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# Refinement monoids and adaptable separated graphs

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## Abstract

We define a subclass of separated graphs, the class of *adaptable separated graphs*, and study their associated monoids. We show that these monoids are primely generated conical refinement monoids, and we explicitly determine their associated *I*-systems. We also show that any finitely generated conical refinement monoid can be represented as the monoid of an adaptable separated graph. These results provide the first step toward an affirmative answer to the Realization Problem for von Neumann regular rings, in the finitely generated case.

**Keywords** Refinement monoid · Separated graph · *I*-system · Primely generated

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## Introduction

The structure of commutative refinement monoids is generally very intricate, and it is difficult to rephrase their architecture in terms of combinatorial data. These monoids appear naturally in different contexts, such as non-stable  $K$ -theory of exchange rings and real rank zero  $C^*$ -algebras (see e.g. [9,19]), classification of Boolean algebras (see e.g. [18,21]), the realization problem for von Neumann regular rings (see below), and the theory of type semigroups (see e.g. [22,24]). In this paper, based on the work developed in [11,12], we provide a concrete and useful description of a subclass of all primely generated conical refinement monoids, which contains all the finitely generated ones, in terms of a specific type of separated graphs.

Recall that a separated graph [8] is a pair  $(E, C)$ , where  $E$  is a directed graph and  $C$  is a partition of the set of edges of  $E$  which is finer than the partition induced by the source map  $s: E^1 \rightarrow E^0$ . Visually one may think of a separated graph as a directed graph where the edges have been given different colours. Several interesting algebras and  $C^*$ -algebras have been attached to these combinatorial objects, some of them having exotic behaviour (see for instance [7,8]). Given a separated graph  $(E, C)$ , one can naturally associate a monoid  $M(E, C)$  to it [8]. However, it is not always true that  $M(E, C)$  is a refinement monoid [8, Section 5].

Generalizing earlier work by Dobbertin [15] and Pierce [21], the first and third-named authors have completely determined in [11] the structure of primely generated conical refinement monoids. The main ingredient of this characterization is the notion of an  $I$ -system, which is a certain poset of semigroups generalizing the posets of groups used by Dobbertin in [15] (see Definition 1.1 below). Using this description, a characterization of the finitely generated conical refinement monoids which are isomorphic to a graph monoid  $M(E)$  for a (non-separated) directed graph  $E$  has been obtained in [12]. In particular, we stress the fact that *not* all such monoids are isomorphic to graph monoids. It is the purpose of this paper to show that a large class of primely generated conical refinement monoids, including all the finitely generated ones, can be obtained as monoids of the form  $M(E, C)$  for  $(E, C)$  belonging to a particularly well-behaved class of separated graphs, the *adaptable separated graphs* (see Definition 1.4 below).

Concretely, the main result of this paper (Theorem 2.1) is the following:

**Theorem** *The following two statements hold:*

- (1) *If  $(E, C)$  is an adaptable separated graph, then  $M(E, C)$  is a primely generated conical refinement monoid.*
- (2) *For any finitely generated conical refinement monoid  $M$ , there exists an adaptable separated graph  $(E, C)$  such that  $M \cong M(E, C)$ .*

We now outline some of the applications of the results obtained in this note. Concretely, we use the structure of an adaptable separated graph in order to get two realization results. The first application is given in [5], where the authors, jointly with A. Sims, attach to each adaptable separated graph  $(E, C)$  an  $E^*$ -unitary inverse semigroup  $S(E, C)$ . Moreover, using techniques developed by Paterson [20] and Exel [16], they build from this inverse semigroup  $S(E, C)$  an ample Hausdorff étale topological groupoid  $\mathcal{G}(E, C)$  satisfying

$$\text{Typ}(\mathcal{G}(E, C)) \cong M(E, C).$$

In particular, we see from Theorem 2.1(2) that all finitely generated conical refinement monoids arise as type semigroups of this well-behaved class of topological groupoids. The second application concerns the *Realization Problem for von Neumann regular rings*, posed by Goodearl in [17]. This wonders which refinement monoids appear as a  $\mathcal{V}(R)$  for a von Neumann regular ring  $R$ , where the latter stands for the monoid of isomorphism classes of finitely generated projective (left, say)  $R$ -modules, with the operation induced from direct sum (see [2] for a survey on this problem). For an adaptable separated graph  $(E, C)$  and an arbitrary field  $K$ , we build in [4] a von Neumann regular  $K$ -algebra  $Q_K(E, C)$ , which is a certain universal localization of the Steinberg algebra  $A_K(\mathcal{G}(E, C))$  (see [23] for its definition) of the above groupoid  $\mathcal{G}(E, C)$ , and which satisfies that

$$\mathcal{V}(Q_K(E, C)) \cong M(E, C).$$

Again, Theorem 2.1(2) gives that the realization problem for von Neumann regular  $K$ -algebras has a positive answer for any finitely generated conical refinement monoid. This construction extends at once the constructions given in [3,6].

The paper is organized as follows. In the first section we introduce background material needed for our results. We have split this in three subsections, concerning commutative monoids, primely generated refinement monoids, and separated graphs, respectively. In Sect. 2, we prove our results. We have divided this section into two subsections, in each of which we prove one of the statements of our Theorem.

## 1 Preliminaries

### 1.1 Basics on commutative monoids

All semigroups and monoids considered in this paper are commutative. We will denote by  $\mathbb{N}$  the semigroup of positive integers, and by  $\mathbb{Z}^+$  the monoid of non-negative integers.

Given a commutative monoid  $M$ , we set  $M^* := M \setminus \{0\}$ . We say that  $M$  is *conical* if  $M^*$  is a semigroup, that is, if, for all  $x, y$  in  $M$ ,  $x + y = 0$  only when  $x = y = 0$ .

We say that a monoid  $M$  is *separative* provided  $2x = 2y = x + y$  always implies  $x = y$ ; there are a number of equivalent formulations of this property, see e.g. [9, Lemma 2.1]. We say that  $M$  is a *refinement monoid* if, for all  $a, b, c, d$  in  $M$  such that  $a + b = c + d$ , there exist  $w, x, y, z$  in  $M$  such that  $a = w + x, b = y + z, c = w + y$  and  $d = x + z$ . A basic example of refinement monoid is the monoid  $M(E)$  associated to a countable row-finite graph  $E$  [10, Proposition 4.4].

If  $x, y \in M$ , we write  $x \leq y$  if there exists  $z \in M$  such that  $x + z = y$ . Note that  $\leq$  is a translation-invariant pre-order on  $M$ , called the *algebraic pre-order* of  $M$ . All inequalities in commutative monoids will be with respect to this pre-order. An element  $p$  in a monoid  $M$  is a *prime element* if  $p$  is not invertible in  $M$ , and, whenever  $p \leq a + b$  for  $a, b \in M$ , then either  $p \leq a$  or  $p \leq b$ . The monoid  $M$  is *primely generated* if every non-invertible element of  $M$  can be written as a sum of prime elements.

An element  $x \in M$  is *regular* if  $2x \leq x$ . An element  $x \in M$  is an *idempotent* if  $2x = x$ . An element  $x \in M$  is *free* if  $nx \leq mx$  implies  $n \leq m$ . Any element of a separative monoid is either free or regular. In particular, this is the case for any primely generated refinement monoid, by [14, Theorem 4.5]. Furthermore, every finitely generated refinement monoid is primely generated [14, Corollary 6.8].

A subset  $S$  of a monoid  $M$  is called an *order-ideal* if  $S$  is a subset of  $M$  containing 0, closed under taking sums and summands within  $M$ . An order-ideal can also be described as a submonoid  $I$  of  $M$ , which is hereditary with respect to the canonical pre-order  $\leq$  on  $M$ :  $x \leq y$  and  $y \in I$  imply  $x \in I$ . A non-trivial monoid is said to be *simple* if it has no non-trivial order-ideals.

