# Sharp Remez inequality

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ABSTRACT. Let an algebraic polynomial $P_n(\zeta)$ of degree $n$ be such that $|P_n(\zeta)| \leqslant 1$ for $\zeta \in E \subset \mathbb{T}$ and $|E| \geqslant 2\pi - s$. We prove the sharp Remez inequality

$$\sup_{\zeta \in \mathbb{T}} |P_n(\zeta)| \leqslant \mathfrak{T}_n\left(\sec \frac{s}{4}\right),$$

where $\mathfrak{T}_n$ is the Chebyshev polynomial of degree $n$. The equality holds if and only if

$$P_n(e^{iz}) = e^{i(nz/2+c_1)}\mathfrak{T}_n\left(\sec \frac{s}{4} \cos \frac{z-c_0}{2}\right), \quad c_0, c_1 \in \mathbb{R}.$$

This gives the solution of the long-standing problem on the sharp constant in the Remez inequality for trigonometric polynomials.

## 1. Introduction

Let $\mathfrak{T}_n(x)$ be the Chebyshev polynomial of degree $n$, i.e.,

$$\mathfrak{T}_n(x) = \frac{1}{2}\left(\left(x + \sqrt{x^2-1}\right)^n + \left(x - \sqrt{x^2-1}\right)^n\right)$$

for every $x \in \mathbb{R}$ and $|B|$ denote the Lebesgue measure of a measurable set $B$.

The Remez inequality [22] for algebraic polynomials $P_n$ asserts that

$$\max_{x \in [-1,1]} |P_n(x)| \leqslant \mathfrak{T}_n\left(\frac{2+s}{2-s}\right),$$

for every $P_n$ satisfying

$$\left|\{x : [-1,1] : |P_n(x)| \leqslant 1\}\right| \geqslant 2 - s, \qquad 0 < s < 2.$$

Moreover, the equality holds if and only if

$$P_n(x) = \pm\mathfrak{T}_n\left(\frac{\pm 2x + s}{2-s}\right).$$

The Remez inequality plays an important role in many problems in approximation theory, harmonic and functional analysis (see, for example, [3, 4, 5, 25] and references therein).

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A similar question for trigonometric polynomials is a widely studied and open problem. In more detail, consider the space of (complex) trigonometric polynomials of degree at most $n \in \mathbb{N}$, i.e.,

$$\mathfrak{N}_n = \left\{ Q_n : \quad Q_n(x) = \sum_{|k| \leqslant n} c_k e^{ikx}, \quad c_k \in \mathbb{C}, \quad x \in [0, 2\pi) \right\}.$$

PROBLEM A. *How large can* $\|Q_n\|_{L_\infty([0,2\pi))}$ *be if*

(1.1)
$$\left| \left\{ x \in [0, 2\pi) : |Q_n(x)| \leqslant 1 \right\} \right| \geqslant 2\pi - s$$

*holds for some* $0 < s < 2\pi$?

In other words, we study the best constant in the Remez inequality

(1.2)    $$\|Q_n\|_{C[0,2\pi)} \leqslant C(n,s) \|Q_n\|_{C([0,2\pi) \setminus B)}, \qquad Q_n \in \mathfrak{N}_n, \qquad |B| = s.$$

The problem of finding the sharp constant in the Remez inequality, or at least of obtaining some suitable bounds on this constant, has been extensively studied starting from the 1990s (see, e.g., [1, 2, 3, 8, 9, 11, 13, 14, 15, 20]) but some estimates were established much earlier [24, 26].

The sharp constant in (1.2) is known only in the case of an interval, that is, (see, e.g., [8])

(1.3)    $$\|Q_n\|_{C[0,2\pi)} \leqslant \mathfrak{T}_{2n} \left( \csc \frac{a}{2} \right) \|Q_n\|_{C[-a,a]}, \qquad [-a,a] \subset [-\pi, \pi].$$

In the general case the problem remains open. Erdélyi [8] proved that

$$C(n,s) \leqslant \exp(4ns), \qquad s \leqslant \frac{\pi}{2}.$$

Later on, this result was slightly improved in [13] as follows $C(n,s) \leqslant \exp(2ns)$. Moreover, it is known [20] that, for a fixed $n$,

$$C(n,s) = 1 + \frac{(ns)^2}{8} + O(s^4) \qquad \text{as} \quad s \to 0.$$

Several papers studied the behavior of the constant $C(n,s)$ in (1.2) for $B$ with large measure ($|B| = s \geqslant \frac{\pi}{2}$), see [9, 13, 18]. In this case, we have

$$C(n, |B|) \leqslant \left( \frac{A}{2\pi - s} \right)^{2n},$$

where $A$ is an absolute constant which can be taken as 17.

