_t,$$

for some constant $C_1 > 0$ depending only on n and α. Multiplying this inequality by $H[\Omega_t](x)$ and integrating in $x \in \partial\Omega_t$, we get

$$|\partial\Omega_t|^{-\frac{\alpha}{n}} \int_{\partial\Omega_t} H[\Omega_t]\, dx \leqslant C_1 \int_{\partial\Omega_t} H_\alpha[\Omega_t]H[\Omega_t]\, dx. \tag{5.2}$$

We now use the classical Michael–Simon inequality (Theorem 1.1) with $u \equiv 1 = p$ if $n \geqslant 2$, or the Gauss–Bonnet formula for curves: $2\pi = \int_{\partial\Omega_t} H[\Omega_t](x)\, dx$ if $n = 1$. Either way, we have

$$|\partial\Omega_t|^{\frac{n-1}{n}} \leqslant C_2 \int_{\partial\Omega_t} H[\Omega_t]\, dx \tag{5.3}$$

for some constant $C_2 > 0$ depending only on n.

Finally, using (5.3), (5.2), and (5.1), we deduce that

$$\begin{aligned}
|\partial\Omega_t|^{\frac{n-(1+\alpha)}{n}} &= |\partial\Omega_t|^{\frac{n-1}{n}}|\partial\Omega_t|^{-\frac{\alpha}{n}} \leqslant C_2|\partial\Omega_t|^{-\frac{\alpha}{n}} \int_{\partial\Omega_t} H[\Omega_t]\, dx \\
&\leqslant C_1C_2 \int_{\partial\Omega_t} H_\alpha[\Omega_t]H[\Omega_t]\, dx = -C_1C_2\frac{d}{dt}|\partial\Omega_t|.
\end{aligned}$$

That is, $\frac{d}{dt}|\partial\Omega_t|^{\frac{1+\alpha}{n}} \leqslant -\delta$, for some constant $\delta > 0$ depending only on n and α. By integrating this relation, we obtain that $|\partial\Omega_t|^{\frac{1+\alpha}{n}} \leqslant |\partial\Omega_0|^{\frac{1+\alpha}{n}} - \delta t$. This shows that the maximal time of existence must satisfy $T^* \leqslant \delta^{-1}|\partial\Omega_0|^{\frac{1+\alpha}{n}}$, as claimed by the theorem. ■

A. Proof of the Rosenthal–Szász-type inequality

In this section we denote by B_1^n the open unit ball of $\mathbb{R}^n$ centered at the origin, that is $B_1^n := \{x \in \mathbb{R}^n : |x| < 1\}$. Here we give a proof of the first inequality in Proposition 3.6 (the isodiametric inequality for perimeter), which states that

$$|\partial\Omega| \leqslant |\mathbb{S}^n| \frac{\operatorname{diam}(\Omega)^n}{2^n} \quad \text{for every bounded convex set } \Omega \subset \mathbb{R}^{n+1}. \tag{A.1}$$

Observe that the inequality is optimal, i.e., there is equality for balls. This inequality was first proved by Rosenthal and Szász [36] in the plane. The version in higher dimensions can be found in [7, Section 44] as inequality (6). The proof however is scattered over several sections of [7], of which many steps are in greater generality than actually needed to prove (A.1), making the proof unnecessarily long and complicated if one is only interested in the Rosenthal–Szász inequality. We have not found a better reference and, thus, we present a quick proof here. It is based on two better-known results: Cauchy's surface area formula (Proposition A.2 below) and the isodiametric inequality for volume. This last result – see, e.g., [25, Theorem 1 in Section 2.2] – states that

$$|E| \leqslant |B_1^n| \frac{\operatorname{diam}(E)^n}{2^n}, \tag{A.2}$$

where $|\cdot|$ indicates the n-dimensional Lebesgue measure and $E \subset \mathbb{R}^n$ is any measurable set – here convexity is not needed. In [25] it is proved using Steiner symmetrizations. As for (A.1), in (A.2) equality is achieved for balls. Observe that the isodiametric inequality for perimeter does not hold in general if the convexity assumption is relaxed. Consider for example a domain with oscillating boundary – giving an arbitrarily large perimeter – contained in a ball of a given diameter.

