Title:	*Doubling condition at the origin for non-negative positive definite functions*
Journal Information:	*Proceedings of the American Mathematical Society*,
Author(s):	Gorbachev D., Tikhonov S..
Volume, pages:	*147 10*, DOI:[10.1090/proc/14191]

DOUBLING CONDITION AT THE ORIGIN FOR NON-NEGATIVE POSITIVE DEFINITE FUNCTIONS

DMITRY GORBACHEV AND SERGEY TIKHONOV

Abstract. We study upper and lower estimates as well as the asymptotic behavior of the sharp constant $C = C_n(U, V)$ in the doubling-type condition at the origin

$$\frac{1}{|V|}\int_V f(x)\,dx \le C\,\frac{1}{|U|}\int_U f(x)\,dx,$$

where $U, V \subset \mathbb{R}^n$ are 0-symmetric convex bodies and f is a non-negative positive definite function.

1. Introduction

Very recently, answering the question posed by Konyagin and Shteinikov related to a problem from number theory [13], the first author proved [1] that for any positive definite function $f\colon \mathbb{Z}_q \to \mathbb{R}_+$ and for any $n \in \mathbb{Z}_+$ one has

$$\sum_{0\le k\le 2n} f(k) \le C \sum_{0\le k\le n} f(k),$$

where the positive constant C does not depend on n, f, and q. More precisely, it was proved that $C \le \pi^2$.

In this paper we study similar inequalities for a non-negative positive definite function f defined on $\mathbb{R}^n$, $n \ge 1$, i.e.,

$$\int_{|x|\le 2R} f(x)\,dx \le C \int_{|x|\le R} f(x)\,dx, \quad R > 0, \tag{1.1}$$

for some $C > 1$. The latter is the well-known doubling condition at the origin. The doubling condition plays an important role in harmonic and functional analysis, see, e.g., [14]. Note that very recently inequality (1.1) in the one-dimensional case was studied in [3].

Definition 1. A positive definite function $f\colon \mathbb{R}^n \to \mathbb{R}_+$ is called double positive definite function (denoted $f \succeq 0$).

Date: March 8, 2018.

1991 *Mathematics Subject Classification.* 42A82, 42A38.

Key words and phrases. non-negative positive definite functions, doubling property, Wiener property.

The first author was partially supported by the RFBR (no. 16-01-00308) and the Ministry of Education and Science of the Russian Federation (no. 5414GZ). The second author was partially supported by MTM 2014-59174-P and 2014 SGR 289.

As usual [11, Chap. 1], a continuous function $f \in C(\mathbb{R}^n)$ is positive definite if for every finite sequence $X \subset \mathbb{R}^n$ and every choice of complex numbers $\{c_a\colon a \in X\}$, we have

$$\sum_{a,b\in X} c_a\overline{c_b} f(a-b) \ge 0.$$

By Bochner's theorem [11, Chap. 1], $f \in C(\mathbb{R}^n)$ is positive definite if and only if there is a non-negative finite Borel measure μ such that

$$f(x) = \int_{\mathbb{R}^n} e(\xi x)\, d\mu(\xi), \quad \xi \in \mathbb{R}^n, \tag{1.2}$$

where $e(t) = \exp(2\pi i t)$. For $f \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ it is equivalent to the fact that the Fourier transform of f

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e(-\xi x)\, dx$$

is non-negative. Note also that since any positive definite f satisfies $f(-x) = \overline{f(x)}$, a double positive definite function is even.

Throughout the paper we assume that $U, V \subset \mathbb{R}^n$ be 0-symmetric closed convex bodies. For any function $f \succeq 0$ we study the inequality

$$\frac{1}{|V|}\int_V f(x)\, dx \le C\, \frac{1}{|U|}\int_U f(x)\, dx, \tag{1.3}$$

where $|A|$ is the volume of A or the cardinality of A if A is a finite set. By $C_n(U,V)$ we denote the sharp constant in (1.3), i.e.,

$$C_n(U,V) := \sup_{f\succeq 0,\ f\ne 0} \frac{\frac{1}{|V|}\int_V f(x)\, dx}{\frac{1}{|U|}\int_U f(x)\, dx}.$$

The fact that $C_n(U,V) < \infty$ for any U and V will follow from Theorem 1 below.