The solution of the aforementioned Problem A is given as follows.

THEOREM 1.1. *Let* $Q \in \mathfrak{N}_n$ *be such that* (1.1) *holds for some* $0 < s < 2\pi$. *Then*

(1.4)    $$\|Q\|_{C[0,2\pi)} \leqslant \mathfrak{T}_{2n} \left( \sec \frac{s}{4} \right).$$

*Moreover the equality holds if and only if*

$$Q(x) = e^{ic_1} \mathfrak{T}_{2n} \left( \sec \frac{s}{4} \cos \frac{x - c_0}{2} \right), \quad c_0, c_1 \in \mathbb{R}.$$

Estimate (1.4) confirms a conjecture by Erdélyi [**9**], also stated in [**13**].

The proof of Theorem 1.1 is based on a solution of the following more general problem.

PROBLEM B. *Let*

$$P_n(\zeta) = A_0 + A_1\zeta + \cdots + A_n\zeta^n, \qquad \zeta = e^{iz},$$

*and*

$$\mathcal{P}_n(E) := \left\{ P_n(\zeta) : \ |P_n(\zeta)| \leqslant 1, \zeta \in E \right\},$$

*where $E \subset \mathbb{T}$ is a measurable set. For a fixed $s \in (0, 2\pi)$ find*

$$\sup\{ \max_{\substack{P_n \in \mathcal{P}_n(E) \\ c \in [0, 2\pi)}} |P_n(e^{ic})| : |E| \geqslant 2\pi - s\}.$$

Since for a trigonometric polynomial $Q_n(x) = \sum_{-n \leqslant k \leqslant n} c_k e^{ikx}$ one has $Q_n(x)e^{inx} = P_{2n}(e^{ix})$ with $\|Q_n\|_C = \|P_{2n}\|_C$ and $|Q_n(x_0)| = |P_{2n}(e^{ix_0})|$, where

$$\max_{\substack{P_n \in \mathcal{P}_n(E) \\ c \in [0, 2\pi)}} |P_{2n}(e^{ic})| = |P_{2n}(e^{ix_0})|,$$

Theorem 1.1 will follow from the corresponding solution of Problem B for even-degree polynomials. For the partial case of even trigonometric polynomials, i.e., having the form $P_n(\cos t)$, see also the recent preprint [**10**]. We point out that we solve Problem B *for all integer $n$.*

Now we can state our main result.

THEOREM 1.2. *Let $P_n(\zeta)$ be an algebraic polynomial of degree $n$ bounded by one on a subset $E$ of the unit circle $\mathbb{T}$. If $|E| \geqslant 2\pi - s$ for some $0 < s < 2\pi$, then*

$$(1.5) \qquad \sup_{\zeta \in \mathbb{T}} |P_n(\zeta)| \leqslant \mathfrak{T}_n\left(\sec\frac{s}{4}\right).$$

*Moreover, the equality holds if and only if*

$$(1.6) \qquad P_n(e^{iz}) = e^{i(nz/2 + c_1)} \mathfrak{T}_n\left(\sec\frac{s}{4}\cos\frac{z - c_0}{2}\right), \qquad c_0, c_1 \in \mathbb{R}.$$

We reduce Problem B to Problems C and D below.

PROBLEM C. *Let $E$ be a closed proper subset of $\mathbb{T}$,*

$$(1.7) \qquad E = \mathbb{T}\backslash \bigcup_{j=0}^{g} (e^{ia_j}, e^{ib_j}), \quad 0 \leqslant g \leqslant \infty.$$

*Let*

$$P_n(\zeta) = A_0 + A_1\zeta + \cdots + A_n\zeta^n, \qquad \zeta = e^{iz},$$

*and*

$$\mathcal{P}_n(E) := \left\{ P_n(\zeta) : \ |P_n(\zeta)| \leqslant 1, \zeta \in E \right\}.$$

*Find*

$$\max_{\substack{P_n \in \mathcal{P}_n(E) \\ c \in (a_0, b_0)}} |P_n(e^{ic})|.$$

