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Contents

The notes contained in this booklet were printed directly from files supplied by the authors before the course.

Foreword

The present volume contains the notes for the Advanced Course *Building Bridges Between Algebra and Topology* organized by the Centre de Recerca Matemàtica (CRM) in Bellaterra from April 13 to 17, 2015. This advanced course consists of three different series of four lectures each one, delivered by Wojciech Chacholski, (KTH) on *Idempotent Symmetries in Algebra and Topology*, by John Greenlees (University of Sheffield) on *Homotopy Invariant Commutative Algebra: Algebra, Topology and Representations* and by Greg Stevenson (Universität Bielefeld) on *Support Theory for Triangulated Categories.*

Topology and algebra, specifically homological algebra, have developed hand in hand during the last decades. Recent notions as that of "brave new rings", the dissemination of triangulated structures in many areas and the importance therein of localization and cellularization methods coming directly from questions in stable homotopy theory still reinforce this bond between the two areas. The aim of the Advanced Course and of the subsequent notes is to present three domains in which the interplay between algebra and topology at the level of ideas and methods has proven to be particularly rich. In spite of the variety of subjects touched in these lectures the reader will find many common objects of study. As an example we single out the the idea of "best approximation of an object by a fixed class of objects of interest". This appears in the form of "cellular cover" in the lectures by W. Chacholsky, it is at the heart of the different notions of "smallness" in the lectures by J. Greenlees and is the notion motivating the study of localizing subcategories as presented in G. Stevenson's lectures.

Idempotent Symmetries in Algebra and Topology by Wojciech Chacholski.

The importance of idempotent functors in topology can not be underestimated; they provide tools to study how objects can be deformed. An important example of the power of these is the proof of Ravenel's conjectures in the early eighties and their reinterpretation in the form of the classification of the homotopy idempotent functors of spectra that commute with telescopes by Devintaz-Hopkins-Smith. Nevertheless, classifying idempotent functors in other contexts, like groups or modules over a ring seems, in this generality, out of reach. This course will present tools to study the *orbit* of an object under idempotent functors without having to actually compute them. The main example in the applications will be the category of (finite) groups where these methods have proven to be particularly fruitful.

Homotopy Invariant Commutative Algebra: Algebra, Topology and Representations by John Greenlees.

The main aim is to illustrate how powerful it is to formulate ideas of commutative algebra in a homotopy invariant form; in particular shows how robust concepts are, in the sense that they are invariant under deformations. This formulation also offers a framework in which notions from homotopy theory and algebra can migrate from one domain to the other, among these we may point out Morita theory, closely related to cellularity properties, and the Gorenstein property. The course includes many striking new examples that show the richness of these ideas when applied to topology, group cohomology and module theory.

Support Theory for Triangulated Categories by Greg Stevenson.

These lectures will provide an introduction and an overview of the developing field of tensor triangular geometry and support varieties. This is a conceptual framework which unites earlier results in various examples such as perfect complexes over schemes, stable categories of modular representations, and the finite stable homotopy category. Support theories allow, in many cases, to classify localizing subcategories of a triangulated category, but a drawback in the usual approach is that it relies heavily on having a nice monoidal structure compatible with the triangular one; however by using the notion of an action of a triangulated category one can to some extent walk around this limitation. This general machinery applies, among other examples, to the problem of classifying localizing subcategories in the singularity category of an affine hypersurface.

The course is one of the scientific events of the Research Program *IRTATCA: Interactions between Representation Theory, Algebraic Topology and Commutative Algebra*, held in the CRM from January 7 to June 30, 2015. The scientific committee of the program is composed by William Dwyer (University of Notre Dame), Dolors Herbera (Universitat Autònoma de Barcelona), Srikanth B. Iyengar (University of Utah), Henning Krause (Universität Bielefeld), Bernard Leclerc (Université de Caen), Wolfgang Pitsch (Universitat Autònoma de Barcelona), and Santiago Zarzuela (Universitat de Barcelona).

The program is being made possible not only by the support of the CRM, but also by the support from Bielefeld University (Collaborative Research Centre 701 "Spectral Structures and Topological Methods in Mathematics"), the Simons Foundation, the National Science Foundation and the Institut de Matemàtiques (IMUB) from the Universitat de Barcelona. We also acknowledge the financial support from the research projects *Estructura y clasificación de anillos, módulos y C^*-álgebras* (DGI MICIIN MTM2011-28992-C02-01, Spain) and *Análisis local en grupos y espacios topológicos* (DGI MICIIN MTM2010-20692, Spain).

Bellaterra, March 25th, 2015

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Idempotent symmetries in algebra and topology

IDEMPOTENT SYMMETRIES IN ALGEBRA AND TOPOLOGY

WOJCIECH CHACHÓLSKI

1. Introduction

The proofs of Ravenel's conjectures [23] and their reinterpretations in the form of the classification of the homotopy idempotent functors of spectra that commute with telescopes by Devintaz-Hopkins-Smith [7, 15] were a culmination of a few decades of progress achieved in stable homotopy theory. The simplicity of this classification is remarkable. For each prime p, the category of restrictions of these functors to p-local finite spectra is isomorphic to the poset of natural numbers. The obtained invariant is called the Morava-Hopkins type. The stable classification was generalized to the classification of so called Bousfield localizations of finite p-local spaces [3]. This unstable classification is also remarkably simple. Bousfield showed that in addition to the Morava-Hopkins type invariant, connectivity determines such functors. After these remarkable classifications of Devintaz-Hopkins-Smith and Bousfield, there was a hope (even published wrong results) that a similarly explicit classification might be true for all idempotent functors of topological spaces that commute with telescopes. This however turned out to be a failed hope as the category of idempotent functors of topological spaces surjects onto the lattice of all ideals in the ring of stable homotopy groups of the sphere. Thus one should not expect any simple classification result in the case of topological spaces. Any such classification has to include the ring of stable homotopy groups of the sphere. To understand possible difficulties one strategy has been to ask analogous classification questions in other settings with a hope that the results might shed some light on possible difficulties in the case of topological spaces. That led to extensive research particularly in the algebraic settings of the derived categories of rings, stable module categories, and the category of groups, see for example [1, 8, 11, 12, 16, 17, 18, 19, 21, 22].

I believe we should change the goal and the direction in which to study idempotent functors. I believe that understanding globally the category of such functors of topological spaces or groups is simply out of reach. The simplicity of the stable analog was misleading and one should not expect to be able to classify them. Instead we should focus on their action on various spaces (finite complexes, classifying spaces of groups etc.) and try to describe the orbits of this action:

$$\mathrm{Idem}(X) := \{\text{equivalence classes of } \phi(X) \mid \phi \text{ is an idempotent functor}\}$$

I believe it is a much more reasonable task to enumerate or parametrize such an orbit. The main evidence for that is the fact that in the category of groups many such orbits surprisingly can be described explicitly and the category of groups tends to reflect well a lot of phenomena of topological spaces. My aim for these notes is to describe this story for groups and to some extent R-modules in elementary way. These notes for example are in principle self contained.

Here is a short summary of the content. To understand global symmetry of groups one can study functors $\phi\colon \text{Groups} \to \text{Groups}$. We think about them as operations on groups. To study how such an operation deforms groups we look at natural transformations $\epsilon_X\colon \phi(X) \to X$. A choice of an operation ϕ and a comparison ϵ is called an augmented functor. By iterating the augmentation we obtain two homomorphisms $\epsilon_{\phi(X)}\colon \phi^2(X) \to \phi(X)$ and $\phi(\epsilon_X)\colon \phi^2(X) \to \phi(X)$. The augmented functor (ϕ, ϵ) is called idempotent if this iteration process does not produce anything new and the homomorphisms $\epsilon_{\phi(X)}$ and $\phi(\epsilon_X)$ are isomorphisms for any group X. For example, the universal central extension of the maximal perfect subgroup of X, with the natural projection as augmentation, is an idempotent functor. This description is typical. Idempotent functors are often universal with respect to certain properties.

Our goal now is not to describe the category of idempotent functors of groups, but rather to focus on their action and ask if it is possible to enumerate its orbits in terms of some classical invariants. We are going to concentrate on finite groups. It turns out that for any idempotent functor (ϕ, ϵ), the group $\phi(X)$ is finite if X is. Thus finite groups are acted upon by idempotent functors. How complicated is this action? To measure it, as we indicated before, we study its orbits $\mathrm{Idem}(X) := \{\text{isomorphism class of } \phi(X) \mid (\phi, \epsilon) \text{ is idempotent}\}$. Although the collection of idempotent functors does not even form a set, the number of different values idempotent functors can take on a finite group is finite: If X is a finite group, then $\mathrm{Idem}(X)$ is a finite set. This is the first indication to why it might be reasonable to study orbits of the action of idempotent functors on finite groups. Our next step should be to enumerate the orbit $\mathrm{Idem}(X)$. This turns out to be possible if X is finite and simple for which we prove that $\mathrm{Idem}(X)$ is very small. Except for the projective linear groups, such an orbit can have at most 7 elements. This statement requires the classification of finite simple groups. Here is how to enumerate this orbit. Recall that by functoriality $\mathrm{Aut}(X)$ acts on the second integral homology $H_2(X)$ of X (the so called Schur multiplier). Let $\mathrm{InvSub}(H_2(X))$ denote the set of all subgroups of $H_2(X)$ which are invariant (not necessarily pointwise fixed) under this action.

Theorem. *Let X be a finite simple group. There is a bijection between* $\mathrm{Idem}(X)$ *and the set* $\{0\} \coprod \mathrm{InvSub}(H_2(X))$.

This theorem illustrates well why our change in the strategy to study idempotent functors is worthwhile. Although it is not reasonable to expect an explicit description of the category of idempotent functors of finite groups, it is within reach to give an explicit description of the orbits of their action on finite groups. Furthermore classical invariants, such as the Schur multiplier, can be used to determine such orbits.

These note are based on the following articles: [2, 4, 5, 6, 9, 10].

2. Cellular properties

Let $\mathcal{C}$ be a category. A **property** in $\mathcal{C}$, by definition, is a full subcategory $\mathcal{D} \subset \mathcal{C}$ closed under isomorphisms: if X belongs to $\mathcal{C}$, then so does any object that is isomorphic to X. An object is said to satisfy $\mathcal{D}$ if it belongs to $\mathcal{D}$. An object can satisfy $\mathcal{D}$ or not. We however would like to be able to know more. We would like to be able to measure how far an object is from satisfying $\mathcal{D}$. We would like to be able to estimate the failure of an object to belong to $\mathcal{D}$. We start with discussing a general framework of how this can be done.

2.1. **Definition.** Let $\mathcal{D} \subset \mathcal{C}$ be a property. A $\mathcal{D}$-cover of a X is a homomorphism $c\colon \mathrm{cov}_{\mathcal{D}}X \to X$ which fulfills the following requirements:

(a) $\mathrm{cov}_{\mathcal{D}}X$ belongs to $\mathcal{D}$;
(b) for any Y that satisfies $\mathcal{D}$, the following function of sets is a bijection:

$$\mathrm{mor}(Y, c)\colon \mathrm{mor}(Y, \mathrm{cov}_{\mathcal{D}}X) \to \mathrm{mor}(Y, X)$$

It is a direct consequence of the definition that if $c_0\colon Y_0 \to X$ and $c_1\colon Y_1 \to X$ are $\mathcal{D}$-covers of X, than there is a unique isomorphism $h\colon Y_0 \to Y_1$ for which $c_1 h = c_0$. In this sense, if it exists, a $\mathcal{D}$-cover is unique and hence will be called the $\mathcal{D}$-cover of X. Because of this uniqueness, we will often call simply the object $\mathrm{cov}_{\mathcal{D}}X$ the $\mathcal{D}$-cover of X.

If X satisfies $\mathcal{D}$, then $\mathrm{id}\colon X \to X$ is the $\mathcal{D}$-cover. On the other hand if $\mathrm{id}\colon X \to X$ is the $\mathcal{D}$-cover, then X must satisfy $\mathcal{D}$. Thus the cover $c\colon \mathrm{cov}_{\mathcal{D}}X \to X$ can be used to measure how close X is to satisfy $\mathcal{D}$. If it is an isomorphism, then X belongs to $\mathcal{D}$. Otherwise we can try to estimate its "kernel" and "quotient" to understand the failure of X to satisfy $\mathcal{D}$.

To catalog behavior of an object X with respect to various properties we consider:

$$\mathrm{Cov}(X) := \{\text{iso class of } \mathrm{cov}_{\mathcal{D}}X \mid \mathcal{D} \text{ is a property for which } \mathrm{cov}_{\mathcal{D}}X \text{ exists}\}.$$

For example in [2] it was shown that in the category of groups the collection $\mathrm{Cov}(X)$ is finite if X is finite (see 10.8). Furthermore in the case X is a finite and simple group and not linear, then $\mathrm{Cov}(X)$ has at most 7 elements.

The strategy to enumerate the collection $\mathrm{Cov}(X)$ is to notice that its elements can be characterize locally. Let Y be an object in $\mathcal{C}$ and $\mathrm{Iso}(Y)$ be the property consisting of all objects isomorphic to Y. Then the $\mathrm{Iso}(Y)$-cover of X is a morphism $c\colon Y \to X$ for which $\mathrm{mor}(Y, c)\colon \mathrm{mor}(Y, Y) \to \mathrm{mor}(Y, X)$ is a bijection. We call any object Y, for which the $\mathrm{Iso}(Y)$-cover of X exists, a **cover** of X. We use the symbol $\mathcal{L}(X)$ to denote the collection of all covers of X:

$$\mathcal{L}(X) := \{\text{iso class of } Y \mid \text{there is } c\colon Y \to X \text{ for which } \mathrm{Hom}(c, Y) \text{ is a bijection}\}.$$

Since for any property $\mathcal{D}$, if $c_X\colon \mathrm{cov}_{\mathcal{D}}X \to X$ exists, then $\mathrm{cov}_{\mathrm{Iso}(\mathrm{cov}_{\mathcal{D}}X)}$ and $\mathrm{cov}_{\mathcal{D}}X$ are isomorphic:

2.2. **Proposition.** $\mathrm{Cov}(X) = \mathcal{L}(X)$.

To understand global symmetries of the whole category $\mathcal{C}$, we are however not interested in properties such as $\mathrm{Iso}(Y)$ for which only very limited selection of objects admit covers. We are interested in properties for which any object has the cover:

2.3. **Definition.** A property $\mathcal{D} \subset \mathcal{C}$ is called **cellular** if all objects in $\mathcal{C}$ admit the $\mathcal{D}$-cover.

3. Cellular properties and idempotent deformations

Let $\mathcal{C}$ be a category. One way to understand symmetries of $\mathcal{C}$ is to look for what acts on $\mathcal{C}$. In this way we study symmetries of $\mathcal{C}$ by considering endofunctors $\phi\colon \mathcal{C} \to \mathcal{C}$. To understand how such an operation ϕ deforms objects in $\mathcal{C}$ we consider natural transformations $\epsilon_X\colon \phi(X) \to X$. A choice of a functor $\phi\colon \mathcal{C} \to \mathcal{C}$ and a natural transformation $\epsilon_X\colon \phi(X)\to X$ is called an augmented functor and denoted by (ϕ,ϵ). For example, let $\mathcal{D}\subset\mathcal{C}$ be a cellular property. For any X in $\mathcal{C}$ let us choose its cover $c_X\colon \mathrm{cov}_{\mathcal{D}}X\to X$. For any morphism $f\colon X\to Y$, since $\mathrm{cov}_{\mathcal{D}}X$ belongs to $\mathcal{D}$, there is a unique morphism $\mathrm{cov}_{\mathcal{D}}f\colon \mathrm{cov}_{\mathcal{D}}X\to \mathrm{cov}_{\mathcal{D}}Y$ for which the following square commutes:

$$\begin{tikzcd}
\mathrm{cov}_{\mathcal{D}}X \arrow[r, "\mathrm{cov}_{\mathcal{D}}f"] \arrow[d, "c_X"'] & \mathrm{cov}_{\mathcal{D}}Y \arrow[d, "c_Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

The uniqueness implies that $\mathrm{cov}_{\mathcal{D}}(fg)=(\mathrm{cov}_{\mathcal{D}}f)(\mathrm{cov}_{\mathcal{D}}g)$ and $\mathrm{cov}_{\mathcal{D}}\mathrm{id}=\mathrm{id}$. This means that $X\mapsto \mathrm{cov}_{\mathcal{D}}X$ and $f\mapsto \mathrm{cov}_{\mathcal{D}}f$ define a functor $\mathrm{cov}_{\mathcal{D}}\colon\mathcal{C}\to\mathcal{D}$ and the morphisms $c_X\colon \mathrm{cov}_{\mathcal{D}}X\to X$ a natural transformation.

Two augmented functors (ϕ,ϵ) and (ψ,μ) are said to be **isomorphic** if there is a natural isomorphism $\alpha_X\colon\phi(X)\to\psi(X)$ which makes the following triangle commutative:

$$\begin{tikzcd}
\phi(X) \arrow[rr, "\alpha_X"] \arrow[dr, "\epsilon_X"'] & & \psi(X) \arrow[dl, "\mu_X"] \\
& X &
\end{tikzcd}$$

Let (ϕ,ϵ) be an augmented functor. A morphism $f\colon X\to Y$ is called a ϕ-**equivalence**, if $\phi(f)$ is an isomorphism. The symbol $\mathrm{Cell}(\phi)$ denotes the full subcategory in $\mathcal{C}$ given by all the objects X for which $\epsilon_X\colon\phi(X)\to X$ is an isomorphism. Note that if X and Y belong to $\mathrm{Cell}(\phi)$, then $f\colon X\to Y$ is a ϕ-equivalence if and only if it is an isomorphism. For example if $\mathcal{D}\subset\mathcal{C}$ is a cellular property, then $\mathrm{Cell}(\mathrm{cov}_{\mathcal{D}})=\mathcal{D}$ and the morphism $c_X\colon\mathrm{cov}_{\mathcal{D}}X\to X$ is a $\mathrm{cov}_{\mathcal{D}}$-equivalence. The first equality follows from the definition. To see why c_X is a $\mathrm{cov}_{\mathcal{D}}$-equivalence consider the following commutative square:

$$\begin{tikzcd}
\mathrm{cov}^2_{\mathcal{D}}X \arrow[r, "\mathrm{cov}_{\mathcal{D}}c_X"] \arrow[d, "c_{\mathrm{cov}_{\mathcal{D}}X}"'] & \mathrm{cov}_{\mathcal{D}}X \arrow[d, "c_X"] \\
\mathrm{cov}_{\mathcal{D}}X \arrow[r, "c_X"] & X
\end{tikzcd}$$

Since $c_{\mathrm{cov}_{\mathcal{D}}X}$ is an isomorphism ($\mathrm{cov}_{\mathcal{D}}X$ belongs to $\mathcal{D}$), we can form the following commutative triangles:

$$\begin{tikzcd}
& \mathrm{cov}_{\mathcal{D}}X \arrow[d, "c_X"] & \\
\mathrm{cov}_{\mathcal{D}}X \arrow[ur, "(\mathrm{cov}_{\mathcal{D}}c_X)(c_{\mathrm{cov}_{\mathcal{D}}X})^{-1}"] \arrow[r, "c_X"] & X & X \arrow[l, "c_X"'] \arrow[ul, "\mathrm{id}"']
\end{tikzcd}$$

By the definition of the cover we then get $(\mathrm{cov}_{\mathcal{D}}c_X)(c_{\mathrm{cov}_{\mathcal{D}}X})^{-1}=\mathrm{id}$. It follows that $\mathrm{cov}_{\mathcal{D}}c_X=c_{\mathrm{cov}_{\mathcal{D}}X}$ and hence $\mathrm{cov}_{\mathcal{D}}c_X$ is an isomorphism.

Let (ϕ,ϵ) be an augmented functor. By iterating the augmentation we obtain two morphisms $\epsilon_{\phi(X)}\colon\phi^2(X)\to\phi(X)$ and $\phi(\epsilon_X)\colon\phi^2(X)\to\phi(X)$. Among all augmented functors (ϕ,ϵ) there are the **idempotent** ones for which this iteration process does not produce anything new and the homomorphisms $\epsilon_{\phi(X)}$ and $\phi(\epsilon_X)$ are isomorphisms for any X. Note that by definition (ϕ,ϵ) is idempotent if and only if, for any X, $\epsilon_X\colon \phi(X)\to X$ is a ϕ-equivalence and $\phi(X)$ belongs to $\mathrm{Cell}(\phi)$. For example, if $\mathcal{D}\subset\mathcal{C}$ is a cellular property, then we have just shown that $(\mathrm{cov}_{\mathcal{D}}, c)$ is an idempotent functor. It turns out that all idempotent functors are of the form $(\mathrm{cov}_{\mathcal{D}}, c)$:

3.1. **Proposition.** *The following are equivalent:*

(1) *An augmented functor (ϕ,ϵ) is idempotent.*
(2) $\mathrm{Cell}(\phi)\subset\mathcal{C}$ *is a cellular property.*
(3) *The inclusion* $\mathrm{Cell}(\phi)\subset\mathcal{C}$ *has a right adjoint.*

Furthermore if any of the above conditions hold, then the right adjoin to $\mathrm{Cell}(\phi)\subset\mathcal{C}$ *is given by ϕ, and (ϕ,ϵ) and $(\mathrm{cov}_{\mathrm{Cell}(\phi)}, c)$ are isomorphic augmented functors.*

3.2. **Lemma.** *Assume that $\mathcal{D}\subset\mathcal{C}$ is a property such that this inclusion has a right adjoin $\phi\colon\mathcal{C}\to\mathcal{D}$. Let $\epsilon_X\colon\phi(X)\to X$ be the morphism which is adjoin to the identity* $\mathrm{id}\colon\phi(X)\to\phi(X)$*. Then the composition of $\phi\colon\mathcal{C}\to\mathcal{D}$ and the inclusion $\mathcal{D}\subset\mathcal{C}$ together with the natural transformation $\epsilon_X\colon\phi(X)\to X$ is idempotent. Furthermore* $\mathrm{Cell}(\phi)=\mathcal{D}$.

Proof. By definition any $f\colon Y\to X$ with Y in $\mathcal{D}$ can be uniquely fit into the following commutative triangle:

$$\begin{tikzcd} & \phi(X) \arrow[d, "\epsilon_X"] \\ Y \arrow[ur, "\overline{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

We can apply this to the morphisms $\mathrm{id}\colon Y\to Y$ and $\epsilon_Y\colon\phi(Y)\to Y$, for any Y in $\mathcal{D}$ to get the following commutative triangles:

$$\begin{tikzcd} & \phi(Y) \arrow[d, "\epsilon_Y"] \\ Y \arrow[ur, "\overline{\mathrm{id}}"] \arrow[r, "\mathrm{id}"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & & \phi(Y) \arrow[dd, "\epsilon_Y"] \\ & Y \arrow[ur, "\overline{\mathrm{id}}"] & \\ \phi(Y) \arrow[ur, "\epsilon_Y"] \arrow[rr, "\epsilon_Y"'] & & Y \end{tikzcd} \qquad \begin{tikzcd} & \phi(Y) \arrow[d, "\epsilon_Y"] \\ \phi(Y) \arrow[ur, "\mathrm{id}"] \arrow[r, "\epsilon_Y"'] & Y \end{tikzcd}$$

It follows that, for any Y in $\mathcal{D}$, the morphism $\epsilon_Y\colon\phi(Y)\to Y$ is an isomorphism.

Since ϵ is a natural transformation, for any X in $\mathcal{C}$, we have a commutative square:

$$\begin{tikzcd} \phi^2(X) \arrow[r, "\phi(\epsilon_X)"] \arrow[d, "\epsilon_{\phi(X)}"'] & \phi(X) \arrow[d, "\epsilon_X"] \\ \phi(X) \arrow[r, "\epsilon_X"'] & X \end{tikzcd}$$

As the compositions $\epsilon_X\phi(\epsilon_X)=\epsilon_X\epsilon_X\phi(X)$ fit into the following commutative diagram:

$$\begin{tikzcd} & & \phi(X) \arrow[d, "\epsilon_X"] & & \\ \phi^2(X) \arrow[r, "\phi(\epsilon_X)"] & \phi(X) \arrow[ur, "\mathrm{id}"] \arrow[r, "\epsilon_X"] & Y & \phi(Y) \arrow[ul, "\mathrm{id}"'] \arrow[l, "\epsilon_X"'] & \phi^2(X) \arrow[l, "\epsilon_{\phi(X)}"'] \end{tikzcd}$$

we can conclude that $\epsilon_{\phi(X)}\colon \phi^2(X) \to \phi(X)$ and $\phi(\epsilon_X)\colon \phi^2(X) \to \phi(X)$ are the same isomorphisms. It follows that (ϕ, ϵ) is an idempotent functor. $\square$

Proof of 3.1. Assume (1), i.e., (ϕ, ϵ) is idempotent. The values of ϕ lies in $\mathrm{Cell}(\phi)$ and hence ϕ induces a functor which we denote by the same symbol $\phi\colon \mathcal{C} \to \mathrm{Cell}(\phi)$. Consider the function $\mathrm{mor}(Y, \epsilon_X)\colon \mathrm{mor}(Y, \phi(X)) \to \mathrm{mor}(Y, X)$ for any Y in $\mathrm{Cell}(\phi)$ and any X in $\mathcal{C}$. We claim that this function is a bijection which would imply that ϵ_X is the $\mathrm{Cell}(\phi)$-cover, proving (2). Consider the following commutative square:

$$\begin{tikzcd} \phi(Y) \arrow[r, "\phi(f)"] \arrow[d, "\epsilon_Y"'] & \phi(X) \arrow[d, "\epsilon_X"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

Since $\epsilon_Y$is an isomorphism, the morphism $\phi(f)\epsilon_Y^{-1}\colon Y \to \phi(X)$ composed with ϵ_X is f. It follows that is $\mathrm{mor}(Y, \epsilon_X)$ is surjective. It reminds to show that this map is also injective. Let $g, h\colon Y \to \phi(X)$ be such that $\epsilon_X g = \epsilon_X h$ and consider the following commutative diagram:

$$\begin{tikzcd} \phi(Y) \arrow[r, shift left, "\phi(g)"] \arrow[r, shift right, "\phi(h)"'] \arrow[d, "\epsilon_Y"'] & \phi^2(X) \arrow[r, "\phi(\epsilon_X)"] \arrow[d, "\epsilon_{\phi(X)}"] & \phi(X) \arrow[d, "\epsilon_X"] \\ Y \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & \phi(X) \arrow[r, "\epsilon_X"] & X \end{tikzcd}$$

As $\epsilon_X g = \epsilon_X h$, then $\phi(\epsilon_X)\phi(g) = \phi(\epsilon_X g) = \phi(\epsilon_X h) = \phi(\epsilon_X)\phi(h)$. Since $\phi(\epsilon_X)$ is an isomorphism (here we use the assumption that (ϕ, ϵ) is idempotent), we get $\phi(g) = \phi(h)$. Since ϵ_Y and $\epsilon_{\phi(X)}$ are also isomorphisms, commutativity of the left square in the above diagram implies $g = h$. We can conclude $\mathrm{mor}(Y, \epsilon_X)$ is injective.

Assume (2), i.e., $\mathrm{Cell}(\phi) \subset \mathcal{C}$ is a cellular property. By definition, for any Y in $\mathrm{Cell}(\phi)$, the map $\mathrm{mor}(Y, c_X)\colon \mathrm{mor}(Y, \mathrm{cov}_{\mathrm{Cell}(\phi)} X) \to \mathrm{mor}(Y, X)$ is a bijection. Thus the right adjoint to $\mathrm{Cell}(\phi) \subset \mathcal{C}$ is given by $\mathrm{cov}_{\mathrm{Cell}(\phi)}$ and we proved (3).

Assume (3). Let $\psi\colon \mathcal{C} \to \mathrm{Cell}(\phi)$ be the right adjoint to the inclusion $\mathrm{Cell}(\phi) \subset \mathcal{C}$. If $\mu_X\colon \psi(X) \to X$ is adjoin to the identity $\mathrm{id}\colon \psi(X) \to \psi(X)$, then according to 3.2, (ψ, μ) is an idempotent functor. To finish the proof of the proposition it would be enough to show that ψ and ϕ are naturally isomorphic. For any Y in $\mathrm{Cell}(\phi)$, any $f\colon Y \to X$ can be uniquely fit into the following commutative triangle:

$$\begin{tikzcd} & \psi(X) \arrow[d, "\mu_X"] \\ Y \arrow[ur, "\overline{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

Thus since $\phi(X)$ is in $\mathrm{Cell}(\phi)$, we can fit $\epsilon_X \colon \phi(X) \to X$ into the following commutative diagram:

$$\begin{tikzcd}
 & \psi(X) \arrow[d, "\mu_X"] & \phi\psi(X) \arrow[l, "\epsilon_{\psi(X)}"'] \arrow[d, "\phi(\mu_X)"] \\
\phi(X) \arrow[ur, "\overline{\epsilon_X}"] \arrow[r, "\epsilon_X"] \arrow[rr, bend right, "\mathrm{id}"'] & X & \phi(X) \arrow[l, "\epsilon_X"']
\end{tikzcd}$$

The object $\psi(X)$ belongs to $\mathrm{Cell}(\phi)$ and hence $\epsilon_{\psi(X)} \colon \phi(\psi(X)) \to \psi(X)$ is an isomorphism. In this way we obtain two natural transformations $\overline{\epsilon_X} \colon \phi(X) \to \psi(X)$ and $\phi(\mu_X)\epsilon^{-1}_{\psi(X)} \colon \psi(X) \to \phi(X)$. The commutativity of the above diagram shows $\phi(\mu_X)\epsilon^{-1}_{\psi(X)}\overline{\epsilon_X} = \mathrm{id}_{\phi(X)}$ and $\mu_X\overline{\epsilon_X}\phi(\mu_X)\epsilon^{-1}_{\psi(X)} = \mu_X$. Note that obviously $\mu_X \mathrm{id}_{\psi(X)} = \mu_X$ and therefore, together with the previous equality, we get $\overline{\epsilon_X}\phi(\mu_X)\epsilon^{-1}_{\psi(X)} = \mathrm{id}_{\psi(X)}$. We can conclude that $\overline{\epsilon_X}$ and $\phi(\mu_X)\epsilon^{-1}_{\psi(X)}$ are inverse isomorphisms. $\square$

One of our aims to understand the action of idempotent functors. To do that we are going to study the orbits $\mathrm{Idem}(X)$ of this action on an object X in $\mathcal{C}$:

$$\mathrm{Idem}(X) := \{\text{iso class of } \phi(X) \mid (\phi, \epsilon) \text{ is an idempotent functor}\}.$$

Proposition 3.1 says that idempotent functors of $\mathcal{C}$ can be identified with cellular properties of $\mathcal{C}$. That is what we will do from now on. One way of encoding this identification is:

3.3. **Corollary.** *An augmented functor (ϕ, ϵ) is idempotent if and only if, for any Y in $\mathrm{Cell}(\phi)$ and any X in $\mathcal{C}$, the map $\mathrm{mor}(Y, \epsilon_X) \colon \mathrm{mor}(Y, \phi(X)) \to \mathrm{mor}(Y, X)$ is a bijection.*

Here are some further consequences of Proposition 3.1:

3.4. **Corollary.**
1. *If (ϕ, ϵ) is idempotent, then $\phi(\epsilon_X) \colon \phi^2(X) \to \phi(X)$ and $\epsilon_{\phi(X)} \colon \phi^2(X) \to \phi(X)$ are the same isomorphisms.*
2. *Let $\phi \colon \mathcal{C} \to \mathcal{C}$ be a functor and $\epsilon_X, \mu_X \colon \phi(X) \to X$ be natural transformations. If (ϕ, ϵ) and (ϕ, μ) are idempotent, then (ϕ, ϵ) and (ϕ, μ) are isomorphic.*

We finish this section with describing basic properties of cellular properties: cellular properties are closed under retracts and colimits:

3.5. **Proposition.** *Let $\mathcal{D} \subset \mathcal{C}$ be a cellular property.*

1. *If X is in $\mathcal{D}$, then so is any retract of X.*
2. *Let $F \colon I \to \mathcal{D}$ be a functor. If $\mathrm{colim}_I F$ exists in $\mathcal{C}$, then $\mathrm{colim}_I F$ belongs to $\mathcal{D}$.*

Proof. Let (ϕ, ϵ) be an idempotent functor such that $\mathrm{Cell}(\phi) = \mathcal{D}$.

(1): Consider the following commutative diagram:

$$\begin{tikzcd}
\phi(Y) \arrow[r, "\phi(s)"] \arrow[d, "\epsilon_Y"] \arrow[rr, bend left, "\mathrm{id}"] & \phi(X) \arrow[r, "\phi(r)"] \arrow[d, "\epsilon_X"] & \phi(Y) \arrow[d, "\epsilon_Y"] \\
Y \arrow[r, "s"] \arrow[rr, bend right, "\mathrm{id}"'] & X \arrow[r, "r"] & Y
\end{tikzcd}$$

Since a retract of an isomorphism is an isomorphism, if ϵ_X is an isomorphism, then so is ϵ_Y. That proves (1).

(2): Consider the augmentation $\epsilon\colon \phi(\mathrm{colim}_I F) \to \mathrm{colim}_I F$. According to 3.3, for any i in I, the map $\mathrm{mor}(F(i), \epsilon)\colon \mathrm{mor}(F(i), \phi(\mathrm{colim}_I F)) \to \mathrm{mor}(F(i), \mathrm{colim}_I F)$ is a bijection. As inverse limit of bijections is a bijection we will obtain that $\lim_{i\in I^{\mathrm{op}}} \mathrm{mor}(F(i), \epsilon)$ is a bijection. This inverse limit is the map:

$$\mathrm{mor}(\mathrm{colim}_I F, \epsilon)\colon \mathrm{mor}(\mathrm{colim}_I F, \phi(\mathrm{colim}_I F)) \longrightarrow \mathrm{mor}(\mathrm{colim}_I F, \mathrm{colim}_I F)$$

This shows that there is a morphisms $\alpha\colon \mathrm{colim}_I F \to \phi(\mathrm{colim}_I F)$ whose composition with ϵ is the identity. consequently $\mathrm{colim}_I F$ is a retract of $\phi(\mathrm{colim}_I F)$. We can then use statement (1) to conclude that $\mathrm{colim}_I F$ belongs to $\mathrm{Cell}(\phi)$. $\square$

Proposition 3.5.(2) implies that the category of sets has only two cellular properties: the one that contains all the sets or the one that contains only the empty set. These are the only properties of sets closed under colimits. Thus the category of sets has, up to isomorphism, only two idempotent functors: the identity and the one that assign to any set the empty set. In this case, for any set X, $\mathrm{Cov}(X) = \mathrm{Idem}(X) = \{X, \emptyset\}$.