First, we list the following simple properties of $C_n(U,V)$.

(1) A trivial lower bound

$$C_n(U,V) \ge 1, \tag{1.4}$$

since $1 \succeq 0$;

(2) The homogeneity property

$$C_n(\lambda U, \lambda V) = C_n(U,V), \quad \lambda > 0, \tag{1.5}$$

since $f_\lambda(x) = f(\lambda x) \succeq 0$ if and only if $f \succeq 0$;

(3) The homogeneity estimate

$$C_n(U, \lambda V) \ge \lambda^{-n} C_n(U,V), \qquad \lambda \ge 1, \tag{1.6}$$

since $V \subset \lambda V$;

(4) $C_n(U,U) = 1$ and if $V \subset U$, then

$$C_n(U,V) \le \frac{|U|}{|V|};$$

(5) The multiplicative estimate

$$C_n(U,V) \le C_n(\lambda^k U, V)(C_n(U, \lambda U))^k, \quad \lambda \ge 1,\ k \in \mathbb{Z}_+,$$

which follows from the chain of inequalities

$$\begin{aligned} C_n(U,V) &\le C_n(\lambda U,V)C_n(U,\lambda U)\\ &\le C_n(\lambda^2 U,V)C_n(\lambda U,\lambda^2 U)C_n(U,\lambda U)\\ &= C_n(\lambda^2 U,V)(C_n(U,\lambda U))^2 \le \dots\\ &\le C_n(\lambda^k U,V)(C_n(U,\lambda U))^k; \end{aligned}$$

(6) A trivial upper bound for the doubling constant: for fixed $\lambda > 1$ and any $r > \lambda$

$$C_n(U,rU) \le (C_n(U,\lambda U))^{\log_\lambda r}. \tag{1.7}$$

which follows from the multiplicative estimate.

Bellow we will obtain the upper bound for the constant $C_n(U,rU)$, which depends only on n.

We will use the following notation. Let $A+B$ be the Minkowski sum of sets A and B, λA be the product of A and the number λ, and $B_R := \{x \in \mathbb{R}^n : |x| \le R\}$ be the Euclidean ball.

2. The upper estimates

In what follows, we set

$$H := \tfrac{1}{2}U \quad \text{and} \quad K := V + H.$$

Theorem 1. *Let $X \subset \mathbb{R}^n$ be a finite set of points such that*

$$K \subseteq H + X. \tag{2.8}$$

Then

$$C_n(U,V) \le \frac{|X|\,|U|}{|V|}.$$

From the geometric point of view, condition (2.8) means that the translates $\{H + a : a \in X\}$ of the set H covers the set K.

Example 1 ([3]). If $n = 1$ and $r \in \mathbb{N}$, then

$$C_1(r) := C_1([-1,1],[-r,r]) \le 2 + \frac{1}{r}.$$

Indeed, take $H = [-\frac{1}{2},\frac{1}{2}]$, $X = \{-r,-r+1,\dots,r-1,r\}$, and $K = [-r-\frac{1}{2}, r+\frac{1}{2}] = H + X$.

Let $n \in \mathbb{N}$. There holds ([10, (6)])

$$N(K,H) \le \frac{|K-H|}{|H|}\,\theta(H). \tag{2.9}$$

Here $N(K,H)$ denotes the smallest number of translates of H required to cover K and

$$\theta(H) = \inf_{X \subset \mathbb{R}^n} \theta(H,X), \tag{2.10}$$

where $\theta(H,X)$ is the covering density of $\mathbb{R}^n$ by translates of H [9, p.16]. In other words, for a discrete set X such that $\mathbb{R}^n \subseteq H + X$ one has $|X \cap A|\,|H|/|A| = \theta(H,X)(1 + o(1))$ for a convex body A such that $|A| \to \infty$.