PROBLEM D. *Let $E$ and $\mathcal{P}_n(E)$ be given as in Problem C. Find*

$$\max_{P_n \in \mathcal{P}_n(E)} |P_n(e^{ic})|, \qquad c \in (a_0, b_0), \quad c \text{ is fixed.}$$

In the next Section we reveal the structure of extremal polynomials in Problem D. We describe them by means of certain conformal mappings on the comb domains, see Theorem 2.1. Particularly, this allows us to define the so-called $n$-regular extension for a given set $E$ and to give a relation between the solutions of Problems C and D. In Section 3 we prove our main theorem. First, in Lemma 3.1 we show that an extremal configuration of a set $E$ for Problem B belongs to the class of $n$-regular sets (a set which coincides with its $n$-regular extension). Second, in Lemma 3.2 we prove that among these sets a single-arc set is extremal. Finally, we obtain an explicit formula for extremal polynomials using some elementary conformal mappings.

## 2. Comb domains and solutions of Problems D and C

Comb-domains and the corresponding conformal mappings were introduced by Akhiezer and Levin, see [16]. Nowadays they are actively used in spectral theory (see, e.g., [17]) and approximation theory (see, e.g., [23]). For a modern presentation, see [12]. In this paper we will employ only the periodic $n$-regular comb domains.

DEFINITION 1. *Let $g \in \mathbb{N}$ and $\{h_k\}_{k=0}^g$ be a collection of positive numbers. The periodic $n$-regular comb domain is of the form*

$$\begin{aligned}
\Pi(h_0, \cdots, h_g) &= \Pi(h_0, \cdots, h_g; \omega_0, \cdots, \omega_g) \\
&= \mathbb{C}_+ \backslash \bigcup_{k=0}^g \bigcup_{m \in \mathbb{Z}} \big\{ z = \omega_k + 2\pi m + iy \ (0 < y \leqslant h_k) \big\},
\end{aligned}$$

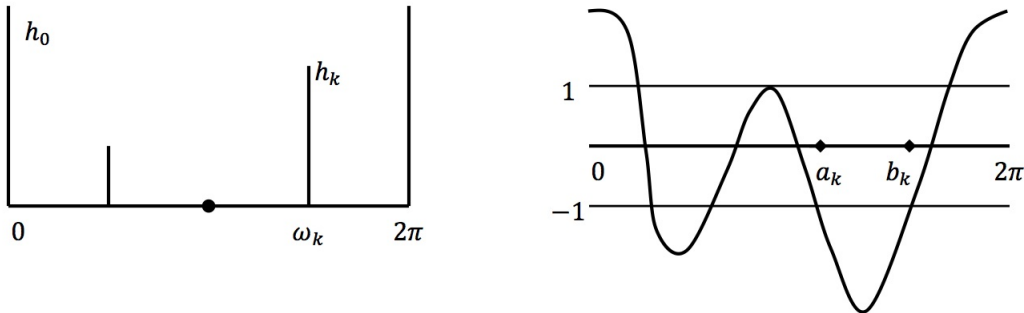*where $\omega_k = 2\pi \frac{j_k}{n}$, $0 \leqslant j_k < n$; see Fig. 1.*



FIGURE 1. Comb domain and graph of the function $\cos \frac{n}{2}\theta(z)$ on the period

For $\Pi = \Pi(h_0, \cdots, h_g)$, consider the conformal mapping $\theta$ defined as follows:

$$(2.1) \qquad\qquad \theta : \mathbb{C}_+ \longrightarrow \Pi, \qquad \theta(iy) \sim iy \quad \text{as} \quad y \to \infty.$$

We say that a set $E$ is $n$-regular if

$$E = \left\{ e^{iz} : z \in \theta^{-1}(\mathbb{R}) \right\},$$

where $\theta$ is given by (2.1). Note that in this case the endpoints of the gaps in (1.7) are given by

$$a_k = \theta^{-1}(\omega_k - 0), \qquad 0 \leqslant k \leqslant g;$$
$$b_k = \theta^{-1}(\omega_k + 0), \qquad 0 \leqslant k \leqslant g.$$

THEOREM 2.1. *For an extremizer* $T_n(e^{iz}, e^{ic_0}, E)$ *of Problem D,* $c_0 \in (a_0, b_0)$, *there exists the representation*

$$(2.2) \qquad T_n(e^{iz}) = e^{\frac{inz}{2}} \cos \frac{n}{2}\theta(z),$$

*where* $\theta$ *is a conformal mapping* $\mathbb{C}_+ \to \Pi$ *with the normalization*

$$(2.3) \qquad \theta(iy) \sim iy \qquad as \quad y \to \infty.$$

*Moreover,*

$$(2.4) \qquad \theta(a_0) = 0-, \qquad \theta(b_0) = 0+.$$

*Consequently, the extremizer does not depend on a position of* $c_0$ *in the given gap* $(a_0, b_0)$.