Being closed under colimits is going to be key for us to understand cellular properties. Properties that are closed under colimits are rather special and we are going to discuss them in the category of groups in the next sections. A fundamental question for us is which objects can be obtained from a fixed object A using the colimit constructions. Such objects are build in stages, each one involving taking colimits of diagrams whose values are objects from the previous stages.

3.6. Here is a typical example of a property closed under colimits. Let $f\colon Y \to X$ be a morphism in $\mathcal{C}$. Then the full subcategory of $\mathcal{C}$ given by all the objects A such that $\mathrm{mor}(A, f)\colon \mathrm{mor}(A, Y) \to \mathrm{mor}(A, X)$ is a bijection is a property that is closed under colimits.

4. Colimits in the category of groups

In this section we go through various examples of functors of groups and their colimits and use it to give the first example of a cellular property in the category of groups.

4.1. Let I be a discrete category (i.e., just a set). A functor $F\colon I \to \mathrm{Groups}$ is simply a sequence of groups $F(i)$ indexed by I. Its colimit $\mathrm{colim}_I F$ is the free product $\coprod_{i\in I} F(i)$.

4.2**.** Let $f\colon X \to A$ and $g\colon A \to X$ be homomorphisms for which $fg = \mathrm{id}_X$. Set $I = \{*\}$ with $\mathrm{mor}(*,*) = \{\mathrm{id}, \epsilon\}$, where $\epsilon^2 = \mathrm{id}$. Let $F\colon I \to$ Groups be the functor $F(*) = A$ and $F(\epsilon) = gf$. Then $\mathrm{colim}_I F$ is the quotient of A by the normal subgroup generated by the relation $a \sim f(g(a))$. It is easy to check that this normal subgroup is given by the kernel of g and hence $\mathrm{colim}_I F$ is isomorphic to X.

4.3**.** Let X and Y be groups. Let I be the category with two objects $\{*_0, *_1\}$ and morphisms given by:

$$\mathrm{mor}(*_0,*_0) = \{id\} \quad \mathrm{mor}(*_1,*_1) = \{id\} \quad \mathrm{mor}(*_1,*_0) = \emptyset \quad \mathrm{mor}(*_0,*_1) = X$$

Define $F\colon I \to$ Groups as follows: $F(*_0) = Y$, $F(*_1) = X \coprod Y$ and, for any morphism $x \in X = \mathrm{mor}(*_0, *_1)$, $F(x)\colon Y \to X \coprod Y$ is the composition of the standard inclusion $Y \subset X \coprod Y$ with the conjugation by x. The group $\mathrm{colim}_I F$ is isomorphic to the quotient of $X \coprod Y$ by the normal subgroup generated by the relation $g \sim xgx^{-1}$. Thus $\mathrm{colim}_I F = X \times Y$.

4.4**.** Let I be a set and $\{X_i\}_{i\in I}$ be a sequence of groups. For any fine subset $J \subset I$, define $F(J) := \prod_{j\in J} X_j \subset \prod_{i\in I} X_i$. These subgroups of the product $\prod_{i\in I} X_i$ form a functor indexed by the poset of finite subsets of I. The colimit of this functor $\mathrm{colim}_{J\subset I,\ |J|<\infty} F(J)$ is a subgroup of $\prod_{i\in I} X_i$ consisting of sequences for which only finitely many coordinates are not identities and hence it coincides with the direct sum $\bigoplus_{i\in I} X_i$.

4.5**.** Let K be a normal subgroup of X. Let $I = \{*\}$ with $\mathrm{mor}(*,*) = K$. Let $F(*) = X$ and for $k \in K$ let $F(k)$ be the inner automorphism of X induced by k. Then $\mathrm{colim}_I F$ is the quotient of X by the normal subgroup generated by the relation $x \sim kxk^{-1}$, for all $x \in X$ and $k \in K$. This colimit can be therefore identified with $X/[K, X]$.

We can summarize the consequence of the above examples in:

4.6. **Proposition.** *Let $\mathcal{D} \subset$ Groups be a property closed under colimits. Then:*

1. *$\mathcal{D}$ is preserved by retracts and products.*
2. *$\bigoplus_{i\in I} X_i$ belongs to $\mathcal{D}$ if and only if, for any i, X_i belongs to $\mathcal{D}$.*
3. *If X is in $\mathcal{D}$, then so is $X/[K, X]$ for any normal subgroup $K \subset X$. In particular, $H_1(X) := X/[X, X]$ and more generally $X/\Gamma_i(X)$ belongs to $\mathcal{D}$.*
4. *If X belongs to $\mathcal{D}$, then so does $H_1(X) \otimes_{\mathbb{Z}} M$ for any abelian group M.*
5. *If X is abelian and belongs to $\mathcal{D}$, then so does X/K for any finite subgroup $K \subset X$.*
6. *Let $\overline{\mathcal{D}}$ be the smallest property of groups containing $\mathcal{D}$ and closed under colimits, quotients, and extensions. Let Y be in $\mathcal{D}$ and $f\colon Y \to X$ be an epimorphism whose kernel belongs to $\overline{\mathcal{D}}$. Then X belongs to $\mathcal{D}$.*
7. *Let $f\colon Y \to X$ be a homomorphism such that Y belongs to $\mathcal{D}$. Then f can be factored as:*
$$Y \xrightarrow{f_1} W \xrightarrow{f_2} X$$
where f_1 is an epimorphism, W belongs to $\mathcal{D}$, and, for any Z in $\mathcal{D}$, the map $\mathrm{Hom}(Z, f_2)$ is a monomorphism. Moreover, such a factorization can be made functorial with respect to f.

Proof. (1): This follows from 4.2 and 4.3.

(2): This follows from 4.4.

(3): This follows from 4.5.

(4): Choose an exact sequence $\bigoplus \mathbb{Z} \to \bigoplus \mathbb{Z} \to M \to 0$ (an abelian presentation of M). By tensoring this exact sequence with $H_1(X)$ we get a new exact sequence: $\bigoplus H_1(X) \to \bigoplus H_1(X) \to H_1(X) \otimes_{\mathbb{Z}} M \to 0$. If X is in $\mathcal{D}$, then according to (2) and (3) so are $H_1(X)$ and $\bigoplus H_1(X)$. As the cokernel of a homomorphism between groups that belong to $\mathcal{D}$, also $H_1(X) \otimes_{\mathbb{Z}} M$ belongs to $\mathcal{D}$.

(5): It is enough to show the statement for $K = \mathbb{Z}/p$. Let X in $\mathcal{D}$ be abelian. If $X \otimes_{\mathbb{Z}} \mathbb{Z}/p \neq 0$, then $\mathbb{Z}/p$ is its retract and, by (1) and (4), $\mathbb{Z}/p$ is in $\mathcal{D}$. It is then clear that $X/\mathbb{Z}/p$ also belongs to $\mathcal{D}$, as $\mathcal{D}$ is preserved by colimits. If $X \otimes_{\mathbb{Z}} \mathbb{Z}/p = 0$, then X is p-divisible and the inclusion $\mathbb{Z}/p \subset X$ factors as $\mathbb{Z}/p \subset \mathbb{Z}/p^\infty \subset X$. Hence $X \cong \mathbb{Z}/p^\infty \oplus Y$ and $X/(\mathbb{Z}/p) \cong (\mathbb{Z}/p^\infty)/(\mathbb{Z}/p) \oplus Y \cong X$, so $X/(\mathbb{Z}/p)$ belongs to $\mathcal{D}$.

(6): Since $\mathcal{D}$ is preserved by colimits, the trivial group 0 is in $\mathcal{D}$, as 0 is the colimit of the empty functor. Define $\mathbf{D}$ to be the collection of all groups Y such that, for any X in $\mathcal{D}$ and any homomorphism $f\colon Y \to X$, $X/f(Y)$ is in $\mathcal{D}$. The group $X/f(Y)$ is isomorphic to $\mathrm{colim}(0 \leftarrow Y \xrightarrow{f} X)$. Thus if Y is in $\mathcal{D}$, then so does $X/f(Y)$. This means that $\mathcal{D} \subset \mathbf{D}$. It is clear that $\mathbf{D}$ is a property. Since $\overline{\mathcal{D}}$ is the smallest property containing $\mathcal{D}$ which is preserved by colimits, quotients and extensions, to prove the proposition it is enough to show that $\mathbf{D}$ is preserved by colimits, quotients and extensions.

Let $F\colon I \to$ Groups be a functor whose values belong to $\mathcal{D}$. Let X belong in $\mathcal{D}$ and $f\colon \mathrm{colim}_I F \to X$ be a homomorphism. Note that $X/f(\mathrm{colim}_I F)$ is isomorphic to $\mathrm{colim}_I\big(X/f(F(i))\big)$. As $X/f(F(i)$ is in $\mathcal{D}$ for any i, same is true for $\mathrm{colim}_I\big(X/f(F(i))\big)$ and $\mathrm{colim}_I F$ is in $\mathcal{D}$. Therefore $\mathbf{D}$ is preserved by colimits.

Let $g\colon Y \to Z$ be an epimorphism and Y is in $\mathbf{D}$. For any homomorphism $f\colon Z \to X$, the groups $X/f(Z)$ and $X/f(g(Y))$ are isomorphic. If X is in $\mathcal{D}$, then since $Y \in \mathbf{D}$, $X/f(Z)$ belongs to $\mathcal{D}$ and Z is in $\mathbf{D}$. The collection $\mathbf{D}$ is therefore preserved by quotients.

Let N be a normal subgroup of Y such that N and Y/N belong to $\mathbf{D}$. Let X be in $\mathcal{D}$ and $f\colon Y \to X$ be a homomorphism. The groups $X/f(Y)$ and $\big(X/f(N)\big)/f(Y/N)$ are isomorphic. Since N is in $\mathbf{D}$, $X/f(N)$ belongs to $\mathcal{D}$. As Y/N is in $\mathbf{D}$, $\big(X/f(N)\big)/f(Y/N)$ also belongs to $\mathcal{D}$. The group $X/f(Y)$ therefore is in $\mathcal{D}$ and $\mathbf{D}$ is preserved by extensions.

(7): Consider the set S_f of all normal subgroups H of Y such that they are contained in the kernel of f and Y/H is in $\mathcal{D}$. This set is non empty as it contains the trivial subgroup. We claim that S contains a unique maximal element with respect to the inclusion. Let $H_0 \subset H_1 \subset \cdots$ be a chain of elements of S and set $H = \bigcup H_i$. Then H is a normal subgroup of Y which is contained in the kernel of f. Moreover, since $Y/H = \mathrm{colim}_i(Y/H_i)$ and $\mathcal{D}$ is preserved by colimits, Y/H is in $\mathcal{D}$ and $H \in S_f$. By the Kuratowski-Zorn lemma S_f contains a maximal element. Let H_f be a maximal element of S_f. For any $K \in S_f$, define L to be the normal subgroup of Y generated by H_f and K. This subgroup clearly is contained in the kernel of f as H_f and K are. Moreover $\mathrm{colim}(Y/H_f \leftarrow Y \to Y/K)$ is isomorphic to Y/L. Thus Y/L is in $\mathcal{D}$ and $L \in S_f$. Since $H_f \subset L$ and H_f is maximal in S_f, $H_f = L$ and K must be contained in H_f. The group H_f is therefore the unique maximal element of S_f. We claim that the factorization $Y \xrightarrow{f_1} Y/H_f \xrightarrow{f_2} X$, where f_1 is the quotient and f_2 is induced by f, satisfies the desired requirements.

We need to show the functoriality of this factorization and that, for any Z in $\mathcal{D}$, $\mathrm{Hom}(Z, f_2)$ is a monomorphism. Let $\alpha, \beta\colon Z \to Y/H_f$ be homomorphisms such that $\alpha f_2 = \beta f_2$. We can then form the following commutative diagram:

$$\begin{tikzcd} Z \coprod Z \arrow[r, "\alpha \coprod \beta"] \arrow[d, "\mathrm{id} \coprod \mathrm{id}"'] & Y/H_f \arrow[d, "f_2"] \\ Z \arrow[r, "\alpha f_2"] & X \end{tikzcd}$$

Commutativity of this diagram implies that f_2 can be factored as $Y/H_f \to P \to X$, where $P = \mathrm{colim}(Z \xleftarrow{\mathrm{id} \coprod \mathrm{id}} Z \coprod Z \xrightarrow{\alpha \coprod \beta} Y/H_f)$ and the homomorphisms are induced by the universal property of the push-out. The group P is in $\mathcal{D}$, as $\mathcal{D}$ is preserved by colimits. Moreover, the homomorphism $Y/H_f \to P$ is an epimorphism since $\mathrm{id} \coprod \mathrm{id}\colon Z \coprod Z \to Z$ is so. It follows that the kernel of $Y \xrightarrow{g} Y/H_f \to P$ belongs to S_f. Since it contains H_f it has to coincide with H_f and $Y/H_f \to P$ is an isomorphism. The homomorphisms α and β must therefore be equal as they are after composing with $Y/H_f \to P$. This means that $\mathrm{Hom}(Z, f_2)$ is a monomorphism.

For the functoriality, we have to show that, for any commutative square:

$$\begin{tikzcd} Y \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ X \arrow[r, "\beta"] & A \end{tikzcd}$$

where Y and B are in $\mathcal{D}$, α takes H_f into H_g. The homomorphism g can be factored as $B \to P \to A$, where $P = \mathrm{colim}(Y/H_f \leftarrow Y \xrightarrow{\alpha} B)$ and the homomorphisms are induced by the universal property of the push-out. Note that the kernel of $B \to P$ is contained in S_g and contains $\alpha(H_f)$. Since H_g is the maximal element of S_g, $\alpha(H_f) \subset H_g$. □

We use 4.6.(7) to give our first example of a cellular property.

4.7. **Proposition.** *Let A be a group. The smallest property* $\mathrm{Cell}(A)$ *of groups containing A and closed under colimits is cellular.*

Proof. Apply 4.6.(7) to the property $\mathrm{Cell}(A)$ and the homomorphism:

$$f := \coprod_{\alpha\colon A \to X} \alpha\colon \coprod_{\alpha\colon A \to X} A \longrightarrow X.$$

We claim that the obtain homomorphism $f_2\colon W \to X$ is the $\mathrm{Cell}(A)$-cover of X. According 4.6.(7), the group W belongs to $\mathrm{Cell}(A)$. We need to show that, for any Y in $\mathrm{Cell}(A)$, $\mathrm{Hom}(Y, f_2)\colon \mathrm{Hom}(Y, W) \to \mathrm{Hom}(Y, X)$ is a bijection. According to 4.6.(7), this map is a monomorphism. Further more, by definition, $\mathrm{Hom}(A, f_2)$ is also a surjection. Since the full subcategory of all groups Y for which $\mathrm{Hom}(Y, f_2)$ is a bijection is a property closed under colimits, we can conclude that $\mathrm{Hom}(Y, f_2)$ is indeed a bijection for any group in $\mathrm{Cell}(A)$, as $\mathrm{Cell}(A)$ is the smallest property containing A and closed under colimits. □

If a group belongs to $\mathrm{Cell}(A)$, then we say that it is **A-cellular**. Note that this relation is transitive, i.e., if X is B cellular and B is A-cellular, then X is A-cellular. We also denote the cover $\mathrm{cov}_{\mathrm{Cell}(A)}$ simply by cov_A.

Let p be a prime. Theorem of R. Flores [10] states that a group G is an object in $\mathcal{C}(\mathbb{Z}/p^n)$ if and only if it is not only generated by elements of order p^n but also whose second integral homology $H_2(G,\mathbb{Z})$ is p-torsion. For example recall that for an alternating group A_n, we have:

$$H_2(A_n,\mathbb{Z}) = \begin{cases} 0 & \text{if } n \leq 3 \\ \mathbb{Z}/6 & \text{if } n = 6 \text{ or } n = 7 \\ \mathbb{Z}/2 & \text{for all other } n \end{cases}$$

Thus for $n \geq 4$, A_n belongs to $\mathcal{C}(\mathbb{Z}/2)$ if and only if $n \neq 6$ or $n \neq 7$.

4.8. **Corollary.** *Let X be a group. Then:*

$$\{\textit{iso class of } \mathrm{cov}_A X \mid A \textit{ is a group}\} = \mathrm{Idem}(X) = \mathrm{Cov}(X) = \mathcal{L}(X)$$

Proof. Let $\mathcal{E}(X) := \{\text{iso class of } \mathrm{cov}_A X \mid A \text{ is a group}\}$. From the definitions and 2.2 we have inclusions $\mathcal{E}(X) \subset \mathrm{Idem}(X) \subset \mathrm{Cov}(X) = \mathcal{L}(X)$. It remains to show $\mathcal{L}(X) \subset \mathcal{E}(X)$. Let Y be in $\mathcal{L}(X)$ and $c\colon Y \to X$ be a map for which $\mathrm{Hom}(Y,c)$ is a bijection. Since the collection of groups Z, for which $\mathrm{Hom}(Z,c)$ is a bijection is closed under colimits, for any Y-cellular group Z, $\mathrm{Hom}(Z,c)$ is also a bijection. It follows that Y is isomorphic to $\mathrm{cov}_Y X \to X$ and hence Y is in $\mathcal{E}$. $\square$

Our goal is to enumerate the collection $\mathrm{Cov}(X)$ for a given group X. One way to get some idea what kind of groups belong to $\mathrm{Cov}(X)$, is to understand which properties of X are preserved by idempotent functors. For example if X is finite, or abelian, or solvable, or nilpotent, are its covers also such groups? Answering these questions is part of what we aim for in these notes.

5. Structures preserved by Idempotent functors of groups

In this section we recall a general machinery of describing certain structures on groups. We also give necessary assumptions on when such structures are preserved by an idempotent functor. These structures are described by triples. Recall, that a triple on a category $\mathcal{C}$ is a functor $F\colon \mathcal{C} \to \mathcal{C}$ together with two natural transformations $\epsilon_X\colon X \to F(X)$ and $\mu_X\colon F(F(X)) \to F(X)$ such that $F(\mu_X)\mu_X = \mu_{F(X)}\mu_X$, $\epsilon_{F(X)}\mu_X = \mathrm{id}_{F(X)}$, and $F(\epsilon_X)\mu_X = \mathrm{id}_{F(X)}$, for any object X (see [20]).

Let (F,ϵ,μ) be a triple on $\mathcal{C}$. An F-algebra is an object X and a morphism $m\colon F(X) \to X$ such that $\epsilon_X m = \mathrm{id}_X$ and $\mu_X m = F(m)m$. A morphism of F-algebras $m_X\colon F(X) \to X$ and $m_Y\colon F(Y) \to Y$ is a morphism $f\colon X \to Y$ such that $m_X f = F(f)m_Y$ (see [20]).

A fundamental observation is:

5.1. **Proposition.** *Let (F,ϵ,μ) be a triple on the category of groups.*

1. *Assume $\mathcal{D}$ is a cellular property of groups such that if Z satisfies it then so does $F(Z)$. Then, for any F-algebra $m_X\colon F(X) \to X$, there is a unique F-algebra $m_{\mathrm{cov}_{\mathcal{D}}X}\colon F(\mathrm{cov}_{\mathcal{D}}X) \to \mathrm{cov}_{\mathcal{D}}X$ for which $c_X\colon \mathrm{cov}_{\mathcal{D}}X \to X$ is an F-algebra morphism.*
2. *Assume that, for any group A, $F(A)$ is A-cellular. Then, for any F-algebra $m_X\colon F(X) \to X$ and any group A, the cover $c_{A,X}\colon \mathrm{cov}_A X \to X$ is*

isomorphic to $c_{F(A),X}\colon \mathrm{cov}_{F(A)}X \to X$. In particular an F-algebra X is A-cellular if and only if it is $F(A)$-cellular.

Proof. (1): Since $F(\mathrm{cov}_{\mathcal{D}}X)$ satisfies $\mathcal{D}$, there is a unique morphism $m_{\mathrm{cov}_{\mathcal{D}}X}$ for which the following diagram commutes:

$$\begin{tikzcd}
F(\mathrm{cov}_{\mathcal{D}}X) \arrow[r, "m_{\mathrm{cov}_{\mathcal{D}}X}"] \arrow[d, "F(c_X)"'] & \mathrm{cov}_{\mathcal{D}}X \arrow[d, "c_X"] \\
F(X) \arrow[r, "m_X"] & X
\end{tikzcd}$$

Verification that the homomorphism $m_{\mathrm{cov}_{\mathcal{C}}X}$ is an F-algebra is left to the reader. Commutativity of necessary diagrams is a straight forward consequence of the fact that c_X is a $\mathcal{D}$-equivalence and F takes objects of $\mathcal{D}$ into $\mathcal{D}$.

(2): To prove that $c_{F(A),X}$ and $c_{A,X}$ are isomorphic we need to show that $\mathrm{cov}_{F(A)}X$ is A-cellular and that for any Y in $\mathrm{Cell}(A)$, $\mathrm{Hom}(Y, c_{F(A),X})$ is a bijection. The first requirement follows from the assumption that $F(A)$ is A-cellular and the fact that cellularity is a transitive relation. For the second one we need to prove that $\mathrm{Hom}(A, c_{F(A),X})$ is a bijection. Since X is an F algebra, any homomorphism $f\colon A \to X$ can be factored as:

$$A \xrightarrow{\epsilon_A} F(A) \xrightarrow{F(f)} F(X) \xrightarrow{m_X} X$$

Since $\mathrm{Hom}(F(A), c_{F(A),X})$ is a bijection, the homomorphism $F(f)m_X$ can be factored as $F(A) \to \mathrm{cell}_{F(A)}X \xrightarrow{c_{F(A),X}} X$. This shows that $\mathrm{Hom}(A, c_{F(A),X})$ is a surjection.

We now show that $\mathrm{Hom}(A, c_{F(A),X})$ is a monomorphism. For any B, if X is B-cellular, then $F(X)$ being X-cellular is also B-cellular. Thus F takes B-cellular groups into B-cellular groups. In particular this holds for $B = F(A)$ and, according to statement (1), there is a unique F-algebra $m\colon F(\mathrm{cov}_{F(A)}X) \to \mathrm{cov}_{F(A)}X$ for which the cover $c_{F(A),X}\colon \mathrm{cov}_{F(A)}X \to X$ is an F-algebra morphism.

Let $f, g\colon A \to \mathrm{cov}_{F(A)}X$ be homomorphism such that $fc_{F(A),X} = gc_{F(A),X}$. Note that $\epsilon_{A\coprod A}$ can be factored as $A\coprod A \xrightarrow{\epsilon_A \coprod \epsilon_A} F(A)\coprod F(A) \xrightarrow{i} F(A\coprod A)$. Consider next the following commutative diagram:

$$\begin{tikzcd}
A\coprod A \arrow[rr, "{(f,g)}"] \arrow[dr, "\epsilon_A\coprod\epsilon_A"] \arrow[ddr, "\epsilon_{A\coprod A}"'] \arrow[ddd, "{(\mathrm{id},\mathrm{id})}"'] & & \mathrm{cov}_{F(A)}X \arrow[dr, "\epsilon_{\mathrm{cov}_{F(A)}X}"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddd, "c_{F(A),X}"] & & \\
& F(A)\coprod F(A) \arrow[rr, "{(F(f),F(g))}"] \arrow[d, "i"] & & F(\mathrm{cov}_{F(A)}X) \arrow[r, "m"'] \arrow[ddd, "F(c_{F(A),X})"] & \mathrm{cov}_{F(A)}X \arrow[ddd, "c_{F(A),X}"] \\
& F(A\coprod A) \arrow[urr, "{F(f,g)}"] \arrow[dd, "{F(\mathrm{id},\mathrm{id})}"] & & & \\
A \arrow[rr] \arrow[dr, "\epsilon_A"'] & & X \arrow[dr, "\epsilon_X"'] \arrow[drr, bend left, "\mathrm{id}"] & & \\
& F(A) \arrow[rr] & & F(X) \arrow[r, "m_X"'] & X
\end{tikzcd}$$

Since $\mathrm{Hom}(F(A), c_{F(A),X})$ is a bijection, the homomorphisms $F(f)m\colon F(A) \to \mathrm{cov}_{F(A)}X$ and $F(g)m\colon F(A) \to \mathrm{cov}_{F(A)}X$ are the same as they coincide after composition with $c_{F(A),X}$. It follows that the homomorphisms $f = \epsilon_A F(f)m$ and $g = \epsilon_A F(g)m$ are also the same and we can conclude that $\mathrm{Hom}(A, c_{F(A),X})$ is a monomorphism. $\square$

6. Covers of nilpotent groups

Let i be a positive natural number. Recall that $\Gamma_i(X)$ denotes the i-th term of the lower central series of X. For example $\Gamma_1(X) = [X,X]$ and $\Gamma_2(X) = [[X,X],X]$. Define $F(Y) := Y/\Gamma_i(Y)$, ϵ_Y to be the quotient homomorphism $Y \to Y/\Gamma_i(Y) = F(Y)$, and μ_Y to be the identity homomorphism $F(F(Y)) = \big(Y/\Gamma_i(Y)\big)/\Gamma_i\big(Y/\Gamma_i(Y)\big) = Y/\Gamma_i(Y) = F(Y)$. Then (F,ϵ,μ) forms a triple on the category of groups. An F algebra is simply a group Y such that $\Gamma_i(Y) = 0$, i.e., an i-nilpotent group. Any group homomorphism between F-algebras is an F-algebra morphism. Note that according to 4.6.(3), if Z belongs to a cellular property $\mathcal{D}$, then so does $Z/\Gamma_i(Z)$. In particular $A/\Gamma_i(A)$ is A-cellular for any A. It follows that assumptions of 5.1 are satisfied and hence we get:

6.1. **Proposition.** *Let X be an i-nilpotent group and A a group.*

1. *For any idempotent functor (ϕ,ϵ) of groups, $\phi(X)$ is also i-nilpotent.*
2. *The covers $\mathrm{cov}_A X$ and $\mathrm{cov}_{A/\Gamma_i(A)}X$ are isomorphic.*
3. *X is A-cellular if and only if it is $A/\Gamma_i(A)$-cellular.*

For finitely generated nilpotent groups we can say a lot more:

6.2. **Theorem.** *Let X be a finitely generated nilpotent group, (ψ,ϵ) an idempotent functor of groups, and A a group. Then:*

1. *$\epsilon_X : \psi(X) \to X$ is a monomorphism.*
2. *X is A-cellular if and only if it is a quotient of a free product of A's.*
3. *$\mathrm{cov}_A X$ is the subgroup generated by the images of all homomorphisms $A \to X$.*

For example if X is a finite p-group. Then since it is nilpotent, according to 6.2, its covers are subgroups of X. In particular $\mathrm{Cov}(X)$ is a finite set. It consists of all these subgroups $Y \subset X$ such that any homomorphism $f : Y \to X$ has image in this subgroup $Y \subset X$. One can write a computer program to identify such subgroups for a given p-groups. As of writing these notes I am not aware of any conceptual description of these subgroups that would lead for example to the estimation of the size of the set $\mathrm{Cov}(X)$.

To prove Theorem 6.2 we need:

6.3. **Lemma.** *Let $\mathcal{D}$ be a property of groups which is closed under colimits, quotients, and extensions.*

1. *If X is nilpotent and satisfies $\mathcal{D}$, then so does $\Gamma_i(X)$ for any i.*
2. *Let $i > 0$. The abelianization $H_1(X)$ satisfies $\mathcal{D}$ if and only if $X/\Gamma_i(X)$ does.*
3. *Let X be a nilpotent group satisfying $\mathcal{D}$. If K is a normal subgroup of X, such that X/K is finitely generated, then K satisfies $\mathcal{D}$.*

Proof. (1): Recall that the map of sets:

$$H_1(X)^i = (X/[X,X])^i \ni (x_1[X,X], \ldots, x_i[X,X]) \longmapsto$$

$$\longmapsto [x_1,\ldots,x_i]\Gamma_{i+1}(X) \in \Gamma_i(X)/\Gamma_{i+1}(X)$$

is a homomorphism in each variable and its image generates $\Gamma_i(X)/\Gamma_{i+1}(X)$. Hence $\Gamma_i(X)/\Gamma_{i+1}(X)$ is a quotient of a free product of $H_1(X)$ and consequently a quotient of a free product of X. This shows that $\Gamma_i(X)/\Gamma_{i+1}(X)$ satisfies $\mathcal{D}$. As $\mathcal{D}$ is closed under extensions, and X is nilpotent, we can conclude that $\Gamma_i(X)$ is in $\mathcal{D}$.

(2): As $H_1(X)$ is a quotient of $X/\Gamma_i(X)$, if $X/\Gamma_i(X)$ satisfies $\mathcal{D}$ then so does $H_1(X)$. Assume that $H_1(X)$ satisfies $\mathcal{D}$. It follows that so does $\Gamma_i(X)/\Gamma_{i+1}(X)$ as it is a quotient if a free product of $H_1(X)$. Again, as $\mathcal{D}$ is closed under extensions, we can conclude that $X/\Gamma_i(X)$ is in $\mathcal{D}$.

(3): If X/K is infinite, then for some i, $\Gamma_i(X/K)/\Gamma_{i+1}(X/K)$ is infinite and, as it is also finitely generated, $\mathbb{Z}$ is its quotient. As $\Gamma_i(X/K)/\Gamma_{i+1}(X/K)$ satisfies $\mathcal{D}$ (see the proof of (1) above), $\mathbb{Z}$ also satisfies $\mathcal{D}$. It follows that all the groups belong to $\mathcal{D}$ as any group is a quotient of a free group, in particular K.

Let X/K be finite. Assume in addition that X is abelian. We are going to use induction on the order of X/K to show the proposition. If $|X/K|$ is not prime, we can find a proper subgroup H of X that properly contains K. By the inductive assumption both H and K must satisfy $\mathcal{D}$. Assume now that $|X/K| = p$ is a prime. Multiplication by p, $X \xrightarrow{p} X$, factors through $K \subset X$. The cokernel of the obtained homomorphism $f\colon X \to K$ is a $\mathbb{Z}/p$ vector space, and thus satisfies $\mathcal{D}$ (since $\mathbb{Z}/p$ is a quotient of X). Thus the image of f and the cokernel of f both satisfy $\mathcal{D}$ and hence so does K.

Assume X is nilpotent. Set $\Gamma_i := \Gamma_i(X)$. Note that for any i, the subgroup $(\Gamma_i \cap K)/(\Gamma_{i+1} \cap K) \subset \Gamma_i/\Gamma_{i+1}$, is of finite index. By the abelian case, $(\Gamma_i \cap K)/(\Gamma_{i+1} \cap K)$ therefore satisfies $\mathcal{D}$. As $\mathcal{D}$ is preserved by extensions, we can conclude that K also satisfies $\mathcal{D}$. $\square$

6.4. **Corollary.** *Let X be a nilpotent group and K its normal subgroup such that X/K is finitely generated. Let Y be such that* $\mathrm{Hom}(X,Y)=0$. *Then* $\mathrm{Hom}(\Gamma_i X, Y) = 0$ *and* $\mathrm{Hom}(K,Y) = 0$. *Furthermore, if* $\mathrm{Hom}(K,L) \neq 0$, *then* $\mathrm{Hom}(H_1(X),L) \neq 0$.

Proof. Note that the collections of all groups X for which $\mathrm{Hom}(X,Y) = 0$ is a property closed under colimits, quotients and extensions (compare with 3.6). The first statement of the corollary follows now direct from 6.3. To prove the second statement, assume $\mathrm{Hom}(H_1(X),L) = 0$ for some quotient L of K. Since X is nilpotent, $\mathrm{Hom}(X,L) = 0$ (see 6.3.(2)). By the first statement we would then get $\mathrm{Hom}(K,L) = 0$ which contradicts the assumption. $\square$

Proof of Theorem 6.2. (1): Let $\mathcal{D} = \mathrm{Cell}(\phi)$. Recall that $\mathcal{D}$ is a cellular property and that (ψ,ϵ) and $(\mathrm{cov}_{\mathcal{D}}, c)$ are isomorphic augmented functors (see 3.1). Consider the image $I \subset X$ of the the cover $c_X\colon \mathrm{cov}_{\mathcal{D}} X \to X$. We claim that I belongs to $\mathcal{D}$. Since X is finitely generated and nilpotent, then so is I. By Lemma 6.3.(2), the kernel of c_X must therefore belong to $\overline{\mathcal{D}}$ (the smallest property containing $\mathcal{D}$ and closed under colimits, quotients, and extensions). We can now conclude, using Proposition 4.6.(6), that the image I of c_X satisfies $\mathcal{D}$. To finish, for Y in $\mathcal{D}$, consider the maps:

$$\begin{tikzcd} \mathrm{Hom}(Y, \mathrm{cov}_{\mathcal{D}} X) \arrow[r, "\mathrm{Hom}(Y, c_X)"] \arrow[rr, bend right, "\simeq"'] & \mathrm{Hom}(Y, I) \arrow[r, hook] & \mathrm{Hom}(Y, I) \end{tikzcd}$$

Since the composition is a bijection and the right map is a monomorphism, all the maps in this diagram are bijections and hence $I \subset X$ is the $\mathcal{D}$-cover of X.

(2)&(3): These are the direct consequences of (1). $\square$

It turns out that $\mathrm{Cov}(X)$ is a finite set not only for a finite nilpotent group X. This set is finite for any finite group X (we prove it in Section 10, see Corollary 10.8). Furthermore, if X is finite, the elements of $\mathrm{Cov}(X)$ are finite groups. Preservation of being finite and solvable is what we aim to prove in the following section. Being finite is not a property detected by a triple, like for example being i-nilpotent is. Although being solvable is detected by a triple, the triple involved does not satisfy the assumptions of Proposition 5.1. Thus we can not use this proposition to prove the preservation of being finite or solvable. We need to use other techniques.

7. Generalized subgroups and covers of finite groups

Let X be a group. Recall that our aim is to classify elements of $\mathrm{Idem}(X)$ or $\mathrm{Cov}(X)$ using some classical invariants of X. This is equivalent to the enumeration of (see 4.8):

$$\mathcal{L}(X) := \{Y \mid \text{there is } c\colon Y \to X \text{ for which } \mathrm{Hom}(c, Y) \text{ is a bijection}\}$$

Unfortunately we are unable to enumerate $\mathcal{L}(X)$ in general. It turns out however that in the case X is finite, it is easier to give a classification of elements of a bigger collection whose elements we call **generalized subgroups** of X. In this section we define this bigger collection and discuss some properties of its elements.