From (2.9), taking into account that $H=-H$, $K-H=V+2H=V+U$, and $|U|=2^n|H|$, we obtain that

$$N(K,H)\le 2^n\,\frac{|V+U|}{|U|}\,\theta(H).$$

Moreover, it is clear that the best possible result in Theorem 1 is when X is such that $|X|=N(K,H)$. Therefore, we have

Corollary 1. *For $n\ge 1$ and any U and V, we have*

$$C_n(U,V)\le 2^n\,\frac{|V+U|}{|V|}\,\theta(H).$$

In particular, for $r\ge 1$

$$C_n(U,rU)\le 2^n(1+r^{-1})^n\theta(H). \tag{2.11}$$

Estimate (2.11) substantially improves (1.7). For $n=1$ and $r\ge 1$, we have that $\theta([-\frac{1}{2},\frac{1}{2}])=1$ and $C_1(r)\le 2(1+r^{-1})$, which is similar to the estimate from Example 1.

Note that Rogers [8] proved that

$$\theta(H)\le n\ln n+n\ln\ln n+5n,\quad n\ge 2. \tag{2.12}$$

Estimate (2.12) was slightly improved in [4] as follows

$$\theta(H)\le n\ln n+n\ln\ln n+n+o(n)\quad \text{as}\quad n\to\infty.$$

Therefore, we obtain

Corollary 2. *We have*

$$C_n(U,V)\le 2^n(n\ln n+n\ln\ln n+n+o(n))\,\frac{|V+U|}{|V|}\quad \textit{as}\quad n\to\infty.$$

In particular, taking $V=rU$, $r\ge 1$, we arrive at the following example.

Example 2. We have

$$C_n(U,rU)\le 2^n(n\ln n+n\ln\ln n+n+o(n))(1+r^{-1})^n\quad \text{as}\quad n\to\infty. \tag{2.13}$$

Proof of Theorem 1. Consider the function

$$\varphi:=\varphi_H=|H|^{-1}\cdot\chi_H*\chi_H,$$

where χ_H is the characteristic function of H and $(f*g)(x)=\int_{\mathbb{R}^n}f(x-y)g(y)\,dy$ is the convolution of f and g.

Since $\varphi\succeq 0$, $\operatorname{supp}\varphi\subset U$, and $\varphi\le\varphi(0)=1$, we have for any $f\succeq 0$

$$I:=\int_{\mathbb{R}^n}f(x)\varphi(x)\,dx=\int_U f(x)\varphi(x)\,dx\le\int_U f(x)\,dx.$$

Let $X\subset\mathbb{R}^n$ be a finite set and

$$S(x)=\frac{1}{|X|}\sum_{a\in X}\varphi(x-a).$$

Then $S\ge 0$ and $\widehat{S}=\widehat{\varphi}D$, where

$$D(\xi)=\frac{1}{|X|}\sum_{a\in X}e(a\xi)$$

is the Dirichlet kernel with respect to X.

Let us estimate the integral I from below. Using $f(x) = f(-x)$, we get

$$\int_V f(x)S(x)\,dx \le \int_{\mathbb{R}^n} f(x)S(x)\,dx = \int_{\mathbb{R}^n} f(x)S_0(x)\,dx := I_1,$$

where $S_0(x) = 2^{-1}(S(x) + S(-x))$. Taking into account that

$$\widehat{S_0}(\xi) = \widehat{\varphi}(\xi)\,\frac{D(\xi) + D(-\xi)}{2} = \widehat{\varphi}(\xi)\,\frac{1}{|X|}\sum_{a\in X}\cos(2\pi a\xi) \le \widehat{\varphi}(\xi), \quad \xi \in \mathbb{R}^n,$$

and using (1.2), we obtain

$$I_1 = \int_{\mathbb{R}^n} \widehat{S_0}(\xi)\,d\mu(\xi) \le \int_{\mathbb{R}^n} \widehat{\varphi}(\xi)\,d\mu(\xi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx = I,$$

provided that f and φ are even.

Let $K = V + H \subseteq H + X$. This means that for any points $x \in V$ and $y \in H$ there is $a \in X$ such that $x + y \in H + a$. Hence,

$$\sum_{a\in X} \chi_H(x + y - a) \ge 1.$$

Using $H = -H$, we have

$$\varphi(x) = \frac{1}{|H|}\int_H \chi_H(x + y)\,dy.$$

Therefore, for any $x \in V$

$$\begin{aligned} S(x) &= \frac{1}{|X|}\sum_{a\in X}\frac{1}{|H|}\int_H \chi_H(x - a + y)\,dy \\ &\ge \frac{1}{|X||H|}\int_H \sum_{a\in X}\chi_H(x - a + y)\,dy \\ &\ge \frac{1}{|X||H|}\int_H dy = \frac{1}{|X|}. \end{aligned}$$