PROOF. The proof is based on Markov's method of correction of an extremal function [**23**, Chapter 7], see also [**7**, Theorem 3.2].

Let $c_0 \in (a_0, b_0)$ and $T_n(e^{iz}) = T_n(e^{iz}, e^{ic_0}, E)$ be an extremizer in Problem D. We write it as follows

$$T_n(e^{iz}) = e^{i\frac{zn}{2}} F(z),$$

where $F$ is an entire function such that $F(z + 2\pi) = (-1)^n F(z)$. Without loss of generality we assume that $F(c_0) > 0$. Noting that the function $e^{i\frac{zn}{2}}(F(z) + \overline{F(\overline{z})})/2$ is also an extremizer in Problem D, we may suppose that $F(z) \in \mathbb{R}$ whenever $z \in \mathbb{R}$. We divide the rest of the proof into 6 steps.

Step 1. We show that $F$ has no complex zeros. Assume that $F(z_0) = 0$, Im $z_0 > 0$, which gives that $F(\overline{z_0}) = 0$. Define the Markov correction function ($\delta > 0$)

$$Q(\zeta) = T_n(\zeta) \left( 1 - \delta \frac{(\zeta - e^{ic_0})(1 - \zeta e^{-ic_0})}{(\zeta - e^{iz_0})(1 - \zeta e^{-i\overline{z_0}})} \right),$$

which is a polynomial of degree at most $n$ satisfying $T_n(e^{ic_0}) = Q(e^{ic_0})$. Since the expression in the brackets is equal to

$$1 - \delta \left| \frac{\zeta - e^{ic_0}}{\zeta - e^{iz_0}} \right|^2$$

whenever $\zeta \in \mathbb{T}$, we have $\max_{\zeta \in E} |T_n(\zeta)| > \max_{\zeta \in E} |Q(\zeta)|$ for small enough $\delta$. This contradicts that $T_n$ is an extremizer.

Step 2. We prove that zeros of $F$ are simple. Here we use the similar correction function ($\delta > 0$)

$$Q(\zeta) = T_n(\zeta)\left(1 - \delta\frac{(\zeta - e^{ic_0})(1 - \zeta e^{-ic_0})}{(\zeta - e^{iz_0})(1 - \zeta e^{-iz_0})}\right)$$

and follow the proof of the previous step.

Step 3. Let us show that between two consecutive zeros of $F$ (take, for example, $F(z_1) = F(z_2) = 0$) there is $y$ such that $|F(y)| = 1$ and $e^{iy} \in E$. Assume the reverse:

(i) $\{e^{iy} : y \in (z_1, z_2)\} \cap E = \varnothing$ or (ii) $\max_{y \in (z_1, z_2)} |F(y)| < 1$.

Then we define the correction function ($\delta > 0$)

$$Q(\zeta) = e^{i\frac{nz}{2}}G(\zeta), \qquad G(\zeta) = F(\zeta)\left(1 - \delta\frac{\sin^2\frac{z-c_0}{2}}{\sin\frac{z-z_1}{2}\sin\frac{z-z_2}{2}}\right).$$

On the set $I_1 = (z_1 - \varepsilon, z_1 + \varepsilon) \cup (z_2 - \varepsilon, z_2 + \varepsilon)$ taking sufficiently small $\varepsilon$ and $\delta$, we have $|G(z)| < 1$. In the case (i) the function $|G(z)|$ on the set $I_2 = (z_1 + \varepsilon, z_2 - \varepsilon)$, should not be restricted. In the case (ii) since $\max_{z \in I_2} |F(z)|$ has a fixed value less then one, for sufficiently small $\delta$ we obtain $|G(z)| < 1$. Finally, on the set $I_3 = [-\pi, \pi] \setminus (I_1 \cup I_2)$ we always have $\max_{z \in I_3} |G(z)| < 1$. Then $\max_{\zeta \in E} |T_n(\zeta)| > \max_{\zeta \in E} |Q(\zeta)|$ gives a contradiction.