If  $(S_k)_{k \in \Lambda}$  is a family of (commutative) semigroups,  $\bigoplus_{k \in \Lambda} S_k$  (resp.  $\prod_{k \in \Lambda} S_k$ ) stands for the coproduct (resp. the product) of the semigroups  $S_k, k \in \Lambda$ , in the category of commutative semigroups. If the semigroups  $S_k$  are subsemigroups of a semigroup  $S$ , we will denote by  $\sum_{k \in \Lambda} S_k$  the subsemigroup of  $S$  generated by  $\bigcup_{k \in \Lambda} S_k$ . Note that  $\sum_{k \in \Lambda} S_k$  is the image of the canonical map  $\bigoplus_{k \in \Lambda} S_k \rightarrow S$ . We will use the notation  $\langle X \rangle$  to denote the semigroup generated by a subset  $X$  of a semigroup  $S$ .

Given a semigroup  $M$ , we will denote by  $G(M)$  the Grothendieck group of  $M$ . There exists a semigroup homomorphism  $\psi_M: M \rightarrow G(M)$  such that for any semigroup homomorphism  $\eta: M \rightarrow H$  to a group  $H$  there is a unique group homomorphism  $\tilde{\eta}: G(M) \rightarrow H$  such that  $\tilde{\eta} \circ \psi_M = \eta$ .  $G(M)$  is abelian and it is generated as a group by  $\psi(M)$ . If  $M$  is already a group then  $G(M) = M$ . If  $M$  is a semigroup of the form  $\mathbb{N} \times G$ , where  $G$  is an abelian group, then  $G(M) = \mathbb{Z} \times G$ . In this case, we will view  $G$  as a subgroup of  $\mathbb{Z} \times G$  by means of the identification  $g \leftrightarrow (0, g)$ .

Let  $M$  be a conical commutative monoid, and let  $x \in M$  be any element. The *archimedean component* of  $M$  generated by  $x$  is the subsemigroup

$$G_M[x] := \{a \in M : a \leq nx \text{ and } x \leq ma \text{ for some } n, m \in \mathbb{N}\}.$$

For any  $x \in M$ ,  $G_M[x]$  is a simple semigroup. If  $M$  is separative, then  $G_M[x]$  is a cancellative semigroup; if moreover  $x$  is a regular element, then  $G_M[x]$  is an abelian group.

### 1.2 Primely generated refinement monoids

The structure of primely generated refinement monoids has been recently described in [11]. We recall here some basic facts.

Given a poset  $(I, \leq)$ , we say that a subset  $A$  of  $I$  is a *lower set* if  $x \leq y$  in  $I$  and  $y \in A$  implies  $x \in A$ . For any  $i \in I$ , we will denote by  $I \downarrow i = \{x \in I : x \leq i\}$  the lower subset generated by  $i$ . We will write  $x < y$  if  $x \leq y$  and  $x \neq y$ .

The following definition is crucial for this work:

**Definition 1.1** ([11, Definition 1.1]) Let  $I = (I, \leq)$  be a poset. An *I-system*

$$\mathcal{J} = (I, \leq, (G_i)_{i \in I}, \varphi_{ji} (i < j))$$

is given by the following data:

- (a) A partition  $I = I_{free} \sqcup I_{reg}$  (we admit one of the two sets  $I_{free}$  or  $I_{reg}$  to be empty).
- (b) A family  $\{G_i\}_{i \in I}$  of abelian groups. We adopt the following notation:
  - (1) For  $i \in I_{reg}$ , set  $M_i = G_i$ , and  $\widehat{G}_i = G_i = M_i$ .
  - (2) For  $i \in I_{free}$ , set  $M_i = \mathbb{N} \times G_i$ , and  $\widehat{G}_i = \mathbb{Z} \times G_i$

Observe that, in any case,  $\widehat{G}_i$  is the Grothendieck group of  $M_i$ .

- (c) A family of semigroup homomorphisms  $\varphi_{ji}: M_i \rightarrow G_j$  for all  $i < j$ , to which we associate, for all  $i < j$ , the unique extension  $\widehat{\varphi}_{ji}: \widehat{G}_i \rightarrow G_j$  of  $\varphi_{ji}$  to a group homomorphism from the Grothendieck group of  $M_i$  to  $G_j$  (we look at these maps as maps from  $\widehat{G}_i$  to  $\widehat{G}_j$ ). We require that the family  $\{\varphi_{ji}\}$  satisfies the following conditions:
  - (1) The assignment

$$\left\{ \begin{array}{l} i \mapsto \widehat{G}_i \\ (i < j) \mapsto \widehat{\varphi}_{ji} \end{array} \right\}$$

defines a functor from the category  $I$  to the category of abelian groups (where we set  $\widehat{\varphi}_{ii} = \text{id}_{\widehat{G}_i}$  for all  $i \in I$ ).

- (2) For each  $i \in I_{free}$  we have that the map

$$\bigoplus_{k < i} \varphi_{ik} : \bigoplus_{k < i} M_k \rightarrow G_i$$

is surjective.

We say that an  $I$ -system  $\mathcal{J} = (I, \leq, (G_i)_{i \in I}, \varphi_{ji} (i < j))$  is *finitely generated* in case  $I$  is a finite poset and all the groups  $G_i$  are finitely generated.

To every  $I$ -system  $\mathcal{J}$  one can associate a primely generated conical refinement monoid  $M(\mathcal{J})$ , and conversely to any primely generated conical refinement monoid  $M$ , we can associate an  $I$ -system  $\mathcal{J}$  such that  $M \cong M(\mathcal{J})$ , see Sects. 1 and 2 of [11] respectively.

### 1.3 Separated graphs

Here, we recall definitions and properties about separated graphs that will be needed in the sequel. In particular, we define the notion of *adaptable* separated graph, which is crucial for this paper. We refer the reader to [1,8] for more information and general notation about (separated) graphs.

Let  $E$  be a directed graph, and let  $\leq$  be the preorder on  $E^0$  determined by  $w \geq v$  if there is a path in  $E$  from  $w$  to  $v$ . Let  $I$  be the antisymmetrization of  $E^0$ , with the partial order  $\leq$  induced by the order on  $E^0$ . Thus, denoting by  $[v]$  the class of  $v \in E^0$  in  $I$ , we have  $[v] \leq [w]$  if and only if  $v \leq w$ .

For  $v \in E^0$ , we refer to the set  $[v]$  as the component of  $v$ , and we will denote by  $E[v]$  the restriction of  $E$  to  $[v]$ , that is, the graph with  $E[v]^0 = [v]$  and  $E[v]^1 = \{e \in$

$E^1 \mid s(e) \in [v] \text{ and } r(e) \in [v]$ . If  $J$  is a lower subset of  $I$ , we will denote by  $E|_J$  the restriction of the graph  $E$  to the set of vertices  $\{v \in E^0 \mid [v] \in J\}$ .

We now describe our graphs.

**Definition 1.2** ([8, Definition 2.1]) A *separated graph* is a pair  $(E, C)$  where  $E$  is a directed graph,  $C = \bigsqcup_{v \in E^0} C_v$ , and  $C_v$  is a partition of  $s^{-1}(v)$  (into pairwise disjoint nonempty subsets) for every vertex  $v$ . (In case  $v$  is a sink, we take  $C_v$  to be the empty family of subsets of  $s^{-1}(v)$ ).

If all the sets in  $C$  are finite, we shall say that  $(E, C)$  is a *finitely separated graph*.

From now on, we will assume that all our separated graphs are finitely separated graphs without any further comment.

Following [8], we associate the following monoid to any finitely separated graph.

**Definition 1.3** ([8, Definition 4.1]) Given a finitely separated graph  $(E, C)$ , we define the monoid of the separated graph  $(E, C)$ , to be

$$M(E, C) = \left\langle a_v \ (v \in E^0) : a_v = \sum_{\{e \in X\}} a_{r(e)} \text{ for every } X \in C_v, v \in E^0 \right\rangle. \quad (1.1)$$

Remind that a directed graph is said to be *transitive* if any two vertices can be connected by a finite directed path.

The following definition is the milestone of the current paper. We arrived at it via a distillation of the methods used in [11, 12], with the help of the insight provided by [8, 13]. Recall that it was shown in [13] (see also [12]) that not all finitely generated refinement monoids arise as graph monoids, and it was shown in [8] that not all monoids  $M(E, C)$ , for a separated graph  $(E, C)$ , are refinement monoids. Hence, in order to accomplish our aim of obtaining a combinatorial description of all finitely generated conical refinement monoids, we are forced to downsize the class of separated graphs under consideration. The main idea is to find a class of separated graphs which, on one hand, is big enough to represent all the finitely generated conical refinement monoids, and, on the other hand, is well behaved in the sense that all the graph monoids of separated graphs in the class are primely generated refinement monoids. This is achieved by the subsequent notion of *adaptable* separated graph.