Thus, combining the estimates above, we arrive at the inequality

$$\frac{1}{|X|}\int_V f(x)\,dx \le \int_V f(x)S(x)\,dx \le I \le \int_U f(x)\,dx,$$

which is the desired result. □

3. The lower estimates

Our goal is to improve the trivial lower estimate (1.4). The idea is to consider the functions $\sum_{a,b\in X\cap B_R}\delta(x + a - b)$, where X is a packing of $\mathbb{R}^n$ by H and $R \gg 1$ (see also [2, 3]).

First we consider the one-dimensional result, partially given in Example 1.

Theorem 2 ([3]). *For $r \in \mathbb{N}$, we have*

$$2 - \frac{1}{r} \le C_1(r) \le 2 + \frac{1}{r},$$

and $\lim_{r\to\infty} C_1(r) = 2$.

This is one of the main results of the paper [3]. The upper bound is given in Example 1. The lower bound follows from Theorem 3 below for $U = [-1, 1]$, $V = [-r, r]$, and $\Lambda = \mathbb{Z}$. The fact that $\lim_{r\to\infty} C_1(r) = 2$ follows from estimates of $C_1(r)$ for integers r and (1.6).

Now we consider the general case $n \geq 1$. Our aim is to improve the trivial lower bound (1.4) respect to n.

Let

$$\delta_L(H) = \sup_{\Lambda\subset\mathbb{R}^n} \delta(H, \Lambda),$$

where $\delta(H, \Lambda)$ is the packing density of $\mathbb{R}^n$ by lattice translates of H [9, Intr.]. In other words, $\Lambda = M\mathbb{Z}^n \subset \mathbb{R}^n$ is a lattice of rank n ($M \in \mathbb{R}^{n\times n}$ is a generator matrix of Λ, $\det M \neq 0$) such that $a - b \notin \operatorname{int}(2H)$ for any $a, b \in \Lambda$, $a \neq b$, and $|\Lambda \cap A|\,|H|/|A| = \delta(H, \Lambda)(1 + o(1))$ for a convex body A such that $|A| \to \infty$. Note that in this case $H+\Lambda$ is *a lattice packing* of H [6, Sect. 30.1]. Recall that $H = \frac{1}{2}U$.

Theorem 3. *Let $H + \Lambda$ be a lattice packing of H. Then*

$$C_n(U, V) \geq \frac{|\Lambda \cap \operatorname{int} V|\,|U|}{|V|}. \tag{3.14}$$

In particular,

$$C_n(U, V) \geq 2^n \delta_L(H)(1 + o(1)) \quad \text{as} \quad |V| \to \infty. \tag{3.15}$$

Proof of Theorem 3. Let Λ be an lattice with the packing density $\delta(H, \Lambda)$. Denote $\Lambda_N = \Lambda \cap B_N$ for $N > 0$. Let B_r be the smallest ball that contained V. Assume that $R \geq r$ is sufficiently large number and ε is sufficiently small. Define $\varphi_\varepsilon = \varphi_{B_\varepsilon}$.

We consider the function

$$f(x) = \sum_{a,b\in\Lambda_R} \varphi_\varepsilon(x + a - b).$$

It is easy to see that

$$f(x) = \sum_{c\in\Lambda_{2R}} N_c\varphi_\varepsilon(x + c),$$

where

$$N_c = \sum_{a-b=c} 1 = \sum_{a\in\Lambda_R\cap(\Lambda_R+c)} 1 = |\Lambda_R \cap (\Lambda_R + c)|.$$

Since Λ is a lattice, we have $\Lambda = \Lambda + c$ for any $c \in \Lambda$. Hence, $N_0 = |\Lambda_R|$ and $N_c \geq |\Lambda_{R-r}|$ for $|c| \leq r$, provided $\Lambda_{R-r} \subset \Lambda_R \cap (\Lambda_R + c)$.