Step 4. The following condition holds: $F(a_0) = F(b_0) = 1$. In our normalization this corresponds to (2.4). Here we assume that $F(b_0) < 1$ and take the first zero $F(\xi) = 0$, $\xi > b_0$. Considering the function ($\delta > 0$)

$$G(\zeta) = F(\zeta)\left(1 - \delta\frac{\sin\frac{z-c_0}{2}}{\sin\frac{z-\xi}{2}}\right)$$

implies a contradiction.

Step 5. Let $\{z_k\}_{k=1}^n$ be zeros of the function $F(z)$ in a period, respectively, $\{e^{iz_k}\}_{k=1}^n$ are zeros of $T_n(\zeta)$. Between each two of them (including the pair $(z_n, z_{n+1})$, with the agreement $z_{n+1} := z_1 + 2\pi$) there is at least one critical point $y_k$ such that $F'(y_k) = 0$. Since

$$F'(z) = e^{i(-n/2)z}\left(i\zeta T_n'(\zeta) - i\frac{n}{2}T_n(\zeta)\right),$$

the total number of these points in a period is at most $n$. Thus all critical points of the function $F(z)$ are real. Also, by step 3, in each interval $(z_k, z_{k+1})$ there is a point $x_k$, $e^{ix_k} \in E$, such that $|F(x_k)| = 1$. Therefore at the critical point we have $|F(y_k)| \geqslant 1$. An example of such a function is shown in Fig. 1. According to the Marchenko–Ostrovskii theorem, see, e.g., [**23**, Section 7.3], the function $F(z)$ possesses the comb representation $F(z) = \cos\frac{n}{2}\theta(z)$ with $\theta(z)$ normalized by (2.3). Since $F(z)$ is periodic, $F(z + 2\pi) = (-1)^n F(z)$, the corresponding comb domain is periodic. That is, (2.2) is proved.

Step 6. We prove the last assertion of the theorem. Let $c$ be an arbitrary point in $(a_0, b_0)$. We will show that

(2.5)                          $|P_n(e^{ic})| \leqslant |T_n(e^{ic})|, \quad \forall P_n(\zeta) \in \mathcal{P}_n(E).$

First we note that there is a polynomial $Q_n(\zeta) \in \mathcal{P}_n(E)$ of the form

$$Q_n(e^{iz}) = e^{inz/2}G(z), \quad G(z) = \overline{G(\bar{z})}$$

such that $|Q_n(e^{ic})| = |P_n(e^{ic})|$. For an arbitrary positive $\varepsilon$ consider the function

$$H_\varepsilon(z) = (1 + \varepsilon)F(z) - G(z).$$

This function attains positive and negative values at points $\{x_k\}_{k=1}^n$, since $F(x_k) = \pm 1$, see step 5. Therefore each interval $(x_k, x_{k+1})$ (with our agreement $x_{n+1} := x_1 + 2\pi$) contains at most one zero of $H_\varepsilon(z)$. In particular, the interval $(a_0, b_0)$ may have at most one zero of this function. But, due to step 4,

$$H_\varepsilon(a_0) = (1 + \varepsilon) - G(a_0) > 0 \quad \text{and} \quad H_\varepsilon(b_0) = (1 + \varepsilon) - G(a_0) > 0.$$

Thus, for all $\varepsilon > 0$, $H_\varepsilon(z)$ is positive on this interval and we obtain

$$|P_n(e^{ic})| = |Q_n(e^{ic})| = |G(e^{ic})| \leqslant F(e^{ic}) = |T_n(e^{ic})|,$$

that is, (2.5) holds.

$\square$

REMARK 2.2. *In fact, we proved that*

$$T_n(e^{ic}, e^{ic_0}, E) = e^{inc/2} \sup_{P_n \in \mathcal{P}_n(E)} |P_n(e^{ic})|, \quad \forall c \in (a_0, b_0),$$

*which also implies the uniqueness of the extremal polynomial. Since the extremizer does not depend on the position of $c_0 \in (a_0, b_0)$, in what follows we write $T_n(\zeta, (a_0, b_0), E)$ instead of $T_n(\zeta, e^{ic_0}, E)$.*