**Definition 1.4** Let  $(E, C)$  be a finitely separated graph and let  $(I, \leq)$  be the antisymmetrization of  $(E^0, \leq)$ . We say that  $(E, C)$  is *adaptable* if  $I$  is finite, and there exist a partition  $I = I_{\text{free}} \sqcup I_{\text{reg}}$ , and a family of subgraphs  $\{E_p\}_{p \in I}$  of  $E$  such that the following conditions are satisfied:

- (1)  $E^0 = \bigsqcup_{p \in I} E_p^0$ , where  $E_p$  is a transitive row-finite graph if  $p \in I_{\text{reg}}$  and  $E_p^0 = \{v^p\}$  is a single vertex if  $p \in I_{\text{free}}$ .
- (2) For  $p \in I_{\text{reg}}$  and  $w \in E_p^0$ , we have that  $|C_w| = 1$  and  $|s_{E_p}^{-1}(w)| \geq 2$ . Moreover, all edges departing from  $w$  either belong to the graph  $E_p$  or connect  $w$  to a vertex  $u \in E_q^0$ , with  $q < p$  in  $I$ .
- (3) For  $p \in I_{\text{free}}$ , we have that  $s^{-1}(v^p) = \emptyset$  if and only if  $p$  is minimal in  $I$ . If  $p$  is not minimal, then there is a positive integer  $k(p)$  such that  $C_{v^p} = \{X_1^{(p)}, \dots, X_{k(p)}^{(p)}\}$ .

Moreover, each  $X_i^{(p)}$  is of the form

$$X_i^{(p)} = \{\alpha(p, i), \beta(p, i, 1), \beta(p, i, 2), \dots, \beta(p, i, g(p, i))\},$$

for some  $g(p, i) \geq 1$ , where  $\alpha(p, i)$  is a loop, i.e.,  $s(\alpha(p, i)) = r(\alpha(p, i)) = v^p$ , and  $r(\beta(p, i, t)) \in E_q^0$  for  $q < p$  in  $I$ . Finally, we have  $E_p^1 = \{\alpha(p, 1), \dots, \alpha(p, k(p))\}$ .

The edges connecting a vertex  $v \in E_p^0$  to a vertex  $w \in E_q^0$  with  $q < p$  in  $I$  will be called *connectors*. □

## 2 Adaptable separated graphs and their associated monoids

In this section we show the main result of the paper:

**Theorem 2.1** *The following two statements hold:*

- (1) *If  $(E, C)$  is an adaptable separated graph, then  $M(E, C)$  is a primely generated conical refinement monoid.*
- (2) *For any finitely generated conical refinement monoid  $M$ , there exists an adaptable separated graph  $(E, C)$  such that  $M \cong M(E, C)$ .*

We have divided the proof in two parts. First we show statement (1) (Proposition 2.6), and, subsequently, we show the realization result stated in (2) (Theorem 2.11).

### 2.1 The monoid of an adaptable separated graph

We show below that the monoid  $M(E, C)$  associated to an adaptable separated graph  $(E, C)$  is a primely generated conical refinement monoid. As a consequence, we obtain from [11, Theorem 2.7] that there is a poset  $\mathbb{P}$ , with a partition  $\mathbb{P} = \mathbb{P}_{\text{free}} \sqcup \mathbb{P}_{\text{reg}}$ , and a  $\mathbb{P}$ -system  $\mathcal{J}$  such that  $M(E, C) \cong M(\mathcal{J})$ . We will explicitly determine this system.

To show our results, we will need the “confluence” property of the congruence associated to our separated graphs  $(E, C)$ . This was established for all graph monoids  $M(E)$  of ordinary row-finite graphs in [10, Lemma 4.3]. Amongst other things, this enables us to show the refinement property of the monoids  $M(E, C)$ , when  $(E, C)$  is an adaptable separated graph.

Let  $(E, C)$  be an adaptable separated graph, and  $F$  be the free commutative monoid on the set  $E^0$ . The nonzero elements of  $F$  can be written in a unique form up to permutation as  $\sum_{i=1}^n v_i$ , where  $v_i \in E^0$  (repetition of elements is allowed). Now we will give a description of the congruence on  $F$  generated by the relations (1.1) (see Definition 1.3) on  $F$ .

It will be convenient to introduce the following notation. For  $X \in C_v$  ( $v \in E^0$ ), write

$$\mathbf{r}(X) := \sum_{e \in X} r(e) \in F.$$



With this new notation, the relations in (1.1) become  $v = \mathbf{r}(X)$  for every  $v \in E^0$  and every  $X \in C_v$ .

**Definition 2.2** Define a binary relation  $\rightarrow_1$  on  $F \setminus \{0\}$  as follows. Let  $\sum_{i=1}^n v_i \in F \setminus \{0\}$ , and let  $X \in C_{v_j}$  for some  $j \in \{1, 2, \dots, n\}$ . Then  $\sum_{i=1}^n v_i \rightarrow_1 \sum_{i \neq j} v_i + \mathbf{r}(X)$ . Let  $\rightarrow$  be the transitive and reflexive closure of  $\rightarrow_1$  on  $F \setminus \{0\}$ , that is,  $\alpha \rightarrow \beta$  if and only if there is a finite string  $\alpha = \alpha_0 \rightarrow_1 \alpha_1 \rightarrow_1 \dots \rightarrow_1 \alpha_n = \beta$ .

Let  $\sim$  be the congruence on  $F$  generated by the relation  $\rightarrow_1$  (or, equivalently, by the relation  $\rightarrow$ ). Namely  $\alpha \sim \alpha'$  for all  $\alpha \in F$  and, for  $\alpha, \beta \neq 0$ , we have  $\alpha \sim \beta$  if and only if there is a finite string  $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$ , such that, for each  $i = 0, \dots, n - 1$ , either  $\alpha_i \rightarrow_1 \alpha_{i+1}$  or  $\alpha_{i+1} \rightarrow_1 \alpha_i$ . The number  $n$  above will be called the *length* of the string. □

It is clear that  $\sim$  is the congruence on  $F$  generated by relations (1.1), and so  $M(E, C) = F/\sim$ .

The *support* of an element  $\gamma$  in  $F$ , denoted  $\text{supp}(\gamma) \subseteq E^0$ , is the set of basis elements appearing in the canonical expression of  $\gamma$ .

The proof of the following easy lemma is similar to the one of [10, Lemma 4.2].

**Lemma 2.3** (cf. [10, Lemma 4.2]) *Let  $(E, C)$  be any finitely separated graph. Let  $\rightarrow$  be the binary relation on  $F$  defined above and  $\alpha, \beta \in F \setminus \{0\}$ . Assume that  $\alpha = \alpha_1 + \alpha_2$  and  $\alpha \rightarrow \beta$ . Then  $\beta$  can be written as  $\beta = \beta_1 + \beta_2$ , with  $\alpha_1 \rightarrow \beta_1$  and  $\alpha_2 \rightarrow \beta_2$ .*

We are now ready to obtain the crucial lemma that gives the important “confluence” property of the congruence  $\sim$  on the free commutative monoid  $F$ .

**Lemma 2.4** *Let  $(E, C)$  be an adaptable separated graph. Let  $\alpha$  and  $\beta$  be nonzero elements in  $F$ . Then  $\alpha \sim \beta$  if and only if there is  $\gamma \in F$  such that  $\alpha \rightarrow \gamma$  and  $\beta \rightarrow \gamma$ .*

**Proof** The proof is similar to the proof of [10, Lemma 4.3]; however, we provide the majority of details by completeness.