7.1. **Definition.** Let X be a group.

(1) A homomorphism $a\colon Y \to X$ is called a **generalized subgroup** of X if $\mathrm{Hom}(Y, a)\colon \mathrm{Hom}(Y, Y) \to \mathrm{Hom}(Y, X)$ is an injection of sets (but not necessarily a bijection as it is in the case of a cover of X).
(2) Two generalized subgroups $a\colon Y \to X$ and $b\colon Z \to X$ are defined to be equivalent if there is an isomorphism $h\colon Y \to Z$ for which $bh = a$.
(3) The symbol $\mathrm{Sub}(X)$ denotes the collection of equivalence classes of generalized subgroups of X.

We start by giving a direct characterization of generalized subgroups of X:

7.2. **Proposition.** *A homomorphism $a\colon Y \to X$ is a generalized subgroup of X if and only if $\mathrm{Ker}(a)$ is a central subgroup of Y and $\mathrm{Hom}(Y, \mathrm{Ker}(a)) = 0$.*

Proof. Assume first that $a\colon Y \to X$ is a generalized subgroup of X. Let $y \in \mathrm{Ker}(a)$. Consider the identity $\mathrm{id}\colon Y \to Y$ and the conjugation $c_y\colon Y \to Y$. Then $ac_y = a\mathrm{id}$. It follows that $c_y = \mathrm{id}$ and hence y is in the center of Y. This shows (a).

Consider now the trivial homomorphism $Y \to Y$ and the composition of some $f\colon Y \to \mathrm{Ker}(a)$ with the inclusion $\mathrm{Ker}(a) \subset Y$. The compositions of these homomorphisms with a are equal to the trivial homomorphism. Thus any such f must be trivial and consequently $\mathrm{Hom}(Y, \mathrm{Ker}(a)) = 0$ which is requirement (b).

Assume that conditions (a) and (b) are satisfied. We need to show injectivity of $\mathrm{Hom}(Y, a)\colon \mathrm{Hom}(Y, Y) \to \mathrm{Hom}(Y, X)$. Let $f, g\colon Y \to Y$ be homomorphisms. Assume $af = ag$. This means that, for any $y \in Y$, $f(y)g(y)^{-1}$ belongs to $\mathrm{Ker}(a)$. We claim that the function $Y \ni y \mapsto f(y)g(y)^{-1} \in \mathrm{Ker}(a)$ is a group homomorphism. This follows from the fact that $\mathrm{Ker}(a)$ is central in Y:

$$f(xy)g(xy)^{-1} = f(x)f(y)g(y)^{-1}g(x)^{-1} = f(x)g(x)^{-1}f(y)g(y)^{-1}.$$

Since $\mathrm{Hom}(Y, \mathrm{Ker}(a)) = 0$, we can conclude that $f(y)g(y)^{-1}$ is the identity element for any $y \in Y$. Consequently $f = g$. □

We can use this direct characterization to show that generalized subgroups of X inherit certain properties of X.

7.3. **Proposition.** *Let $a\colon Y \to X$ be a generalized subgroup.*

(1) *If X is nilpotent of class n, respectively solvable, then so is Y.*
(2) *If X is finite, then so is Y. Moreover $\mathrm{Ker}(a) \subset \Gamma_i(Y)$ for any i.*
(3) *If X is finitely generated and nilpotent, then $a\colon Y \to X$ is an injection. In particular Y is also finitely generated.*

7.4. **Lemma.** *Let $K \trianglelefteq Y$ be a normal subgroup such that:*

(a) *K is nilpotent,*
(b) *$\mathrm{Hom}(Y, K) = 0$,*
(c) *Y/K is finitely generated,*
(d) *for some $j \geq 1$, $K \cap \Gamma_j(Y)$ is finite.*

Then K is finite and $K \subset \Gamma_i(Y)$ for any $i \geq 1$.

Proof. Note that to prove the corollary it is enough to show $K = K \cap \Gamma_i(Y)$ for any $i \geq j$. Let $X := Y/\Gamma_i(Y)$ and consider $Q := K/\big(K \cap \Gamma_i(Y)\big)$. Then Q is (isomorphic to) a subgroup of X and X/Q is finitely generated. Assume that Q is non-trivial. Then, by 6.4, $\mathrm{Hom}(X, Q) \neq 0$, and then also $\mathrm{Hom}(Y, Q) \neq 0$. On the other hand, by hypothesis (b) $\mathrm{Hom}(Y, K) = 0$. As $K \cap \Gamma_i(Y)$ is finite and K is nilpotent Corollary 6.4, implies that $\mathrm{Hom}(Y, Q) = 0$. This contradiction completes the proof. □

7.5. **Lemma.** *Let Y be group and $Z(Y)$ be its center. If $Y/Z(Y)$is finite, then the comparator subgroup $[Y, Y]$ is also finite.*

Proof. See Theorem [24, 10.1.4, p. 287]. □

Proof of Proposition 7.3. (1): Assume X is nilpotent and $\Gamma_n(X) = 0$. We claim that $\Gamma_n(Y) = 0$. The assumption $\Gamma_i(X) = 0$ implies that $\Gamma_i(Y)$ is in the kernel of a and hence, according to 7.2, it is central in Y. It follows that $\Gamma_{i+1}(Y) = 0$ and Y is a nilpotent group. We can now use Corollary 6.4. If $\Gamma_i(Y)$ were non-trivial, there would be a non-trivial homomorphism $Y \to \Gamma_i(Y)$. The composition of this homomorphism with the inclusion $\Gamma_i(Y) \subset \mathrm{Ker}(a)$ would be then also non-trivial. This contradicts the fact that $\mathrm{Hom}(Y, \mathrm{Ker}(a)) = 0$ (see 7.2). Consequently $\Gamma_i(Y) = 0$.

Similar argument works for solvable groups. If $X^{(i)} = 0$, then $Y^{(i)} \subset \mathrm{Ker}(a)$ and hence $Y^{(i)}$ is central in Y. This implies that $Y^{(i+1)} = 0$ and consequently Y is solvable.

(2): Assume X is finite. We apply 7.4 to the subgroup $K_a := \mathrm{Ker}(a) \trianglelefteq Y$ to prove that K_a is finite. It would then follow that Y is also finite. Since K_a is central in Y, it is abelian and hence nilpotent. This is hypothesis (a) of 7.4. Hypothesis (b) of 7.4 is the second condition in 7.2. As X is finite, then so is its subgroup $a(Y) \cong Y/K_a$.

In particular this quotient is finitely generated and we get hypothesis (c) of 7.4. As Y/K_a is finite and K_a is central in Y, the quotient $Y/Z(Y)$ is also finite, where $Z(Y)$ denotes the center of Y. It follows that the commutator $[Y,Y]$ is finite (see 7.5). In particular $K_a \cap [Y,Y]$ is finite and we get hypothesis (d) of 7.4. We can then conclude that K_a is a finite group and $K_a \subset \Gamma_i(Y)$ for any i.

(3): Assume X is finitely generated and nilpotent. As in (2) we will apply 7.4 to the subgroup $K_a = \mathrm{Ker}(a) \trianglelefteq Y$. Hypotheses (a) and (b) of 7.4 are clear. Since X is finitely generated and nilpotent, then so is any of its subgroups. In particular Y/K_a is finitely generated. This shows that hypothesis (c) of 7.4 holds. As X is nilpotent, there is i for which $\Gamma_i(X) = 0$. It then follows that $\Gamma_i(Y)$ is also trivial (see the proof of part (1)). In particular $K_a \cap \Gamma_i(Y)$ is finite. We can conclude that $K_a \subset \Gamma_i(Y) = 0$ and hence a is an injection. □

7.6. **Corollary.** *Let (ϕ, ϵ) be an idempotent functor of groups. If G is s-nilpotent, or solvable, or finite, or finitely generated and s-nilpotent, then so is $\phi(G)$.*

An inclusion $Y \subset X$ is of course an example of a generalized subgroup of X. In the case X is finitely generated and nilpotent all the generalized subgroups are inclusions (see 7.3(3)). In this case $\mathrm{Sub}(X)$ is simply the collection of all the subgroups of X. For example the set $\mathrm{Sub}(\mathbb{Z}/n)$, of subgroups of the cyclic group $\mathbb{Z}/n$ $(n > 0)$, can be enumerated by the set $\{k \in \mathbb{Z} \mid k > 0 \text{ and } k \text{ divides } n\}$ of all positive divisors of n. For any such k, the corresponding subgroup is generated by n/k and is isomorphic to $\mathbb{Z}/k$. Note that the inclusion $\mathbf{Z}/k \subset \mathbf{Z}/n$ is not only a generalized subgroup but it is also a cellular cover. Thus in this case we have an equality $\mathrm{Cov}(\mathbf{Z}/n) = \mathrm{Sub}(\mathbb{Z}/n)$. Thus the cardinality of $\mathrm{Cov}(\mathbf{Z}/n)$ is given by the number of different positive divisors of n. More generally let A be a finite abelian group. Recall that, for an integer k, the k-torsion subgroup of A consists of all $a \in A$ for which $ka = 0$.

7.7. **Proposition.** *If A is a finite abelian group, then:*

$$\mathrm{Cov}(A) = \{Y \in \mathrm{Sub}(A) \mid Y \textit{ is the } k\textit{-torsion subgroup of } A \textit{ for some } k\}$$

Proof. If $Y \subset A$ is the k-torsion subgroup, then Y is k-torsion. Since any homomorphism $f\colon Y \to A$ takes the k-torsion elements to the k-torsion elements, the image of f sits in the subgroup $Y \subset A$. This means that $Y \subset A$ is a cellular cover.

Let $Y \subset A$ be a cellular cover and k be the exponent of Y, i.e., the smallest positive integer k for which $kY = 0$. Since Y is finite, there is a surjection $Y \twoheadrightarrow \mathbb{Z}/k$. For any k-torsion element $x \in A$, consider the composition of this surjection $Y \twoheadrightarrow \mathbb{Z}/k$ and a homomorphism $\mathbb{Z}/k \to A$ that maps some generator to the element x. Since $Y \subset A$ is a cellular cover, the image of this composition has to lie in Y. It follows that Y contains all the k-torsion elements of A. As Y consists of k-torsion elements, Y is the k-torsion subgroup of A. □

8. Covers of R-modules

Let R be a ring with 1. In this section we discuss cellular properties, idempotent functors and covers in the category of R-modules. We start with a basic example of a cellular property of R-modules, analogous to 4.7.

8.1. **Proposition.** *Let R be a ring with 1 and A be an R-module. The smallest property $\mathrm{Cell}_R(A)$ of R-modules containing A and closed under colimits is cellular.*

To prove this proposition we need a version of 4.6.(6) and 4.6.(7) for the category of R-modules:

8.2. **Lemma.** *Let $\mathcal{D}$ be a property of R-modules closed under colimits. Then:*

(1) *Let $\overline{\mathcal{D}}$ be the smallest property of R-modules containing $\mathcal{D}$ and closed under colimits, quotients, and extensions. Let Y be in $\mathcal{D}$ and $f\colon Y \to X$ be an epimorphism whose kernel belongs to $\overline{\mathcal{D}}$. Then X belongs to D.*

(2) *Let $f\colon Y \to X$ be a homomorphism of R-modules such that Y belongs to $\mathcal{D}$. Then f can be factored as:*

$$Y \xrightarrow{f_1} W \xrightarrow{f_2} X$$

where f_1 is an epimorphism, W belongs to $\mathcal{D}$, and, for any Z in $\mathcal{D}$, the map $\mathrm{Hom}_R(Z, f_2)$ is a monomorphism. Moreover, such a factorization can be made functorial with respect to f.

Proof. The proof of this lemma is almost exactly the same as the proofs of 4.6.(6) and 4.6.(7). Here we just give a sketch of it leaving the details to the reader.
(1): Since $\mathcal{D}$ is preserved by colimits, the trivial module 0 is in $\mathcal{D}$, as 0 is the colimit of the empty functor. Define $\mathbf{D}$ to be the collection of all R-modules Y such that, for any X in $\mathcal{D}$ and any homomorphism $f\colon Y \to X$, $X/f(Y)$ is in $\mathcal{D}$. The module $X/f(Y)$ is isomorphic to $\mathrm{colim}(0 \leftarrow Y \xrightarrow{f} X)$. Thus if Y is in $\mathcal{D}$, then so does $X/f(Y)$. This means that $\mathcal{D} \subset \mathbf{D}$. It is clear that $\mathbf{D}$ is a property. Since $\overline{\mathcal{D}}$ is the smallest property containing $\mathcal{D}$ which is preserved by colimits, quotients and extensions, to prove the proposition it is enough to show that $\mathbf{D}$ is preserved by colimits, quotients and extensions. Thus can be done is an exactly same way as in the proof of 4.6.(6).

(2): Consider the set S_f of all submodules H of Y such that they are contained in the kernel of f and Y/H is in $\mathcal{D}$. This set is non empty as it contains the trivial module. By the same argument as in as the proof of 4.6.(7), S contains the unique maximal element H_f with respect to the inclusion. We claim that the factorization $Y \xrightarrow{f_1} Y/H_f \xrightarrow{f_2} X$, where f_1 is the quotient and f_2 is induced by f, satisfies the desired requirements. $\square$

Proof of Proposition 8.1. Let X be an R-module. Apply 8.2 to the property $\mathrm{Cell}_R(A)$ and the homomorphism:

$$f := \bigoplus_{\alpha\colon A \to X} \alpha\colon \bigoplus_{\alpha\colon A \to X} A \longrightarrow X$$

We claim that the obtain function $f_2\colon W \to X$ is the $\mathrm{Cellc}_R(A)$-cover of X. According 8.2, the R-module W belongs to $\mathrm{Cell}_R(A)$. We need to show for any Y in $\mathrm{Cell}_R(A)$, $\mathrm{Hom}_R(Y, f_2)\colon \mathrm{Hom}_R(Y, W) \to \mathrm{Hom}_R(Y, X)$ is a bijection. According to 8.2, this map is a monomorphism. Further more, by definition, $\mathrm{Hom}_R(A, f_2)$ is also a surjection. Since the full subcategory of all R-modules Y for which $\mathrm{Hom}_R(Y, f_2)$ is a bijection is a property closed under colimits, we can conclude that $\mathrm{Hom}_R(Y, f_2)$ is indeed a bijection for any R-module Y in $\mathrm{Cell}_R(A)$, as $\mathrm{Cell}_R(A)$ is the smallest property containing A and closed under colimits. $\square$

Let X be an R-module. We can form the following collections:

$\mathcal{E}_{\mathrm{Gr}}(X)$	$=$	$\left\{\right.$ iso class of $\mathrm{cov}_A X$ in groups	$\mid$	A is a group $\left.\right\}$
$\mathrm{Idem}_{\mathrm{Gr}}(X)$	$=$	$\left\{\right.$ iso class of $\phi(X)$ in groups	$\mid$	(ϕ, ϵ) is an idempotent functor of groups $\left.\right\}$
$\mathrm{Cov}_{\mathrm{Gr}}(X)$	$=$	$\left\{\right.$ iso class of $\mathrm{cov}_{\mathcal{D}} X$ in groups	$\mid$	$\mathcal{D}$ is a property of groups for which $\mathrm{cov}_{\mathcal{D}} X$ exists $\left.\right\}$
$\mathcal{L}_{\mathrm{Gr}}(X)$	$=$	$\left\{\right.$ iso class of Y in groups	$\mid$	there is $c\colon Y \to X$ in Groups s.t. $\mathrm{Hom}(c, Y)$ is a bijection $\left.\right\}$
$\mathcal{E}_R(X)$	$=$	$\left\{\right.$ iso class of $\mathrm{cov}_A X$ in R-mod	$\mid$	A is R-mod $\left.\right\}$
$\mathrm{Idem}_R(X)$	$=$	$\left\{\right.$ iso class of $\phi(X)$ in R-mod	$\mid$	(ϕ, ϵ) is an idempotent functor of R-mod $\left.\right\}$
$\mathrm{Cov}_R(X)$	$=$	$\left\{\right.$ iso class of $\mathrm{cov}_{\mathcal{D}} X$ in R-mod	$\mid$	$\mathcal{D}$ is a property of R-mod for which $\mathrm{cov}_{\mathcal{D}} X$ exists $\left.\right\}$
$\mathcal{L}_R(X)$	$=$	$\left\{\right.$ iso class of Y in R-mod	$\mid$	there is $c\colon Y \to X$ in R-mod s.t. $\mathrm{Hom}_R(c, Y)$ is a bijection $\left.\right\}$

We already know that the first four collections are the same (see 4.8). It turns out that all the above collections are equal. Thus it does not matter if we form the covers of an R-module in the category of groups or R-modules. To prove this, define $F(Y) := H_1(Y)\otimes_{\mathbb{Z}} R$, ϵ_Y to be given by $Y \ni y \mapsto [y]\otimes 1 \in H_1(Y)\otimes_{\mathbb{Z}} R = F(Y)$, and μ_Y to be given by $F(F(Y)) = (H_1(Y) \otimes_{\mathbb{Z}} R) \otimes_{\mathbb{Z}} R \ni ([y] \otimes r_1) \otimes r_2 \mapsto [y] \otimes r_1 r_2 \in H_1(Y) \otimes_{\mathbb{Z}} R = F(Y)$. Then (F, ϵ, μ) forms a triple on the category of groups. An F-algebra is simply an R-module and an F-algebra morphism is an R-module homomorphism. Note that according to 4.6.(4), if Z belongs to a cellular property $\mathcal{D}$, then so does $H_1(Z) \otimes_{\mathbb{Z}} R$. In particular $H_1(A) \otimes_{\mathbb{Z}} R$ is A-cellular for any A. It follows that assumptions of 5.1 are satisfied and hence we get:

8.3. **Proposition.** *Let R be a ring with 1, X an R-module, and A a group.*

1. *For any idempotent functor (ϕ, ϵ) of groups, $\phi(X)$ has a unique R-module structure for which ϵ_X is an R-module homomorphism.*
2. *The covers $\mathrm{cov}_A X$ and $\mathrm{cov}_{H_1(A)\otimes_{\mathbb{Z}} R} X$ are isomorphic.*
3. *X is A-cellular if and only if it is $H_1(A) \otimes_{\mathbb{Z}} R$-cellular.*

8.4. **Corollary.** *Let X be an R-module. Then:*

$$\begin{aligned}\mathcal{E}_{\mathrm{Gr}}(X) &= \mathrm{Idem}_{\mathrm{Gr}}(X) = \mathrm{Cov}_{\mathrm{Gr}}(X) = \mathcal{L}_{\mathrm{Gr}}(X)\\ &= \mathcal{E}_R(X) = \mathrm{Idem}_R(X) = \mathrm{Cov}_R(X) = \mathcal{L}_R(X).\end{aligned}$$

Furthermore an R-module Y belongs to $\mathrm{Cell}_R(X)$ if and only if it belongs to $\mathrm{Cell}(X)$.

Proof. The first three equalities are given by 4.8. This together with 8.3 gives an inclusion $\mathcal{L}_{\mathrm{Groups}}(X) \subset \mathcal{E}_R(X)$. The inclusions $\mathcal{E}_R(X) \subset \mathrm{Idem}_R(X) \subset \mathrm{Cov}_R(X) \subset \mathcal{L}_R(X)$ follows directly from the definitions of these collections. It remains to prove $\mathcal{L}_R(X) \subset \mathcal{E}_{\mathrm{Groups}}(X)$. This follows using the very same argument as in the proof of 4.8 and two facts. First, for any group Z and any R-modules T, we have en equality $\mathrm{Hom}(Z, T) = \mathrm{Hom}_R(H_1(Z) \otimes_{\mathbb{Z}} R, T)$. Second, $H_1(Z) \otimes_{\mathbb{Z}} R$ is Z-cellular for any group Z (see 4.6.(4)). These two facts also imply the last claim of the corollary. $\square$

Similarly to finitely generated nilpotent groups (see 6.2), in the case the ring R is commutative and Noetherian and X is a finitely generated R-module, we get more:

8.5. **Theorem.** *Let R be commutative with 1 and Noetherian, M a finitely generated R-module, (ψ, ϵ) an idempotent functor of R-modules, and A an R-module. Then:*

(1) $\epsilon_X : \psi(X) \to X$ *is a monomorphism.*
(2) *X is A-cellular in the category of R-modules if and only if it is a quotient of a direct sum of A.*
(3) $\mathrm{cov}_A X$ *is the submodule of X generated by all the images of all R-module homomorphisms $A \to X$.*

Our strategy to prove the theorem is the same as in the proof of 6.2. We start with:

8.6. **Lemma.** *Let $\mathcal{D}$ be a property of R-modules closed under colimits, quotients, and extensions. Let X be an R-module satisfying $\mathcal{D}$ and $K \subset X$ its submodule. If X/K is a finitely generated R-module, then K satisfies $\mathcal{D}$.*

Proof. We first prove the lemma in the case the quotient X/K is a cyclic module isomorphic to R/I for some ideal I in R. Let $L \subset K$ be the submodule generated by all the images of all homomorphism $Z \to K$ where Z is in $\mathcal{D}$. Since $\mathcal{D}$ is closed under colimits and quotient, L belongs to $\mathcal{D}$. We claim that for any Z in $\mathcal{D}$, $\mathrm{Hom}_R(Z, K/L) = 0$. Let $g \colon Z \to K/L$ be a homomorphism and consider the following commutative diagram where the right square is a pull-back square:

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d, "\mathrm{id}"'] & P \arrow[r, two heads] \arrow[d] & Z \arrow[d, "g"] \\ L \arrow[r, hook] & K \arrow[r, two heads] & K/L \end{tikzcd}$$

As $\mathcal{D}$ is closed under extensions, P belongs to $\mathcal{D}$. It thus follows that the image of $P \to K$ is in L. This can happen only if g is the 0 homomorphism.

We claim also that X/L is an R/I-module. Let r be in I and consider the following diagram where $r \colon X/L \to X/L$ is multiplication by r homomorphism (here we use the fact that R is commutative):

$$\begin{tikzcd} & X/L \arrow[d, "r"] & \\ K/L \arrow[r, hook] & X/L \arrow[r, two heads] & R/I \end{tikzcd}$$

The composition of the multiplication by r homomorphism $r \colon X/L \to X/L$ and the quotient $X/L \to R/I$ is the trivial homomorphism. Therefore the image of the multiplication homomorphism $r \colon X/L \to X/L$ is in K/L. As K/L does not receive any homomorphism out of any element in $\mathcal{D}$, the multiplication homomorphism $r \colon X/L \to X/L$ is trivial. This means that X/L is an R/I-module and hence so is K/L. Note that R/I belongs to $\mathcal{D}$ as it is a quotient of X. The only R/I module that does not receive any homomorphism out if R/I is the trivial module, and we can conclude $L = K$. This means that K is in $\mathcal{D}$.

We proceed by induction on number of generators of X/K. Let $n > 1$ and assume the lemma is true if X/K can be generated by strictly less than$n$ generators. Assume X/K can be generated by n elements $g_1, \ldots, g_n$. Let $Y \subset X/K$ be the submodule of X/K generated by $g_1, \ldots, g_{n-1}$. Note that quotient of X/K by Y is a cyclic module isomorphic to R/I for some ideal I in R. Consider the following commutative diagram where the top right square is a pull-back square:

$$\begin{tikzcd}
K \arrow[r, hook] \arrow[d, "\mathrm{id}"'] & \overline{X} \arrow[r, two heads] \arrow[d, hook] & Y \arrow[d, hook] \\
K \arrow[r, hook] & X \arrow[r, two heads] \arrow[d] & X/K \arrow[d] \\
 & R/I \arrow[r, "\mathrm{id}"] & R/I
\end{tikzcd}$$

By what we have already proved, $\overline{X}$ belongs to $\mathcal{D}$. By the inductive assumption, since Y is generated by $n-1$ elements, we can conclude that K belongs to $\mathcal{D}$ too. ☐

Proof. Proof of Theorem 8.5 (1): Let $\mathcal{D} = \mathrm{Cell}(\phi)$. Recall that $\mathcal{D}$ is a cellular property and that (ψ, ϵ) and $(\mathrm{cov}_{\mathcal{D}}, c)$ are isomorphic augmented functors (see 3.1). Consider the image $I \subset X$ of the the cover $c_X \colon \mathrm{cov}_{\mathcal{D}} X \to X$. We claim that I belongs to $\mathcal{D}$. Since X is finitely generated and R is Noetherian, then I is also a finitely generated module. By Lemma 8.6, the kernel of c_X must therefore belong to $\overline{\mathcal{D}}$ (the smallest property containing $\mathcal{D}$ and closed under colimits, quotients, and extensions). We can now conclude, using 8.2.(2), that the image I of c_X satisfies $\mathcal{D}$. To finish, for Y in $\mathcal{D}$, consider the maps:

$$\begin{tikzcd}
\mathrm{Hom}_R(Y, \mathrm{cov}_{\mathcal{D}} X) \arrow[r, "\mathrm{Hom}_R(Y, c_X)"] \arrow[rr, bend right, "\simeq"'] & \mathrm{Hom}_R(Y, I) \arrow[r, hook] & \mathrm{Hom}_R(Y, I)
\end{tikzcd}$$

Since the composition is a bijection and the right map is a monomorphism, all the maps in this diagram are bijections and hence $I \subset X$ is the $\mathcal{D}$-cover of X.

(2)&(3): These are the direct consequences of (1). ☐

8.7. **Corollary.** *Let R be a commutative Noetherian ring with* 1 *and M and N be finitely generated modules. If $\mathrm{Cell}_R(M) \subset \mathrm{Cell}_R(N)$, then $\mathrm{ann}(N) = \mathrm{ann}(M)$. In particular for two ideals I and J in R, $\mathrm{Cell}_R(R/I) \subset \mathrm{Cell}_R(R/J)$ if and only if $J \subset I$.*

In the case of the rings $\mathbb{Z}[S^{-1}]$ and $\mathbb{Z}/n$ we can get a bit more and be more explicit:

8.8. **Proposition.** *Let R be either $\mathbb{Z}[S^{-1}]$ or $\mathbb{Z}/n$. Let A and X be abelian groups.*

(1) *If A is an R-module, then so is any abelian A-cellular group.*
(2) *X is $A \otimes_{\mathbb{Z}} R$-cellular if and only if X is an A-cellular group and an R-module. In particular, X is R-cellular if and only if it is an R-module.*
(3) *The evaluation at* 1 *homomorphism $e \colon \hom(R, X) \to X$ is isomorphic to the R-cellular cover of X.*

Proof. (1): Assume A is a $\mathbb{Z}[S^{-1}]$-module and Y an abelian group which is A-cellular. Since Y is abelian, the set $\mathrm{Hom}(Z, Y)$is an abelian group for any group Z. Let $\mathcal{D}$ be the property of all groups Z, for which $\mathrm{Hom}(Z, Y)$ is a $\mathbb{Z}[S^{-1}]$-module. Note that $\mathcal{D}$ is a property closed under colimits. Since it contains A, it also contains Y. $\mathrm{Hom}(Y, Y)$ is therefore a $\mathbb{Z}[S^{-1}]$-module. This can happen only if Y is a $\mathbb{Z}[S^{-1}]$-module.

Assume that A is a $\mathbb{Z}/n$-module. Any A-cellular group Y is generated by the images of all homomorphisms $A \to Y$, and hence it is generated by n-torsion elements. In the case Y is abelian this means that Y is a $\mathbb{Z}/n$-module.

(2): If X is $A \otimes_{\mathbb{Z}} R$-cellular then, by Proposition 4.6.(4), it is also A-cellular, as cellularity is a transitive relation. Moreover by (1), X is an R-module. If X is A-cellular and an R-module, then it is isomorphic to $\mathrm{cov}_A X$, which, by Proposition 8.3, is isomorphic to $\mathrm{cov}_{A\otimes_{\mathbb{Z}} R} X$. Consequently X is $A \otimes_{\mathbb{Z}} R$-cellular.

(3): Since X is abelian, then so is $\mathrm{cov}_R X$. We can then form the following commutative square, where the vertical homomorphisms are evaluations at 1:

$$\begin{tikzcd}
\mathrm{Hom}(R, \mathrm{cell}_R X) \arrow[r, "\mathrm{Hom}(R, c_X)"] \arrow[d, "e'"'] & \mathrm{Hom}(R, X) \arrow[d, "e"] \\
\mathrm{cov}_R X \arrow[r, "c_X"] & X
\end{tikzcd}$$

The top homomorphism $\mathrm{Hom}(R, c_X)$ is an isomorphism. Moreover, according to (1), $\mathrm{cov}_R X$ is an R-module. Therefore the evaluation e' is also an isomorphism. We can conclude that e and c_X are isomorphic. $\square$

8.9. **Corollary.** *Let G be a group. A cyclic group is G-cellular if and only if it is a quotient of G.*

Our next goal is to show that $\mathrm{Idem}(X)$ is a finite set if X is a finite group. For that it would be enough to show that the set of generalized subgroups $\mathrm{Sub}(X)$ (see 7.1) is finite. Recall that any element in $\mathrm{Sub}(X)$ is represented by a homomorphism $Y \to X$ whose kernel is central in Y. To enumerate $\mathrm{Sub}(X)$ we therefore need to study central extensions. In the next section we review briefly how central extensions are enumerated and their relation to second (co)homology of X.

9. Central extensions and the initial cover

In this section we recall how to classify central extensions. We will use this classification to construct an extension which in a sense is a universal cover of a finite group. We call it the initial cover. It is a generalization of the well known universal cover of a perfect group. The initial cover will be used in our classification results in the following sections.

Recall that, for two finite abelian groups A and B, the groups $A\otimes B$, $\mathrm{Hom}(A, B)$, $\mathrm{Hom}(B, A)$, $B \otimes A$, $\mathrm{Ext}^1(A, B)$, and $\mathrm{Ext}^1(B, A)$ are isomorphic. Thus all these groups are zero if and only if the orders of A and B are relatively prime.

For a group X and abelian group K, $H_n(X)$ and $H^n(X, K)$ denote respectively the n-th integral homology group and n-th cohomology group with coefficient in K of the classifying space BX. The first homology group $H_1(X)$ is the abelianization $X/[X, X]$ of X. The second homology $H_2(X)$ is also called the Schur multiplier of

X. Recall that if X is finite, then, for any n, $H_n(X)$ is a finite abelian group whose exponent divides $|X|$. If X is finite and cyclic, then $H_2(X) = 0$.

Here are some very basic homotopy tools one can use to study properties of the homology groups of a group:

- If $K \trianglelefteq Y \twoheadrightarrow X$ is an exact sequence of groups, then the induced sequence of classifying spaces $BK \to BY \to BX$ is a fibration sequence.
- **Ganea's theorem.** Let $F \to E \to B$ be a fibration sequence. Then the homotopy fiber of the induced map $\mathrm{Cof}(F \to E) \to B$ is weakly equivalent to $S^1 \wedge F \wedge \Omega B$. This is a particular case of:
- **Puppe's theorem** Consider the following commutative diagram and the induced map of spaces:

$$\begin{tikzcd} C \arrow[dr, "f"'] & A \arrow[l] \arrow[r] \arrow[d, "g"] & B \arrow[dl, "h"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} \mathrm{hocolim}(A \leftarrow B \to C) \arrow[d, "{(f,g,h)}"] \\ X \end{tikzcd}$$

  Then the homotopy fibre $\mathrm{Fib}\Big((f, g, h)\colon \mathrm{hocolim}(A \leftarrow B \to C) \to X)\Big)$ is weakly equivalent to $\mathrm{hocolim}\Big(\mathrm{Fib}(f) \leftarrow \mathrm{Fib}(g) \to \mathrm{Fib}(h)\Big)$.

I would encourage anyone to use these tools to prove the following theorem of Hopf (you can also use this theorem as an alternative definition of $H_2(X)$):

9.1. **Theorem.** *Let $F \twoheadrightarrow X$ be a free presentation of X, i.e., a surjective homomorphism with F a free group. Let K be the kernel of this presentation. Then $H_2(X)$ is isomorphic to $(K \cap [F, F])/[K, F]$.*

Since $H_2(X)$ depends on X, and not its presentation, as a corollary of the Hopf theorem one gets that the group $(K \cap [F, F])/[K, F]$ is independent from the choice of the free presentation $F \to X$.

For an abelian group K and a group X, a central extension of X by K is a group Y containing K in its center and a surjective homomorphism $f\colon Y \twoheadrightarrow X$ for which $\mathrm{Ker}(f) = K$. Two such central extensions $f\colon Y \twoheadrightarrow X$ and $f'\colon Y' \twoheadrightarrow X$ are equivalent if there is a homomorphism $h\colon Y \to Y'$ for which $f'h = f$ and h restricted to K is the identity. Such h necessarily has to be an isomorphism. Let us recall that the equivalence classes of central extensions of X by K form a set which can be identified with the second cohomology group $H^2(X, K)$ (see [24, 11.1.4, p. 318]). An effective tool to study the group $H^2(X, K)$ is the universal coefficient exact sequence ([24, 11.4.18, p.349]):

$$0 \to \mathrm{Ext}^1(H_1(X), K) \to H^2(X, K) \xrightarrow{\mu} \mathrm{Hom}(H_2(X), K) \to 0$$

where $\mathrm{Ext}^1(H_1(X), K) \subset H^2(X, K)$ can be identified with abelian extensions of X by K. This universal coefficient exact sequence measures the difference between abelian and central extensions. The difference is given by $\mathrm{Hom}(H_2(X), K)$.

If $f\colon Y \twoheadrightarrow X$ represents an equivalence class of a central extension of X by K, then the homomorphism $\mu(f)\colon H_2(X) \to K$ is called the **differential** of f. This differential fits into the following exact sequence, called the exact sequence of f:

$$H_2(Y) \xrightarrow{H_2(f)} H_2(X) \xrightarrow{\mu(f)} K \xrightarrow{\alpha} H_1(Y) \xrightarrow{H_1(f)} H_1(X) \to 0 \tag{1}$$

where the homomorphism α is given by $K \ni y \mapsto y[Y,Y] \in Y/[Y,Y] = H_1(Y)$. I would encourage everyone to obtain this sequence using the basic homotopical tools listed above.