*Moreover, from the proof we see that a polynomial of the presented form (2.2) is extremal on every set $\tilde{E}$, which contains at least one of possibly two different points $e^{i\theta^{-1}(2\pi k/n \pm 0)}$ for $k = 1, \cdots, n - 1$, together with the endpoints $e^{ia_0}$ and $e^{ib_0}$. Such collections of points form the so-called maximal Chebyshev sets [23, Section 7.2] for the given extremal function. Note also the fact that every periodic comb generates a periodic function is shown in [6, Appendix A].*

DEFINITION 2. *Let $E$ be a closed proper subset of $\mathbb{T}$. Let $T_n(\zeta, (a_0, b_0), E)$ be the extremizer associated to the gap $(a_0, b_0)$ presented in the form (2.2). We will call*

$$\widehat{E} = \left\{ e^{iz} : z \in \theta^{-1}(\mathbb{R}) \right\}$$

*the n-regular extension of the set $E$ associated to the gap $(a_0, b_0)$. This is the maximal possible set on which $T_n(\zeta, (a_0, b_0), E)$ remains extremal in the sense of Problem D.*

REMARK 2.3. *Since the n-regular extension does not depend on $c \in (a_0, b_0)$ we obtain a connection between the solutions of Problems C and D. Let $c_* = \theta^{-1}(ih_0)$. Then*

$$\max_{\substack{P_n \in \mathcal{P}_n(E) \\ c \in (a_0, b_0)}} |P_n(e^{ic})| = \max_{P_n \in \mathcal{P}_n(E)} |P_n(e^{ic_*})| = |T_n(e^{ic_*}, (a_0, b_0), \widehat{E})| = \cosh \frac{n}{2}h_0.$$

*Without loss of generality, we will assume that $c_* = 0$ and impose the third normalization condition for $\theta$ (see (2.3)) given by*

(2.6) $$\theta(0) = ih_0.$$

## 3. Proof of Theorem 1.2

We start with the following two lemmas.

LEMMA 3.1. *For any*

$$E = \mathbb{T} \backslash \bigcup_{j=0}^{g} (e^{ia_j}, e^{ib_j}), \qquad 1 \leqslant g \leqslant \infty$$

*there is an $n$-regular set*

$$E^* = \mathbb{T} \backslash \bigcup_{j=0}^{g^*} (e^{ia_j^*}, e^{ib_j^*}), \quad 1 \leqslant g^* < \infty,$$

*such that $|E^*| = |E|$ and moreover,*

$$\max_{c \in (a_0, b_0)} |T_n(e^{ic}, (a_0, b_0), E)| = |T_n(1, (a_0, b_0), E)| \leqslant |T_n(1, (a_0^*, b_0^*), E^*)|.$$

PROOF. In the proof we deal with the gap corresponding to $j = 0$, so the dependence of $(a_0, b_0)$ will be dropped.

Let $E$ be not $n$-regular. Consider $\widehat{E}$. Since the extension is proper, we have $|\widehat{E}| > |E|$ and moreover, $T_n(\zeta, E) = T_n(\zeta, \widehat{E})$. Note that $\widehat{E}$ has a finite number of gaps, i.e., $\widehat{g} < \infty$. This is because a number of critical points (where the derivative of $e^{-\frac{inz}{2}} T_n(e^{iz})$ is zero) on a period is finite and each gap contains a critical point, see Fig. 1.

Let $\Pi = \Pi(h_0, h_1, \cdots, h_{\widehat{g}})$ be the comb associated to the extremal polynomial $T_n(e^{iz}, \widehat{E})$. Let also $\Pi_h = \Pi(h_0 + h, h_1, \cdots, h_{\widehat{g}})$ with $h \geqslant 0$ and $\widehat{\theta}_h$ be the corresponding conformal mapping normalized exactly as (2.3), (2.6) and respectively $T_n(e^{iz}, \widehat{E_h}) = \cos \frac{n}{2} \widehat{\theta}_h(z) e^{\frac{inz}{2}}$ (by Theorem 2.1).

We will show that $|\widehat{E_h}|$ is decreasing with $h$ and tends to zero as $h \to \infty$. In the same time, $T_n(1, \widehat{E_h}) = \cosh \frac{n}{2}(h + h_0)$, increasing with $h$. Therefore, for some $h_* > 0$, we will have $|\widehat{E_{h_*}}| = |E|$ and $\cosh \frac{n}{2}(h_* + h_0) > \cosh \frac{n}{2} h_0$. Thus, one can set $E^* = \widehat{E_{h_*}}$.