Assume that  $\alpha \sim \beta$ . Then there exists a finite string  $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$  such that, for each  $i = 0, \dots, n - 1$ , either  $\alpha_i \rightarrow_1 \alpha_{i+1}$  or  $\alpha_{i+1} \rightarrow_1 \alpha_i$ . We proceed by induction on  $n$ . If  $n = 0$ , then  $\alpha = \beta$  and there is nothing to prove. Assume the result is true for strings of length  $n - 1$ , and let  $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$  be a string of length  $n$ . By induction hypothesis, there is  $\lambda \in F$  such that  $\alpha \rightarrow \lambda$  and  $\alpha_{n-1} \rightarrow \lambda$ . Now there are two cases to consider. If  $\beta \rightarrow_1 \alpha_{n-1}$ , then  $\beta \rightarrow \lambda$  and we are done. Assume that  $\alpha_{n-1} \rightarrow_1 \beta$ . By definition of  $\rightarrow_1$ , there is a basis element  $v \in E^0$  in the support of  $\alpha_{n-1}$  and  $X \in C_v$  such that  $\alpha_{n-1} = v + \alpha'_{n-1}$  and  $\beta = \mathbf{r}(X) + \alpha'_{n-1}$ . By Lemma 2.3, we have  $\lambda = \lambda(v) + \lambda'$ , where  $v \rightarrow \lambda(v)$  and  $\alpha'_{n-1} \rightarrow \lambda'$ . If the length of the string from  $v$  to  $\lambda(v)$  is positive, then we have  $\mathbf{r}(Y) \rightarrow \lambda(v)$  for some  $Y \in C_v$ . If  $[v] \in I_{\text{reg}}$ , then  $X = Y$  and the proof is the same as in [10, Lemma 4.3]. If  $[v] \in I_{\text{free}}$ , then  $X$  may be distinct from  $Y$ , but in this case we play with the special form of the sets in  $C_v$ . Indeed, assume that  $[v] \in I_{\text{free}}$ . In this case, write  $\lambda'' := \lambda + (\mathbf{r}(X) - v)$ . Then we have

$$\begin{aligned} \beta &= \mathbf{r}(X) + \alpha'_{n-1} = v + (\mathbf{r}(X) - v) + \alpha'_{n-1} \\ &\rightarrow_1 \mathbf{r}(Y) + (\mathbf{r}(X) - v) + \alpha'_{n-1} \end{aligned}$$

$$\begin{aligned} &\rightarrow \lambda(v) + \alpha'_{n-1} + (\mathbf{r}(X) - v) \\ &\rightarrow \lambda(v) + \lambda' + (\mathbf{r}(X) - v) \\ &= \lambda + (\mathbf{r}(X) - v) = \lambda''. \end{aligned}$$

On the other hand, since  $v + \alpha'_{n-1} \rightarrow \lambda$  and since  $[v] \in I_{\text{free}}$ , it follows easily by induction on the length of this string that  $v \in \text{supp}(\lambda)$  and thus  $\lambda \rightarrow_1 \lambda + (\mathbf{r}(X) - v) = \lambda''$ . Hence  $\alpha \rightarrow \lambda \rightarrow \lambda''$  and  $\beta \rightarrow \lambda''$ , as desired.

In the remaining case that  $v = \lambda(v)$ , set  $\gamma = \mathbf{r}(X) + \lambda'$ . Then we have  $\lambda \rightarrow_1 \gamma$  and so  $\alpha \rightarrow \gamma$ , and also  $\beta = \mathbf{r}(X) + \alpha'_{n-1} \rightarrow \mathbf{r}(X) + \lambda' = \gamma$ . This concludes the proof.  $\square$

Now, exactly the same proof as in [10, Proposition 4.4] (using Lemmas 2.3 and 2.4) gives the following result.

**Proposition 2.5** *Let  $(E, C)$  be an adaptable separated graph. Then the monoid  $M(E, C)$  is a refinement monoid.*

We now show that, for any adaptable separated graph, the monoid  $M(E, C)$  is a primely generated monoid.

**Proposition 2.6** *Let  $(E, C)$  be an adaptable separated graph and let  $(I, \leq)$  be the antisymmetrization of  $E^0$  with respect to the path-way pre-order. Then  $M(E, C)$  is a primely generated conical refinement monoid.*

**Proof** By [8, Lemma 4.2],  $M(E, C)$  is a nonzero, conical monoid whenever  $(E, C)$  is an arbitrary finitely separated graph such that  $E^0$  is non-empty.

Suppose now that  $(E, C)$  is an adaptable separated graph. By Proposition 2.5,  $M(E, C)$  is a refinement monoid. We now show that  $M(E, C)$  is primely generated. For this, it is enough to observe that each generator  $a_v$ , with  $v \in E^0$ , is prime in  $M(E, C)$ . For this purpose, we work in the free monoid  $F$  generated by  $E^0$  and we use the notation introduced above. We denote by  $\bar{\alpha}$  the class of an element  $\alpha \in F$  in  $M(E, C) = F/\sim$ . Note that  $a_v = \bar{v}$  for  $v \in E^0$ . We have to show that if we have a relation  $\bar{v} + \bar{\delta} = \bar{\alpha}_1 + \bar{\alpha}_2$  in  $F/\sim = M(E, C)$ , then there is  $i \in \{1, 2\}$  such that  $\bar{v} \leq \bar{\alpha}_i$ . Now since  $v + \delta \sim \alpha_1 + \alpha_2$  in  $F$ , we have by Lemma 2.4 that there is  $\gamma \in F$  such that  $v + \delta \rightarrow \gamma$  and  $\alpha_1 + \alpha_2 \rightarrow \gamma$ . We claim that there is  $\gamma' \in F$  such that  $\gamma \rightarrow \gamma'$  and  $v$  belongs to the support of  $\gamma'$ . If  $[v] \in I_{\text{free}}$ , this is clear already for  $\gamma' = \gamma$ , because each  $X \in C_v$  contains a loop. If  $p := [v] \in I_{\text{reg}}$ , then since  $|s_{E_p}^{-1}(w)| \geq 1$  for all  $w \in E_p^0$ , it follows that the support of  $\gamma$  contains a vertex  $w \in E_p^0$ . Now using that  $E_p$  is transitive, we see that there is a finite path in  $E_p$  joining  $w$  to  $v$ , and using this path, we find  $\gamma' \in F$  with  $\gamma \rightarrow \gamma'$  and  $v \in \text{Supp}(\gamma')$ .

Hence, replacing  $\gamma$  with  $\gamma'$  if necessary, we may assume that  $v$  belongs to the support of  $\gamma$ . By Lemma 2.3, we can write  $\gamma = \gamma_1 + \gamma_2$  with  $\alpha_i \rightarrow \gamma_i$  for  $i = 1, 2$ . Therefore, we get that  $v$  belongs to the support of  $\gamma_i$  for some  $i \in \{1, 2\}$ . We can thus write  $\gamma_i = v + \gamma'_i$  for some  $\gamma'_i \in F$  and so

$$\bar{\alpha}_i = \bar{\gamma}_i = \bar{v} + \bar{\gamma}'_i,$$

showing that  $\bar{v} \leq \bar{\alpha}_i$  for some  $i \in \{1, 2\}$ , as desired.  $\square$

It follows from Proposition 2.6 and [11, Theorem 2.7] that for any adaptable separated graph  $(E, C)$  there exists a poset  $\mathbb{P}$ , a partition  $\mathbb{P} = \mathbb{P}_{\text{free}} \sqcup \mathbb{P}_{\text{reg}}$ , and a  $\mathbb{P}$ -system  $\mathcal{J}$  such that  $M(E, C) \cong M(\mathcal{J})$ . We close this subsection by explicitly computing this system. Together with our main result in the next subsection (Theorem 2.11), this allows us to express all the structure of a finitely generated conical refinement monoid in terms of the information contained in a representing adaptable separated graph.

Let  $(E, C)$  be an adaptable separated graph and let  $(I, \leq)$  be the antisymmetrization of  $E^0$  with respect to the path-way pre-order. In order to neatly express our result, we first define a certain  $I$ -system and then we will show it is isomorphic to the system corresponding to  $M(E, C)$ .