The sequence (1) is functorial, which means that, for two central extensions $f\colon Y \twoheadrightarrow X$ of X by K_f and $g\colon H \twoheadrightarrow G$ of G by K_g that fit into the following commutative diagram:

$$\begin{tikzcd}
K_f \arrow[r, hook] \arrow[d, "h_1"] & Y \arrow[r, two heads, "f"] \arrow[d, "h_1"] & X \arrow[d, "h"] \\
K_g \arrow[r, hook] & H \arrow[r, two heads, "g"] & G
\end{tikzcd}$$

the following diagram of their exact sequences also commutes:

$$\begin{tikzcd}
H_2(Y) \arrow[r, "H_2(f)"] \arrow[d, "H_2(h_1)"'] & H_2(X) \arrow[r, "\mu(f)"] \arrow[d, "H_2(h)"] & K_f \arrow[r] \arrow[d, "h_1"] & H_1(Y) \arrow[r, "H_1(f)"] \arrow[d, "H_1(h_1)"] & H_1(X) \arrow[r] \arrow[d, "H_1(h)"] & 0 \\
H_2(H) \arrow[r, "H_2(g)"] & H_2(G) \arrow[r, "\mu(g)"] & K_g \arrow[r] & H_1(H) \arrow[r, "H_1(g)"] & H_1(G) \arrow[r] & 0
\end{tikzcd}$$

Under the assumption that Y is finite (it is actually enough to assume that only K is finite), the exact sequence of f can be extended one step further to an exact sequence, called also the exact sequence of f:

$$H_1(Y) \otimes K \to H_2(Y) \xrightarrow{H_2(f)} H_2(X) \xrightarrow{\mu(f)} K \xrightarrow{\alpha} H_1(Y) \xrightarrow{H_1(f)} H_1(X) \to 0.$$

Again one can use the basic homotopical tools above to prove the existence of this longer sequence in the case K is finite.

9.2. **Definition.** For a finite group X, $H_{2\backslash 1}(X)$ denotes the localization $H_2(X)[S^{-1}]$ where S is the set of primes dividing the order of $H_1(X)$.

The group $H_{2\backslash 1}(X)$ is simply the quotient of $H_2(X)$ by the S-torsion, and the localization $H_2(X) \to H_2(X)[S^{-1}] = H_{2\backslash 1}(X)$ is the quotient homomorphism. Since the orders of $H_2(X)[S^{-1}]$ and $H_1(X)$ are coprime, the group $\mathrm{Ext}^1(H_1(X), H_{2\backslash 1}(X))$ is trivial. The homomorphism $\mu\colon H^2(X, H_{2\backslash 1}(X)) \to \mathrm{Hom}(H_2(X), H_{2\backslash 1}(X))$ is therefore an isomorphism. It follows that there is a unique central extension $e_X\colon E \twoheadrightarrow X$ of X by $H_{2\backslash 1}(X)$ whose differential $\mu(e_X)$ is the localization homomorphism:

$$\begin{tikzcd}
H_2(X) \arrow[r, "\text{localization}"] \arrow[rr, bend right, "\mu(e_X)"'] & H_2(X)[S^{-1}] \arrow[r, equal] & H_{2\backslash 1}(X)
\end{tikzcd}$$

We call the extension $e_G\colon E \twoheadrightarrow X$ the **initial extension** of X. In the case X is perfect, i.e., if $H_1(X) = 0$, then $H_{2\backslash 1}(X) = H_2(X)$ and the initial extension is the universal central extension of X.

The key property of the initial extension $e_G\colon E \twoheadrightarrow X$ of a finite group X is that its differential $\mu(e_X)\colon H_2(X) \to H_{2\backslash 1}(X)$ is a surjection.

9.3. **Proposition.** *Let X be a finite group and $f\colon Y \twoheadrightarrow X$ be a central extension of X by $H_{2\backslash 1}(X)$ whose differential $\mu(f)\colon H_2(X) \to H_{2\backslash 1}(X)$ is a surjection. Then:*

(1) *$H_1(f)\colon H_1(Y) \to H_1(X)$ is an isomorphism.*

(2) *The following is an exact sequence:*

$$0 \to H_2(Y) \xrightarrow{H_2(f)} H_2(X) \xrightarrow{\mu(f)} H_{2\backslash 1}(X) \to 0$$

(3) $H^2(Y, H_{2\backslash 1}(X)) = 0$.

(4) $\mathrm{Hom}(Y, f)\colon \mathrm{Hom}(Y, Y) \to \mathrm{Hom}(Y, X)$ *is a bijection (Y is a cover of X).*

(5) *There is an isomorphism $\phi\colon Y \to E$ which makes the following triangle commutative:*

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr, "f"'] & & E \arrow[dl, "e_X"] \\ & X & \end{tikzcd}$$

Proof. Since X is finite, $H_2(X)$ is finite and so is its quotient $H_{2\backslash 1}(X)$. The group Y is then also finite and we have the following exact sequence:

$$\begin{tikzcd} H_1(Y) \otimes H_{2\backslash 1}(X) \arrow[r] & H_2(Y) \arrow[r, "H_2(f)"] & H_2(X) \arrow[r, "\mu(f)"] & H_{2\backslash 1}(X) \arrow[d, "\alpha"] \\ & 0 & H_1(X) \arrow[l] & H_1(Y) \arrow[l, "H_1(f)"'] \end{tikzcd}$$

(1): As $\mu(f)$ is a surjection, the homomorphism α, in the above sequence, is trivial, and hence $H_1(f)\colon H_1(Y) \to H_1(X)$ is an isomorphism. This is statement (1).

(2): The orders of $H_1(X)$ and $H_{2\backslash 1}(X)$ are coprime and thus $H_1(X) \otimes H_{2\backslash 1}(X) = 0$. Using statement (1), we then get $H_1(Y) \otimes H_{2\backslash 1}(X) = 0$. The homomorphism $H_2(f)$ is therefore an injection which proves statement (2).

(3): By the universal coefficient exact sequence, to show the statement, we need to prove that $\mathrm{Ext}^1(H_1(Y), H_{2\backslash 1}(X)) = 0$ and $\mathrm{Hom}(H_2(Y), H_{2\backslash 1}(X)) = 0$. The triviality of $\mathrm{Ext}^1(H_1(Y), H_{2\backslash 1}(X))$ follows from the fact that the orders of $H_1(Y) = H_1(X)$ and $H_{2\backslash 1}(X)$ are coprime.

Since $H_{2\backslash 1}(X)$ is the localization $H_2(X)[S^{-1}]$, where S is the set of primes that divide the order of $H_1(X)$, the homomorphism $\mu(f)$ factors uniquely as:

$$\begin{tikzcd} H_2(X) \arrow[r, "\text{localization}"] \arrow[rr, bend right, "\mu(f)"'] & H_{2\backslash 1}(X) \arrow[r, "h"] & H_{2\backslash 1}(X) \end{tikzcd}$$

The surjectivity of $\mu(f)$ implies the surjectivity of h. As a surjective homomorphism between finite groups, h is an isomorphism. The kernel of $\mu(f)$, which by (2) is given by $H_2(Y)$, is therefore isomorphic to the kernel of the localization homomorphism $H_2(X) \to H_2(X)[S^{-1}]$. The primes dividing the order of $H_2(Y)$ are thus among the primes dividing the order of $H_1(X)$. This means that the orders of $H_2(Y)$ and $H_{2\backslash 1}(X)$ are coprime and hence the group $\mathrm{Hom}(H_2(Y), H_{2\backslash 1}(X))$ is also trivial.

(4): We need to show $\mathrm{Hom}(Y, f)\colon \mathrm{Hom}(Y, Y) \to \mathrm{Hom}(Y, X)$ is a bijection. The kernel $H_{2\backslash 1}(X)$ of $f\colon Y \twoheadrightarrow X$ is central in Y. Moreover, as the orders of $H_{2\backslash 1}(X)$ and $H_1(Y)$ are relatively prime, $\mathrm{Hom}(Y, H_{2\backslash 1}(X)) = \mathrm{Hom}(H_1(Y), H_{2\backslash 1}(X)) = 0$. The injectivity of $\mathrm{Hom}(Y, f)$ follows then from 7.2.

It remains to prove that $\mathrm{Hom}(Y, f)\colon \mathrm{Hom}(Y, Y) \to \mathrm{Hom}(Y, X)$ is also surjective. Let $g\colon Y \to X$ be an arbitrary homomorphism. Consider the following commutative diagram, where the right hand square is a pull-back square:

$$\begin{tikzcd}
H_{2\backslash 1}(X) \arrow[r, hook] \arrow[d, equal] & P \arrow[r, two heads, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
H_{2\backslash 1}(X) \arrow[r, hook] & Y \arrow[r, two heads, "f"] & X
\end{tikzcd}$$

Note that $f' \colon P \twoheadrightarrow Y$ represents a central extension of Y by $H_{2\backslash 1}(X)$. According to statement (3) any such central extension is split. Let $s \colon Y \to P$ be its section. The composition $g's \colon Y \to Y$ is then a homomorphism for which $fg's = g$. This shows surjectivity of $\mathrm{Hom}(Y, f)$.

(5): The argument to show that there is an isomorphism $\phi \colon Y \to E$ for which $e_X\phi = f$ is the same as the proof of the surjectivity in the previous statement. Consider the following commutative diagram, where the bottom right square is a pull-back square:

$$\begin{tikzcd}
& H_{2\backslash 1}(X) \arrow[r, equal] \arrow[d, hook] & H_{2\backslash 1}(X) \arrow[d, hook] \\
H_{2\backslash 1}(X) \arrow[r, hook] \arrow[d, equal] & P \arrow[r, two heads, "f'"] \arrow[d, hook, "e'"] & E \arrow[d, hook, "e_G"] \\
H_{2\backslash 1}(X) \arrow[r, hook] & Y \arrow[r, two heads, "f"] & X
\end{tikzcd}$$

Both $e' \colon P \twoheadrightarrow Y$ and $f' \colon P \twoheadrightarrow E$ represent central extensions. Statement (3) implies that these extensions are split. Using their sections we can construct homomorphisms $h \colon Y \to E$ and $g \colon E \to Y$ for which $e_G h = f$ and $fg = e_G$. It follows that $e_G hg = e_G$ and $fgh = f$. As e_G and f are covers of X (this is the statement (4)), we can conclude $hg = \mathrm{id}_E$ and $gh = \mathrm{id}_Y$. This proves (5). □

Let $e_X : E \to X$ be the initial extension of a finite X. Then according to 9.3.(4), E belongs to $\mathrm{Cov}(X)$. We are also going to use the term **initial cover** of X to call the element in $\mathrm{Cov}(X)$ represented by the homomorphism $e_X : E \to X$. this initial extension.

10. Generalized subgroups of a finite group

The collection $\mathrm{Cov}(X)$ is a subcollection of $\mathrm{Sub}(X)$. Thus to show for example that X has finitely many covers it is enough to show that $\mathrm{Sub}(X)$ is a finite set. The aim of this section is to do that under the assumption that X is a finite group.

For a homomorphism $a \colon Y \to X$, we use the symbol I_a to denote its image $\mathrm{im}(a)$. This is one of the two invariants we use to enumerate generalized subgroups of X. Note that if generalized subgroups $a \colon Y \to X$ and $b \colon Z' \to X$ are equivalent, then they have the same images. Thus the function $a \mapsto I_a$ is well defined on the collection $\mathrm{Sub}(X)$ of equivalence classes of generalized subgroups. Furthermore it is immediate from the definition that a homomorphism $a \colon Y \to X$ is a generalized subgroup of X if and only if $a \colon Y \twoheadrightarrow I_a$ is a generalized subgroup of I_a. Thus any generalized subgroup is a composition of a surjective generalized subgroup and an inclusion. This is the reason why surjective generalized subgroups are important for us. We use the symbol $\mathrm{SurSub}(G)$ to denote the collection of equivalence classes of generalized subgroups of X which are represented by surjective homomorphisms.

For any subgroup I of X, let $\mathrm{in}_I : \mathrm{SurSub}(I) \subset \mathrm{Sub}(X)$ be the function that assigns to an equivalence class of a surjective generalized subgroup $a\colon Y \twoheadrightarrow I$ of I the equivalence class of the composition $a\colon Y \twoheadrightarrow I \subset X$. By summing up these inclusions over all the subgroups of X, it is then clear that we get a bijection:

$$\coprod_{I\subset X} \mathrm{in}_I\colon \coprod_{I\subset X} \mathrm{SurSub}(I) \to \mathrm{Sub}(X).$$

To enumerate $\mathrm{Sub}(X)$ it thus suffices to enumerate $\mathrm{SurSub}(I)$ for all subgroups I of X.

We proceed to enumerate $\mathrm{SurSub}(X)$ for any finite group X. We start with:

10.1. **Definition.** Let A be an abelian group.

(1) Two surjections $\sigma\colon A \twoheadrightarrow K$ and $\tau\colon A \twoheadrightarrow L$ are defined to be equivalent if there is an isomorphism $h\colon K \to L$ such that $h\sigma = \tau$ (such an isomorphism, if it exists, is necessary unique).
(2) The symbol $\mathrm{Quot}(A)$ denotes the set of equivalence classes of surjections out of A.

Note that the subgroup of A given by the kernel of a surjection $\sigma\colon A \twoheadrightarrow K$ depends only on the equivalence class of σ in $\mathrm{Quot}(A)$. It is then clear that the function that assigns to an element $[\sigma]$ in $\mathrm{Quot}(A)$ the subgroup $\mathrm{Ker}(\sigma)$ of A is a bijection between $\mathrm{Quot}(A)$ and the set of all the subgroups of A which, in the case A is finitely generated, coincides with the set $\mathrm{Sub}(A)$. Thus for a finitely generated abelian group A, we shall identify $\mathrm{Quot}(A)$ with $\mathrm{Sub}(A)$. For example let k be a positive integer. The element of $\mathrm{Quot}(A)$ that corresponds to the k-torsion subgroup of A is denoted by q_k. It is represented by the surjection, denoted by the same symbol:

$$q_k\colon A \twoheadrightarrow A/(k\text{-torsion}),$$

that maps an element $a \in A$ to its coset. In the case of the cyclic group $\mathbb{Z}/n$ ($n > 0$), these are all the elements of $\mathrm{Quot}(\mathbb{Z}/n)$. For any $k > 0$ dividing n, the k-torsion subgroup of $\mathbb{Z}/n$ is the subgroup generated by n/k. It is the unique subgroup isomorphic to $\mathbb{Z}/k$. In this way $\mathrm{Quot}(\mathbb{Z}/n)$ is in bijection with the set of all positive divisors of n.

We enumerate $\mathrm{SurSub}(X)$ using the set $\mathrm{Quot}(H_{2\setminus 1}(X))$ (recall that $H_{2\setminus 1}(X)$ denotes the localization $H_2(X)[S^{-1}]$, where S is the set of primes dividing the order of $H_1(X)$, see 9.2). We define two functions:

$$\mu\colon \mathrm{SurSub}(X) \to \mathrm{Quot}(H_{2\setminus 1}(X)), \qquad \Psi\colon \mathrm{Quot}(H_{2\setminus 1}(X)) \to \mathrm{SurSub}(X),$$

and show that their compositions $\mu\Psi$ and $\Psi\mu$ are the identities. For a surjective generalized subgroup $a\colon Y \twoheadrightarrow X$, the value $\mu(a) \in \mathrm{Quot}(H_{2\setminus 1}(X))$ is called the differential of a. Recall that according to 7.2, the kernel $K_a := \mathrm{Ker}(a)$ of a is central in Y. Thus the homomorphism $a\colon Y \twoheadrightarrow X$ represents a central extension of X by K_a. We use the corresponding element in $H^2(X, K_a)$ to define the differential. First we need a generalization of 9.3:

10.2. **Proposition.** *Let $a\colon Y \twoheadrightarrow X$ be a surjective generalized subgroup of a finite group X. Then:*

(1) *$H_1(a)\colon H_1(Y) \to H_1(X)$ is an isomorphism.*

(2) $\mathrm{Ext}^1(H_1(X), K_a) = H_1(Y) \otimes K_a = 0$.
(3) $\mu\colon H^2(X, K_a) \to \mathrm{Hom}(H_2(X), K_a)$ *is an isomorphism.*
(4) $0 \to H_2(Y) \xrightarrow{H_2(a)} H_2(X) \xrightarrow{\mu(a)} K_a \to 0$ *is an exact sequence.*
(5) *If* $Z \subset Y$ *is a subgroup such that* $a(Z) = X$*, then* $Y = Z$*.*

Proof. Since X is finite, by 7.3(2), Y is also finite and we have en exact sequence of the central extension $a\colon Y \twoheadrightarrow X$:

$$H_1(Y) \otimes K_a \to H_2(Y) \xrightarrow{H_2(a)} H_2(X) \xrightarrow{\mu(a)} K_a \xrightarrow{\alpha} H_1(Y) \xrightarrow{H_1(a)} H_1(X) \to 0.$$

(1): The finiteness of X implies also that $K_a \subset [Y, Y]$ (see 7.3(2)). The homomorphism $\alpha\colon K_a \to H_1(Y)$, in the above exact sequence, is then trivial and $H_1(a)$ is an isomorphism. This proves (1).

(2): According to 7.2, $\mathrm{Hom}(H_1(Y), K_a) = \mathrm{Hom}(Y, K_a) = 0$. The orders of $H_1(Y)$ and K_a are therefore relatively prime numbers. As $H_1(Y)$ and $H_1(X)$ are isomorphic (statement (1)), we get $\mathrm{Ext}^1(H_1(X), K_a) = H_1(Y) \otimes K_a = 0$ which is statement (2).

(3): This is a consequence of the universal coefficient exact sequence and the triviality of $\mathrm{Ext}^1(H_1(X), K_a)$ (statement (2)).

(4): This follows from the exact sequence of the central extension $a\colon Y \twoheadrightarrow X$ above and the triviality of $H_1(Y) \otimes K_a$ (statement (2)).

(5): We have $Y = ZK_a$ and since K_a is central in Y we get that $[Z, Z] = [Y, Y]$. However, as we observed earlier in the proof, $K_a \subset [Y, Y]$ and it follows that $K_a \subset Z$ and so $Y = Z$. □

The differential of a surjective generalized subgroup. If $a\colon Y \twoheadrightarrow X$ is a surjective generalized subgroup, then according to 10.2(3), the homomorphism $\mu\colon H^2(X, K_a) \to \mathrm{Hom}(H_2(X), K_a)$ is an isomorphism. The extension $a\colon Y \twoheadrightarrow X$, which is an element of $H^2(X, K_a)$, can be then identified with the homomorphism $\mu(a)\colon H_2(X) \to K_a$. According to 10.2(4) such homomorphisms associated with generalized subgroups are surjections. Furthermore, as $H_1(X) \otimes K_a = 0$ (see 10.2(2)), the primes that divide the order of $H_1(X)$ do not divide the order of K_a. This means that the localization $K_a \to K_a[S^{-1}]$ is an isomorphism, where S is the set of primes that divide the order of $H_1(X)$. Consequently $\mu(a)\colon H_2(X) \twoheadrightarrow K_a$ factors uniquely as:

$$\begin{tikzcd} H_2(X) \arrow[r, "\text{localization}"] \arrow[rr, bend right, two heads, "\mu(a)"'] & H_2(X)[S^{-1}] = H_{2\backslash 1}X) \arrow[r, two heads] & K_a \end{tikzcd}$$

We will use the same symbol $\mu(a)\colon H_{2\backslash 1}(X) \twoheadrightarrow K_a$ to denote the surjection in the above factorization. We can now define:

10.3. **Definition.** Let $a\colon Y \twoheadrightarrow X$ be a surjective generalized subgroup of X. The element in $\mathrm{Quot}(H_{2\backslash 1}(X))$ represented by the surjection $\mu(a)\colon H_{2\backslash 1}(X) \twoheadrightarrow K_a$ is called the differential of a and is denoted also by the same symbol $\mu(a)$.

Assume now that $a\colon Y \twoheadrightarrow X$ and $b\colon Z \twoheadrightarrow X$ are equivalent surjective generalized subgroups of X and $h\colon Y \to Z$ is an isomorphism for which $bh = a$. By the naturality of the exact sequence of a central extension, we get a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H_2(Y) \arrow[r, "H_2(a)"] \arrow[d, "H_2(h)"'] & H_2(X) \arrow[r, "\mu(a)"] \arrow[d, "\mathrm{id}"] & K_a \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & H_2(Z) \arrow[r, "H_2(b)"] & H_2(X) \arrow[r, "\mu(b)"] & K_b \arrow[r] & 0
\end{tikzcd}$$

After localizing with respect to the set S of primes that divide the order of $H_1(X)$, we get then that $\mu(b)\colon H_{2\backslash 1}(X) \twoheadrightarrow K_b$ is the composition of $\mu(a)\colon H_{2\backslash 1}(X) \twoheadrightarrow K_a$ and the isomorphism $h\colon K_a \to K_b$. The surjections $\mu(a)$ and $\mu(b)$ are thus equivalent and define the same element in $\mathrm{Quot}(H_{2\backslash 1}(X))$. It follows that the differential is well defined on the collection $\mathrm{SurSub}(X)$ of equivalence classes of generalized subgroups. In this way we get a well-defined function denoted by μ:

$$\mathrm{SurSub}(X) \ni [a\colon Y \to X] \xmapsto{\quad\mu\quad} \mu(a) \in \mathrm{Quot}(H_{2\backslash 1}(X)).$$

Next we define its inverse, which we denote by $\Psi\colon \mathrm{Quot}(H_{2\backslash 1}(X)) \to \mathrm{SurSub}(X)$. Let us choose a surjection $\sigma\colon H_{2\backslash 1}(X) \twoheadrightarrow K$ that represents a given element in $\mathrm{Quot}(H_{2\backslash 1}(X))$. Recall that $e_X\colon E \twoheadrightarrow X$ denotes the initial central extension of X by $H_{2\backslash 1}(X)$ (see Section 9). Define $a\colon Y \twoheadrightarrow X$ to be the homomorphism that fits into the following commutative diagram where the left square is a push-out square:

$$\begin{tikzcd}
H_{2\backslash 1}(X) \arrow[r, hook] \arrow[d, two heads, "\sigma"'] & E \arrow[r, two heads, "e_X"] \arrow[d, "\pi"] & X \arrow[d, equal, "\mathrm{id}"] \\
K \arrow[r, hook] & Y \arrow[r, two heads, "a"] & X
\end{tikzcd}$$

Note that Y is isomorphic to the quotient of E by the kernel of $\sigma\colon H_{2\backslash 1}(X) \twoheadrightarrow K$. The homomorphism $a\colon Y \twoheadrightarrow X$ represents a central extension of X by K, as K is in the center of Y. The above diagram therefore leads, by the naturality of the exact sequence of a central extension, to a commutative diagram of homology groups:

$$\begin{tikzcd}
H_2(X) \arrow[r, "\mu(e_X)"] \arrow[d, "\mathrm{id}"'] & H_{2\backslash 1}(X) \arrow[r] \arrow[d, two heads, "\sigma"] & H_1(E) \arrow[r, "H_1(e_X)"] \arrow[d] & H_1(X) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
H_2(X) \arrow[r, "\mu(a)"] & K \arrow[r, "\alpha"] & H_1(Y) \arrow[r, "H_1(a)"] & H_1(X) \arrow[r] & 0
\end{tikzcd}$$

As $\mu(e_X)$ and σ are surjections, then so is $\mu(a)$. The homomorphism α is therefore trivial and consequently $H_1(a)\colon H_1(Y) \to H_1(X)$ is an isomorphism. Since K is a quotient of $H_{2\backslash 1}(X)$, the primes that divide the order of $H_1(X)$ do not divide the order of K. It follows that $\mathrm{Hom}(X, K) = \mathrm{Hom}(H_1(X), K) = 0$. Thus according to 7.2, $a\colon Y \twoheadrightarrow X$ is a generalized subgroup of X. We define $\Psi([\sigma])$ to be the element of $\mathrm{SurSub}(X)$ given by the equivalence class represented by this surjective generalized subgroup $a\colon Y \twoheadrightarrow X$. It is straight forward to check that $\Psi([\sigma])$ does not depend on the choice of a surjection $\sigma\colon H_{2\backslash 1}(X) \twoheadrightarrow K$ representing the given element in $\mathrm{Quot}(H_{2\backslash 1}(X))$. In this way we have a well defined function $\Psi\colon \mathrm{Quot}(H_{2\backslash 1}(X)) \to \mathrm{SurSub}(X)$. Note further that $\sigma = \mu(a)$. This means that $\mu\Psi = \mathrm{id}$.

To show that $\Psi\mu$ is also the identity, let us choose a surjective generalized subgroup $a\colon Y \twoheadrightarrow X$. Let $b\colon Z \twoheadrightarrow X$ be a surjective generalized subgroup representing

$\Psi\mu(a)$. We need to show that a and b are equivalent. Since $\mu\Psi$ is the identity:

$$\mu(b) = \mu\Psi\mu(a) = \mu(a)$$

This means that the differential $\mu(b)$ is equivalent to $\mu(a)$. As in this case the differential determines the central extension it comes from, a and b are indeed equivalent generalized subgroups. We just have shown:

10.4. **Theorem.** *If X is a finite group, then the differential $\mu\colon \mathrm{SurSub}(X) \to \mathrm{Quot}(H_{2\backslash 1}(X))$ is a bijection.*

10.5. To summarize: any surjective generalized subgroup $a\colon Y \twoheadrightarrow X$ of a finite group X fits into the following commutative ladder of short exact sequences with the left square being a push-out:

$$\begin{tikzcd} H_{2\backslash 1}(X) \arrow[r, hook] \arrow[d, two heads, "\mu(a)"'] & E \arrow[r, two heads, "e_X"] \arrow[d, two heads, "\pi"] & X \arrow[d, equal, "\mathrm{id}"] \\ K_a \arrow[r, hook] & Y \arrow[r, two heads, "a"] & X \end{tikzcd}$$

Theorem 10.4 can be used to enumerate all the generalized subgroups of G.

10.6. **Definition.** Define $\mathrm{In}(X)$ to be the set of pairs (I, σ) where I is a subgroup of X and $\sigma \in \mathrm{Quot}(H_{2\backslash 1}(I))$.

As a corollary of 10.4 we get:

10.7. **Corollary.** *For a finite group X, the following function is a bijection between $\mathrm{Sub}(X)$ and $\mathrm{In}(X)$:*

$$\mathrm{Sub}(G) \ni [a\colon Y \to X] \longmapsto (I_a, \mu(a\colon Y \twoheadrightarrow I_a)) \in \mathrm{In}(G)$$

Since $\mathrm{Cov}(G)$ is a subcollection of $\mathrm{Sub}(G)$ and $\mathrm{In}(G)$ is finite, 10.7 implies:

10.8. **Corollary.** *For a finite group X, the collections $\mathrm{Sub}(X)$, $\mathrm{Cov}(X)$, and $\mathrm{Idem}(X)$ are finite sets.*

11. Surjective cellular covers of finite groups

Let X be a finite group. Now that we know $\mathrm{Idem}(X)$ is a fine set, we attempt to enumerate it. We use the same strategy as in the case of generalized subgroups in the previous section. According to 10.7, a generalized subgroup of X is determined by two invariants: its image and its differential. For a given subgroup $I \subset X$ the differential classifies all possible generalized subgroups of X whose image is I, or equivalently generalized subgroups of I which are represented by surjective homomorphisms $Y \twoheadrightarrow I$. Thus to classify covers of X we need to determine first the subgroups of X which are images of covers and then, for any such subgroup I, identify these surjective generalized subgroups of I which are covers of X. Unfortunately we can not say much about the first step in this process (see comment after Theorem 6.2). We do not know how to identify subgroups of X which are images of covers. However we can deal with the second step: the covers of finite groups which are represented by surjective homomorphisms in several important cases. This is the aim of this section.

11.1. **Definition.** $\mathrm{SurCov}(X)$ denotes the subcollection of $\mathrm{Cov}(X)$ consisting of covers represented by surjective homomorphisms.

$H_{2\backslash 1}$ as a functor. To describe surjective covers we use functorial properties of $H_{2\backslash 1}$ (see 9.2). Any homomorphism $f\colon Y \to X$ induces a homomorphism of the Schur multipliers $H_2(f)\colon H_2(Y) \to H_2(X)$ (the Schur multiplier is a functor). In general this homomorphism $H_2(f)$ however does not induce a homomorphism between $H_{2\backslash 1}(Y)$ and $H_{2\backslash 1}(X)$. For that we need to assume that both Y and X are finite and that the set S_Y of primes that divide the order of $H_1(Y)$ is a **subset** of the set S_X of primes that divide the order of $H_1(X)$. In this case there is a unique homomorphism:

$$H_{2\backslash 1}(f)\colon H_{2\backslash 1}(Y) = H_2(Y)[S_Y^{-1}] \to H_2(X)[S_X^{-1}] = H_{2\backslash 1}(X)$$

for which the following diagram commutes:

$$\begin{tikzcd} H_2(Y) \arrow[r, "\text{localization}"] \arrow[d, "H_2(f)"'] & H_2(Y)[S_Y^{-1}] \arrow[r, equal] & H_{2\backslash 1}(Y) \arrow[d, "H_{2\backslash 1}(f)"] \\ H_2(X) \arrow[r, "\text{localization}"] & H_2(X)[S_X^{-1}] \arrow[r, equal] & H_{2\backslash 1}(X) \end{tikzcd}$$

This is because $H_{2\backslash 1}(X)$ is uniquely divisible by the primes in S_Y. Observe further the uniqueness implies $H_{2\backslash 1}(fg) = H_{2\backslash 1}(f)H_{2\backslash 1}(g)$, for any two homomorphisms $g\colon Z \to Y$ and $f\colon Y \to X$ for which the inclusions $S_Z \subset S_Y \subset S_X$ of sets of prime numbers that divide the corresponding orders of the abelianizations hold.

We conclude that $H_{2\backslash 1}$ is a functor on the full subcategory of finite groups with a fixed isomorphism type of H_1. For example let Y and X be finite groups for which there is a surjective homomorphism $c\colon Y \twoheadrightarrow X$ which is a generalized subgroup. By 10.2(1), $H_1(Y)$ and $H_1(X)$ are isomorphic. Thus, for such groups Y and X, any homomorphism $f\colon Y \to X$ induces $H_{2\backslash 1}(f)\colon H_{2\backslash 1}(Y) \to H_{2\backslash 1}(X)$.

Since X is finite, according to 10.2(4), a surjective generalized subgroup $c\colon Y \twoheadrightarrow X$ induces an exact sequence:

$$0 \to H_2(Y) \xrightarrow{H_2(c)} H_2(X) \xrightarrow{\mu(c)} K_c \to 0.$$

After localization with respect to the set S of primes that divide the order of $H_1(Y) = H_1(X)$ (see 10.2(1)), we get again an exact sequence:

$$0 \to H_{2\backslash 1}(Y) \xrightarrow{H_{2\backslash 1}(c)} H_{2\backslash 1}(X) \xrightarrow{\mu(c)} K_c \to 0.$$

Thus the kernel of the differential $\mu(c)\colon H_{2\backslash 1}(X) \twoheadrightarrow K_c$ is given by the image of $H_{2\backslash 1}(c)\colon H_{2\backslash 1}(Y) \subset H_{2\backslash 1}(X)$ which we simply denote by $H_{2\backslash 1}(Y)$.

To enumerate the set $\mathrm{SurCov}(X)$ of surjective covers of X we need to understand for which surjective generalized subgroups $c\colon Y \twoheadrightarrow X$ the function $\mathrm{Hom}(Y,c)\colon \mathrm{Hom}(Y,Y) \to \mathrm{Hom}(Y,X)$ is a surjection. We start by determining the image of $\mathrm{Hom}(Y,c)$. This image consists of homomorphisms $f\colon Y \to X$ that can be lifted through c and expressed as compositions of some $s\colon Y \to Y$ and $c\colon Y \to X$. The following proposition describes such homomorphisms:

11.2. **Proposition.** *Let $c\colon Y \twoheadrightarrow X$ be a surjective generalized subgroup. A homomorphism $f\colon Y \to X$ can be expressed as a composition of $s\colon Y \to Y$ and $c\colon Y \twoheadrightarrow X$ if and only if the image of $H_{2\backslash 1}(f)$ lies in the image $H_{2\backslash 1}(c)$.*

Proof. Clearly if $f = cs$ for some $s\colon Y \to Y$, then $H_{2\backslash 1}(f) = H_{2\backslash 1}(c)H_{2\backslash 1}(s)$ and hence the image of $H_{2\backslash 1}(f)$ is in the image of $H_{2\backslash 1}(c)$.

Assume that $H_{2\backslash 1}(f)$ is in the image of $H_{2\backslash 1}(c)$. Recall that $c\colon Y \twoheadrightarrow X$ fits into the following commutative diagram where the left bottom square is a push-out square and $e_G\colon E \twoheadrightarrow X$ is the initial extension (see the comment after Theorem 10.4):

$$\begin{tikzcd}
H_{2\backslash 1}(Y) \arrow[r, equal, "\mathrm{id}"] \arrow[d, hook, "H_{2\backslash 1}(c)"'] & H_{2\backslash 1}(Y) \arrow[d, hook] & \\
H_{2\backslash 1}(X) \arrow[r, hook] \arrow[d, two heads, "\mu(c)"'] & E \arrow[r, two heads, "e_G"] \arrow[d, two heads, "\pi"] & X \arrow[d, equal, "\mathrm{id}"] \\
K_c \arrow[r, hook] & Y \arrow[r, two heads, "c"] & X
\end{tikzcd}$$

To prove the proposition we construct the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ker}(\pi) \arrow[r, equal] \arrow[d, "f'"'] & H_{2\backslash 1}(Y) \arrow[r, hook] \arrow[d, "H_{2\backslash 1}(f)"] & E \arrow[r, two heads, "\pi"] \arrow[d, "\bar{f}"] & Y \arrow[d, "s"] \arrow[dr, "f"] & \\
H_{2\backslash 1}(Y) \arrow[r, hook, "H_{2\backslash 1}(c)"] & H_{2\backslash 1}(X) \arrow[r, hook] & E \arrow[r, two heads, "\pi"] \arrow[rr, bend right, "e_X"'] & Y \arrow[r, two heads, "c"] & X
\end{tikzcd}$$

Since $e_X\colon E \twoheadrightarrow X$ is a cover (see 9.3(4)), there is a unique $\bar{f}$ for which $e_X\bar{f} = f\pi$. By the naturality of the differentials we have a commutative square:

$$\begin{tikzcd}
H_2(Y) \arrow[r, two heads, "\mu(\pi)"] \arrow[d, "H_2(f)"'] & H_{2\backslash 1}(Y) \arrow[d, "\bar{f}"] \\
H_2(X) \arrow[r, two heads, "\mu(e_X)"] & H_{2\backslash 1}(X)
\end{tikzcd}$$

This implies that the restriction of $\bar{f}$ to $H_{2\backslash 1}(Y)$ is given by $H_{2\backslash 1}(f)$. By the assumption the image of $H_{2\backslash 1}(f)$ lies in the image of $H_{2\backslash 1}(c)$ which gives the homomorphism f'. As $\bar{f}\colon E \to E$ maps the kernel $\mathrm{Ker}(\pi) \subset E$ to itself, we get the desired homomorphism $s\colon Y \to Y$. $\square$

11.3. **Corollary.** *A surjective generalized subgroup $c\colon Y \twoheadrightarrow X$ is a cover if and only if, for any homomorphism $f\colon Y \to X$, the image of $H_{2\backslash 1}(f)\colon H_{2\backslash 1}(Y) \to H_{2\backslash 1}(X)$ lies in the image of $H_{2\backslash 1}(c)$.*

We can use this corollary to prove:

11.4. **Proposition.** (1) *Let k be a positive divisor of the order of $H_{2\backslash 1}(X)$. If $c\colon Y \twoheadrightarrow X$ is a surjective generalized subgroup whose differential is given by the quotient homomorphism $q_k\colon H_{2\backslash 1}(X) \to H_{2\backslash 1}(X)/(k\text{-torsion})$, then c is a cover.*

(2) *If $H_{2\backslash 1}(X)$ is cyclic, then all surjective generalized subgroups of X are covers.*

Proof. (1): Recall that we have an exact sequence:

$$0 \to H_{2\backslash 1}(Y) \xrightarrow{H_{2\backslash 1}(c)} H_{2\backslash 1}(X) \xrightarrow{\mu(c)} H_{2\backslash 1}(X)/(k\text{-torsion}) \to 0.$$

Thus, $H_{2\backslash 1}(Y)$ is the k-torsion subgroup of $H_{2\backslash 1}(X)$. Let $f\colon Y \to X$ be a homomorphism. Since $H_{2\backslash 1}(Y)$ is k-torsion, the image of $H_{2\backslash 1}(f)\colon H_{2\backslash 1}(Y) \to H_{2\backslash 1}(X)$

lies in the k-torsion subgroup of $H_{2\backslash 1}(X)$ which is the image of $H_{2\backslash 1}(c)$. According to 11.3 a is a cover.