If one does not need the best constant in the Rosenthal–Szász inequality (A.1) – as in our case – then a weaker inequality follows more easily from the inclusion $\Omega \subset \overline{B}_{\operatorname{diam}(\Omega)}(x)$, where x is any point in $\overline{\Omega}$, and the monotonicity of the perimeter with respect to the inclusion of convex sets. This monotonicity property follows, for instance, from Cauchy's surface area formula, stated later in Proposition A.2. Given our statement of this result, one also needs to approximate the convex set by polytopes, as we do in the proof of Proposition 3.6 below.

For the proof of (A.1) we need to introduce the notion of polytopes. A bounded open set $K \subset \mathbb{R}^{n+1}$ is called a polytope if its boundary ∂K is the finite union of sets P_i, for $i = 1, \ldots, m_K$, with each P_i being contained in an n-dimensional affine hyperplane. The P_i are the n-dimensional faces of K. In this section $K \subset \mathbb{R}^{n+1}$ always denotes a convex polytope. Now, given a unit vector $\sigma \in \mathbb{S}^n$, let K_σ be the projection of K onto the hyperplane orthogonal to σ. Obviously, we have

$$|\partial K| = \sum_{i=1}^{m_K} |P_i|. \tag{A.3}$$

Denote by ξ_i a unit normal vector on P_i. Note that projecting the n-dimensional faces P_i onto the hyperplane orthogonal to σ and then taking the union over i also coincides with K_σ. At the same time, by the convexity of K, the preimage of a.e. $x \in K_\sigma$ under this projection consists of exactly two points lying on two different faces. Thus, we obtain the identity

$$2|K_\sigma| = \sum_{i=1}^{m_K} |P_i|\,|\langle \xi_i, \sigma\rangle|. \tag{A.4}$$

We will use the following lemma.

Lemma A.1. *Let $\tau \in \mathbb{S}^n$ be a unit vector in $\mathbb{R}^{n+1}$. Then*

$$\int_{\mathbb{S}^n} |\langle \sigma, \tau\rangle|\, d\sigma = 2|B_1^n|.$$

Proof. After a rotation, we can assume $\tau = e_{n+1} = (0, \dots, 0, 1)$. Using the parametrization $\varphi\colon B_1^n \to \mathbb{S}^n$, given by $\varphi(x_1, \dots, x_n) = (x_1, \dots, x_n, \sqrt{1-|x|^2})$, we see that

$$\int_{\mathbb{S}^n} |\langle \sigma, \tau\rangle|\, d\sigma = 2\int_{\mathbb{S}^n \cap \{\sigma_{n+1}>0\}} \sigma_{n+1}\, d\sigma = 2\int_{B_1^n} \sqrt{1-|x|^2}\sqrt{1+|\nabla\varphi^{n+1}(x)|^2}\, dx.$$

The claim follows as $\sqrt{1+|\nabla\varphi^{n+1}(x)|^2} = 1/\sqrt{1-|x|^2}$ for every $x \in B_1^n$. ∎

As a result of the previous considerations, we have the following identity for the perimeter of K, which is known as Cauchy's formula (see for instance [23, page 89]).

Proposition A.2 (Cauchy's surface area formula). *Let $K \subset \mathbb{R}^{n+1}$ be a convex polytope. Then it holds that*

$$|\partial K| = \frac{1}{|B_1^n|}\int_{\mathbb{S}^n} |K_\sigma|\, d\sigma.$$

Proof. Integrate (A.4) with respect to σ over $\mathbb{S}^n$, then apply Lemma A.1 (with $\tau = \xi_i$), and finally use (A.3). ∎

We can finally give the following proof:

Proof of Proposition 3.6. To prove (A.1) we can assume by approximation that Ω is a convex polytope K (see for instance [33, Section 22] on approximations by polytopes). For any direction $\sigma \in \mathbb{S}^n$ it follows from the isodiametric inequality for volume (A.2) that

$$|K_\sigma| \leqslant |B_1^n| \frac{\operatorname{diam}(K_\sigma)^n}{2^n}.$$

Now using that $\operatorname{diam}(K_\sigma) \leqslant \operatorname{diam}(K)$ and Proposition A.2, we get

$$|\partial K| \leqslant \left(\int_{\mathbb{S}^n} d\sigma\right) \frac{\operatorname{diam}(K)^n}{2^n} = |\mathbb{S}^n| \frac{\operatorname{diam}(K)^n}{2^n},$$

and the proposition is proved. ∎

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