It is left to verify that $|\widehat{E_h}|$ is decreasing with $h$ and that $|\widehat{E_h}| \to 0$ as $h \to \infty$. First, define $w_h(\zeta) = e^{i\widehat{\theta}_h(z)}$, $\zeta = e^{iz}$. Since $\widehat{\theta}_h(z)$ is $2\pi$-periodic this map is well-defined. This is a conformal mapping of the unit disk on the radial slit domain (the unit disk with the system of radial slits), that is,

$$w_h : \mathbb{D} \to \mathbb{D} \backslash \bigcup_{k=0}^{\widehat{g}} \left\{ w = e^{i\omega_k - y}, \ \ 0 < y \leqslant h + h_0, \ 0 < y \leqslant h_k, 1 \leqslant k \leqslant \widehat{g} \right\} =: \Omega_h.$$

According to the principle of harmonic measure [**19**, Chapter 3], the harmonic measure $\omega(0, \mathbb{T}, \Omega_h)$ is monotonic with $h$. On the other hand, since $\widehat{E_h} = w_h^{-1}(\mathbb{T})$ and $w_h(0) = 0$, we have

$$\frac{1}{2\pi}|\widehat{E_h}| = \omega(0, \widehat{E_h}, \mathbb{D}) = \omega(0, \mathbb{T}, \Omega_h).$$

Second, consider the conformal mapping $\widetilde{\theta}_h : \mathbb{C}_+ \to \Pi_h$, which satisfies another normalization

$$\widetilde{\theta}_h(-1) = 0, \quad \widetilde{\theta}_h(1) = 2\pi, \quad \widetilde{\theta}_h(\infty) = \infty.$$

We define

$$-C_-(h) := \widetilde{\theta}_h^{-1}(i(h + h_0)), \qquad C_+(h) := \widetilde{\theta}_h^{-1}(2\pi + i(h + h_0)).$$

By Carathéodory kernel convergence theorem [**21**, p. 28], in this normalization the sequence of conformal mappings has a limit, and $\lim_{h\to\infty} C_\pm(h) = \infty$. Comparing $\theta$ and $\widetilde{\theta}$, we obtain

$$\frac{1}{2\pi}|\widehat{E_h}| \leqslant \frac{2}{C_-(h) + C_+(h)} \longrightarrow 0 \qquad \text{as} \qquad h \longrightarrow \infty.$$

$\square$

LEMMA 3.2. *Let*

$$E = \mathbb{T}\backslash \bigcup_{j=0}^{g}(e^{ia_j}, e^{ib_j}), \qquad 1 \leqslant g < \infty$$

*be an n-regular set. Then there exists*

$$\widehat{E} = \mathbb{T}\backslash \bigcup_{j=0}^{g-1}(e^{i\widehat{a}_j}, e^{i\widehat{b}_j})$$

*such that* $|\widehat{E}| = |E|$ *and* $|T_n(1, E)| \leqslant |T_n(1, \widehat{E})|$.

PROOF. Let $\Pi = \Pi(h_0, h_1, \cdots, h_g)$. Using again the principle of harmonic measure with respect to the length of the slit $h_g$ for the set $E_{\check{h}_g}$ associated to the comb

$$\Pi_{\check{h}_g} = \Pi(h_0, h_1, \cdots, h_{g-1}, \check{h}_g), \ \check{h}_g < h_g,$$

we have that $|E_{\check{h}_g}| > |E|$ and

$$|T_n(1, E)| = \cosh\frac{n}{2}h_0 = |T_n(1, E_{\check{h}_g})|.$$

In this way, decreasing $\check{h}_g$, we can delete one of the gaps, i.e., to obtain $\check{h}_g = 0$ and, respectively, the set $\check{E} := E_0$. Thus, the number of gaps is now one less.

Repeating the proof of Lemma 3.1 with respect to $\check{E}$, we increase the value of $h$ in $\Pi(h_0 + h, h_1, \cdots, h_{g-1})$. Then, finally, we obtain for some $h_* > 0$ the comb $\widehat{\Pi} := \Pi(h_0 + h_*, h_1, \cdots, h_{g-1})$ and the set $\widehat{E}$ such that $|\widehat{E}| = |E|$ and $|T_n(1, E)| \leqslant |T_n(1, \widehat{E})|$.