(2): If $H_{2\backslash 1}(X)$ is cyclic, then any of its subgroups is the k-torsion subgroup for some k. Statement (2) follows then from statement (1). $\square$

The action of Out(G)**.** According to 10.4 a surjective generalized subgroup $c\colon Y \twoheadrightarrow X$ is determined by its differential $\mu(c)\colon H_{2\backslash 1}(X) \twoheadrightarrow K_c$ which in turn is determined by its kernel $H_{2\backslash 1}(Y)$. The following functions are bijections:

$$\begin{array}{ccccc} \mathrm{SurSub}(X) & \longrightarrow & \mathrm{Quot}(H_{2\backslash 1}(X)) & \longrightarrow & \mathrm{Sub}(H_{2\backslash 1}(X)); \\ (c\colon Y \twoheadrightarrow X) & \longmapsto & [\mu(c)\colon H_{2\backslash 1}(X) \twoheadrightarrow K_c] & \longmapsto & H_{2\backslash 1}(Y). \end{array}$$

In this way surjective generalized subgroups of X are enumerated by subgroups of $H_{2\backslash 1}(X)$. To enumerate the set $\mathrm{SurCov}(X)$, we need to identify these elements in $\mathrm{Quot}(H_{2\backslash 1}(X))$, or equivalently in $\mathrm{Sub}(H_{2\backslash 1}(X))$, which are differentials of surjective cellular covers. For that we look at the action of $\mathrm{Out}(X)$ on these sets. Let $h\colon X \to X$ be an automorphism. Consider the induced isomorphism $H_2(h)\colon H_2(X) \to H_2(X)$ and its localization $H_{2\backslash 1}(h)\colon H_{2\backslash 1}(X) \to H_{2\backslash 1}(X)$ with respect to the set S of primes that divide the order of $H_1(X)$. The function:

$$\mathrm{Sub}(H_{2\backslash 1}(X)) \times \mathrm{Aut}(X) \ni (H, h) \longmapsto H_{2\backslash 1}(h)^{-1}(H) \in \mathrm{Sub}(H_{2\backslash 1}(X))$$

defines a right action of $\mathrm{Aut}(X)$ on the set $\mathrm{Sub}(H_{2\backslash 1}(X))$ of all subgroups of $H_{2\backslash 1}(X)$. Since inner automorphisms induce the identity on homology, this action induces an action of $\mathrm{Out}(X)$ on $\mathrm{Sub}(H_{2\backslash 1}(X))$.

The corresponding right action of $\mathrm{Out}(X)$ on $\mathrm{Quot}(H_{2\backslash 1}(X))$ can be described as follows. Let $h\colon X \to X$ be an automorphism. For a surjective homomorphism $\sigma\colon H_{2\backslash 1}(X) \twoheadrightarrow K$, the composition $\sigma H_{2\backslash 1}(h)\colon H_{2\backslash 1}(X) \to K$ is also surjective. Note further if $\sigma\colon H_{2\backslash 1}(X) \twoheadrightarrow K$ and $\tau\colon H_{2\backslash 1}(X) \twoheadrightarrow K$ define the same element in $\mathrm{Quot}(H_{2\backslash 1}(X))$, then so do their compositions $\sigma H_{2\backslash 1}(h)$ and $\tau H_{2\backslash 1}(h)$. The following induced function defines a right action of $\mathrm{Out}(X)$ on $\mathrm{Quot}(H_{2\backslash 1}(X))$:

$$\mathrm{Quot}(H_{2\backslash 1}(X)) \times \mathrm{Out}(X) \ni ([\sigma], [h]) \longmapsto \big[\sigma H_{2\backslash 1}(h)\big] \in \mathrm{Quot}(H_{2\backslash 1}(X))$$

Moreover the bijection that assigns to an element σ in $\mathrm{Quot}(H_{2\backslash 1}(X))$ its kernel $\mathrm{Ker}(\sigma)$, which is an element in $\mathrm{Sub}(H_{2\backslash 1}(X))$, is an equivariant isomorphism.

We will be interested in the fixed points of these actions:

11.5. **Definition.** The fixed points of the actions of $\mathrm{Out}(X)$ described above are denoted by:

$$\mathrm{InvQuot}(H_{2\backslash 1}(X)) \subset \mathrm{Quot}(H_{2\backslash 1}(X)) \qquad \mathrm{InvSub}(H_{2\backslash 1}(X)) \subset \mathrm{Sub}(H_{2\backslash 1}(X))$$

The reason we are interested in these fixed points is:

11.6. **Proposition.** *If $c\colon Y \twoheadrightarrow X$ is a surjective cellular cover, then its differential $[\mu(c)\colon H_{2\backslash 1}(X) \twoheadrightarrow K_c] \in \mathrm{Quot}(H_{2\backslash 1}(X))$ and $H_{2\backslash 1}(Y) \in \mathrm{Sub}(H_{2\backslash 1}(X))$ are fixed by the action of* $\mathrm{Out}(X)$.

Proof. Let $h\colon X \to X$ be an automorphism. Since $c\colon Y \twoheadrightarrow X$ is a cover, there is a unique homomorphism $h'\colon Y \to Y$ that fits into the following commutative square on the left which by the naturality induces a commutative square on the right:

$$\begin{tikzcd} Y \arrow[r, "h'"] \arrow[d, "c"'] & Y \arrow[d, "c"] \\ X \arrow[r, "h"] & X \end{tikzcd} \qquad \begin{tikzcd} H_{2\backslash 1}(X) \arrow[r, "H_{2\backslash 1}(h)"] \arrow[d, "\mu(c)"'] & H_{2\backslash 1}(X) \arrow[d, "\mu(c)"] \\ K_c \arrow[r, "h'"] & K_c \end{tikzcd}$$

It follows that $H_{2\backslash 1}(h)$ maps the kernel of $\mu(c)$ to itself. This means that, as an element of $\mathrm{Sub}(H_{2\backslash 1}(X))$, this kernel is invariant under the action of $\mathrm{Out}(X)$. $\square$

We can then use both 11.4 and 11.6 to get:

11.7. **Corollary.** (1) *Let $k > 0$ be a divisor of the exponent of $H_{2\backslash 1}(X)$. Then the k-torsion subgroup of $H_{2\backslash 1}(X)$ is fixed by* $\mathrm{Out}(X)$.
(2) *If $H_{2\backslash 1}(X)$ is cyclic, then the action of* $\mathrm{Out}(X)$ *on the sets* $\mathrm{Sub}(H_{2\backslash 1}(X))$ *and* $\mathrm{Quot}(H_{2\backslash 1}(X))$ *is trivial.*

According to what has been proven we have the following sequence of inclusions:

$$\begin{tikzcd} k \arrow[dd, mapsto] & & \{k \in \mathbb{Z} \mid k > 0 \text{ and } k \text{ divides the order of } H_{2\backslash 1}(X)\} \arrow[d, hook] & \\ & & \mathrm{SurCov}(X) \arrow[dl, hook, "\mu"'] \arrow[dr, hook] & \\ q_k & \mathrm{InvQuot}(H_{2\backslash 1}(X)) \arrow[dr, hook] & & \mathrm{SurSub}(X) \arrow[dl, equal, "\mu"'] \\ & & \mathrm{Quot}(H_{2\backslash 1}(X)) & \end{tikzcd}$$

Proposition 11.4 can be rephrased as:

11.8. **Corollary.** *Let X be a group for which $H_{2\backslash 1}(X)$ is cyclic. Then all the inclusions in the above diagram are bijections. In particular:*

(1) *The differential $\mu\colon \mathrm{SurCov}(X) \to \mathrm{Quot}(H_{2\backslash 1}(X))$ is a bijection.*
(2) *Any surjective generalized subgroup of X is a cellular cover.*
(3) *Let $c\colon Y \twoheadrightarrow X$ be a surjection. Then $\mathrm{Hom}(Y, c)\colon \mathrm{Hom}(Y, Y) \to \mathrm{Hom}(Y, X)$ is a bijection if and only if it is an injection.*

12. Cellular covers of finite simple groups

The aim of this section is to classify cellular covers of finite simple groups. A simple group G has a trivial abelianization and thus $H_{2\backslash 1}(G) = H_2(G)$. According to 11.6 the differential induces an inclusion $\mu : \mathrm{SurCov}(G) \subset \mathrm{InvQuot}(H_2(G))$. Our key result is:

12.1. **Theorem.** *If G is a finite simple group, then $\mu\colon \mathrm{SurCov}(G) \subset \mathrm{InvQuot}(H_2(G))$ is a bijection.*

12.2. **Corollary.** *Let G be a finite simple group. Then the sets* $\mathrm{Cov}(G)$ *and* $\mathrm{Idem}(G)$ *are in bijection with* $\{0\} \coprod \mathrm{InvSub}(H_2(G))$.

Proof. Recall that $\mathrm{Cov}(G)$ and $\mathrm{Idem}(G)$ are in bijection with each other. Let $c\colon Y \to G$ be a cellular cover. Since the image of c is a normal subgroup of G, this image is either the trivial group or the whole G. In the first case Y has to be trivial. In the second case c is a surjective cellular cover of G. Thus according to 12.1 the assignment that maps the trivial cellular cover to the element 0 and a surjective cellular cover $c\colon Y \twoheadrightarrow G$ to the image of $H_2(c) : H_2(Y) \subset H_2(G)$, is the desired bijection between $\mathrm{Cov}(G)$ and $\{0\} \coprod \mathrm{InvSub}(H_2(G))$. ☐

The key property of finite simple groups used to prove the above theorem is:

12.3. **Lemma.** *Let $c\colon Y \twoheadrightarrow G$ be a surjective generalized subgroup of a finite simple group G. Then any non-trivial homomorphism $f\colon Y \to G$ can be expressed as a composition of $c\colon Y \twoheadrightarrow G$ and some automorphism $G \to G$.*

Proof. Let $K_f = \mathrm{Ker}(f)$ and $K_c = \mathrm{Ker}(c)$. Consider the following commutative diagram:

$$\begin{tikzcd}
K_f \cap K_c \arrow[r, hook] \arrow[d, hook] & K_c \arrow[d, hook] & \\
K_f \arrow[r, hook] \arrow[dr, "g"'] & Y \arrow[r, "f"] \arrow[d, "c"] & G \\
& G &
\end{tikzcd}$$

The image of g is a normal subgroup of G. Since G is simple, there are two possibilities. Either g is a surjection or it is the trivial homomorphism.

Assume that g is trivial. In this case K_f is a subgroup of K_c and we have the following commutative diagram:

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "c"'] \arrow[dr] & G \\
G & Y/K_f \arrow[l, two heads] \arrow[u, hook]
\end{tikzcd}$$

Finiteness of G implies that the surjection $Y/K_f \twoheadrightarrow G$ and the injection $Y/K_f \hookrightarrow G$ in the above diagram have to be isomorphisms. We can then use the commutativity of this diagram to conclude that f can be expressed as a composition of $c\colon Y \twoheadrightarrow G$ and some automorphism $G \to G$.

We will show that under the assumption that f is non-trivial, the homomorphism g can not be surjective. Assume to the contrary that g is surjective. In this case we can use 10.2(5) to get equality $K_f = Y$. Consequently the surjectivity of g implies the triviality of f. ☐

Proof of 12.1. Let $c\colon Y \twoheadrightarrow G$ be a surjective generalized subgroup whose differential $\mu(c)\colon H_2(G) \twoheadrightarrow K_c$ represents an element in $\mathrm{InvQuot}(H_2(G))$. We will use 11.3 to prove the theorem. According to 12.3, any non-trivial homomorphism $f\colon Y \to G$ is a composition of $c\colon Y \twoheadrightarrow G$ and an automorphism $h\colon G \to G$. Consequently $H_2(f) = H_2(h)H_2(c)$. As the image of $H_2(c)$ is fixed by the action of $\mathrm{Out}(G)$ on $\mathrm{Sub}(H_2(G))$, we have:

$$\mathrm{image}(H_2(f)) = \mathrm{image}\,(H_2(h)H_2(c)) = \mathrm{image}(H_2(c))$$

By 11.3, $c\colon Y \twoheadrightarrow G$ is then a cellular cover. ☐

We finish this section with:

12.4. **Corollary.** *Let G be a finite simple group and $Y \in \mathrm{Idem}(G)$. Then any non-trivial homomorphism $f\colon Y \to G$ is a cellular cover of G.*

Proof. Let $Y \in \mathrm{Idem}(G)$. Then there is a cover. $c\colon Y \to G$ is a cover. The image of c is a normal subgroup of G. It is then either the trivial group or the whole group G. In the first case Y is the trivial group and there is no non-trivial homomorphisms from Y to G. In the second case we can use 12.3, to conclude that $f\colon Y \to G$ can be expressed as a composition of $c\colon Y \twoheadrightarrow G$ and some automorphism $G \to G$. It is then clear that $\mathrm{Hom}(Y, f)\colon \mathrm{Hom}(Y, Y) \to \mathrm{Hom}(Y, G)$ is a bijection and therefore f is a cellular cover. □

13. Cellularity with respect to finite nilpotent groups

Let A be a finite nilpotent group. Recall that this means that A is a product of finite p-groups for various primes p. Let S be the set of prime numbers that divide the order of A. The set S consists of these prime numbers that divide the order of $H_1(A)$. In this section we determined which finite groups belong to $\mathrm{Cell}(A)$, i.e., which finite groups can be constructed out of A by repeatedly taking colimits.

Let X be a finite group. We now describe how to construct the $\mathrm{Cell}(A)$-cover of X. Consider the homomorphism:

$$\pi = \coprod_{g\colon A\to X} g\colon F = \coprod_{g\colon A\to X} A \longrightarrow X$$

Note that:

- F is A-cellular;
- $\mathrm{Hom}(A, \pi)\colon \mathrm{Hom}(A, F) \to \mathrm{Hom}(A, X)$ is a surjective function.

Let $L \subset F$ be the kernel of π. Our next step is to form the following commutative diagram, where the vertical homomorphisms are the quotient homomorphisms and the left square is a push-out square:

$$\begin{tikzcd}
L \arrow[r, hook] \arrow[d, two heads] & F \arrow[r, two heads, "\pi"] \arrow[d] & X \arrow[d, equal] \\
L/[F,L] \arrow[r, hook] & F/[F,L] \arrow[r, "\pi_1"] & X
\end{tikzcd}$$

Note that:

- $L/[F, L]$ is in the center of $F/[F, L]$ and hence $L/[F, L]$ is abelian.
- $F/[F, L]$ is A-cellular by 4.6.(3).
- $\mathrm{Hom}(A, \pi')\colon \mathrm{Hom}(A, F/[F, L]) \to \mathrm{Hom}(A, X)$ is a surjective function, since $\mathrm{Hom}(A, \pi)$ is a surjection.
- π is an epimorphism if and only if π_1 is an epimorphism.

Let $T \subset L/[F, L]$ be the subgroup generated by all elements of order p^n for any $n \geq 1$ and any p in S. Since, for any p in S, the cyclic group $\mathbb{Z}/p$ is a quotient of A, then for any n, the group $\mathbb{Z}/p^n$ belongs to the smallest collection $\overline{\mathrm{Cell}(A)}$ of groups containing $\mathrm{Cell}(A)$ and closed under colimits, extensions and quotients. It follows that T is also a member of $\overline{\mathrm{Cell}(A)}$. In the next step set $K := (L/[F, L])/T$ and $Y := (F/[F, L])/T$ and extend the above diagram further to the following commutative diagram, where the vertical homomorphisms are the quotient homomorphisms and the left squares are push-out squares:

(2)
$$\begin{tikzcd}
L \arrow[r, hook] \arrow[d, two heads] & F \arrow[r, two heads, "\pi"] \arrow[d, two heads] & X \arrow[d, equal] \\
L/[F,L] \arrow[r, hook] \arrow[d, two heads] & F/[F,L] \arrow[r, "\pi_1"] \arrow[d, two heads] & X \arrow[d, equal] \\
K \arrow[r, hook] & Y \arrow[r, "f"] & X
\end{tikzcd}$$

Note that again:

- K is in the center of Y and hence K is abelian. Moreover K has no elements whose order is a power of a prime in S.
- Since T belongs to $\overline{\mathrm{Cell}(A)}$ and $F/[F,L]$ to $\mathrm{Cell}(A)$, according to 4.6.(6), the group $Y = (F/[F,L])/T$ is A-cellular.
- $\mathrm{Hom}(A, f)\colon \mathrm{Hom}(A, Y) \to \mathrm{Hom}(A, X)$ is a surjective function, since the map $\mathrm{Hom}(A, \pi')$ is a surjection.
- $\mathrm{Hom}(A, K) = 0$. This follows from the fact K has no elements whose order is a power of a prime in S.
- f is an epimorphism if and only if π is an epimorphism.
- $\mathrm{Hom}(Y, K) = 0$. This follows from the fact that Y is A-cellular.

13.1. **Proposition.** *The homomorphism $f\colon Y \to X$ is the* $\mathrm{Cell}(A)$*-cover.*

Proof. We already know that Y is A-cellular and $\mathrm{Hom}(A, f)$ is a surjection. We need to show that $\mathrm{Hom}(A, f)$ is also an injection. This is a consequence of the fact that $\mathrm{Hom}(A, K) = 0$ and that $K \subset Y$ is a central subgroup. □

We can then conclude that X belongs to $\mathrm{Cell}(A)$ if and only if:

(1) f is an epimorphism.
(2) $K = 0$.

The first of the above requirements is equivalent to π being an epimorphism, which can be rephrased as X being a quotient of a coproduct of A's. Let us assume that about X. In this case we get an exact sequence $L \subset F \twoheadrightarrow X$. We can then use Hopf Theorem 9.1 to obtain an exact sequence of homologies:

$$H_2(F) \xrightarrow{H_2(\pi)} H_2(X) \longrightarrow L/[F,L] \longrightarrow H_1(F).$$

Observe that the second requirement for X being in $\mathrm{Cell}(A)$ is equivalent to $L/[F,L]$ being generated by elements of order p^n for $n \geq 1$ and p in S. Since $H_2(F) = \bigoplus_{g\colon A \to X} H_2(A)$ is also generated by elements of order p^n for $n \geq 1$ and p in S, we can conclude that if X belongs to $\mathrm{Cell}(A)$, then $H_2(X)$ is generated by elements of order p^n for $n \geq 1$ and p in S. On the other hand if $H_2(X)$ is generated by elements of order p^n for $n \geq 1$ and p in S, then from the above exact sequence $L/[F,L]$ would also have the same property. We can thus conclude that this second requirement is equivalent to $H_2(X)$ having only torsion from S. We have just proved R. Flores type of theorem:

13.2. **Theorem.** *Let A be a finite and nilpotent group. Then a finite group X belongs to* $\mathrm{Cell}(A)$ *if and only if it is a quotient of a coproduce of A's and the primes that divide the order of $H_2(X)$ have to divide the order of A.*

14. The covers of finite simple groups

In this section we will illustrate Theorem 12.1 and its Corollary 12.2. We let G denote a finite simple group. We use the symbol $\exp(H_2(G))$ to denote the exponent of $H_2(G)$ and $\sigma_0(G)$ the number of different positive divisors of $\exp(H_2(G))$. Recall that the exponent of a finite abelian group A is the least positive integer k for which $kA = 0$. Let $e_G\colon E \twoheadrightarrow G$ be the initial extension of G which coincides with the universal central extension of G. The center of E is isomorphic to $H_2(G)$.

According to 12.2, $\mathrm{Idem}(G)$ is in bijection with the set $\{0\} \coprod \mathrm{InvQuot}(H_2(G))$. Explicitly, the element 0 corresponds to the trivial group in $\mathrm{Idem}(G)$. Any non-trivial element in $\mathrm{Idem}(G)$ is the quotient of E by an $\mathrm{Out}(G)$-invariant subgroup of its center $H_2(G)$. A basic example of such a subgroup is given by the k-torsion subgroup for some k dividing the exponent $\exp(H_2(G))$ of $H_2(G)$ (see 11.7(1)). The number of such basic invariant subgroups of $H_2(G)$ is therefore given by $\sigma_0(G)$. Thus the set $\mathrm{Idem}(G)$ contains at least $\sigma_0(H_2(G)) + 1$ elements. The question is if there are any other invariant subgroups of $H_2(G)$? For example in the case $H_2(G)$ is cyclic, since the action of $\mathrm{Out}(G)$ on $\mathrm{Sub}(H_2(G))$ is trivial (see 11.7), all the subgroups of $H_2(G)$ are invariant. In this case the set $\mathrm{Idem}(G)$ has exactly $\sigma_0(H_2(G))+1$ elements. It turns out that this happens for almost all simple groups. The only exceptions are the groups $D_n(q)$ ($n \geq 3$) for q odd and n even. In this case the Schur multiplier is $\mathbb{Z}/2\oplus\mathbb{Z}/2$ and hence its exponent is 2 and consequently $\sigma_0(D_n(q)) = 2$. However, the number of invariant subgroups in the Schur multiplier turns out to be 3 and hence $\mathrm{Idem}(D_n(q))$ has 4 elements.

14.1. **Proposition.** *The following table lists the size of* $\mathrm{Idem}(G)$ *(fifth column) for all finite simple groups G. In the first column the boxed entries contain the names of the groups with restrictions on relevant indices that are required for the groups to be simple or to avoid some repetition. The notation is taken from* [13]. *The constraints below the boxes distinguish between different Schur multipliers. Schur multipliers are the content of the second column. The third column contains* $\exp(H_2(G)$ *and the forth $\sigma_0(G)$. In the first column, we write* $\bullet\boxed{G}\bullet$ *to denote these groups G for which the Schur multiplier is not cyclic and yet* $\mathrm{Idem}(G)$ *has $\sigma_0(H_2(G))+1$ elements. We use* $\bullet\bullet\boxed{G}\bullet\bullet$ *to denote the cases for which* $\mathrm{Idem}(G)$ *has more than $\sigma_0(H_2(G)) + 1$ elements.*

G	$H_2(G)$	$\exp(H_2(G))$	$\sigma_0(G)$	$\lvert\mathrm{Idem}(G)\rvert$
Cyclic groups of prime order				
$\boxed{\mathbb{Z}/p}$	0	1	1	2
Alternating groups				
$\boxed{A_n,\ n \geq 5}$ $n \neq 6,\ n \neq 7$	$\mathbb{Z}/2$	2	2	3
$\boxed{A_6,\ A_7}$	$\mathbb{Z}/6$	6	4	5

G	$H_2(G)$	$\exp(H_2(G))$	$\sigma_0(G)$	$\|\mathrm{Idem}(G)\|$
Linear groups				
$\boxed{\begin{array}{l} A_n(q),\ n \geq 1 \\ (n,q) \neq (1,2) \\ (n,q) \neq (1,3) \end{array}}$ $(n,q) \neq (1,4)$ $(n,q) \neq (1,9)$ $(n,q) \neq (2,2)$ $(n,q) \neq (2,4)$ $(n,q) \neq (3,2)$	$\mathbb{Z}/(n+1, q-1)$	$(n+1, q-1)$	σ_0	$\sigma_0 + 1$
$\boxed{A_3(2)}$	$\mathbb{Z}/2$	2	2	3
$\bullet\ \boxed{A_2(4)}\ \bullet$	$\mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$	12	6	7
Unitary groups				
$\boxed{\begin{array}{l} {}^2A_n(q),\ n \geq 2 \\ (n,q) \neq (2,2) \end{array}}$ $(n,q) \neq (3,2)$ $(n,q) \neq (3,3)$ $(n,q) \neq (5,2)$	$\mathbb{Z}/(n+1, q+1)$	$(n+1, q+1)$	σ_0	$\sigma_0 + 1$
$\boxed{{}^2A_3(2)}$	$\mathbb{Z}/2$	2	2	3
$\bullet\ \boxed{{}^2A_3(3)}\ \bullet$	$\mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$	12	6	7
$\bullet\ \boxed{{}^2A_5(2)}\ \bullet$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$	6	4	5
Suzuki Groups				
$\boxed{{}^2B_2(2^{2n+1}),\ n \geq 1}$ $n > 1$	0	1	1	2
$\bullet\ \boxed{{}^2B_2(8)}\ \bullet$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	2	2	3
Symplectic groups				
$\boxed{C_n(q),\ n \geq 3,\ q \text{ odd}}$	$\mathbb{Z}/2$	2	2	3

G	$H_2(G)$	$\exp(H_2(G))$	$\sigma_0(G)$	$\lvert\mathrm{Idem}(G)\rvert$
Orthogonal groups of type B				
$\boxed{B_n(q),\ n \geq 2\ (n,q) \neq (2,2)}$ q *odd* $(n,q) \neq (3,3)$	$\mathbb{Z}/2$	2	2	3
$\boxed{B_3(3)}$	$\mathbb{Z}/6$	6	4	5
$\boxed{B_n(q),\ n \geq 2\ (n,q) \neq (2,2)}$ q *even,* $(n,q) \neq (3,2)$	0	1	1	2
$\boxed{B_3(2)}$	$\mathbb{Z}/2$	2	2	3
Orthogonal groups of type D				
$\boxed{D_n(q),\ n \geq 4}$ q *even,* $(n,q) \neq (4,2)$	0	1	1	2
$\bullet$ $\boxed{D_4(2)}$ $\bullet$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	2	2	3
$\boxed{D_n(q),\ n \geq 5}$ q *odd,* n *odd*	$\mathbb{Z}/4$	4	3	4
$\bullet\bullet$ $\boxed{D_n(q),\ n \geq 4}$ $\bullet\bullet$ q *odd,* n *even*	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	2	2	4
$\boxed{{}^2D_n(q),\ n \geq 4}$ q *even*	0	1	1	2
$\boxed{{}^2D_n(q),\ n \geq 4}$ q *odd*	$\mathbb{Z}/2$	2	2	3

G	$H_2(G)$	$\exp(H_2(G))$	$\sigma_0(G)$	$\|\mathrm{Idem}(G)\|$
Exceptional groups of Lie type				
$\boxed{G_2(q),\ q \neq 2}$ $q \neq 3,\ q \neq 4$	0	1	1	2
$\boxed{G_2(3)}$	$\mathbb{Z}/3$	3	2	3
$\boxed{G_2(4)}$	$\mathbb{Z}/2$	2	2	3
$\boxed{{}^2G_2(3^{2n+1}),\ n \geq 1}$	0	1	1	2
$\boxed{F_4(q)}$ $q \neq 2$	0	1	1	2
$\boxed{F_4(2)}$	$\mathbb{Z}/2$	2	2	3
$\boxed{{}^2F_2(2^{2n+1}),\ n \geq 1}$	0	1	1	2
$\boxed{{}^2F_2(2)'}$	0	1	1	2
$\boxed{F_4(2)}$ $q \not\equiv 1\ mod\ 3$	0	1	1	2
$\boxed{E_6(q)}$ $q \equiv 1\ mod\ 3$	$\mathbb{Z}/3$	3	2	3
$\boxed{{}^2E_6(q)}$ $q \not\equiv -1\ mod\ 3$	0	1	1	2
$\boxed{{}^2E_6(q)}$ $q \equiv -1\ mod\ 3,$ $q \neq 2$	$\mathbb{Z}/3$	3	2	3
$\bullet$ $\boxed{{}^2E_6(2)}$ $\bullet$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$	6	4	5
$\boxed{E_7(q)}$ $q\ even$	0	1	1	2
$\boxed{E_7(q)}$ $q\ odd$	$\mathbb{Z}/2$	2	2	3
$\boxed{E_8(q)}$	0	1	1	2

G	$H_2(G)$	$\exp(H_2(G))$	$\sigma_0(G)$	$\lvert\mathrm{Idem}(G)\rvert$
Sporadic groups				
$\boxed{M_{11}}$	0	1	1	2
$\boxed{M_{12}}$	$\mathbb{Z}/2$	2	2	3
$\boxed{M_{22}}$	$\mathbb{Z}/12$	12	6	7
$\boxed{M_{23},\ M_{24},\ J_1}$	0	1	1	2
$\boxed{J_2}$	$\mathbb{Z}/2$	2	2	3
$\boxed{J_3}$	$\mathbb{Z}/3$	3	2	3
$\boxed{J_4}$	0	1	1	2
$\boxed{HS}$	$\mathbb{Z}/2$	2	2	3
$\boxed{He}$	0	1	1	2
$\boxed{Mc}$	$\mathbb{Z}/3$	3	2	3
$\boxed{Suz}$	$\mathbb{Z}/6$	6	4	5
$\boxed{Ly}$	0	1	1	2
$\boxed{Ru}$	$\mathbb{Z}/2$	2	2	3
$\boxed{O'N}$	$\mathbb{Z}/3$	3	2	3
$\boxed{Co_1}$	$\mathbb{Z}/2$	2	2	3
$\boxed{Co_2,\ Co_3}$	0	1	1	2
$\boxed{Fi_{22}}$	$\mathbb{Z}/6$	6	4	5
$\boxed{Fi_{23}}$	0	1	1	2
$\boxed{Fi'_{24}}$	$\mathbb{Z}/3$	3	2	3
$\boxed{F_5,\ F_3}$	0	1	1	2
$\boxed{F_2}$	$\mathbb{Z}/2$	2	2	3
$\boxed{F_1}$	0	1	1	2

Proof. The proposition can be derived by elementary arguments from [14, 6.3.1]. For self containment we present these elementary arguments below.

We need to explain the table only for the groups whose Schur multiplier is not cyclic. There are just seven such cases.

Cases: ${}^2\mathrm{A}_5(2)$, ${}^2\mathrm{B}_2(8)$, $\mathrm{D}_4(2)$, ${}^2\mathrm{E}_6(2)$. In all of these cases the Schur multiplier is of the form $\mathbb{Z}/2\oplus\mathbb{Z}/2$ or $\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$. To prove the proposition we need to find all the invariant subgroups of the 2-torsion part. In all of these cases, according to [14, 6.3.1], the group $\mathrm{Out}(G)$ contains $\mathbb{Z}/3$ which acts faithfully on the 2-torsion part $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. However if $\psi : \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is an automorphism such that $\psi \neq \mathrm{id}$ and $\psi^3 = \mathrm{id}$, then ψ has no eigenvectors. Consequently the only subgroups of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ which are invariant under ψ are the trivial subgroup and the whole group.

Case: $\mathrm{A}_2(4)$. In this case the Schur multiplier is isomorphic to $\mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3$. To prove the proposition we need to understand the invariant subgroups of the 4-torsion part $V := \mathbb{Z}/4 \oplus \mathbb{Z}/4$. Again according to [14, 6.3.1], the group $\mathrm{Out}(\mathrm{A}_2(4))$ contains $\mathbb{Z}/3$ which acts faithfully on $\mathbb{Z}/4 \oplus \mathbb{Z}/4$. Let $\psi\colon V \to V$ be an automorphism of order 3. We claim that the only ψ-invariant subgroups in V are the trivial subgroup, the Frattini subgroup $\Phi \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and the whole group V. We have $V = [V, \psi] \oplus C_V(\psi)$ and so if $C_V(\psi)$ is non-trivial, then $[V, \psi] \cong \mathbb{Z}/4$ contradicting the fact that ψ is faithful on $[V, \psi]$. Hence $C_V(\psi)$ is trivial. Let K be a ψ-invariant subgroup. If $|K| = 2$ or 4 and $K \neq \Phi$, then K is cyclic and hence it is centralized by ψ, a contradiction. If $|K| = 8$, then $K \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$, so ψ centralizes the Frattini subgroup of K, a contradiction.

Case: ${}^2\mathrm{A}_3(3)$. In this case the Schur multiplier is isomorphic to $\mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$. To prove the proposition we need to understand the invariant subgroup of the 3-torsion part. According to [14, 6.3.1], the group $\mathrm{Out}({}^2\mathrm{A}_3(3))$ contains $\mathbb{Z}/4$ which acts faithfully on $V := \mathbb{Z}/3 \oplus \mathbb{Z}/3$. Let $\psi\colon V \to V$ be an automorphism of order 4 acting faithfully. If there would be a proper non-trivial ψ-invariant subspace $W \subset V$, then V would split as a direct sum $V = W \oplus U$ with U ψ-invariant. But then ψ^2 would centralize V, a contradiction. Hence the only ψ-invariant subspaces are the trivial one and V.

Case: $\mathrm{D}_n(q)$, $n \geq 4$, q odd, n even. In this case the Schur multiplier is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. According to [14, 6.3.1], after an appropriate choice of a base, automorphisms of $\mathrm{D}_n(q)$ act on the Schur multiplier $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ either as the identity or the transposition. Moreover there is an element that does act as a transposition. It follows that, with this choice of a base, the invariant subgroups are: the trivial subgroup, the diagonal, and the whole group. □

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Homotopy invariant commutative algebra

HOMOTOPY INVARIANT COMMUTATIVE ALGEBRA

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Abstract. These lectures give some basic homotopy invariant definitions in commutative algebra and illustrate their interest by giving a number of examples.