**Definition 2.7** Let  $(E, C)$  be an adaptable separated graph, let  $(I, \leq)$  be the antisymmetrization of  $E^0$ , and let  $I = I_{\text{free}} \sqcup I_{\text{reg}}$  be the canonical partition of  $I = E^0 / \sim$  (see Definition 1.4). Define an  $I$ -system  $\mathcal{J}'' = (I, \leq, (G''_p)_{p \in I}, \varphi''_{p,q} (q < p))$  as follows: (1) For each  $p \in I_{\text{free}}$  minimal, define  $G''_p := \{0\}$  (i.e.  $M_p = \mathbb{N}$ ). Now for each non-minimal  $p \in I_{\text{free}}$ , consider the abelian group  $G''_p$  generated by elements  $x''_w$ , where  $w$  is a vertex in  $E$  such that  $[w] < p = [v^p]$ , subject to the relations

$$x''_w = \sum_{e \in s_E^{-1}(w)} x''_{r(e)}, \quad [w] \in I_{\text{reg}}, \tag{2.1}$$

and

$$\sum_{j=1}^{g(q,i)} x''_{r(\beta(q,i,j))} = 0, \quad (i = 1, \dots, k(q)) \text{ for } q \in I_{\text{free}}, q \leq p. \tag{2.2}$$

(2) For  $p \in I_{\text{reg}}$ , we let  $G''_p$  be the abelian group with generators  $x''_w$ , where  $w$  is a vertex in  $E$  such that  $[w] \leq p$ , and with relations (2.1) for every  $w \in E^0$  such that  $[w] \in I_{\text{reg}}$  and  $[w] \leq p$ , and (2.2) for every  $q \in I_{\text{free}}$  with  $q < p$ .

Recalling that  $M''_p = G''_p$  if  $p \in I_{\text{reg}}$  and  $M''_p = \mathbb{N} \times G''_p$  if  $p \in I_{\text{free}}$ , we now define the connecting homomorphisms  $\varphi''_{p,q} : M''_q \rightarrow G''_p$ , for  $q < p$ , as follows:

$$\varphi''_{p,q}(x''_w) = x''_w, \quad \text{if } q \in I_{\text{reg}},$$

and

$$\varphi''_{p,q} \left( n, \sum_{w < v^q} c_w x''_w \right) = n x''_{v^q} + \sum_{w < v^q} c_w x''_w, \quad \text{if } q \in I_{\text{free}}$$

where  $n \in \mathbb{N}$ , and  $c_w \in \mathbb{Z}$  are almost all 0. It is straightforward to show that  $\mathcal{J}''$  is an  $I$ -system.

**Remark 2.8** Note that, in case  $p \in I_{\text{reg}}$ , the relations in  $G''_p$  can be expressed in the form  $x''_w = \sum_{e \in X} x''_{r(e)}$ , for each  $X \in C_w$  and each  $w \in E^0$  such that  $[w] \leq p$ . The resulting group is therefore the Grothendieck group of the monoid  $M(E_H, C^H)$ , where  $(E_H, C^H)$  is the restriction of the separated graph  $(E, C)$  to the hereditary set

$H := \{w \in E^0 : [w] \leq p\}$ . However, this is not the case when  $p \in I_{\text{free}}$ , due to the fact that, in that case, we are only considering generators  $x_w^p$  for  $w \in E^0$  such that  $[w] < p$ .

We now recall the crucial steps in the construction of the canonical  $\mathbb{P}$ -system associated to a primely generated conical refinement monoid  $M$ , see [11] for details. The set of primes of  $M$  is denoted by  $\mathbb{P}(M)$ . Let  $\overline{M}$  be the antisymmetrization of  $M$ , i.e. the quotient monoid of  $M$  by the congruence given by  $x \equiv y$  if and only if  $x \leq y$  and  $y \leq x$ . We will denote the class of an element  $x$  of  $M$  in  $\overline{M}$  by  $\overline{x}$ . The monoid  $\overline{M}$  is an antisymmetric primely generated refinement monoid and an element  $p$  in  $M$  is prime (resp. free, resp. regular) in  $M$  if and only if  $\overline{p}$  is prime (resp. free, resp. regular) in  $\overline{M}$  ([11, Lemma 2.1]).

For  $x, y \in M$ , we will write  $x <^* y$  when  $\overline{x} < \overline{y}$  in  $\overline{M}$ . We write  $x \leq^* y$  if either  $x <^* y$  or  $x = y$ .

We are ready to define the  $\mathbb{P}$ -system associated to a primely generated conical refinement monoid  $M$ :

(1) We choose, for each prime  $\overline{p}$  of  $\overline{M}$ , a representative  $p$  of  $\overline{p}$  in  $M$ , and we consider the set  $\mathbb{P}$  formed by the set of all the elements  $p$  obtained in this way. The chosen poset is  $\mathbb{P}$  endowed with the partial order  $\leq^*$ . Note that  $(\mathbb{P}, \leq^*)$  is order-isomorphic with  $(\overline{\mathbb{P}}, \leq) = (\mathbb{P}(\overline{M}), \leq)$ . We have a partition  $\mathbb{P} = \mathbb{P}_{\text{free}} \sqcup \mathbb{P}_{\text{reg}}$  into free and regular primes.

(2) For each  $p \in \mathbb{P}$ , let  $M_p$  be the archimedean component of  $p$ . We separate two cases:

- (i) If  $p$  is regular then  $M_p$  is an abelian group (see e.g. [14, Lemma 2.7]), denoted by  $G_p$ . In this case, we choose as the canonical representative of  $\overline{p}$  the only idempotent element lifting  $\overline{p}$ , i.e. the unit of the group  $M_p$ .
- (ii) If  $p$  is free, we define

$$G'_p = \{p + \alpha : \alpha \in M \text{ and } p + \alpha \leq p\}.$$

Then  $G'_p$  is a group with respect to the operation  $\circ$  given by:

$$(p + \alpha) \circ (p + \beta) = p + (\alpha + \beta)$$

(see [14, Definition 2.8 & Lemma 2.9]). We have that  $M_p \cong \mathbb{N} \times (G'_p, \circ)$  ([11, Lemma 2.4]). Moreover we can identify the group  $(G'_p, \circ)$  with the subgroup  $G_p := \{(p + \alpha) - p : p + \alpha \in G'_p\}$  of the Grothendieck group  $G(M_p)$  and thus we get  $M_p \cong \mathbb{N} \times G_p$  ([11, Remark 2.5]).

This gives the description of the family of abelian groups  $\{G_p\}_{p \in \mathbb{P}}$  for our  $\mathbb{P}$ -system. Finally the maps  $\varphi_{pq} : M_q \rightarrow G_p$  for  $q <^* p \in \mathbb{P}$  are defined by

$$\varphi_{pq}(x) = (p + x) - p \in G_p,$$

for  $x \in M_q$ . This completes the definition of the  $\mathbb{P}$ -system  $\mathcal{J}_M = (\mathbb{P}, \leq^*, (G_p)_{p \in \mathbb{P}}, \varphi_{pq} (q <^* p))$  canonically associated to  $M$ .

**Proposition 2.9** *Let  $(E, C)$  be an adaptable separated graph, let  $I = I_{\text{free}} \sqcup I_{\text{reg}}$  be the canonical partition of  $I = E^0 / \sim$ , and let  $\mathcal{J}''$  be the  $I$ -system of Definition 2.7. Let  $\mathbb{P} = \mathbb{P}_{\text{free}} \sqcup \mathbb{P}_{\text{reg}}$  be the poset associated to  $M(E, C)$ , and  $\mathcal{J} = (\mathbb{P}, \leq, (G_p)_{p \in \mathbb{P}}, \varphi_{p,q} (q < p))$  be the corresponding  $\mathbb{P}$ -system. Then there is an isomorphism of systems  $\mathcal{J}'' \cong \mathcal{J}$ . In particular*

$$M(E, C) \cong M(\mathcal{J}) \cong M(\mathcal{J}'').$$

**Proof** Since  $M(E, C)$  is a primely generated conical refinement monoid, there is a  $\mathbb{P}$ -system  $\mathcal{J}$  such that  $M(E, C) \cong M(\mathcal{J})$ . We follow the description of this  $\mathbb{P}$ -system given above.