$\square$

PROOF OF THEOREM 1.2. In light of Lemma 3.1 the extremal configuration cor-
responds to a regular set $E$. In Lemma 3.2 we proved that the extremal configuration
corresponds to the case of a single gap, i.e., to the set of the form

$$E_s = \mathbb{T} \setminus (e^{-is/2}, e^{is/2}),$$

which corresponds to the comb $\Pi = \Pi(h_0)$. Due to the extremality, we can claim that
for an arbitrary polynomial $P_n(\zeta)$ such that

$$\left|\left\{x \in [0, 2\pi) : |P_n(e^{ix})| \leqslant 1\right\}\right| \geqslant 2\pi - s,$$

we have

(3.1) $$\sup_{\zeta \in \mathbb{T}} |P_n(\zeta)| \leqslant |T_n(1, E_s)| = \cosh n\frac{h_0}{2}.$$

It remains to find the relation between the value $h_0$ and the length of the gap $s$.

In term of the variables

$$w(\zeta) = e^{i\theta(z)}, \quad \zeta = e^{iz},$$

we have a conformal mapping of the unit disk $\mathbb{D}$ on the domain

$$\Omega = \mathbb{D} \setminus \{w = u : u \in (e^{-h_0}, 1)\}$$

such that $w(0) = 0$ and, by symmetry, $w(\zeta)$ is real for real $\zeta$. We point out that the
preimage of the radial slit is the arc $(e^{-is/2}, e^{is/2})$.

By a standard change of variables

(3.2) $$\lambda = i\frac{1 - \zeta}{1 + \zeta}, \quad \mu(\lambda) = i\frac{1 - w(\zeta)}{1 + w(\zeta)},$$

we pass to a conformal mapping of the upper half-plane $\mathbb{C}_+$ to the upper half plane
with a cut along a single vertical interval

$$\mathbb{C}_+ \setminus \{\mu = i\eta : \eta \in (0, \kappa)\},$$

where

(3.3) $$\kappa = \frac{1 - e^{-h_0}}{1 + e^{-h_0}} = \tanh\frac{h_0}{2}.$$

Respectively, the preimage of this vertical interval is the interval $(-\lambda_0, \lambda_0) \subset \mathbb{R}$, where

(3.4) $$\lambda_0 = i\frac{1 - e^{is/2}}{1 + e^{is/2}} = \tan\frac{s}{4}.$$

It is well known (and easy to check directly) that this conformal mapping is of the form

$$\mu(\lambda) = C\sqrt{\lambda^2 - \lambda_0^2}, \quad C > 0.$$

To find $C$, we use the normalization condition $\mu(i) = i$. We have

$$\mu(\lambda) = \frac{\sqrt{\lambda^2 - \lambda_0^2}}{\sqrt{1 + \lambda_0^2}} \quad \text{and particularly} \quad \mu(0) = i\frac{\lambda_0}{\sqrt{1 + \lambda_0^2}} = i\kappa.$$

By (3.3) and (3.4) we have

$$\frac{\tan \frac{s}{4}}{\sqrt{1 + \tan^2 \frac{s}{4}}} = \sin \frac{s}{4} = \tanh \frac{h_0}{2},$$

or $\cosh \frac{h_0}{2} = \sec \frac{s}{4}$. We substitute this value of $h_0$ in (3.1), as the result we obtain (1.5).

Further, since

$$\mu^2 + 1 = (\lambda^2 + 1) \cos^2 \frac{s}{4},$$

using (3.2), we obtain

$$\frac{4w}{(1+w)^2} = \frac{4\zeta}{(1+\zeta)^2} \cos^2 \frac{s}{4}.$$

Thus

$$\frac{w^{1/2} + w^{-1/2}}{2} = \sec \frac{s}{4} \frac{\zeta^{1/2} + \zeta^{-1/2}}{2} = \sec \frac{s}{4} \cos \frac{z}{2}$$

and, finally,

$$T_n(\zeta, E_s) = e^{inz/2} \frac{(w^{1/2})^n + (w^{-1/2})^n}{2} = e^{inz/2} \mathfrak{T}_n \left( \sec \frac{s}{4} \cos \frac{z}{2} \right).$$

Due to the fixed normalizations $c_* = 0$ and $T_n(1, E_s) > 0$, this extremal polynomial is unique, see Remarks 2.2 and 2.3. Generally, we can choose an arbitrary normalization point $c_* = c_0 \in [0, 2\pi)$ and multiply the extremal polynomial by a unimodular constant. Thus, we obtain (1.6). $\square$

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