Contents

I am grateful to the CRM and the Simons Foundation for the opportunity to spend time in the stimulating environment at the CRM during the IRTATCA programme, and to give these lectures.

Part 1. **Context**

1. Introduction

1.A. **The lectures.** The purpose of these lectures is to illustrate how powerful it can be to formulate ideas of commutative algebra in a homotopy invariant form. The point is that it shows the concepts are robust, in the sense that they are invariant under deformations. They can then be applied to derived categories of rings, but of course derived categories of rings are just one example of the homotopy category of a model category. We will focus on the fact that this includes many striking new examples from topology.

Indeed, our principal example will be that arising from a topological space X, which is to say we will work over the ring $R = C^*(X; k)$ of cochains on a space X with coefficients in a field k. Of course this is a rather old idea, applied in rational homotopy theory to give a very rich theory and many interesting examples. The key is that R needs to be commutative: classically, one needs to work over the rationals so that there is a commutative model for cochains (coming from piecewise polynomial differential forms). It is well known that there is no natural model for $C^*(X; k)$ which is a commutative differential graded algebra (commutative DGA) if k is not of characteristic 0, since the Steenrod operations (which do not usually vanish) are built out of the non-commutativity. However, there is a way round this, since we can relax our requirements for what we consider a model. Instead of requiring DGAs, we permit R to be a commutative ring spectrum in the sense of homotopy theory (see Appendix A). There is then a commutative model for cochains; we will continue to use the notation $C^*(X;k)$ even though it is now a ring spectrum rather than a DGA.

This may be our principal example, but actually it is a very restricted one. It is like restricting commutative rings to be negatively graded k-algebras. We will look at some other types of spectra, particularly the chromatic ones coming from derived algebraic geometry and those concerned with descent.

The main point of the lectures is to give some interesting examples, and the main source of examples will be modular representation theory and group cohomology. In fact, the focus will be on examples picked out by requiring the ring spectrum to be well behaved in terms of commutative algebra: we look for regular, complete intersection or Gorenstein ring spectra. As one expects from commutative algebra, regular ring spectra are very restricted and can sometimes even be classified. One also expects to be able to parametrise complete intersections in some sense. However, at present we only have a range of examples of hypersurfaces and complete intersections and the full picture for complete intersections is poorly understood. Finally, just as is the case for conventional commutative rings, the class of Gorenstein ring spectra has proved ubiquitous. We shall see examples from representation theory, from chromatic stable homotopy theory and from rational homotopy theory, and one of the distinctive contributions is to emphasize duality.

1.B. **Sources.** Much of this account comes from joint work [25, 26, 39, 40, 62, 63, 73] and other conversations with D. Benson, W.G. Dwyer, K. Hess, S.B. Iyengar, S. Shamir and V. Stojanoska and I would like to thank them for their collaboration and inspiration.

It is worth emphasizing that this is a vast area, and these lectures will just focus on the tiny part of it I have been directly involved with. In particular we will not touch on the Galois theory of J. Rognes, on Brauer groups, or on derived algebraic geometry more generally.

1.C. **Overview.** The point of these lectures is to show that homotopy invariant commutative algebra can be useful in classical algebra, in representation theory of groups, in classical algebraic topology, in chromatic stable homotopy theory, ...

This version of the notes was prepared in advance of the lectures, and the author failed to allow enough time for this process. Some parts are therefore currently either sketchy or completely absent, and the structure, cross referencing and proofreading of the article are poor. It is very likely there are still mistakes and missing hypotheses. I intend to repair these deficiencies in the final version. For the present, I offer my apologies, and hope that this unpolished version may still be of some use.

1.D. **Background.** We summarize some relevant background, but commutative algebraists may find the articles [60] and [61] from the 2004 Chicago Workshop useful.

General background in topology may come from [101] and in conventional commutative algebra from [35, 99].

1.E. **Conventions.** We'll generally denote rings and ring spectra by letters like, Q, R and S. Ring homomorphisms will generally go in reverse alphabetical order, as in $S \longrightarrow R \longrightarrow Q$. Modules will be denoted by letters like $L, M, N, \ldots$.

The ring of integers (initial amongst conventional rings) is denoted $\mathbb{Z}$. The sphere spectrum (initial amongst ring spectra) is denoted $\mathbb{S}$.

After the inital sections we will use the same notation for a conventional ring and the associated Eilenberg-MacLane spectrum. Similarly for modules.

Generally, $M \otimes_R N$ will denote the left derived tensor product, and $\mathrm{Hom}_R(M, N)$ the module of derived homomorphisms. Similarly, fibre sequences, cofibre sequences, pullbacks and pushouts will be derived.

1.F. **Grading conventions.** We will have cause to discuss homological and cohomological gradings. Our experience is that this a frequent source of confusion, so we adopt the following conventions. First, we refer to lower gradings as *degrees* and upper gradings as *codegrees*. As usual, one may convert gradings to cogradings via the rule $M_n = M^{-n}$. Thus both chain complexes and cochain complexes have differentials of degree -1 (which is to say, of codegree $+1$). This much is standard. However, since we need to deal with both homology and cohomology it is essential to have separate notation for homological suspensions (Σ^i) and cohomological suspensions (Σ_i): these are defined by

$$(\Sigma^i M)_n = M_{n-i} \text{ and } (\Sigma_i M)^n = M^{n-i}.$$

Thus, for example, with reduced chains and cochains of a based space X, we have

$$C_*(\Sigma^i X) = \Sigma^i C_*(X) \text{ and } C^*(\Sigma^i X) = \Sigma_i C^*(X).$$

2. Ring spectra

Most of the generalities can take place in any category with suitable formal structure: we need a symmetric monoidal product so that we can discuss rings and modules over them, and we need to be able to form tensor products over a commutative ring, and we will want a notion of homotopy theory so that we can form a triangulated homotopy category. In fact we will work with a category of spectra in the sense of homotopy theory; the construction is sketched in Appendix A, and the present section gives a brief orientation. A more extensive introduction designed for commutative algebraists is given in [60].

2.A. **Additive structure.** We may start from the observation that the homotopy theory of highly connected spaces is simpler than that of general spaces. By suspending a space we may steadily simplify the homotopy theory, but because cohomology theories have suspension isomorphisms, we do not lose any additive cohomological information: spectra capture the limit of this process. Thus spectra are a sort of abelianization of spaces where behaviour has a more algebraic formal flavour. Associated to any based space X there is a suspension spectrum, and arbitrary spectra can be built from those of this form. The other important source of spectra is as the representing objects for cohomology theories. If $E^*(X)$ is a contravariant functor of a based space X satisfying the Eilenberg-Steenrod axioms then there is a spectrum E so that the equation

$$E^*(X) = [X, E]^*$$

holds. This has the usual benefits that one can then apply geometric constructions to cohomology theories and one can argue more easily by universal examples.

2.B. **Multiplicative structure.** Having taken the step of representing cohomology theories by spectra, one may ask if good formal behaviour of the functor $E^*(\cdot)$ is reflected in the representing spectrum E. For our purposes the most important piece of structure is that of being a commutative ring, and we would like to say that a cohomology theory whose values on spaces are commutative rings is represented by a spectrum which is a commutative ring in the category of spectra. This is true remarkably often.

In order to do homotopy theory we need a Quillen model structure on the category of spectra, and to have commutative rings in this setting we need a symmetric monoidal smash product so that the two structures are compatible in a way elucidated by Schwede and Shipley. In retrospect it seems strange that such models were not constructed until the 1990s, but several such models are now known, and they give equivalent theories. We sketch the construction of symmetric and orthogonal spectra in Appendix A.

In this context, it makes sense to ask for a cohomology theory to be represented by a commutative ring spectrum R (i.e., R comes with a map $\mu\colon R \wedge R \longrightarrow R$ making it into a commutative monoid). Many of the important examples do have this structure. The most obvious example from this point of view is the sphere spectrum $\mathbb{S}$, which is obtained as the suspension spectrum of the two point space S^0. This is the initial ring in the category of spectra, and the smash product plays the role of tensor product over $\mathbb{S}$. We will describe a number of examples below, but for the present we continue with the general formalism just assuming that R is a commutative ring spectrum.

2.C. **Modules.** We may consider module spectra over R, and there is a model category structure on R-modules; furthermore, since R is commutive, there is a tensor product of R-modules formed in the usual way from the tensor product over the initial ring $\mathbb{S}$ (i.e., as the coequalizer of the two maps $M\otimes_{\mathbb{S}}R\otimes_{\mathbb{S}}N \longrightarrow M\otimes_{\mathbb{S}}N$). From the good formal properties of the original category, this category of R-modules is again a model category with a compatible symmetric monoidal product. This has an associated homotopy category $Ho(R-\mathrm{mod})$ and will be the context in which we work.

2.D. **Reverse approach.** Commutative algebraists may approach spectra from the algebraic direction. Traditional commutative algebra considers commutative rings A and modules over them, but some constructions make it natural to extend further to considering chain complexes of A-modules; the need to consider robust, homotopy invariant properties leads to the derived category $D(A)$. Once we admit chain complexes, it is natural to consider the corresponding multiplicative objects, differential graded algebras (DGAs). Although it may appear inevitable, the real justification for this process of generalization is the array of naturally occurring examples.

The use of spectra is a natural extension of this process. Shipley has shown that associated to any DGA A there is a ring spectrum HA, so that the category of DG-A-modules is Quillen equivalent to the category of HA-modules and $D(A) \simeq Ho(HA-\mathrm{mod})$. Accordingly we can view ring spectra as generalizations of DGAs and categories of module spectra as flexible generalizations of the derived category. Ring spectra extend the notion of rings, module spectra extend the notion of (chain complexes of) modules over a ring, and the homotopy category of module spectra extends the derived category. Many ring theoretic constructions extend to ring spectra, and thus extend the power of commutative algebra to a vast new supply of naturally occurring examples. Even for traditional rings, the new perspective is often enlightening, and thinking in terms of spectra makes a number of new tools available. Once again the only compelling justification for this inexorable process of generalization is the array of interesting, naturally occurring examples, some of which we will be described later in these lectures.

3. Three classes of examples

The construction of the category of symmetric spectra and the category of orthogonal spectra are sketched in Appendix A, and some basic examples are given in concrete form. The point of the present section is to introduce our three basic classes of examples.

The importance of symmetric and orthogonal spectra is that they each admit symmetric monoidal smash products compatible with the model structrures. Given this, we can start to do algebra with spectra: choose a ring spectrum R (i.e., a monoid in the category of spectra), form the category of R-modules or R-algebras and then pass to homotopy. We may then attempt to use algebraic methods and intuitions to study R and its modules. In turning to examples, we remind the reader that the equivalence results of [97] mean that we are free to choose the category most convenient for each particular application.

3.A. **Classical commutative algebra.** We explain why the classical derived category is covered by the context of spectra.

In the appendix we described a functorial construction for symmetric spectra taking an abelian group M and giving the Eilenberg-MacLane spectrum HM, which is characterised up to homotopy by the property that $\pi_*(HM) = \pi_0(HM) = M$. For symmetric spectra the construction respects tensor product, so that if A is a commutative ring, HA is a commutative ring spectrum. Furthermore the construction gives a functor A-modules $\longrightarrow HA$-modules. Passage to homotopy groups gives a functor $\mathrm{Ho}(HA\text{-mod}) \to Ho(A\text{-modules}) = D(A)$ and in fact the model categories are equivalent.

Theorem 3.1 (Shipley [110])**.** *There is a Quillen equivalence between the category of differential graded $\mathbb{Z}$-algebras and the category of $H\mathbb{Z}$-algebras in spectra.*

If we choose a DGA A and the corresponding $H\mathbb{Z}$-algebra HA, there is a Quillen equivalence between the category of A-modules and the category of HA-modules, and hence in particular a triangulated equivalence

$$D(A) = Ho(A\text{-}modules) \simeq Ho(HA\text{-}modules) = D(HA)$$

of derived categories.

We may sometimes use this identification to excuse the omission of the letter H indicating spectra. In this translation homology in the classical context of chain complexes corresponds to homotopy in the context of spectra: $H_*(M) = \pi_*(HM)$.

Now that we can view classical commutative rings as commutative ring spectra we can attempt to extend the classical notions to ring spectra. From one point of view, we should first attempt to understand the analogues of local rings before attempting to look at more geometrically complicated ones. Accordingly, in most of the lectures we will assume the commutative ring A is local, with residue field k.

3.B. **Cochains on a space.** In the category of spectra, we may solve the commutative cochain problem. More precisely, for any space X and a commutative ring k, we may form the function spectrum $C^*(X;k) = \mathrm{map}(\Sigma^\infty X, Hk)$. It is obviously an Hk-module, but using the diagonal on X it is also a commutative Hk-algebra. The notation is chosen because it is a model for the cochains in the sense

$$\pi_*(\mathrm{map}(\Sigma^\infty X, Hk)) = H^*(X; k).$$

The commutative algebra of $C^*(X; k)$ is one of the main topics for these lectures, and we will omit the coefficients k when it is clear from the context.

We then use algebraic behaviour of this commutative ring to pick out interesting classes of spaces. In accordance with the principle that $C^*(X; k)$ is a sort of ring of functions on X, we simplify terminology and say that X has a property P over k if the commutative ring spectrum $C^*(X; k)$ has the property P.

3.C. **Chromatic examples.** The examples we work with are ones constructed from the theory of formal groups, using the methods of derived algebraic geometry, but this comes about in a curious way. For the purposes of these lectures it is not necessary to understand much about the spectra beyond their homotopy groups, but we give a little more detail here.

One of the first examples of ring spectra was the theory of bordism, where the representing spectra have explicit models as orthogonal spectra based on Thom spaces over Grassmannians which quite easily shows they are commutative rings [108].

Example 3.2. The bordism spectrum MO is a commutative ring spectrum whose nth homotopy group consists of bordism classes of n-manifolds is a commutative $\mathbb{S}$-algebra. Its coefficient ring MO_* is polynomial over $\mathbb{F}_2$ on one generator of each degree ≥ 1 not of the form $2^i - 1$.

Example 3.3. The story really begins with the complex counterpart MU is a commutative $\mathbb{S}$-algebra. Its coefficient ring MU_* is polynomial over $\mathbb{Z}$ on one generator of each positive even degree.

The most important feature of MU is that it is complex oriented, in the sense that it has Thom isomorphisms for complex bundles. Accordingly we may develop a theory of Chern classes. In particular, there is a formula $c_1(\alpha \otimes \beta) = F(c_1(\alpha), c_1(\beta))$ for the first Chern class of a tensor product as a power series in those of the tensor factors. This power series defines a one dimensional commutative formal group law. Quillen proved that the coefficient ring MU_* is naturally isomorphic to Lazard's universal ring for formal group laws. This means that we have a homotopy theoretic object to feed into the derived algebraic geometry (DAG) machine, which can construct moduli spaces of various objects derived from elliptic curves and abelian varieties. The machine then generates numerous new commutative ring spectra as the sections of sheaves of ring spectra over these moduli spaces. Some of them will be mentioned later.

Perhaps the simplest of these is the classical topological K-theory spectrum.

Example 3.4. The complex K-theory spectrum KU is a commutative $\mathbb{S}$-algebra with coefficients $KU_* = \mathbb{Z}[v, v^{-1}]$ with v of degree 2. Its connective cover ku is a commutative $\mathbb{S}$-algebra with coefficient ring $ku_* = \mathbb{Z}[v]$.

The conventional model is as an orthogonal spectrum based on Fredholm operators, but from the point of view of DAG it can be viewed as the ring of functions on the (one point) moduli space of strictly multiplicative formal groups, $F(x, y) = x + y - vxy$.

3.D. **Other examples.** There are many other types of examples, and we mention two because they are related to some of the matters we discuss.

Example 3.5. If G is a group or a monoid. Then the suspension spectrum

$$R = \Sigma^\infty G_+$$

is a monoid, commutative if G is abelian. The case $G = \Omega X$ for a space X is important in geometric topology (here one should use Moore loops to ensure that G is strictly associative).

Example 3.6. There are a number of ring valued functors of rings that have counterparts for ring spectra giving a further supply of ring spectra. We will meet the counterpart of Hochschild homology later, and a version of cyclic homology.

Perhaps more fundamental is the fact that we may apply the algebraic K-theory functor to any ring spectrum R to form a spectrum $K(R)$. If R is a commutative ring spectrum so is $K(R)$.

This generalizes the classical case in the sense that $K(HR) = K(R)$ (where the right hand side is the version of algebraic K-theory based on finitely generated free modules). Another important example comes from geometric topology: $K(\Sigma^\infty \Omega X_+)$ is Waldhausen's $A(X)$ [47, VI.8.2]. The spectrum $A(X)$ embodies a fundamental step in the classification of high dimensional manifolds [114].

4. The standard context

The point of this section is to introduce the general context for all our examples. The essential limitation of all we do is that it is based on *local* rings in commutative algebra. We will not discuss the new and interesting features that can arise when there are many maximal ideals.

Context 4.1. The main input is a map $R \longrightarrow k$ of ring spectra with notation suggested by the case when R is commutative local ring with residue field k. We also write $\mathcal{E} = \mathrm{Hom}_R(k, k)$ for the (derived) endomorphism ring spectrum.

For the most part, we work in the homotopy category $Ho(R\text{-mod})$ of left R-modules.

4.A. **New modules from old.** Three construction principles will be important to us. There is some duplication in terminology, but the flexibility is convenient.

If M is an R-module we say that X is *built* from M (and write $M \vdash X$) if X can be formed from M by completing triangles, taking coproducts and retracts (i.e., X is in the *localizing subcategory* generated by M). We refer to objects built from M as *M-cellular*, and write $\mathrm{Cell}(R, M)$ for the resulting full subcategory of $D(R\text{-mod})$. An *M-cellular approximation* of X is a map $\mathrm{Cell}_M(X) \longrightarrow X$ where $\mathrm{Cell}_M(X)$ is M-cellular and the map is an $\mathrm{Hom}_R(M, \cdot)$-equivalence.

We say that X is *finitely built* from M (and write $M \models X$) if only finitely many steps and finite coproducts are necessary (i.e., X is in the *thick* subcategory generated by M).

Finally, we say that X is *cobuilt* from M if X can be formed from M by completing triangles, taking products and retracts (i.e., X is in the *colocalizing subcategory* generated by M).

4.B. **Finiteness conditions.** We say M is *small* if the natural map

$$\bigoplus_{\alpha}[M, N_\alpha] \longrightarrow [M, \bigoplus_{\alpha} N_\alpha]$$

is an isomorphism for any set of R-modules N_α. Smallness is equivalent to being finitely built from R. It is easy to see that any module finitely built from R is small. For the reverse implication we use the fact that we can build an R-cellular approximation $\mathrm{Cell}(R, M) \longrightarrow M$; this is an equivalence, and by smallness, M is a retract of a finitely built subobject of $\mathrm{Cell}(R, M)$.

We sometimes require that k itself is small, but this is an extremely strong condition on R and it is important to develop the theory under a much weaker condition.

Definition 4.2. [40] We say that k is *proxy-small* if there is an object K with the following properties

- K is small ($R \models K$)
- K is finitely built from k ($k \models K$) and
- k is built from K ($K \vdash k$).

Remark 4.3. Note that the second and third condition imply that the R-module K generates the same category as k using triangles and coproducts: $\mathrm{Cell}(R, K) = \mathrm{Cell}(R, k)$.

It is one of the messages of [40] that the weak condition of proxy-smallness allows one to develop a very useful theory.

We are not ready to illustrate the condition for all of our examples, but we should at least explain the algebraic motivation.

Example 4.4 (Conventional commutative algebra)**.** Take R to be a commutative Noetherian local ring in degree 0, with maximal ideal I and residue field k.

By the Auslander-Buchsbaum-Serre theorem, k is small if and only if R is a regular local ring, confirming that the smallness of k is a very strong condition. On the other hand, k is always proxy-small: we may take $K = K(\boldsymbol{\alpha})$ to be the Koszul complex for a generating sequence $\boldsymbol{\alpha}$ for I (see Appendix B).

It is shown in [39] that $\mathrm{Cell}(R, k)$ consists of objects whose homology is I-power torsion.

Part 2. Morita equivalences.

5. Morita equivalences.

Morita theory studies objects X of a category $\mathbb{C}$ by considering maps from a test object k. More precisely, X is studied by considering $\mathrm{Hom}(k, X)$ as a module over the endomorphism ring $\mathrm{End}(k)$. In favourable circumstances this may give rather complete information.

In the classical situation, $\mathbb{C}$ is an abelian category with infinite sums and k is a small projective generator, and we find $\mathbb{C}$ is equivalent to the category of $\mathrm{End}(k)$-modules [18, II Thm 1.3]. We will work with a model category rather than in an abelian category, and k will not necessarily be either small or a generator. The fact that the objects of the categories are spectra is unimportant except for the formal context it provides. See [106] for an account from the present point of view.

In fact, two separate Morita equivalences play a role. Indeed, two separate categories of modules over a commutative ring are both shown to be equivalent to a category of modules over a non-commutative ring. This is another instance of the philosophy that one gains insights by looking at rings of operations.

This section is based on [39], with augmentations from [40].

5.A. **First variant.** Continuing to let $\mathcal{E} = \mathrm{Hom}_R(k, k)$ denote the (derived) endomorphism ring, we consider the relationship between the derived categories of left R-modules and of right $\mathcal{E}$-modules. We have the adjoint pair

$$T : \mathrm{mod}\!-\!\mathcal{E} \rightleftarrows R\!-\!\mathrm{mod} : E$$

defined by

$$T(X) := X \otimes_{\mathcal{E}} k \text{ and } E(M) := \mathrm{Hom}_R(k, M).$$

Remark 5.1. If k is small, it is easy to see that this adjunction gives equivalence

$$\mathrm{Cell}(R, k) \simeq D(\mathrm{mod}\!-\!\mathcal{E})$$

between the derived category of R-modules built from k and the derived category of $\mathcal{E}$-modules. Indeed, to see the unit $X \longrightarrow ETX = \mathrm{Hom}_R(k, X \otimes_{\mathcal{E}} k)$ is an equivalence, we note it is obviously an equivalence for $X = \mathcal{E}$ and hence for any X built from $\mathcal{E}$, by smallness of k. The argument for the counit is similar.

Remark 5.2. If k is not small, the unit of the adjunction may not be an equivalence. For example if $R = \Lambda(\tau)$ is exterior on a generator of degree 1 then $\mathcal{E} \simeq k[x]$ is polynomial on a generator of degree -2. As an R-module, k is of infinite projective dimension and hence it is not small. In this case all R-modules are k-cellular, so that $\mathrm{Cell}(R, k) = R$-mod. Furthermore, the only subcategories of R-modules closed under coproducts and triangles are the trivial category and the whole category. On the other hand the category of torsion $\mathcal{E}$-modules is a proper non-trivial subcategory closed under coproducts and triangles.

Exchanging roles of the rings, so that $R = k[x]$ and $\mathcal{E} \simeq \Lambda(\tau)$, we see k is small as a $k[x]$-module and $\mathrm{Cell}(k[x], k)$ consists of torsion modules. Thus we deduce

$$\text{tors-}k[x]\text{-mod} \simeq \text{mod-}\Lambda(\tau). \qquad \square$$

In fact the counit

$$TEM = \mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k \longrightarrow M$$

of the adjunction is of interest much more generally. Notice that any $\mathcal{E}$-module (such as $\mathrm{Hom}_R(k, M)$) is built from $\mathcal{E}$, so the domain is k-cellular. We say M is *effectively constructible* from k if the counit is an equivalence, because TEM gives a concrete and functorial model for the cellular approximation to M. Under the much weaker assumption of proxy smallness we obtain a very useful conclusion linking Morita theory to commutative algebra.

Lemma 5.3. *Provided k is proxy-small, the counit*

$$TEM = \mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k \longrightarrow M$$

is k-cellular approximation, and hence in particular any k-cellular object is effectively constructible from k.

Proof: We observed above that the domain is k-cellular. To see the counit is a $\mathrm{Hom}_R(k, \cdot)$-equivalence, consider the evaluation map

$$\gamma \colon \mathrm{Hom}_R(k, X) \otimes_{\mathcal{E}} \mathrm{Hom}_R(Y, k) \longrightarrow \mathrm{Hom}_R(Y, X).$$

This is an equivalence if $Y = k$, and hence by proxy-smallness it is an equivalence if $Y = K$. This shows that the top horizontal in the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_R(k, X) \otimes_{\mathcal{E}} \mathrm{Hom}_R(K, k) & \xrightarrow{\simeq} & \mathrm{Hom}_R(K, X) \\
\simeq \downarrow & & \downarrow = \\
\mathrm{Hom}_R(K, \mathrm{Hom}_R(k, X) \otimes_{\mathcal{E}} k) & \longrightarrow & \mathrm{Hom}_R(K, X)
\end{array}$$

is an equivalence. The left hand-vertical is an equivalence since K is small. Thus the lower horizontal is an equivalence, which is to say that the counit

$$TEX = \mathrm{Hom}_R(k, X) \otimes_{\mathcal{E}} k \longrightarrow X$$

is a K-equivalence. By proxy-smallness, this counit map is a k-equivalence. □

Examples 5.4. (i) If R is a commutative local ring, we recall in Appendix B that the k-cellular approximation of a module M is $\Gamma_{\mathfrak{m}} M = K_\infty(\mathfrak{m}) \otimes_R M$, where $K_\infty(\mathfrak{m})$ is the stable Koszul complex, so we have

$$TEM \simeq K_\infty(\mathfrak{m}) \otimes_R M.$$

(ii) If $R = C^*(X; k)$ it is not easy to say what the k-cellular approximation is in general, but any bounded below module M is cellular.

5.B. **Second variant.** There is a second adjunction between the derived categories of left R-modules and of right $\mathcal{E}$-modules. In the first variant, k played a central role as a left R-module and a left $\mathcal{E}$-module. In this second variant

$$Dk := \mathrm{Hom}_R(k, R)$$

plays a corresponding role: it is a right R-module and a right $\mathcal{E}$-module. We have the adjoint pair

$$E' : R\text{−mod} \rightleftarrows \text{mod−}\mathcal{E} : C$$

defined by

$$E'(M) := Dk \otimes_R M \text{ and } C(X) := \mathrm{Hom}_{\mathcal{E}}(Dk, X).$$

Remark 5.5. If k is small then

$$E'(M) = \mathrm{Hom}_R(k, R) \otimes_R M \simeq \mathrm{Hom}_R(k, M) = EM,$$

so the two Morita equivalences consider the left and right adjoints of the same functor.

The unit of the adjunction $M \longrightarrow CE'(M)$ is not very well behaved, and the functor CE' is not even idempotent in general.

5.C. **Complete modules and torsion modules.** Even when we are not interested in the intermediate category of $\mathcal{E}$-modules, several of the composite functors give interesting endofunctors of the category of R-modules.

Lemma 5.6. *If k is proxy-small then k-cellular approximation is smashing:*

$$\mathrm{Cell}_k M \simeq (\mathrm{Cell}_k R) \otimes_R M.$$

Proof: Both modules come with natural maps to M. Since R builds M, $(\mathrm{Cell}_k R)\otimes_R R$ builds $(\mathrm{Cell}_k R)\otimes_R M$, so that there is a unique map $(\mathrm{Cell}_k R)\otimes_R M \longrightarrow \mathrm{Cell}_k M$.

Assuming k is proxy-small, there is a unique map in the reverse direction, because $\mathrm{Hom}_R(\mathrm{Cell}_k M, \check{C}(k) \otimes_R M) \simeq 0$. Indeed, this obstruction module cobuilt from $\mathrm{Hom}_R(K, \check{C}(k) \otimes_R M)$, and $DK \otimes_R \check{C}(k) \simeq 0$. □

We therefore see by 5.3 and 5.6 that if k is proxy-small

$$\mathrm{Cell}_k(M) = TEM = TE'M.$$

This is the composite of two left adjoints, focusing attention on its right adjoint CEM, and we note that

$$CE(M) = \mathrm{Hom}_R(Dk, \mathrm{Hom}_R(k, M)) = \mathrm{Hom}_R(TER, M).$$

By analogy with Subsection II, we may make the following definition.

Definition 5.7. The *completion* of an R-module M is the map

$$M \longrightarrow \mathrm{Hom}_R(TER, M) = CEM.$$

We say that M is *complete* if the completion map is an equivalence.

Remark 5.8. By 5.6 we see that completion is idempotent.

We adopt the notation

$$\Gamma_k M := TE'M$$

and

$$\Lambda^k M := CEM.$$

This is by analogy with the case of commutative algebra through the approach of Appendix C, where $\Gamma_k = \Gamma_I$ is the total right derived functor of the I-power torsion functor and $\Lambda^k = \Lambda^I$ is the total left derived functor of the completion functor (see [2, 3] for the context of commutative rings).

It follows from the adjunctions described in Section 5 that Γ_k is left adjoint to Λ^k as endofunctors of the category of R-modules:

$$\mathrm{Hom}_R(\Gamma_k M, N) = \mathrm{Hom}_R(M, \Lambda^k N)$$

for R-modules M and N. Slightly more general is the following observation.

Lemma 5.9. *If k is proxy-small, Γ_k and Λ^k give an adjoint equivalence*

$$\mathrm{Cell}(R, k) \simeq \mathrm{comp}{-}R\text{-}mod,$$

where comp$-R$*-mod is the triangulated subcategory of* R*-mod consisting of complete modules.*

Proof: We have

$$TE'M \simeq TEM \simeq \Gamma_k M \simeq \Gamma_K M,$$

and

$$CEM \simeq \mathrm{Hom}_R(\Gamma_k R, M) \simeq \mathrm{Hom}_R(\Gamma_K R, M),$$

so it suffices to prove the result when k is small. When k is small the present adjunction is the composite of two adjoint pairs of equivalences. We have seen this for the first variant, and the second variant is proved similarly by arguing that the unit and counit are equivalences. □

5.D. **Commutative local rings.** In the classical setting, we note that $H^*(\mathrm{Hom}_R(k,k)) = \mathrm{Ext}_R^*(k,k)$, which is a ring whose importance is very familiar. We will see in Section 6 that as a DGA, the endormorphisms $\mathcal{E} = \mathrm{Hom}_R(k,k)$ contain considerably more information.

Provided that k is proxy-small (as is always the case if k is the residue field), the effect of passing to Morita equivalents twice

$$R \longrightarrow \mathrm{Hom}_{\mathcal{E}}(k,k)$$

is simply completion at I where $k = R/I$.

5.E. **Cochains on a space.** In the setting $R = C^*(X;k)$ of cochains on a space, the Eilenberg-Moore spectral theorem shows that

$$\mathcal{E} = \mathrm{Hom}_{C^*(X;k)}(k,k) \simeq C_*(\Omega X;k), H_*(\mathcal{E}) = H_*(\Omega X;k)$$

provided that either (i) X is simply connected or (ii) X is connected, $\pi_1(X)$ is a finite p-group $k = \mathbb{F}_p$ and X is p-complete [38]. This immediately gives a very rich source of examples that we will revisit frequently.

A constant theme will be that properties of $C^*(X)$ are often reflected in $C_*(\Omega X)$, sometimes more visible in one and sometimes in the other.

Provided that k is proxy-small and either (i) X is simply connected or (ii) $k = \mathbb{F}_p$, X is p-complete and $\pi_0(X)$ is a finite p-group, the effect of passing to Morita equivalents twice

$$R \longrightarrow \mathrm{Hom}_{\mathcal{E}}(k,k)$$

is an isomorphism.

5.F. **Chromatic examples.** In the chromatic setting, group rings of operations will play an important role.

[[This subsection is to be expanded]]

6. Exterior algebras

There are many ways to use Morita theory, but there is a very elementary and striking one which illustrates that it is significant in even very simple cases.

6.A. **Exterior algebras over $\mathbb{F}_p$.** We may use Morita theory to give a classification [43].

Theorem 6.1. *Differential graded algebras $\mathcal{E}$ with $H_*(\mathcal{E})$ exterior over $\mathbb{F}_p$ on a single generator of degree -1 (up to quasi-isomoprhism) are in bijective correspondence with complete discrete valuation rings with residue field $\mathbb{F}_p$.*

Proof: The idea is to associate to any such $\mathcal{E}$ the endomorphism DGA $R = \mathrm{Hom}_{\mathcal{E}}(\mathbb{F}_p, \mathbb{F}_p)$. Evidently the spectral sequence for calculating $H_*(R)$ collapses at E_2 with value $\mathbb{F}_p[\overline{x}]$ for an element $\overline{x}$ of total degree 0. This shows R is an ungraded ring with a filtration with this as associated graded ring. One argues that it is commutative and complete with residue field $\mathbb{F}_p$. If one starts with R, one may note that there is a element x and so we may form a complex $\underline{\mathbb{F}}_p = (R \xrightarrow{x} R)$. The DGA $\mathcal{E} = \mathrm{Hom}_R(\underline{\mathbb{F}}_p, \underline{\mathbb{F}}_p)$ has homology exterior on a generator of degree -1, and the double centralizer completion map

$$R \longrightarrow \mathrm{Hom}_{\mathcal{E}}(\mathbb{F}_p, \mathbb{F}_p)$$

is an equivalence. □

6.B. **Classification of free rational G-spectra.** In another direction, working over the rationals we note that for any compact Lie grou G the DGA $C_*(G)$ has homology $H_*(G)$ exterior on odd degree generators. It might not be apparent that it is formal. However, we may consider $R = \mathrm{Hom}_{C_*(G)}(k, k) = C^*(BG)$; since this is commutative and has cohomology which is polynomial on even degree generators it is formal and hence equivalent to $H^*(BG)$. It follows that $C_*(G) \simeq H_*(G)$. This is the starting point for classifying free rational G-spectra [71, 72].

Part 3. Regular rings

This is the first of a series of parts that take particular classes of commutative local rings, and identifies ways of giving homotopy invariant counterparts of the definitions. The justification consists of the dual facts that the new definition reduces to the old in the classical setting and that the new definition covers and illuminates examples in new contexts.

7. Regular ring spectra

In commutative algebra there are three styles for a definition of a regular local ring: ideal theoretic, in terms of the growth of the Ext algebra and a version for modules. We begin in Subsection 7.A by recalling the commutative algebra, where the conditions are equivalent. We then turn to other contexts, where the conditions may differ, and consider which one is most appropriate.

7.A. Commutative algebra.