The first thing we do is to identify  $\mathbb{P}$  with  $I$ . Let us define a relation  $\triangleleft$  on  $I$  as follows. For  $p, q \in I$ , set  $p \triangleleft q$  if  $p < q$  or  $p = q \in I_{\text{reg}}$ . Observe that  $\triangleleft$  is an antisymmetric and transitive relation on  $I$ . Now define the monoid  $M(I, \triangleleft)$  as the commutative monoid with family of generators  $I$  and with relations  $q = q + p$  if  $p \triangleleft q$ . The monoid  $M(I, \triangleleft)$  is an antisymmetric finitely generated refinement monoid, and its set of primes is precisely  $I$ . Moreover, the regular (resp. free) primes of  $M(I, \triangleleft)$  are exactly the elements in  $I_{\text{reg}}$  (resp.  $I_{\text{free}}$ ). Now, it is straightforward, using the defining properties of an adaptable separated graph, to show that the antisymmetrization  $\overline{M(E, C)}$  of  $M(E, C)$  is isomorphic to  $M(I, \triangleleft)$ , sending  $\overline{a_v} \in \overline{M(E, C)}$  to  $[v] \in M(I, \triangleleft)$ . It follows from the description of the poset  $\mathbb{P}$  associated to  $M(E, C)$  and the above observations that  $\mathbb{P}$ , with its canonical partition  $\mathbb{P} = \mathbb{P}_{\text{free}} \sqcup \mathbb{P}_{\text{reg}}$ , can be identified with  $I$ , and its partition  $I = I_{\text{free}} \sqcup I_{\text{reg}}$ . Hence, the construction in [11, Section 2] gives rise to an  $I$ -system  $\mathcal{J} = (I, \leq, \{G_p\}_{p \in I}, \varphi_{p,q} (q < p))$ .

It remains to identify the groups  $G_p$ , for  $p \in I$ , and the maps  $\varphi_{p,q} : M_q \rightarrow G_p$  for  $q < p$ . First, we observe that every hereditary subset of  $E^0$  is  $C$ -saturated, because  $|s_{E_p}^{-1}(v)| \geq 1$  for each non-sink  $v \in E^0$ , and all subgraphs  $E_p$  of  $E$  are transitive. Therefore it follows from [8, Corollary 6.10] that the order-ideal of  $M(E, C)$  generated by a hereditary subset  $H$  of  $E^0$  is generated as a monoid by  $\{a_v : v \in H\}$ .

The group  $G'_p$  is defined for each  $p \in I_{\text{free}}$  to be the set

$$\{a_{v^p} + \alpha : \alpha \in M(E, C) \text{ and } a_{v^p} + \alpha \leq a_{v^p}\},$$

endowed with the product  $(a_{v^p} + \alpha) \circ (a_{v^p} + \beta) = a_{v^p} + (\alpha + \beta)$ . We want to show that  $G''_p \cong G'_p$ . To this end we define a map  $\lambda_p : G''_p \rightarrow G'_p$  by  $\lambda_p(x_w^p) = a_{v^p} + a_w$  for  $w \in E^0$  with  $[w] < p$ . Clearly, the defining relations of  $G''_p$  are preserved by  $\lambda_p$ , so this assignment defines a group homomorphism. Now if  $\alpha \in M(E, C)$  and  $a_{v^p} + \alpha \leq a_{v^p}$ , then  $\alpha$  belongs to the order-ideal of  $M(E, C)$  generated by  $a_{v^p}$ , and by the previous remark, it follows that  $\alpha$  must be a sum of elements of the form  $a_w$  with  $w \leq v^p$ . Now, it follows from the fact that  $a_{v^p}$  is free that  $\alpha$  is a sum of elements of the form  $a_w$  with  $w < v^p$ . This shows that  $\lambda_p$  is surjective. In order to show that  $\lambda_p$  is injective, let  $\sum_{w \in A} n_w x_w^p - \sum_{w' \in B} m_{w'} x_{w'}^p$  be an element in the kernel of  $\lambda_p$ , where  $A \cap B = \emptyset$ , and  $n_w, m_{w'} > 0$ . It then follows that  $a_{v^p} + \sum_{w \in A} n_w a_w = a_{v^p} + \sum_{w' \in B} m_{w'} a_{w'}$  in  $M(E, C)$ . Let  $F$  be the free commutative monoid generated by  $E^0$ . It follows from Lemma 2.4 that there is  $\gamma \in F$  such that  $v^p + \sum_{w \in A} n_w w \rightarrow \gamma$  and  $v^p + \sum_{w' \in B} m_{w'} w' \rightarrow \gamma$  in  $F$ . Note that  $\gamma = v^p + \gamma'$ , where  $\gamma' = \sum_{w < v^p} l_w w$

for some  $l_w \geq 0$ . Now we transform  $\sum_{w \in A} n_w x_w^p$  using corresponding steps to the ones used in the transformation  $v^p + \sum_{w \in A} n_w w \rightarrow \gamma$ , replacing each occurrence of a step  $v^q \rightarrow_1 v^q + \sum_{j=1}^{g(q,i)} r(\beta(q, i, j))$  for some  $i = 1, \dots, k(q)$  by the identity  $0 = \sum_{j=1}^{g(q,i)} x_{r(\beta(q,i,j))}^p$  in  $G''_p$ , for each  $i = 1, \dots, k(q)$ , if  $[w] = q \in I_{\text{free}}$  and  $q \leq p$ , and each occurrence of a step  $w \rightarrow_1 \sum_{e \in s^{-1}(w)} r(e)$  by the identity  $x_w^p = \sum_{e \in s^{-1}(w)} x_{r(e)}^p$  if  $[w] \in I_{\text{reg}}$  and  $[w] < p$ . By using this process, we arrive at the identity  $\sum_{w \in A} n_w x_w^p = \sum_{w < v^p} l_w x_w^p$  in  $G''_p$ . With the same reasoning, we obtain  $\sum_{w' \in B} m_{w'} x_{w'}^p = \sum_{w < v^p} l_w x_w^p$ . So we get that  $\sum_{w \in A} n_w x_w^p - \sum_{w' \in B} m_{w'} x_{w'}^p = 0$ , as desired. Finally the group  $G_p$  is naturally isomorphic to  $G'_p$  through the map  $G'_p \rightarrow G_p, a_{v^p} + \alpha \mapsto (a_{v^p} + \alpha) - a_{v^p}$ , so we get the isomorphism  $G''_p \cong G_p$ , which sends  $x_w^p$  to  $(a_{v^p} + a_w) - a_{v^p}$ .

If  $p = [v] \in I_{\text{reg}}$ , then the archimedean component of  $a_v$  in  $M(E, C)$  is a group, and  $G_p$  is defined to be this group. Let  $e_p$  be the neutral element of  $G_p$ . Then one may check as before that the map  $\lambda_p : G''_p \rightarrow G_p$  given by  $x_w^p \mapsto e_p + a_w$  for  $[w] \leq p$ , is a group isomorphism.

Finally it is straightforward to show that  $\varphi_{p,q} \circ \tilde{\lambda}_q = \lambda_p \circ \varphi''_{p,q}$  whenever  $q < p$  in  $I$ , where  $\tilde{\lambda}_q : M''_q \rightarrow M_q$  is the map induced by  $\lambda_q$ . Hence we get an isomorphism of  $I$ -systems  $\mathcal{J}'' \cong \mathcal{J}$ . Since  $M(\mathcal{J}) \cong M(E, C)$  ([11, Theorem 2.7]), we get the last assertion in the statement. □

### 2.2 Representing finitely generated refinement monoids

In this subsection, given any finitely generated conical refinement monoid  $M$ , we build an adaptable separated graph  $(E, C)$  such that its associated monoid is isomorphic to  $M$ .

To this end recall from Sect. 1 and [11, Sections 1 and 2] that, given any finitely generated conical refinement monoid  $M$ , one canonically associates to it an  $I$ -system

$$\mathcal{J} = (I, \leq, (G_i)_{i \in I}, \varphi_{ji} \ (i < j))$$

such that  $M \cong M(\mathcal{J})$  ([11, Theorem 2.7]). Moreover, the  $I$ -system  $\mathcal{J}$  is finitely generated (meaning that  $I$  is finite and all the abelian groups  $G_i$  are finitely generated, [11, Proposition 2.9]).