On the one hand, since $2H = U$ and $c \notin \operatorname{int} U$ if $c \in \Lambda \setminus \{0\}$, we have

$$\int_{(1-\varepsilon)U} f(x)\,dx = N_0 = |\Lambda_R|.$$

On the other hand, since $V \subset B_r$, we obtain

$$\int_{(1+\varepsilon)V} f(x)\,dx \geq \sum_{c\in\Lambda_{2R}\cap V} N_c \geq |\Lambda_{R-r}|\,|\Lambda \cap V|.$$

Therefore,

$$C_n\big((1-\varepsilon)U, (1+\varepsilon)V\big) \geq \frac{(1-\varepsilon)^n}{(1+\varepsilon)^n}\,\frac{|\Lambda_{R-r}|}{|\Lambda_R|}\,\frac{|\Lambda \cap V|\,|U|}{|V|}.$$

Replacing V by $\frac{1-\varepsilon}{1+\varepsilon}V$ and using (1.5) and (1.6) as above, we arrive at

$$C_n(U,V) \ge \frac{|\Lambda_{R-r}|}{|\Lambda_R|}\,\frac{|\Lambda\cap\frac{1-\varepsilon}{1+\varepsilon}V|\,|U|}{|V|}.$$

Letting $R\to\infty$ and $\varepsilon\to 0$ concludes the proof of (3.14).

Inequality (3.15) follows easily from (3.14) and the definition of $\delta_L(H)$. □

Example 3. We consider the balls $U=B_1$ and $V=B_r$, $r>1$. It is known that

$$\delta_L(B_1)\ge c_n 2^{-n},$$

where $c_n\ge 1$ is the Minkowski constant. It was recently proved in [15] that $c_n > 65963n$ for every sufficiently large n and there exist infinitely many dimensions n for which $c_n\ge 0.5n\ln\ln n$.

Corollary 3. *Let $n\in\mathbb{N}$. We have*

$$C_n(B_1,B_r)\ge c_n(1+o(1)) \quad \text{as} \quad r\to\infty. \tag{3.16}$$

Comparing (2.13) and (3.16) for fixed n and $r\to\infty$, one observes the exponential gap between the upper and lower estimates of $C_n(B_1,B_r)$ with respect to n. Let us give examples of U for which the upper and lower estimates of $C_n(U,V)$ coincide.

Example 4. Let H be a convex body and Λ be a lattice. The set $H+\Lambda$ is lattice tiling if it is both a packing and a covering [6, Sect. 32]. In this case H is *a tile* and $\delta_L(H)=\theta_L(H)=1$, where $\theta_L(H)$ is the lattice covering density, cf. (2.10). To define $\theta_L(H)$, we take the infimum in (2.10) over all lattices $\Lambda\subset\mathbb{R}^n$ of rank n. Note that $\theta(H)\le\theta_L(H)$.

For example, the Voronoi polytop

$$V(\Lambda)=\{x\in\mathbb{R}^n\colon |x|\le|x-a|,\ \forall\, a\in\Lambda\}$$

of a lattice Λ is a tile. In particular, $V(\mathbb{Z}^n)$ is the cube $[-\frac12,\frac12]^n$.

From Corollary 1 and Theorem 3, we have

Theorem 4. *Let $n\in\mathbb{N}$ and U be a tile. We have*

$$C_n(U,V)=2^n(1+o(1)) \quad \text{as} \quad |V|\to\infty.$$

4. Final remarks

1. The inequality

$$\frac{1}{|V|}\int_V f(x)\,dx \le C_n(U,V)\,\frac{1}{|U|}\int_U f(x)\,dx$$

holds for any 1-periodic function $f\succeq 0$. In this case we assume that $U,V\subseteq\mathbb{T}^n$, where $\mathbb{T}=\mathbb{R}/\mathbb{Z}$.

Since a positive definite f is such that $f(-x)=\overline{f(x)}$, then $|f|^p\succeq 0$ for any $p=2k, k\in\mathbb{N}$. Hence, we obtain the following L^p-analogue:

$$\frac{1}{|V|}\int_V |f(x)|^p\,dx \le C_n(U,V)\,\frac{1}{|U|}\int_U |f(x)|^p\,dx.$$