The following definitions are very familiar; we have introduced a slightly more elaborate terminology to smooth the transition to other contexts. The prefix s- signifies that the definition is *structural*, the prefix g- signifies that the definition is in terms of the *growth* rate, and the prefix m- signifies that the definition is in terms of the *module* category.

Definition 7.1. (i) A local Noetherian ring R is *s-regular* if the maximal ideal is generated by a regular sequence.

(ii) A local Noetherian ring R is *g-regular* if $\mathrm{Ext}_R^*(k,k)$ is finite dimensional.

(iii) A local Noetherian ring R is *m-regular* if every finitely generated module is small in the derived category $D(R)$.

It is not hard to see that g-regularity is equivalent to m-regularity or that s-regularity implies g-regularity. Serre proved that g-regularity implies regularity, so the three conditions are equivalent.

7.B. Regularity for ring spectra.

The one definition that is easy to adapt is g-regularity, at least when k is a field. This is a basic input to the entire theory. It is also convenient to have a name for when the coefficient ring is regular.

Definition 7.2. (i) We say that R is c-regular when the coefficient ring $\pi_*(R)$ is regular.

(ii) We say that R is g-regular if $\pi_*(\mathrm{Hom}_R(k,k))$ is a finite rank free k-module.

It is obvious from the spectral sequence

$$\mathrm{Ext}_{R_*}^{*,*}(k,k) \Rightarrow \pi_*(\mathrm{Hom}_R(k,k))$$

that c-regular implies g-regular. The converse is very far from being true, as we shall see shortly.

It is more interesting to consider s-regular ring spectra.

The s-regularity condition on rings states that $\mathfrak{m}$ is generated by a regular sequence $x_1, \ldots, x_r$, and that there are short exact sequences

$$R \xrightarrow{x_1} R \longrightarrow R/(x_1)$$

$$R/(x_1) \xrightarrow{x_2} R/(x_1) \longrightarrow R/(x_1, x_2)$$

and

$$R/(x_1, \ldots, x_{r-1}) \xrightarrow{x_r} R/(x_1, \ldots, x_{r-1}) \longrightarrow R/\mathfrak{m} = k.$$

In other words, we may start with R, and successively factor out an element until we get to k.

If we now remember the degrees, a single instance is the (additive) exact sequence of modules

$$\Sigma^n R \xrightarrow{x} R \longrightarrow R/(x).$$

In other words, the ring $R/(x)$ is equivalent to the Koszul complex $K(x) = (R \xrightarrow{x} R)$. If we think multiplicatively this gives a cofibre sequence [1]

$$R \longrightarrow K(x) \longrightarrow K(x) \otimes_R k = \Lambda(\tau)$$

of rings, where τ is a generator of degree $n+1$. The definition is now obtained by iterating this construction.

Definition 7.3. We say that R is s-regular if there are cofibrations of rings

$$R = R_0 \longrightarrow R_1 \longrightarrow \Lambda_1, R_1 \longrightarrow R_2 \longrightarrow \Lambda_2, \cdots, R_{r-1} \longrightarrow R_r \longrightarrow \Lambda_r,$$

with $R_r \simeq k$ and $\pi_*(\Lambda_i)$ exterior over k on one generator.

Lemma 7.4. *If R is s-regular then it is also g-regular.*

Proof: We need to work our way along the sequence of cofibrations. Since $R_r \simeq k$ is obviously g-regular, it suffices to oberve that given a cofibre sequence $C \longrightarrow B \longrightarrow \Lambda$ with B g-regular, then C is also g-regular. The point here is that $\mathrm{Hom}_C(k,k) \simeq \mathrm{Hom}_k(k \otimes_C k, k)$ and

$$k \otimes_C k \simeq k \otimes_B B \otimes_C k \simeq k \otimes_C \Lambda.$$

Since C is regular, k is small, and so $k \otimes_C \Lambda$ is finitely built from $C \otimes_C \Lambda = \Lambda$ which has finite dimensional homotopy. $\square$

However, there is no good notion of m-regularity for ring spectra. The problem is that we do not have a homotopy invariant definition of finite generation in general. However, we will turn this around, and in Section 8 by showing that one can use the supposed equivalence of g-regularity and the putative m-regularity conditions to *define* finite generation.

[1] If $S \longrightarrow R$ is a map of rings and we have a map $R \longrightarrow k$, the *cofibre* ring is $Q = R \otimes_Q k$. This corresponds to the fact that in algebraic geometry and topology this cofibre ring is often the ring of functions on the geometric fibre.

7.C. **g-regularity for spaces.** Of the commutative algebra definitions, the only one with a straightforward counterpart for $C^*(X)$ is g-regularity. In view of Subsection 5.E, it takes the following form.

Definition 7.5. A space X is *g-regular* if $H_*(\Omega X; k)$ is finite dimensional.

Remark 7.6. Note that for any p-group, the classifying space BG is g-regular over $\mathbb{F}_p$. On the other hand the coefficient ring $H^*(BG)$ is very rarely regular, so this gives may examples of g-regular ring spectra which are not c-regular.

Remark 7.7. One might say that X is globally regular if $\Gamma = \Omega X$ is a finite complex, so that $X = B\Gamma$ is the classifying space of a finite loop space. For instance, if Γ is any compact Lie group we obtain such a space X.

If $k = \mathbb{F}_p$ and X is p-complete and connected, the g-regularity condition that $H_*(\Omega X; \mathbb{F}_p)$ is finite is precisely the condition $\Gamma = \Omega X$ is a p-compact group in the sense of Dwyer and Wilkerson [44] with classifying space $X = B\Gamma$. For example if Γ' is a compact Lie group with component group a p-nilpotent group, the p-completion of $B\Gamma'$ is an example, although there are many examples not of this form.

Indeed, Dwyer and Wilkerson show that connected p-compact groups have a maximal torus. A major programme involving many people has developed the theory along the lines of that for compact Lie groups, and this has finally culminated in the classification of connected p-compact groups in terms of this root data by Anderson, Grodal, Moller and Viruel ([5] for odd primes, and [4] for $p = 2$). It is remarkable that a notion defined purely in terms of a finiteness condition leads to a classification in terms of p-adic analogues of the root data of compact Lie groups.

7.D. **s-regularity for spaces.** Looking back at the definition of s-regularity for ring spectra, the multiplicative sequence

$$R \longrightarrow K(x) \longrightarrow K(x) \otimes_R k = \Lambda(\tau)$$

where τ is a generator of degree $n+1$ and $\Lambda(\tau)$ corresponds $C^*(S^{n+1})$. Thus if $R = C^*(X)$, the Eilenberg-Moore theorem shows this corresponds to a fibration

$$X \longleftarrow X_1 \longleftarrow S^{n+1}.$$

Definition 7.8. A space X is s-regular if there are k-complete fibrations

$$X = X_0 \longleftarrow X_1 \longleftarrow S^{n_1}, X_1 \longleftarrow X_2 \longleftarrow S^{n_2}, \cdots, X_{r-1} \longleftarrow X_r \longleftarrow S^{n_r}$$

with $X_r \simeq *$.

Example 7.9. $X = BU(n)$ is s-regular, in view of the fibrations

$$BU(n) \longleftarrow BU(n-1) \longleftarrow U(n)/U(n-1) \cong S^{2n-1}.$$

Similarly for $BO(n)$.

8. Finite generation

8.A. **Finiteness conditions.** If R is a conventional Noetherian commutative ring and M is a module over it there are a number of natural finiteness conditions. If M has a finite resolution by finitely generated projective then it is finitely built from R and hence small in the derived category; the converse is also true, since one may factor the identity map through a finite truncation of the projective resolution. This gives a satisfactory homotopy invariant finiteness condition.

On the other hand, it is not so clear how to give a homotopy invariant version of the notion of being finitely generated. One way that is useful in practice is to note that if Q is regular, every finitely generated module has a finite resolution by finitely generated projectives, and hence small and finitely generated agree.

8.B. **Normalizable ring spectra.** In commutative algebra, it is natural to assume rings are Noetherian, and one of the most useful consequences for k-algebras is Noether normalization, stating that a Noetherian k-algebra is a finitely generated module over a polynomial subring.

Definition 8.1. We will say that a ring spectrum R is *g-normalizable*, if there is a g-regular ring spectrum S with S_* Noetherian, and a ring map $S \longrightarrow R$ making R into small S-module. In this case $S \longrightarrow R$ is called a *g-normalization* and $R \otimes_S k$ is its *Noether fibre*.

If S can be chosen so that its coefficient ring S_* is regular, we say R is *c-normalizable.*

Lemma 8.2. (i) *If S_* is regular then an S-module N is small if and only if N_* is a finitely generated S-module.*

(ii) *If R is c-normalizable then R_* is Noetherian.*

Example 8.3. (Venkov) If G is a compact Lie group (for example a finite group), then $C^*(BG)$ is c-normalizable. Indeed, we may choose a faithful representation $G \longrightarrow U(n)$, giving a fibration $U(n)/G \longrightarrow BG \longrightarrow BU(n)$.

It is an unpublished consequence of work of Castellana and Ziemianski that every p-compact group has a faithful linear representation, which is to say that if $B\Gamma$ is regular there is a map $B\Gamma \longrightarrow BSU(n)$, for some n, whose homotopy fibre is $\mathbb{F}_p$-finite. This means that every p-compact group is c-normalizable.

8.C. **Finitely generated modules.**

Definition 8.4. Suppose that R is a commutative ring spectrum and M is an R-module. If we are given a normalization $\nu\colon S \longrightarrow R$ we say that M is *ν-finitely generated* if M is small over S.

We say that M is *finitely generated* if M is ν-finitely generated for some normalization ν.

Lemma 8.5. *If R is g-normalizable and M is finitely generated then M_* is a finitely generated R_*-module.*

If R is c-normalizable, then M is finitely generated if and only if M_* is finitely generated over R_*.

Remark 8.6. If R is itself g-regular but not c-regular, it is not clear that every M with M_* finitely generated is small.

Proof: If M is small over S for some normalization, then M_* is finitely generated over S_* and hence over R_*.

If S is a c-normalization of R and M_* is finitely generated over R_* it is finitely generated over S_*. Since S_* is regular, this means M is small over S and hence finitely generated. $\square$

We note that if R is c-normalizable then the condition that a module is finitely generated is entirely independent of the c-normalization.

8.D. **Singularity categories.** In conventional commutative algebra the Buchweitz singularity category $D_{sing}(R)$ is defined to be the quotient of the bounded derived category by the perfect complexes.

If we are given a normalization ν of a commutative ring spectrum R, it is evident that the category of ν-finitely generated modules is a triangulated subcategory of the category of R-modules, so we may take the Verdier quotient of ν-finitely generated modules by small modules. Given a normalization we define the singularity category to be the Verdier quotient

$$D_{sing}(R, \nu) = D_{\nu-fg}(R)/D_{small}(R).$$

This coincides with the Buchweitz category if R is an Eilenberg-MacLane spectrum, but in general it depends on ν.

Example 8.7. Take $X = BC_3$, $k = \mathbb{F}_3$, and $R = C^*(BC_3)$. Since R is g-regular, the identity map is a normalization with trivial singularity category.

We have $H^*(BC_3) = P \otimes \Lambda$ with P polynomial on a codegree 2 class x and Λ exterior on a codegree 1 class τ. We may form an R-module $\tilde{P}$ with $\pi_*(\tilde{P}) = P$. Indeed, we have a map $\Sigma^{-1}R \xrightarrow{\tau} R$, and the cofibre C_1 has homology $P \oplus \Sigma^{-2}P$; next we defined $\Sigma^{-2}R \longrightarrow C_1$ mapping onto $\Sigma^{-2}P$ on homology. We iterate this and pass to colimit to obtain $\tilde{P}$. It is easy to see that $\tilde{P}$ is not small, since for a small module M the k-vector space $\pi_*\mathrm{Hom}_R(M, k)$ is finite dimensional. On the other hand, for $\tilde{P}$ we have a collapsed spectral sequence

$$k[x] = \mathrm{Ext}_{H^*(BC_3)}(P, k) \Rightarrow \pi_* \mathrm{Hom}_{C^*(BC_3)}(P, k)$$

showing the homotopy is infinite dimensional.

On the other hand the inclusion $C_3 \longrightarrow U(1)$ gives a c-normalization in which P is finitely generated. Accordingly this c-normalization has non-trivial singularity category.

Part 4. **Hypersurfaces**

This is the second of a series of parts that take particular classes of commutative local rings, and identifies ways of giving homotopy invariant counterparts of the definitions, which apply to ring spectra. The justification consists of the dual facts that the new definition reduces to the old in the classical setting and that the new definition covers and illuminates examples in new contexts.

9. Complete intersections

To understand the significance of hypersurfaces, we should first say a word about complete intersections. There is a general framework for discussing these in the homotopy invariant context [63, 26], but there are few examples beyond the hypersurface case. The general theory is in some sense obtained by iterating the case of hypersurfaces, but there are a number of different ways of iterating it, and some work better than others.

9.A. **Classical complete intersections.** These lectures will focus on hypersurfaces, but it is helpful to set the more general context with a brief discussion of complete intersections in general. Just as for regular rings there are three styles of definition: (s) a structural one, (g) one involving growth and (m) one involving the homological algebra of modules. These all have counterparts in the homotopy invariant setting which we will introduce in due course.

We start with the structural definition. The best behaved subvarieties of affine space are those which are specified by the right number of equations: if they are of codimension c then only c equations are required. On the same basis, a commutative local ring R is a *complete intersection (ci)* if its completion is the quotient of a regular local ring S by a regular sequence, $f_1, f_2, \ldots, f_c$. We will suppose that R is complete, so that

$$R = S/(f_1, \ldots, f_c).$$

The smallest possible value of c (as S and the regular sequence vary) is called the *codimension* of R. A hypersurface is the special case when $c = 1$ so that a single equation is used.

When it comes to growth, if R is ci of codimension c, one may construct a resolution of any finitely generated module growing like a polynomial of degree $c-1$. In particular the ring $\mathrm{Ext}^*_R(k,k)$ has polynomial growth (we say that R is *gci*). Perhaps the most striking result about ci rings is the theorem of Gulliksen [76] which states that this characterises ci rings so that the ci and gci conditions are equivalent for local rings.

With a little care, one may construct the resolutions in an eventually multiperiodic fashion: the projective resolution eventually has the pattern of the tensor product of c periodic exact sequences. In fact the construction is essentially independent of the module and the calculation can be phrased in terms of the Hochschild cohomology of R. This opens the way to the theory of support varieties for modules over a ci ring [14].

9.B. **Homotopy invariant versions, and Levi's groups.** One may give homotopy invariant versions of all three characterizations of the ci condition:

(sci): the 'regular ring modulo regular sequence' condition,
(mci): the 'modules have eventually multiperiodic resolutions' and
(gci): polynomial growth of the Ext algebra

We note that the Avramov-Quillen characterization of ci rings in terms of André-Quillen homology does not work for cochains on a space in the mod p context since Mandell has shown [95] that the topological André-Quillen cohomology vanishes very generally in this case.

It involves some work to describe the sci and mci definitions, but if we take $R = C^*(X)$ for a p-complete space Subsection 5.E shows $H_*(\mathrm{Hom}_{C^*(X)}(k,k)) = H_*(\Omega X)$, so it is easy to understand the gci condition. Taking $X = (BG)^\wedge_p$ we may consider what this means. Of course if G is a p-group, $H_*(\Omega X^\wedge_p)$ is simply the group ring kG in degree 0. More generally, it is known to be of polynomial growth in certain cases (for instance A_4 or M_{11} in characteristic 2) and R. Levi [87, 88, 89, 90] has proved there is a dichotomy between small growth and large growth, and given examples where the growth is exponential. Evidently groups whose p-completed classifying spaces have loop space homology that has exponential growth cannot be spherically resolvable, so Levi's groups disproved a conjecture of F.Cohen.

Example 9.1. It is amusing to consider the p-completed classifying space $BG^\wedge_p$ for $G = (C_p \times C_p) \rtimes C_3$ where C_3 acts via $(1,0) \mapsto (0,1) \mapsto (-1,-1)$. When $p = 3$, G is a p-group so the space is g-regular. When $p = 2$ the group is the alternating group A_4, which we will see below is a hypersurface. If $p \geq 5$ then Levi shows $H_*(\Omega(BG^\wedge_p))$ has exponential growth.

10. Hypersurfaces in algebra

In this section we recall some standard constructions for hypersurface algebas. We suppose $R = S/(f)$ is a hypersurface ring, where S is a regular ring and f is a nonzero element of degree d. Thus we have a short exact sequence

$$0 \longrightarrow \Sigma^d S \xrightarrow{f} S \longrightarrow R \longrightarrow 0$$

of S-modules for a regular local ring S. There are two basic constructions that we need to generalize.

10.A. **The degree 2 operator.** We describe a construction of a cohomological operator due to Gulliksen [12, 77]. We will do this for a single module, but it is apparent that the construction is essentially independent of the module, and in fact it lifts to Hochschild cohomology.

Given an R-module M we may apply $(\cdot) \otimes_S M$ to the defining sequence to obtain the short exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^S(R, M) \longrightarrow \Sigma^d M \xrightarrow{f} M \longrightarrow R \otimes_S M \longrightarrow 0.$$

Since $f = 0$ in any R-module, we conclude

$$\mathrm{Tor}_1^S(R, M) \cong \Sigma^d M, R \otimes_S M \cong M,$$

and

$$\mathrm{Tor}_i^S(R, M) = 0 \text{ for } i \geq 2.$$

$$\chi_f \in \mathrm{Ext}_R^2(M, \Sigma^d M),$$

or a map

$$\chi_f \colon M \longrightarrow \Sigma^{d+2} M$$

in the derived category of R-modules. In fact this construction lifts to give an element

$$\chi_f \in HH^{d+2}(R|S),$$

and hence in particular that it gives an element of the centre $ZD(R)$ of $D(R)$.

10.B. **The eventually periodic resolution.** Continuing with the above case, we may show that all modules M have free resolutions over R which are eventually periodic of period 2.

Indeed, M has a finite free S-resolution

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

Adding an extra zero term if necessary, we suppose for convenience that n is even. Now apply $(\cdot) \otimes_S R$ to obtain a complex

$$0 \longrightarrow \overline{F}_n \longrightarrow \overline{F}_{n-1} \longrightarrow \cdots \longrightarrow \overline{F}_1 \longrightarrow \overline{F}_0 \longrightarrow M \longrightarrow 0.$$

Since $\mathrm{Tor}_i^S(R,M)=0$ for $i \geq 2$, this is exact except in homological degree 1, where it is $\mathrm{Tor}_1^S(R,M) \cong \Sigma^d M$. Splicing in a second copy of the resolution, we obtain a complex

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{F}_n & \longrightarrow & \overline{F}_{n-1} & \longrightarrow \cdots \longrightarrow & \overline{F}_2 & \longrightarrow & \overline{F}_1 \longrightarrow \overline{F}_0 \longrightarrow M \longrightarrow 0 \\
\oplus & & \oplus & & \oplus & & \oplus & \nearrow & \\
\Sigma^d\overline{F}_{n-1} & \longrightarrow & \Sigma^d\overline{F}_{n-2} & \longrightarrow & \Sigma^d\overline{F}_{n-3} & \longrightarrow \cdots \longrightarrow & \Sigma^d\overline{F}_0 & &
\end{array}$$

which is exact except in the second row, in homological degree 4, where the homology is again M. We may repeatedly splice in additional rows to obtain a free resolution

$$\cdots \longrightarrow G_3 \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

over R. Remembering the convention that n is even, provided the degree is at least n, the modules in the resolution are (up to suspension by a multiple of d)

$$G_{2i} = \overline{F}_n \oplus \Sigma^d\overline{F}_{n-2} \oplus \cdots \oplus \Sigma^{(n-2)d/2}\overline{F}_2 \oplus \Sigma^{nd/2}\overline{F}_0$$

in even degrees and

$$G_{2i+1} = \overline{F}_{n-1} \oplus \Sigma^d\overline{F}_{n-3} \oplus \cdots \oplus \overline{F}_3 \oplus \Sigma^{(n-2)d/2}\overline{F}_1$$

in odd degrees.

10.C. **Smallness.** We may reformulate the eventual periodicity of the previous subsection in homotopy invariant terms.

Lemma 10.1. *If R is a hypersurface $R=S/(f)$ and M is a finitely generated R-module then the mapping cone of $\chi_f : M \longrightarrow \Sigma^{d+2}M$ is small.*

Proof: From the Yoneda interpretation, we notice that

$$\chi_f : M \longrightarrow \Sigma^{d+2}M$$

is realized by the quotient map factoring out the first row subcomplex

$$0 \longrightarrow \overline{F}_n \longrightarrow \overline{F}_{n-1} \longrightarrow \cdots \longrightarrow \overline{F}_1 \longrightarrow \overline{F}_0 \longrightarrow 0.$$

Thus the short exact sequence

$$0 \longrightarrow \overline{F}_\bullet \longrightarrow G_\bullet \longrightarrow \Sigma^{d+2}G_\bullet \longrightarrow 0$$

of R-free chain complexes realizes the triangle

$$\Sigma^1 M/\chi \longrightarrow M \longrightarrow \Sigma^{d+2}M. \qquad \square$$

10.D. **Matrix factorizations.** Because of the importance of matrix factorizations and their prominence in the IRTATCA programme, it is worth a brief note to make the connection. We have considered resolutions of R-modules M, and we note that if M is of projective dimension 1 over S then the S-resolution

$$0 \longleftarrow M \longleftarrow F' \xleftarrow{A} F'' \longleftarrow 0$$

gives a periodic resolution

$$0 \longleftarrow M \longleftarrow \overline{F}' \longleftarrow \overline{F}'' \longleftarrow \overline{F}' \longleftarrow \overline{F}'' \longleftarrow \overline{F}' \longleftarrow \overline{F}'' \longleftarrow \cdots$$

over R.

Since $f = 0$ on M, multiplication by f on the S-resolution is null-homotopic and so there is a diagram

$$\begin{tikzcd}
0 & M \arrow[l] \arrow[d, "0"] & F' \arrow[l] \arrow[d, "f"] & F'' \arrow[l, "A"'] \arrow[d, "f"] \arrow[dl, "B"] & 0 \arrow[l] \\
0 & M \arrow[l] & F' \arrow[l] & F'' \arrow[l, "A"] & 0 \arrow[l]
\end{tikzcd}$$

In short we have two maps $A : F'' \longrightarrow F'$ and $B : F' \longrightarrow F''$ with $AB = f \cdot id, BA = f \cdot id$. Since F' and F'' are free modules, we can choose bases and represent A and B by matrices, so this structure is known as a *matrix factorization.*

If we suppose S is of dimension d, R is of dimension $d-1$. We note that the homological dimension over S is detected in terms of an invariant of R-modules. Indeed, by the Auslander-Buchsbaum formula, M is of depth $d-1$, which is to say it is a maximal Cohen-Macaulay module. For a module M of depth $d-1-p$ (i.e., of projective dimension $p+1$ over S) the above discussion applies to its pth syzygy

11. Bimodules and natural endomorphisms of R-modules

To describe the m-version of the definition of hypersurfaces, we need to briefly discuss bimodules.

11.A. **The centre of the derived category of R-modules.** If R is a commutative Noetherian ring and M is a finitely generated R-module with an eventually n-periodic resolution, there is a map $M \longrightarrow \Sigma^n M$ in the derived category whose mapping cone is a small R-module. If R is a hypersurface, all finitely generated modules have such resolutions. The lesson learnt from commutative algebra is that to use this to characterize hypersurfaces we need to look at all such modules M together, and ask for a natural transformation $1 \longrightarrow \Sigma^n 1$ of the identity functor. By definition the centre $ZD(R)$ is the graded ring of all such natural transformations. There are various ways of constructing elements of the centre, and various natural ways to restrict the elements we consider. Some of these work better than others, and it is the purpose of this section is to introduce these ideas.

11.B. **Bimodules.** We consider a map $S \longrightarrow R$, where S is regular and R is small over S. We may then consider $R^e = R \otimes_S R$, and R^e-modules are $(R|S)$-bimodules. The Hochschild cohomology ring is defined by

$$HH^*(R|S) = \mathrm{Ext}^*_{R^e}(R, R).$$

If $f\colon X \longrightarrow Y$ is a map of $(R|S)$-bimodules, for any R-module M we obtain a map $f \otimes 1\colon X \otimes_R M \longrightarrow Y \otimes_R M$ of (left) R-modules.

The simplest way for us to use this is that if we have isomorphisms $X \cong R$ and $Y \cong \Sigma^n R$ as R-bimodules, the map $f \otimes 1 : M \longrightarrow \Sigma^n M$ is natural in M and therefore gives an element of codegree n in $ZD(R)$: we obtain a map of rings

$$HH^n(R) = \mathrm{Hom}_{R^e}(R, \Sigma^n R) \longrightarrow ZD(R)^n.$$

Continuing, if $X \models_{R^e} Y$ then $X \otimes_R M \models_R Y \otimes_R M$. In particular, if $X = R$ builds a small R^e-module Y then

$$M = R \otimes_R M \models_R Y \otimes_R M =\!| R^e \otimes_R M = R \otimes_S M.$$

Thus if M is finitely generated (i.e., small over S), this shows M finitely builds a small R-module.

We could then restrict the maps permitted in showing that $X \models_{R^e} Y$. We could restrict ourselves to using maps of positive codegree coming from Hochschild cohomology, and say $X \models_{hh} Y$, more generally we could permit any maps of positive codegree from the centre $ZD(R^e)$ and say $X \models_z Y$, or we could relax further and require only that all the maps involved in building are endomorphisms of non-zero degree for some object and say $X \models_e Y$.

Example 11.1. In the topological context the normalization corresponds to a map $X \longrightarrow B\Gamma$. We take $R = C^*(X), S = C^*(B\Gamma)$ and $R^e = C^*(X \times_{B\Gamma} X)$. The associated Hochschild cohomology can be abbreviated

$$HH^*(X|B\Gamma) = \pi_*(\mathrm{Hom}_{C^*(X \times_{B\Gamma} X)}(C^*(X), C^*(X)).$$

12. Hypersurface ring spectra.

We are now ready to describe the s-, g- and m- versions of the hypersurface condition for ring spectra. As usual we begin with the template in commutative algebra, go on to describe it in general and then make it concrete for spaces.

12.A. **The definition in commutative algebra.** In commutative algebra there are three styles for a definition of a hypersurace ring: ideal theoretic, in terms of the growth of the Ext algebra and a derived version. See [25] for a more complete discussion.

Definition 12.1. (i) A local Noetherian ring R is an *s-hypersurface* ring if $R = S/(f)$ some regular ring S and some $f \neq 0$.

(ii) A local Noetherian ring R is a *g-hypersurface* if the dimensions of the k-modules $\mathrm{Ext}^n_R(k,k)$ are bounded.

(iii) A local Noetherian ring R is a *z-hypersurace* if there is an elements $z \in ZD(R)$ of non-zero degree so that M/z is small for all finitely generated modules M. Similarly R is an *hh-hypersurface* if the element z can be chosen to come from Hochschild cohomology.

Remark 12.2. If S is a regular local ring with a map $S \longrightarrow R$ making R into a small R-module we may consider a number of bimodule conditions.

Theorem 12.3 ([12], reformulated in [25]). *For a local Noetherian ring the s-, z- and g-hypersurface conditions are all equivalent.*

12.B. **Definitions for ring spectra.** The appropriate definition of an s-hypersurface can be seen from the discussion of s-regular spectra: the point is that regular elements correspond to maps with exterior cofibres.

Definition 12.4. (i) A ring spectrum R is a *c-hypersurface* if R_* is a hypersurface ring.

(ii) If k is in a single degree, the ring spectrum R is an *s-hypersurface* if there is a normalization $S \longrightarrow R$ with $\pi_*(R \otimes_S k) = \Lambda_k(\tau)$.

(iii) If k is in a single degree, a ring spectrum R is a *g-hypersurface* if the k-modules $\pi_n(\mathrm{Hom}_R(k,k))$ are free over k of bounded rank.

(iv) The c-normalizable ring spectrum R is a *z-hypersurace* if there is an elements $z \in ZD(R)$ of non-zero degree so that M/z is small for all finitely generated modules M. Similarly R is an *hh-hypersurface* if the element z can be chosen to come from Hochschild cohomology.

One immediate source of examples comes from c-hypersurfaces.

Example 12.5. If R is a c-hypersurface then the spectral sequence

$$\mathrm{Ext}^{*,*}_{R_*}(k,k) \Rightarrow \pi_*(\mathrm{Hom}_R(k,k))$$

shows that it is a g-hypersurface.

12.C. **Definitions for spaces.** In view of the fact that regular elements correspond to spherical fibrations, adapting the above definitions for spaces is straightforward.

Definition 12.6. (i) A space X is a *c-hypersurface* if $H^*(X)$ is a hypersurface ring.

(ii) A space X is an *s-hypersurface* (or spherical hypersurface) if it is the total space of a spherical fibration over a connected g-regular space $B\Gamma$: there is a g-regular space $B\Gamma$ and a fibration

$$S^n \longrightarrow X \longrightarrow B\Gamma.$$

(iii) A space X is a *g-hypersurface* space if $H^*(X)$ is Noetherian and the dimensions of the k-vector spaces $H_n(\Omega X)$ are bounded.

(iv) A space X is a *z-hypersurface* space if X is g-normalizable and there is an element $z \in ZD(C^*(X))$ of non-zero degree so that $C^*(Y)/z$ is small for all finitely generated $C^*(Y)$. It is an *hh-hypersurface* if z comes from Hochschild cohomology. (The direct transcription for ring spectra would require that this holds for all finitely generated modules and not just those of the particular form $C^*(Y)$).

Remark 12.7. Other variants have arisen, such as ωsci where we are permitted to use loop spaces on spheres rather than spheres. These conditions arose in Levi's work. This is evidently a weakening of sci which still implies gci.

Over the rationals, s-hypersurfaces, z-hypersurfaces and g-hypersurfaces agree [63]. Over $\mathbb{F}_p$, for g-normalizable spaces X, we have the implications

$$s\text{-hypersurface} \Longrightarrow z\text{-hypersurface} \Longleftrightarrow g\text{-hypersurface}.$$

Three of these are sketched below, but the more difficult one ($\Leftarrow$) is given in [26] and relies on [50].

12.D. **An example.** To start with we may consider the space BA_4 at the prime 2. We will observe directly that it is a hypersurface according to any one of the definitions.

To start with, we note that

$$H^*(BA_4) = H^*(BV_4)^{A_4/V_4} = k[x_2, y_3, z_3]/(r_6)$$

where $r_6 = x_2^3 + y_3^2 + y_3z_3 + z_3^2$. This shows that BA_4 is actually a c-hypersurface and hence also a g-hypersurface. Indeed, the Eilenberg-Moore spectral sequence shows that the loop space homology will eventually have period dividing 4.

In fact we see that BA_4 is an s-hypersurface space at 2. The direct symmetries of a tetrahedron give a homomorphism $A_4 \longrightarrow SO(3)$ and hence a map $BA_4 \longrightarrow BSO(3)$. The fibre is $SO(3)/A_4$, and at the prime 2 this is S^3, so there is a 2-adic fibration

$$S^3 \longrightarrow BA_4 \longrightarrow BSO(3).$$

We will also describe an algebraic approach to showing that BA_4 is a g-hypersurface, which will show that its ultimate period is exactly 4.

12.E. **Squeezed homology.** Since we are working with groups, it is illuminating to recall Benson's purely representation theoretic calculation of the loop space homology $H_*(\Omega(BG_p^\wedge))$ [19]. Benson defines $H\Omega_*(G;k)$ algebraically and proves

$$H_*(\Omega(BG_p^\wedge)) \cong H\Omega_*(G;k),$$

In more detail, $H\Omega_*(G;k)$ is the homology of

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0,$$

a so-called *squeezed resolution* of k. The sequence of projective kG-modules P_i is defined recursively as follows. To start with $P_0 = P(k)$ is the projective cover of k. Now if P_i has been constructed, take $N_i = \ker(P_i \longrightarrow P_{i-1})$ (where we take $P_{-1} = k$), and M_i to be the smallest submodule of N_i so that N_i/M_i is an iterated extension of copies of k. Now take P_{i+1} to be the projective cover of M_i.

12.F. **Trivial cases.** Note that if G is a p-group, we have $\Omega(BG_p^\wedge) \simeq G$ so that the topology focuses on $H_*(\Omega BG) \cong kG$ and since k is the only simple module, $M_0 = 0$ and we again find $H\Omega_*(G) = kG$.

We would expect the next best behaviour to be when $H^*(BG)$ is a hypersurface. Indeed, if $H^*(BG)$ is a polynomial ring modulo a relation of codegree d, the Eilenberg-Moore spectral sequence

$$\mathrm{Ext}^{*,*}_{H^*(BG)}(k,k) \Rightarrow H_*(\Omega(BG_p^\wedge))$$

shows that there is an ultimate periodicity of period $d-2$. The actual period therefore divides $d-2$. We now return to the example of A_4.

12.G. A_4 **revisited.** We described the homotopy theoretic proof that BA_4 is an s-hypersurface space above.

Here we sketch a purely algebraic proof from [19]. To start with, we would like to see algebraically that $H_*(\Omega(BA_4)_2^\wedge)$ is eventually periodic.

This case is small enough to be able to compute products in $H_*(\Omega BG_p^\wedge, k)$ using squeezed resolutions, and we get

$$H_*(\Omega BG_p^\wedge, k) = \Lambda(\alpha) \otimes k\langle \beta, \gamma\rangle/(\beta^2, \gamma^2)$$

with $|\alpha| = 1$ and $|\beta| = |\gamma| = 2$. Beware that β and γ do not commute, so that a k-basis for $H_*(\Omega BG_p^\wedge, k)$ is given by alternating words in β and γ (such as $\beta\gamma\beta$ or the empty word), and α times these alternating words.

There are three simple modules. Indeed, the quotient of A_4 by its normal Sylow 2-subgroup is of order 3; supposing for simplicity that k contains three cube roots of unity $1, \omega, \overline{\omega}$, the simples correspond to how a chosen generator acts. The projective covers of the three simple modules are

$$P(k) = \begin{tikzcd}[column sep=small, row sep=small] & k \arrow[dl, dash] \arrow[dr, dash] & \\ \omega \arrow[dr, dash] & & \bar{\omega} \arrow[dl, dash] \\ & k & \end{tikzcd}, \quad P(\omega) = \begin{tikzcd}[column sep=small, row sep=small] & \omega \arrow[dl, dash] \arrow[dr, dash] & \\ \bar{\omega} \arrow[dr, dash] & & k \arrow[dl, dash] \\ & \omega & \end{tikzcd}, \quad P(\bar{\omega}) = \begin{tikzcd}[column sep=small, row sep=small] & \bar{\omega} \arrow[dl, dash] \arrow[dr, dash] & \\ k \arrow[dr, dash] & & \omega \arrow[dl, dash] \\ & \bar{\omega} & \end{tikzcd}.$$

13. S-HYPERSURFACE SPACES AND Z-HYPERSURFACE SPACES.

In the algebraic setting the remarkable fact is that modules over hypersurfaces have eventually periodic resolutions, and hence that they are hhci of codimension 1. The purpose of this section is to sketch a similar result for spaces.