We now recall the definition of  $M(\mathcal{J})$  for a finitely generated  $I$ -system  $\mathcal{J}$ , and some terminology and facts concerning our monoids. For more details and background the reader is referred to [11, 12, 14, 15]. Let  $A(I)$  be the lattice of lower subsets of the finite poset  $I$ . Then we have  $M(\mathcal{J}) = \bigsqcup_{a \in A(I)} M_a$ , where  $M_a$  are the semigroups defined as follows (adopting the notation from 1.1). For  $a \in A(I)$ , we define  $\widehat{H}_a = \bigoplus_{i \in a} \widehat{G}_i$ . Let  $H_a$  be the subsemigroup of  $\widehat{H}_a$  defined by

$$H_a = \left\{ (z_i)_{i \in a} \in \widehat{H}_a : z_i \in \begin{cases} \mathbb{N} \times G_i & \text{for } i \in \text{Max}(a)_{\text{free}} \\ \{(0, 0_i)\} \cup (\mathbb{N} \times G_i) & \text{for } i \in a_{\text{free}} \setminus \text{Max}(a)_{\text{free}} \end{cases} \right\}.$$

We now define  $M_a := H_a/\equiv$ , where  $\equiv$  is the congruence on  $H_a$  generated by the pairs  $(x + \chi(a, i, \alpha), x + \chi(a, j, \varphi_{ji}(\alpha)))$ , for  $x \in H_a, i < j \in \text{Max}(a)$  and  $\alpha \in M_i$ . (Here for  $k \in a$  and  $\beta \in \widehat{G}_k$ , we denote by  $\chi(a, k, \beta)$  the element of  $\widehat{H}_a$  which has  $\beta$  in the component  $\widehat{G}_k$  and the neutral element  $e_l \in \widehat{G}_l$  in all the other components  $\widehat{G}_l, l \neq k$ .)

For  $i \in I$ , we define the lower cover  $L(I, i)$  of  $i$  in  $I$  as

$$L(I, i) := \{j \in I \mid j < i \text{ and } [j, i] = \{j, i\}\}.$$

Let  $p \in I_{\text{free}}$  and let  $L(I, p) = \{q_1, \dots, q_n\}$  be its lower cover. The archimedean component  $M_p$  of  $p$  has the form  $M_p = \mathbb{N} \times G_p$  for the finitely generated abelian group  $G_p$ . Note that  $M_p = M_{I \downarrow p}$  by [12, Lemma 2.8].

Using the notation established in [12, Section 2], we denote by  $J_p$  the lower subset of  $I$  generated by  $q_1, \dots, q_n$ , and let  $M_{J_p}$  be the associated semigroup, as described above (see also [12, Corollary 2.4]). Then, by [12, Lemma 5.1], there is a surjective semigroup homomorphism

$$\varphi_p: M_{J_p} \rightarrow G_p$$

which is induced by the various maps  $\varphi_{pq}$  for  $q < p$ . Consequently, we obtain a surjective group homomorphism  $G(\varphi_p): G(M_{J_p}) \rightarrow G_p$ . We say that an element  $x$  in  $G(M_{J_p})$  is strictly positive if it belongs to the image of the canonical map  $\iota_{J_p}: M_{J_p} \rightarrow G(M_{J_p})$ . We write  $G(M_{J_p})^{++} = \iota_{J_p}(M_{J_p})$  for the set of strictly positive elements.

With the notation above, we provide the last proposition needed for Theorem 2.11.

**Proposition 2.10** *With the above notation and caveats, we have that the kernel of  $G(\varphi_p)$  is generated by a finite number  $x_1, \dots, x_k$  of strictly positive elements.*

**Proof** Since  $G(M_{J_p})$  is a finitely generated abelian group, we have that the kernel of  $G(\varphi_p)$  is generated by a finite number of elements  $y_1, \dots, y_m$ . So, it is enough to show that the subgroup generated by an element  $y$  in the kernel of  $G(\varphi_p)$  is contained in the subgroup generated by two strictly positive elements in the kernel of  $G(\varphi_p)$ .

Recall that  $L(I, p) = \{q_1, \dots, q_n\}$  is the lower cover of  $p$ . We assume that  $q_1, \dots, q_r$  are free and that  $q_{r+1}, \dots, q_n$  are regular. Now, let  $y \in \ker(G(\varphi_p))$ . Using that the element  $y$  can be expressed as a difference of two elements from  $G(M_{J_p})^{++}$  and [12, Lemma 5.3], we see that there exist positive integers  $n_i, m_i, i = 1, \dots, r$ , and elements  $g_i \in G_{q_i}, i = 1, \dots, n, h_j \in G_{q_j}, j = 1, \dots, r$ , such that

$$y = \iota_{J_p} \left( \sum_{i=1}^r \chi_{q_i}(n_i, g_i) + \sum_{i=r+1}^n \chi_{q_i}(g_i) \right) - \iota_{J_p} \left( \sum_{j=1}^r \chi_{q_j}(m_j, h_j) \right)$$

Since  $\varphi_p$  is surjective and  $G(\varphi_p)(y) = 0$ , there exists  $z \in M_{J_p}$  such that

$$-\varphi_p \left( \sum_{i=1}^r \chi_{q_i}(n_i, g_i) + \sum_{i=r+1}^n \chi_{q_i}(g_i) \right) = -\varphi_p \left( \sum_{j=1}^r \chi_{q_j}(m_j, h_j) \right) = \varphi_p(z).$$

Therefore, if we define the elements  $x_1 = (\sum_{i=1}^r \chi_{q_i}(n_i, g_i) + \sum_{i=r+1}^n \chi_{q_i}(g_i)) + z \in M_{J_p}$  and  $x_2 = (\sum_{j=1}^r \chi_{q_j}(m_j, h_j)) + z \in M_{J_p}$ , then we have  $\iota_{J_p}(x_1), \iota_{J_p}(x_2) \in \ker(\varphi_p) \cap G(M_{J_p})^{++}$ , and  $y = \iota_{J_p}(x_1) - \iota_{J_p}(x_2)$ . This shows the result.  $\square$

**Theorem 2.11** *Let  $M$  be a finitely generated refinement monoid, and let  $\mathcal{J}$  be the associated  $I$ -system, so that  $M \cong M(\mathcal{J})$ . Then there is an adaptable separated graph  $(E, C)$  such that*

$$M(E, C) \cong M(\mathcal{J}) \cong M.$$

**Proof** The proof follows the lines of the proof of [12, Proposition 5.13]. This result says that, if the natural map  $G(\varphi_p): G(M_{J_p}) \rightarrow G_p$  is an almost isomorphism for every free prime  $p$ , then there is a row-finite directed graph  $E$  such that  $M \cong M(E)$ . (In particular, this holds if every prime in  $M$  is regular). We will only outline the point in which the proof has to be adapted, recalling some of the relevant notation.

The proof works by induction. Assume that  $J$  is a lower subset of  $I$  and that an adaptable separated graph  $(E_J, C^J)$  of the desired form has been constructed so that there is a monoid isomorphism

$$\gamma_J: M(J) \rightarrow M(E_J, C^J),$$

where  $M(J)$  is the order-ideal of  $M$  generated by  $J$ , sending the canonical semigroup generators to the corresponding sets of vertices, as specified in [12, p. 113]. In case  $J \neq I$ , let  $p$  be a minimal element of  $I \setminus J$ , and write  $J' = J \cup \{p\}$ . If  $p$  is a regular prime or  $p$  is minimal, proceed as in the proof of [12, Proposition 5.13].

Assume that  $p$  is a non-minimal free prime. By Proposition 2.10, there are a finite number of strictly positive elements  $x_1, \dots, x_k$  which generate the kernel of the map  $G(\varphi_p)$ . Now, using the same arguments as in the proof of [12, Proposition 5.13], we may find elements  $\widehat{x}_i \in M(E_J, C^J)$ ,  $i = 1, \dots, k$ , which are non-negative integer combinations of the vertices of  $E_J$  such that  $\gamma_J(x_i) = \widehat{x}_i$  for  $i = 1, \dots, k$ . Observe that  $\widehat{x}_i \in M_{\gamma_J(J_p)}$ . Now, we introduce the adaptable separated graph  $(E_{J'}, C^{J'})$ . We define  $E_{J'}^0 = E_J^0 \sqcup \{v^p\}$ , and  $C^{J'} \setminus C_{v^p}^{J'} = C^J$ , that is, the structure of  $(E_{J'}, C^{J'})$  is the same as the structure of  $(E_J, C^J)$  when restricted to the vertices of  $E_J$ . For the new vertex  $v^p$  we define  $C_{v^p}^{J'} = \{X_1^{(p)}, \dots, X_k^{(p)}\}$ , where each  $X_i^{(p)}$  has the form described in Definition 1.4(3), and the edges  $\alpha(p, i), \beta(p, i, t)$ ,  $t = 1, \dots, g(p, i)$  are chosen in such a way that the relations

$$v^p = v^p + \widehat{x}_i \tag{2.3}$$

are satisfied in the graph monoid  $M(E_{J'}, C^{J'})$ , for  $i = 1, \dots, k$ . (Here we set  $k(p) = k$ ).