For $U\subset V=\mathbb{T}^n$, this inequality is the well-known Wiener estimate for positive definite periodic functions (see [12, 7, 2]):

$$\int_{\mathbb{T}^n} |f(x)|^p\,dx \le W_{n,p}(U)\,\frac{1}{|U|}\int_U |f(x)|^p\,dx, \tag{4.17}$$

which is valid only for $p = 2k, k \in \mathbb{N}$. Here, $W_{n,p}(U)$ is a sharp constant in (4.17). It is clear that

$$W_{n,2k}(U) \le C_n(U, \mathbb{T}^n).$$

It is interesting to compare the known upper bounds of $W_{n,2k}(U)$ and $C_n(U, \mathbb{T}^n)$. In [2] it was shown that

$$W_{n,2k}(rB_1) \le 2^{(0.401\ldots+o(1))n}, \quad r \in (0, 1/2).$$

On the other hand, by Corollary 2, we obtain that

$$C_n(rB_1, \mathbb{T}^n) \le 2^{n(1+o(1))}(1 + 2r)^n.$$

The exponential gap in the last two bounds is related to the restriction to the class of functions under consideration.

2. If $f \succeq 0$, then $f^p \succeq 0$ for any $p \in \mathbb{N}$. This gives

$$\frac{1}{|V|}\int_V (f(x))^p\,dx \le C_n(U, V)\,\frac{1}{|U|}\int_U (f(x))^p\,dx, \quad p \in \mathbb{N}.$$

It would be of interest to investigate this inequality for any positive p; see in this direction the paper [5].

3. As we showed above, any function $f \succeq 0$ satisfies the doubling property at the origin (1.1). However, taking any nontrivial function $f \succeq 0$ such that $f|_A = 0$, where A is a ball, we can see that the doubling property may fail outside the origin.

References

[1] D. V. Gorbachev, *Certain inequalities for discrete, nonnengative, positive definite functions* (in Russian), Izv. Tul. Gos. Univ. Est. nauki (2015), no. 2, 5–12.

[2] D. V. Gorbachev, S. Yu. Tikhonov, *Wiener's problem for positive definite functions*, arXiv:1604.01302.

[3] A. Efimov, M. Gaal, Sz. Gy. Revesz, *On integral estimates of non-negative positive definite functions*, arXiv:1612.00235.

[4] G. Fejes Tóth, *A note on covering by convex bodies*, Canad. Math. Bull. **52** (2009), no. 3, 361–365.

[5] C. FitzGerald, R. Horn, *On fractional Hadamard powers of positive definite matrices*, J. Math. Anal. Appl. **61** (1977), no. 3, 633–642.

[6] P. M. Gruber, *Convex and Discrete Geometry*, Grundlehren der Mathematischen Wissenschaften, vol. 336, Springer-Verlag, Berlin, 2007.

[7] E. Hlawka, *Anwendung einer Zahlengeometrischen Methode von C. L. Siegel auf Probleme der Analysis*, Comment Math. Helvetici **56** (1981), 66–82.

[8] C. A. Rogers, *A note on coverings*, Mathematika **4** (1957), 1-6.

[9] C. A. Rogers, *Packing and Covering*, Cambridge University Press, 1964.

[10] C. A. Rogers, C. Zong, *Covering convex bodies by translates of convex bodies*, Mathematika **44** (1997), 215–218.

[11] W. Rudin, *Fourier analysis on groups*, Interscience Publ., New York, 1962.

[12] H. S. Shapiro, *Majorant problems for Fourier coefficients*, Quart. J. Math. Oxford Ser. (2) **26** (1975), 9–18.

[13] Yu. N. Shteinikov, *On the set of joint representatives of two congruence Classes*, Proceedings of the Steklov Institute of Mathematics **290** (2015), no. 1, 189–196.

[14] E. M. Stein, *Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals,* Princeton Univ. Press, Princeton, 1993.

[15] A. Venkatesh, *A note on sphere packings in high dimension*, Int. Math. Res. Notices (2013), no. 7, 1628–1642.

D. Gorbachev, Tula State University, Department of Applied Mathematics and Computer Science, 300012 Tula, Russia
E-mail address: dvgmail@mail.ru

S. Tikhonov, Centre de Recerca Matemàtica, Campus de Bellaterra, Edifici C 08193 Bellaterra (Barcelona), Spain; ICREA, Pg. Lluís Companys 23, 08010 Barcelona, Spain, and Universitat Autònoma de Barcelona.
E-mail address: stikhonov@crm.cat