By Proposition 2.6,  $M(E_{J'}, C^{J'})$  is a primely generated conical refinement monoid. Its corresponding system has been determined in Proposition 2.9. In particular, we know that the set of primes of  $M(E_{J'}, C^{J'})$  is  $\mathbb{P}(M(E_J, C^J)) \cup \{v^p\}$  and that  $v^p$  is a free prime in  $M(E_{J'}, C^{J'})$ . Consequently, we have that the archimedean component  $M(E_{J'}, C^{J'})[v^p]$  of  $M(E_{J'}, C^{J'})$  at  $v^p$  satisfies



$$M(E_{J'}, C^{J'})[v^p] = \mathbb{N} \times G'_{v^p}$$

for some abelian group  $G'_{v^p}$ , and that the map  $\phi_p : M(E_{J'}, C^{J'})_{\gamma_J(J_p)} \rightarrow G'_{v^p}$  induced by the various semigroup homomorphisms

$$\begin{aligned} \phi_q^p : M(E_{J'}, C^{J'})_{\gamma_J(q)} &\rightarrow G'_{v^p} \\ y &\mapsto (v^p + y) - v^p \end{aligned}$$

for  $q < p$  is surjective. So, we obtain a surjective group homomorphism

$$G(\phi_p) : G(M(E_{J'}, C^{J'})_{\gamma_J(J_p)}) \rightarrow G'_{v^p}.$$

In order to simplify the notation, we will write  $M(E_{J'}, C^{J'})_{J_p}$  instead of  $M(E_{J'}, C^{J'})_{\gamma_J(J_p)}$ .

It is readily seen that the natural map  $M(E_J, C^J) \rightarrow M(E_{J'}, C^{J'})$  defines a monoid isomorphism from  $M(E_J, C^J)$  onto an order-ideal of  $M(E_{J'}, C^{J'})$ ; hence, we will identify  $M(E_J, C^J)$  with its image without further comment. Moreover, the component  $M(E_{J'}, C^{J'})_{J_p}$  clearly coincides with the component  $M(E_J, C^J)_{J_p}$ .

Now, the monoid isomorphism  $\gamma_J : M(J) \rightarrow M(E_J, C^J)$  restricts to a semigroup isomorphism  $M_{J_p} \rightarrow M(E_J, C^J)_{J_p}$ , which induces a group isomorphism

$$\tilde{\gamma}_{J_p} : G(M_{J_p}) \rightarrow G(M(E_J, C^J)_{J_p})$$

of the Grothendieck groups. Set  $K := \ker(G(\phi_p))$ , and notice that the relation (2.3) implies that  $\tilde{\gamma}_{J_p}(x_i) = \hat{x}_i \in K$  for  $i = 1, \dots, k$ .

Hence, there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \langle x_1, \dots, x_k \rangle & \longrightarrow & G(M_{J_p}) & \xrightarrow{G(\phi_p)} & G_p \longrightarrow 0 \\ & & \downarrow & & \downarrow \tilde{\gamma}_{J_p} & & \downarrow \gamma_p & (2.4) \\ 0 & \longrightarrow & K & \longrightarrow & G(M(E_J, C^J)_{J_p}) & \xrightarrow{G(\phi_p)} & G'_{v^p} \longrightarrow 0, \end{array}$$

where  $\gamma_p : G_p \rightarrow G'_{v^p}$  is the map induced from the cokernel of the inclusion  $\langle x_1, \dots, x_k \rangle \hookrightarrow G(M_J)$  to the cokernel of the inclusion  $K \hookrightarrow G(M(E_J, C^J)_{J_p})$ . Notice that  $\gamma_p$  is an onto map.

We now define the map

$$\gamma_{J'} : M(J') \rightarrow M(E_{J'}, C^{J'})$$

extending the monoid isomorphism  $\gamma_J : M(J) \rightarrow M(E_J, C^J)$ , and defining  $\gamma_{J'}$  on the component  $M_p \cong \mathbb{N} \times G_p$  of  $M(J')$  by the formula

$$\gamma_{J'}(mp + g) = mv^p + \gamma_p(g)$$

for  $m \in \mathbb{N}$  and  $g \in G_p$ . By [11, Corollary 1.8], to show that  $\gamma_{J'}$  is a well-defined monoid homomorphism, it suffices to show that if  $q < p$  and  $y \in G_M[q] = M_q$  then  $\gamma_{J'}(y) + \gamma_{J'}(p) = \gamma_{J'}(\varphi_{p,q}(y) + p)$ , that is,  $\gamma_J(y) + v^p = \gamma_p(\varphi_{p,q}(y)) + v^p$ . For  $x \in M_{J_p}$ , we may define a map

$$\tau_q : M_q \rightarrow G(M_{J_p})$$

by  $\tau_q(y) = (x+y)-x \in G(M_{J_p})$ . The map  $\tau_q$  is a semigroup homomorphism and does not depend on the particular choice of  $x \in M_{J_p}$ . Moreover, we have  $\varphi_{p,q} = G(\varphi_p) \circ \tau_q$ . Analogously, we have a map

$$\tau_{\gamma_J(q)} : M(E_{J'}, C^{J'})_{\gamma_J(q)} \rightarrow G(M(E_{J'}, C^{J'})_{J_p}) = G(M(E_J, C^J)_{J_p})$$

such that  $\phi_{\gamma_J(q)}^{v^p} = G(\phi_p) \circ \tau_{\gamma_J(q)}$ , and clearly  $\tilde{\gamma}_{J_p} \circ \tau_q = \tau_{\gamma_J(q)} \circ \gamma_J|_{M_q}$ .

Using this fact, and the commutativity of (2.4), we have that

$$\begin{aligned} \gamma_p(\varphi_{p,q}(y)) + v^p &= \gamma_p(G(\varphi_p)(\tau_q(y))) + v^p \\ &= G(\phi_p)(\tilde{\gamma}_{J_p}(\tau_q(y))) + v^p \\ &= G(\phi_p)(\tau_{\gamma_J(q)}(\gamma_J(y))) + v^p \\ &= \phi_{\gamma_J(q)}^{v^p}(\gamma_J(y)) + v^p \\ &= ((v^p + \gamma_J(y)) - v^p) + v^p \\ &= v^p + \gamma_J(y), \end{aligned}$$

as desired.

This shows that there is a well-defined monoid homomorphism

$$\gamma_{J'} : M(J') \rightarrow M(E_{J'}, C^{J'})$$

sending the canonical semigroup generators of  $M(J')$  to the corresponding canonical sets of vertices seen in  $M(E_{J'}, C^{J'})$ . In particular,  $\gamma_{J'}$  is an onto map.

In order to prove the injectivity of  $\gamma_{J'}$ , we can build an inverse map  $\delta_{J'} : M(E_{J'}, C^{J'}) \rightarrow M(J')$ , as follows: on  $M(E_J, C^J)$  we define  $\delta_{J'}$  to be  $\gamma_J^{-1}$ , while  $\delta_{J'}(v^p) := p$ . Notice that the only relations on  $M(E_{J'}, C^{J'})$  not occurring already in  $M(E_J, C^J)$  are  $v^p = v^p + \hat{x}_i$ ,  $i = 1, \dots, k$ , where  $\gamma_J(x_i) = \hat{x}_i$ . Thus,  $\delta_{J'}(\hat{x}_i) = x_i$ . But  $x_1, \dots, x_k$  generate the kernel of the map

$$G(\varphi_p) : G(M_{J_p}) \rightarrow G_p \hookrightarrow \widehat{G}_p = \mathbb{Z} \times G_p,$$

so that  $(p + x_i) - p$  equals 0 in  $\widehat{G}_p$ . Hence, the relations  $p = p + x_i$  hold in  $M(J')$ , for  $i = 1, \dots, k$ . Thus,  $\delta_{J'}$  is a well-defined monoid homomorphism, and it is the inverse of  $\gamma_{J'}$ . This completes the proof of the inductive step.  $\square$

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