Theorem 13.1. *If X is an s-hypersurface space with fibre sphere of dimension ≥ 2 then X is a z-hypersurface space.*

13.A. **Split spherical fibrations.** The key in algebra was to consider bimodules, for which we consider the (multiplicative) exact sequence

$$R \longrightarrow R^e \longrightarrow R^e \otimes_R k,$$

where the first map is a monomorphism split by the map μ along which R acquires its structure as an R^e-module structure. This corresponds to the pullback fibration

$$X \longleftarrow X \times_{B\Gamma} X \longleftarrow S^n,$$

split by the diagonal

$$\Delta \colon X \longrightarrow X \times_{B\Gamma} X$$

along which the cochains on X becomes a bimodule. To simplify notation, we consider a more general situation: a fibration

$$B \longleftarrow E \longleftarrow S^n$$

with section $s \colon B \longrightarrow E$. The case of immediate interest is $B = X$, $E = X \times_{B\Gamma} X$, where a $C^*(E)$-module is a $C^*(X)$-bimodule.

Note that by the third isomorphism theorem for fibrations, there is a fibration

$$\Omega S^n \longrightarrow B \xrightarrow{s} E.$$

This gives the required input for the following theorem. The strength of the result is that the cofibre sequences are of $C^*(E)$-modules.

Theorem 13.2. *Suppose given a fibration $\Omega S^n \longrightarrow B \xrightarrow{s} E$ with $n \geq 2$.*

(i) *If n is odd, then there is a cofibre sequence of $C^*(E)$-modules*

$$\Sigma_{n-1} C^*(B) \longleftarrow C^*(B) \longleftarrow C^*(E).$$

(ii) *If n is even, then there are cofibre sequences of $C^*(E)$-modules*

$$C \longleftarrow C^*(B) \longleftarrow C^*(E)$$

and

$$\Sigma_{2n-2}C^*(B) \longleftarrow C \longleftarrow \Sigma_{n-1}C^*(E).$$

In particular the fibre of the composite

$$C^*(B) \longrightarrow C \longrightarrow \Sigma_{2n-2}C^*(B)$$

is a small $C^*(E)$-module constructed with one cell in codegree 0 and one in codegree $n-1$.

Remark 13.3. Note that in either case we obtain a cofibre sequence

$$K \longleftarrow C^*(B) \longleftarrow \Sigma_a C^*(B)$$

of $C^*(E)$-modules with K small.

The strategy is to first prove the counterparts in cohomology by looking at the Serre spectral sequence of the fibration from Part (i) and then lift the conclusion to the level of cochains.

14. Growth of z-hypersurface resolutions

In this section we prove perhaps the simplest implication between the hypersurface conditions: a z-hypersurface is a g-hypersurface.

Lemma 14.1. *If R is a z-hypersurface and k is a field then the vector spaces $\pi_n(\mathcal{E})$ are of bounded dimension and R is a g-hypersurface.*

Proof: Since R is a z-hypersurface, then in particular there is a projective resolution of k which is eventually periodic. In other words there is a triangle

$$\Sigma^n k \longrightarrow k \longrightarrow L$$

with $n \neq 0$ and L small over R. Applying $\mathrm{Hom}_R(\cdot, k)$ we find a triangle

$$\Sigma^{-n}\mathcal{E} \longleftarrow \mathcal{E} \longleftarrow \mathrm{Hom}_R(L, k).$$

Since L is finitely built from R, $\mathrm{Hom}_R(L, k)$ is finite dimensional over k, and hence only nonzero in a finite range of degrees (say $[-N, N]$). Outside that range we have $\pi_{s+n}(\mathcal{E}) \cong \pi_s(\mathcal{E})$, so every homotopy group is isomorphic as a k-vector space to one in the range $[-N-n, N+n]$ and the bound is the largest of these dimensions. $\square$

Remark 14.2. For a general complete intersection, one argues similarly that a cofibre sequence $\Sigma^n A \longrightarrow A \longrightarrow B$ with two terms the same means that the growth rate of A is at most one more than that of B.

Part 5. Gorenstein rings

This is the final part taking a particular classes of commutative local rings, and giving homotopy invariant counterparts of the definitions. The justification consists of the dual facts that the new definition reduces to the old in the classical setting and that the new definition covers and illuminates new examples. The generalization of the Gorenstein condition is perhaps the most successful of the three, since there are so many Gorenstein ring spectra, and this approach provides consequences that are both unexpected and very concrete.

15. The Gorenstein condition

15.A. **Gorenstein local rings.** The usual definition of a Gorenstein local ring is that R is of finite injective dimension over itself. But it is then proved that in fact it is then of injective dimension equal to the Krull dimension and that $\mathrm{Ext}^*_R(k,R) = \mathrm{Ext}^r_R(k,R) = k$. Conversely, if this holds, the ring is Gorenstein. There are various other characterizations of Gorenstein ring spectra, including a duality statement that we will discuss shortly, but this is enough to suggest the definition for ring spectra.

15.B. **Gorenstein ring spectra.** Ultimately, we want to consider duality phenomena modelled on those in commutative algebra of Gorenstein local rings. For ring spectra we provide a parallel development. Corresponding to the Noetherian condition we restricting the class of ring spectra to those which are proxy-small. We begin with the core Gorenstein condition and move onto duality in due course. These definitions come from [40].

15.C. **The Gorenstein condition.** We say that $R \longrightarrow k$ is *Gorenstein* of shift a (and write $\mathrm{shift}(R) = a$) if we have an equivalence

$$\mathrm{Hom}_R(k,R) \simeq \Sigma^a k$$

of R-modules.

Remark 15.1. We will say that it is c-Gorenstein if $R_* \longrightarrow k$ is a Gorenstein local ring. As usual the spectral sequence

$$\mathrm{Ext}^{*,*}_{R_*}(k,R_*) \Rightarrow \pi_*(\mathrm{Hom}_R(k,k))$$

shows that a c-Gorenstein ring spectrum is Gorenstein, but we will give many examples of Gorenstein ring spectra which are not c-Gorenstein.

15.D. **The relative Gorenstein condition.** More generally, we say that $S \longrightarrow R$ is *relatively Gorenstein of shift* a (and write $\mathrm{shift}(R|S) = a$) if

$$\mathrm{Hom}_S(R, S) \simeq \Sigma^a R.$$

Remark 15.2. This is stricter than the usual definition of relatively Gorenstein which would just require $\mathrm{Hom}_S(R, S)$ to be an invertible R-module (rather than a free module), but it covers the main examples we deal with here.

Just as in the classical case, we are interested in proxy-regular rings which satisfy the Gorenstein condition.

16. Gorenstein duality

16.A. **Classical Gorenstein duality.** Although the Gorenstein condition itself is convenient to work with, the real reason for considering it is the duality property that it implies. To formulate this, we use local cohomology in the sense of Grothendieck, and the reader may wish to refer to Appendix B for the basic definitions.

In classical local commutative algebra the Gorenstein duality property is that all local cohomology is in a single cohomological degree, where it is the injective hull $I(k)$ of the residue field. To give a formula, we write $\Gamma_{\mathfrak{m}} M$ for the $\mathfrak{m}$-power torsion in an R-module M, and $H^*_{\mathfrak{m}}(M)$ for the local cohomology of M, recalling Grothendieck's theorem that if R is Noetherian, $H^*_{\mathfrak{m}}(M) = R^*\Gamma_{\mathfrak{m}}(M)$. The Gorenstein duality statement for a local ring of Krull dimension r therefore states

$$H^*_{\mathfrak{m}}(R) = H^r_{\mathfrak{m}}(R) = I(k).$$

If R is a k-algebra, $I(k) = R^\vee = \Gamma_{\mathfrak{m}} \mathrm{Hom}_k(R, k)$.

16.B. **Gorenstein duality for k-algebra spectra.** Turning to ring spectra we will first treat the case that R is a k-algebra. This simplifies things considerably, and covers many interesting examples. The more general case requires a discussion of Matlis lifts, which we hope to include in an expanded version later.

In the case of k-algebras, we may again define $R^\vee = \mathrm{Cell}_k(\mathrm{Hom}_k(R, k))$ and observe this has the Matlis lifting property

$$\mathrm{Hom}_R(T, R^\vee) \simeq \mathrm{Hom}_k(T, k)$$

for any T built from k.

In particular, if R is Gorenstein of shift a we have equivalences of R-modules

$$\mathrm{Hom}_R(k, \mathrm{Cell}_k R) \simeq \mathrm{Hom}_R(k, R) \overset{(G)}{\simeq} \Sigma^a k \overset{(M)}{\simeq} \mathrm{Hom}_R(k, \Sigma^a R^\vee),$$

where the equivalence (G) is the Gorenstein property and the equivalence (M) is the Matlis lifting property. We would like to remove the $\mathrm{Hom}_R(k, \cdot)$ to deduce

$$\mathrm{Cell}_k R \simeq \Sigma^a R^\vee.$$

Such an equivalence is known as *Gorenstein duality*, since $\mathrm{Cell}_k(R)$ is a covariant functor of R and $R^\vee$ is a contravariant functor of R.

Morita theory says that if R is proxy-regular we may make this deduction provided R is orientably Gorenstein in the sense that the right actions of $\mathcal{E} = \mathrm{Hom}_R(k, k)$ on $\Sigma^a k$ implied by the two equivalences (G) and (M) agree.

16.C. **Automatic orientability.** There are a number of important cases where orientability is automatic because $\mathcal{E}$ has a unique action on k, and in this case the Gorenstein condition automatically implies Gorenstein duality.

The first case of this is when R is a classical commutative local ring, although of course we knew already that in this case the Gorenstein condition is equivalent to Gorenstein duality.

From our present point of view, we see this as a consequence of connectivity: $\mathcal{E}$ (whose homology is $\mathrm{Ext}_R^*(k, k)$) has a unique action on k. The same argument applies when the ring spectrum is both a k-algebra and connected.

Proposition 16.1. *Suppose R is a proxy-regular, connected k-algebra and $\pi_*(R)$ is Noetherian with $\pi_0(R) = k$ and maximal ideal $\mathfrak{m}$ of positive degree elements. If R is Gorenstein of shift a, then R it is automatically orientable and so has Gorenstein duality.*

Proof: First we argue that if R is Gorenstein, it is automatically orientable. Indeed, we show that $\mathcal{E}$ has a unique action on k. Since R is a k-algebra, the action of $\mathcal{E}$ on k factors through

$$\mathcal{E} = \mathrm{Hom}_R(k, k) \longrightarrow \mathrm{Hom}_k(k, k) = k,$$

so since k is an Eilenberg-MacLane spectrum, the action is through $\pi_0(\mathcal{E})$. Now we observe that since R is connected, $\mathrm{Ext}^s_{R_*}(k, k)$ is in degrees $\leq -s$, so that the spectral sequence for calculating $\pi_*(\mathrm{Hom}_R(k, k))$ shows $\mathcal{E}$ is coconnective with $\pi_0(\mathcal{E}) = k$ which must act trivially on k. $\square$

We may go a little further to the nilpotent case.

Lemma 16.2. *If X is connected with $\pi_1(X)$ a finite p-group and k is of characteristic p then if $C^*(X)$ is Gorenstein it automatically has Gorenstein duality.*

Proof: Again we find $\mathcal{E}$ has a unique action on k. Since $\mathcal{E}$ is a k-algebra, it acts through $\pi_0(\mathcal{E}) = H_0(\Omega X) = k[\pi_1(X)]$. By the characteristic assumption, this has a unique action on k. $\square$

16.D. **The local cohomology theorem.** In many cases (see Remark E.2) one can give an algebraic description of the k-cellularization and infer algebraic consequences of Gorenstein duality. For simplicity we restrict to local k-algebras, although the methods apply more generally.

Lemma 16.3. *If R_* is a Noetherian k-algebra, with maximal ideal $\mathfrak{m}$ and residue field k, then k-cellularization coincides with the derived $\mathfrak{m}$-power torsion functor. Accordingly, if R has Gorenstein duality, there is a local cohomology spectral sequence*

$$H^*_{\mathfrak{m}}(R_*) \Rightarrow \Sigma^a R_*^\vee.$$

If R has Gorenstein duality, Lemma 16.3 shows that the ring $\pi_*(R)$ has very special properties (even if it falls short of being Gorenstein), studied in [79]. Some of these properties were first observed by Benson and Carlson [20, 21] for group cohomology (corresponding to the special case of the ring spectrum $R = C^*(BG)$, which we will see below has Gorenstein duality).

To start with, we note that the spectral sequence collapses if R_* is Cohen-Macaulay to show $H^r_{\mathfrak{m}}(R_*) \cong \Sigma^{a+r} R_*^\vee$ (where r is the Krull dimension of R_*). Thus the coefficient ring R_* is also Gorenstein.

The spectral sequence also collapses if R_* is of Cohen-Macaulay defect 1, to give an exact sequence

$$0 \longrightarrow H^r_{\mathfrak{m}}(R_*) \longrightarrow \Sigma^{a+r} R_*^\vee \longrightarrow \Sigma H^{r-1}_{\mathfrak{m}}(R_*) \longrightarrow 0.$$

In general, local duality lets one deduce that the cohomology ring R_* is always generically Gorenstein.

The collapse of the local cohomology theorem in the case of Cohen-Macaulay defect ≤ 1 has very concrete consequences in that the Hilbert series of R_* satisfies a suitable pair of functional equations.

Corollary 16.4. [64] *Suppose R has Gorenstein duality of shift a, that $\pi_*(R)$ is Noetherian of Krull dimension r and Hilbert series $p(t) = \sum_s \dim_k(R_s)t^s$.*

If $\pi_*(R)$ is Cohen-Macaulay it is also Gorenstein, and the Hilbert series satisfies

$$p(1/t) = (-1)^r t^{r-a} p(t).$$

If $\pi_*(R)$ is almost Cohen-Macaulay it is also almost Gorenstein, and the Hilbert series satisfies

$$p(1/t) - (-1)^r t^{r-a} p(t) = (-1)^{r-1}(1+t)q(t) \text{ and } q(1/t) = (-1)^{r-1} t^{a-r+1} q(t).$$

In any case $\pi_*(R)$ is Gorenstein in codimension 0 and almost Gorenstein in codimension 1.

17. Ascent, descent and arithmetic of shifts

17.A. **The relatively Gorenstein case.** We make the elementary observation that for any ring map $\theta\colon S \longrightarrow R$

$$\mathrm{Hom}_R(k, \mathrm{Hom}_S(R,S)) \simeq \mathrm{Hom}_R(R \otimes_S k, S) \simeq \mathrm{Hom}_S(k,S).$$

Thus we conclude that if $S \longrightarrow R$ is relatively Gorenstein then R is Gorenstein if and only if S is Gorenstein, and in that case

$$\mathrm{shift}(S) = \mathrm{shift}(R) + \mathrm{shift}(R|S).$$

Example 17.1. (i) The ring map $S = ko \longrightarrow ku = R$ is relatively Gorenstein of shift 2. Indeed, the connective version of Wood's theorem states $ku \simeq ko \wedge (S^0 \cup_\eta e^2)$, so

$$\mathrm{Hom}_{ko}(ku, ko) \simeq \Sigma^{-2} ku.$$

Since $ku_* = \mathbb{Z}[v]$ we see that ku is Gorenstein of shift -4 over $\mathbb{F}_2$, and it follows that ko is Gorenstein of shift -6 over $\mathbb{F}_2$.

(ii) Precisely similar statements hold for tmf. This uses results of Hopkins-Mahowald [82], now written up by Matthew [98].

At the prime 3, there is a map $tmf \longrightarrow tmf_0(2)$. Here $tmf_0(2) \simeq tmf \wedge (S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8)$, so that

$$\mathrm{Hom}_{tmf}(tmf_0(2), tmf) \simeq \Sigma^{-8} tmf_0(2).$$

Since $tmf_0(2)_* = \mathbb{Z}_{(3)}[c_2, c_4]$ (where $|c_i| = 2i$) we see that $tmf_0(2)$ is Gorenstein of shift -15. Hence we deduce by Gorenstein descent that $tmf \longrightarrow \mathbb{F}_3$ is Gorenstein of shift -23.

At the prime 2, there is a map $tmf \longrightarrow tmf_1(3)$. Here $tmf_1(3)$ is a form of $BP\langle 2\rangle$ and $tmf_1(3) \simeq tmf \wedge DA(1)$ so that

$$\mathrm{Hom}_{tmf}(tmf_1(3), tmf) \simeq \Sigma^{-12} tmf_1(3).$$

Since $tmf_1(3)_* = \mathbb{Z}_{(2)}[\alpha_1, \alpha_3]$ (where $|\alpha_i| = 2i$) we see that $tmf_1(3)$ is Gorenstein of shift -11. Hence we deduce by Gorenstein descent that $tmf \longrightarrow \mathbb{F}_2$ is Gorenstein of shift -23.

In general, it can be difficult to decide if $S \longrightarrow R$ is relatively Gorenstein, and we prefer to give conditions depending on the cofibre $Q = R \otimes_S k$.

17.B. **Gorenstein Ascent.** We suppose that $S \longrightarrow R \longrightarrow Q$ is a cofibre sequence of commutative algebras with a map to k, and we now consider the Gorenstein ascent question. When does the fact that S is Gorenstein imply that R is Gorenstein? It is natural to assume that Q is Gorenstein, but it is known this is not generally sufficient.

In effect the Gorenstein Ascent theorem will state that under suitable hypotheses (see Section 17.D) there is an equivalence

$$\mathrm{Hom}_R(k,R) \simeq \mathrm{Hom}_Q(k, \mathrm{Hom}_S(k,S) \otimes_k Q).$$

When this holds (see Subsection 17.D) it follows that if S and Q are Gorenstein, so is R and

$$\mathrm{shift}(R) = \mathrm{shift}(S) + \mathrm{shift}(Q).$$

17.C. **Arithmetic of shifts.** We summarize the behaviour of Gorenstein shifts in the ideal situation when ascent and descent both hold. If all rings and maps are Gorenstein of the indicated shifts

$$\overset{a}{S} \xrightarrow{\lambda} \overset{b}{R} \xrightarrow{\mu} \overset{c}{Q}$$

then $b = a + c, \lambda = -c$ and $\mu = a$

17.D. **When does Gorenstein ascent hold?** The core of our results about ascent come from [40]. Indeed, the proof of [40, 8.6] gives a sufficient condition for Gorenstein ascent in the commutative context.

Lemma 17.2. *If S and R are commutative and the natural map $\nu\colon \mathrm{Hom}_S(k,S)\otimes_S R \longrightarrow \mathrm{Hom}_S(k,R)$ is an equivalence then*

$$\mathrm{Hom}_R(k,R) \simeq \mathrm{Hom}_Q(k, Hom_S(k,S) \otimes_k Q).$$

In this case, if S and Q are Gorenstein, so is R, and the shifts add up: $\mathrm{shift}(R) = \mathrm{shift}(S) + \mathrm{shift}(Q)$. □

Now that we have a sufficient condition for Gorenstein ascent, we want to identify cases in which it is satisfied. It is well known that when R is small over S (or equivalently, when Q is finitely built from k) then ν is an equivalence. This is already a very useful result.

Example 17.3. If we have a fibration $F \longrightarrow E \longrightarrow B$ of spaces to which the Eilenberg-Moore theorem applies in the sense that $C^*(F) = C^*(E) \otimes_{C^*(B)} k$, then if B is Gorenstein and F is an orientable manifold, we deduce that E is Gorenstein.

However ν can be an equivalence more generally. To start with, we emphasize that the hypothesis on ν in Lemma 17.2 only depends on R as a *module* over S. There is useful case where ascent holds when we can express R as an inverse limit of finite S-modules for which ν is an equivalence. We only need a hypothesis to ensure inverse limits and tensor products commute.

Corollary 17.4. *Consider a cofibre sequence $S \longrightarrow R \longrightarrow Q$ of k-algebras. Suppose that $\pi_*(S)$ Noetherian and either connected or simply coconnected and k is a field of of characteristic $p > 0$. If $\pi_*(R)$ and $\pi_*(\mathrm{Hom}_S(k, S))$ are finitely generated $\pi_*(S)$-modules, and $\pi_*(S)$ is concentrated in a finite range of degrees then*

$$\mathrm{Hom}_R(k, R) \simeq \mathrm{Hom}_Q(k, Hom_S(k, S) \otimes_k Q)$$

and Gorenstein ascent holds for the cofibre sequence.

18. Morita invariance of the Gorenstein condition.

We show that the Gorenstein condition is Morita invariant in many useful cases, provided R is a k-algebra. This allows us to deduce striking consequences from well-known examples of Gorenstein rings. For instance we can deduce the local cohomology theorem for finite p-groups from the fact that kG is a Frobenius algebra.

Theorem 18.1. *Suppose R is a k-algebra, and $R^\vee$ is k-cellular and finally that $\mathcal{E}$ and R are Matlis reflexive. Then*

$$\mathrm{Hom}_{\mathcal{E}}(k, \mathcal{E}) \simeq \mathrm{Hom}_R(k, R),$$

and hence

$$\mathcal{E} \text{ is Gorenstein} \iff R \text{ is Gorenstein.}$$

Proof: We use the fact that $E(R^\vee) = k^\vee$, so that we have

$$R^\vee = TE(R^\vee) = Tk^\vee = k^\vee \otimes_{\mathcal{E}} k,$$

and the fact that $\mathcal{E} = \mathrm{Hom}_R(k, k)$ dualizes to give

$$\mathcal{E}^\vee = k \otimes_R k^\vee.$$

Next, note that the expression $k \otimes_R k^\vee \otimes_{\mathcal{E}} k$ makes sense, where the right $\mathcal{E}$-module structure on the first two factors comes from $k^\vee$. The key equality in the proof is simply the associativity isomorphism

$$\mathcal{E}^\vee \otimes_{\mathcal{E}} k = k \otimes_R k^\vee \otimes_{\mathcal{E}} k = k \otimes_R R^\vee.$$

Now we make the following calculation,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{E}}(k, \mathcal{E}) &\simeq \mathrm{Hom}_{\mathcal{E}}(k, (\mathcal{E}^\vee)^\vee) \\
&\simeq \mathrm{Hom}_k(\mathcal{E}^\vee \otimes_{\mathcal{E}} k, k) \\
&\simeq \mathrm{Hom}_k(k \otimes_R k^\vee \otimes_{\mathcal{E}} k, k) \\
&\simeq \mathrm{Hom}_k(k \otimes_R R^\vee, k) \\
&\simeq \mathrm{Hom}_R(k, (R^\vee)^\vee) \\
&\simeq \mathrm{Hom}_R(k, R)
\end{aligned}$$

□

Corollary 18.2. *If G is a finite p-group then kG is Gorenstein of shift 0 and hence $C^*(BG)$ is Gorenstein of shift 0.* $\square$

Combining this with Gorenstein ascent, we have a very general conclusion

Corollary 18.3. *If G is any finite group then $C^*(BG)$ is Gorenstein of shift 0.*

Proof: We may choose a faithful representation $G \longrightarrow U(n)$ and hence obtain a fibration

$$U(n)/G \longrightarrow BG \longrightarrow BU(n).$$

Since $U(n)/G$ is a finite manifold we may apply Gorenstein ascent to the cofibre sequence

$$C^*(U(n)/G) \longleftarrow C^*(BG) \longleftarrow C^*(BU(n)).$$

Since $BU(n)$ and $U(n)/G$ are c-Gorenstein, they are Gorenstein. Since $C^*(U(n)/G)$ is finite dimensional Gorenstein ascent applies to deduce that $C^*(BG)$ is Gorenstein. $\square$

19. Gorenstein duality for group cohomology

This section describes a key example, and deserves an extended account. It will be written later. It may be appropriate to cite [40, 57] for now.

20. Gorenstein duality for rational spaces

This section discusses Gorenstein duality for rational spaces. Félix-Halperin-Thomas [49] have considered the Gorenstein condition at length, and we note here that this implies the Gorenstein duality of [40].

For spaces with finite dimensional cohomology X is Gorenstein if and only if $H^*(X)$ is Gorenstein, but in general there are Gorenstein spaces for which $H^*(X)$ is not Gorenstein and we make explicit the local cohomology theorem and its consequences for $H^*(X)$.

20.A. **Examples.** First we show that there are many familiar examples of Gorenstein DGAs.

Corollary 20.1. [49, 3.6] *If $H^*(A)$ is finite dimensional then A is Gorenstein if and only if $H^*(A)$ is a Poincaré duality algebra.*

Proof: If $H^*(A)$ is a Poincaré duality algebra of formal dimension n then it is a zero dimensional Gorenstein ring with a-invariant $-n$, so A is Gorenstein with shift $-n$ by the previous corollary.

Conversely, if A is Gorenstein of shift a, we have a Gorenstein duality spectral sequence. Since $H^*(A)$ is finite dimensional, it is all torsion. Accordingly, $H^*_{\mathfrak{m}}(H^*(A)) = H^*(A)$, and the spectral sequence reads

$$H^*(A) = \Sigma^a H^*(A)^\vee$$

and $H^*(A)$ is a Poincaré duality algebra of formal dimension $-a$. $\square$

By applying Gorenstein ascent as in Example 17.3 one may use these to construct other examples which are Gorenstein but not c-Gorenstein.

Proposition 20.2. [49, 4.3] *Suppose we have a fibration $F \longrightarrow E \longrightarrow B$ with F finite. If F and B are Gorenstein with shifts f and b then E is Gorenstein with shift $e = f + b$.* $\square$

This allows us to construct innumerable examples. For example any finite Postnikov system is Gorenstein [49, 3.4], so that in particular any sci space is Gorenstein. A simple example will illustrate the duality.

Example 20.3. We construct a rational space X in a fibration

$$S^3 \times S^3 \longrightarrow X \longrightarrow \mathbb{C}P^\infty \times \mathbb{C}P^\infty,$$

so that X is Gorenstein. We will calculate $H^*(X)$ and observe that it is not Gorenstein.

Let V be a graded vector space with two generators u, v in degree 2, and let W be a graded vector space with two generators in degree 4. The two 4-dimensional cohomology classes u^2, uv in $H^*(KV) = \mathbb{Q}[u, v]$ define a map $KV \longrightarrow KW$, and we let X be the fibre, so we have a fibration

$$S^3 \times S^3 \longrightarrow X \longrightarrow KV$$

as required. By [40], this is Gorenstein with shift -4 (being the sum of the shift (viz -6) of $S^3 \times S^3$ and the shift (viz 2) of KV).

It is amusing to calculate the cohomology ring of X. It is $\mathbb{Q}[u, v, p]/(u^2, uv, up, p^2)$ where u, v and p have degrees $2, 2$ and 5. The dimensions of its graded components are
$1, 0, 2, 0, 1, 1, 1, 1, 1, \ldots$ (i.e., its Hilbert series is $p_X(t) = (1 + t^5)/(1 - t^2) + t^2$, where t is of codegree 1).

In calculating local cohomology it is useful to note that $\mathfrak{m} = \sqrt{(v)}$. The local cohomology is $H^0_{\mathfrak{m}}(H^*(X)) = \Sigma_2\mathbb{Q}$ in degree 0 (so that $H^*(X)$ is not Cohen-Macaulay) and as a $\mathbb{Q}[v]$-module $H^1_{\mathfrak{m}}(H^*(X))$ is $\mathbb{Q}[v]^\vee \otimes (\Sigma^{-3}\mathbb{Q} \oplus \Sigma^2\mathbb{Q})$. Since there is no higher local cohomology the local cohomology spectral sequence necessarily collapses, and the resulting exact sequence

$$0 \longrightarrow H^1_{\mathfrak{m}}(H^*(X)) \longrightarrow \Sigma^{-4}H^*(A)^\vee \longrightarrow \Sigma^{-2}\mathbb{Q} \longrightarrow 0$$

is consistent.

Since the Cohen-Macaulay defect here is 1, we have a pair of functional equations

$$p_X(1/t) - (-t)t^{-4}p_X(t) = (1+t)\delta(t)$$

and

$$\delta(1/t) = t^4\delta(t).$$

Indeed, the first equation gives $\delta(t) = t^{-2}$, which is indeed the Hilbert series of $H^0_{\mathfrak{m}}(H^*(X))^\vee$, and it obviously satisfies the second equation.

21. Chromatic examples

[[This section will contain examples of the relative Gorenstein condition in deducing spectra are Gorenstein]]

22. Gorenstein topological Hochschild homology

The purpose of this section is to describe the fact that topological Hochschild homology gives a large supply of Gorenstein ring spectra [62].

22.A. **THH.** Topological Hochschild homology is just Hochschild homology relative to the sphere spectrum. Since $\mathbb{S}$ is initial, for any ring spectrum R we may form $R^e = R \otimes_{\mathbb{S}} R$ and then for any (R,R)-bimodule M

$$THH(R;M) = HH_\bullet(R|\mathbb{S};M) = R \otimes_{R^e} M.$$

If M is an R-algebra this is itself a ring spectrum, and if R and M are commutative, so is $THH(R;M)$.

22.B. **The Gorenstein condition for THH.** The point about this section is that a number of difficult calculations by Bökstedt, McClure-Staffeldt, Ausoni, Rognes, and others had shown that $\pi_*THH(R;k)$ is often Gorenstein. The point of this section is to explain that there is a reason for this, to explain the shift and to give a number of examples.

Theorem 22.1. *If*

(1) $R \longrightarrow k$ *is Gorenstein of shift* a
(2) k *is a field of characteristic* p *and*

(3) *k is small over R*

then $THH(R;k)$ is Gorenstein of shift $-a-3$.

The idea of proof is rather simple. First, we define Q to be the cofibre of $R \longrightarrow k$ (so that $Q = k \otimes_R k$). We may apply Gorenstein ascent to the cofibre sequence, to deduce that Q is Gorenstein of shift $-a$.

Now Dundas's lemma shows there is a cofibre sequence

$$Q \longrightarrow THH(R;k) \longrightarrow THH(k),$$

and we argue that Gorenstein ascent holds again. This is done by showing that $THH(R;k)$ can be approximated appropriately, using the fact that Q is finite dimensional and that k is of characteristic p. Accordingly we deduce that $THH(R;k)$ is Gorenstein and

$$\mathrm{shift}(THH(R;k)) = \mathrm{shift}(Q) + \mathrm{shift}(THH(k)).$$

Since Bökstedt [28] shows $\pi_*(THH(k)) = k[\mu_2]$ where μ_2 is of degree 2, we see that $THH(k)$ is Gorenstein of shift -3, and the result follows.

22.C. **Examples.** We briefly list a number of examples covered by Theorem 22.1.

Example 22.2 (**The map $R = \mathbb{F}_p \longrightarrow \mathbb{F}_p = k$).** We note that since $\mathbb{F}_p \longrightarrow \mathbb{F}_p$ is Gorenstein of shift 0, the theorem predicts $THH(k)$ is Gorenstein of shift -3, which is consistent with Bökstedt's calculation $THH_*(k) = k[\mu_2]$.

Example 22.3 (**The map $R = \mathbb{Z} \longrightarrow \mathbb{F}_p = k$).** We observe that $\mathbb{Z} \longrightarrow \mathbb{F}_p$ is Gorenstein of shift -1, so the theorem predicts $THH(\mathbb{Z};\mathbb{Z}/p)$ is Gorenstein of shift -2.

This is again consistent with Bökstedt's exact calculation

$$THH_*(\mathbb{Z};\mathbb{F}_p) = \mathbb{F}_p[\mu_{2p}] \otimes \Lambda_{\mathbb{F}_p}(\lambda_{2p-1}),$$

since this is Gorenstein of shift $(-2p-1)+(2p-1) = -2$.

Example 22.4 (**The map $R = lu \longrightarrow \mathbb{F}_p = k$).** We consider $THH(lu;\mathbb{F}_p)$ where lu is the Adams summand of p-local connective K-theory with coefficients $lu_* = \mathbb{Z}_{(p)}[v_1]$. It is easy to check that $\mathbb{F}_p$ is small over lu and that $lu \longrightarrow \mathbb{F}_p$ is Gorenstein of shift $-(2p-2)-1-1 = -2p$. The theorem then predicts $THH(lu;\mathbb{F}_p)$ is Gorenstein of shift $(2p-1)+(-2) = 2p-3$.

Again, this is consistent with known exact calculations. McClure-Staffeldt [93] (see also Ausoni-Rognes [10]) have calculated $THH_*(lu;\mathbb{F}_p) = \mathbb{F}_p[\mu_{2p^2}] \otimes \Lambda_{\mathbb{F}_p}(\lambda_{2p-1}, \lambda_{2p^2-1})$. Thus $THH_*(lu;\mathbb{F}_p)$ is Gorenstein of shift $(-2p^2-1) + (2p-1+2p^2-1) = 2p-3$.

Example 22.5 (**The map** $R = ku \longrightarrow \mathbb{F}_p$)**.** If we take $R \longrightarrow k$ to be $ku \longrightarrow \mathbb{F}_p$ we immediately see ku is Gorenstein of shift -4. We deduce $THH(ku;\mathbb{F}_p)$ is Gorenstein of shift $-(-4)-3=1$.

Example 22.6 (**The map** $R = ko \longrightarrow H\mathbb{F}_2 = k$)**.** Using Example 17.1 we apply the theorem to $ko \longrightarrow \mathbb{F}_2$, and conclude that $THH(ko;\mathbb{F}_2)$ is Gorenstein of shift $-(-6)-3=3$

Angeltveit and Rognes [10] show that $THH_*(ko;\mathbb{F}_2) = \Lambda(\lambda_5,\lambda_7)\otimes\mathbb{F}_2[\mu_8]$. This is Gorenstein of shift $5+7-8-1=3$ so $THH(ko;\mathbb{F}_2)$ is c-Gorenstein of shift 3.

Example 22.7 (*tmf localized at 3*)**.** We apply the theorem to $tmf \longrightarrow \mathbb{F}_3$. First, observe that since $tmf_0(3)$ is small over tmf and $\mathbb{F}_3$ is small over $tmf_0(3)$ then $\mathbb{F}_3$ is small over tmf. Since tmf is Gorenstein of shift -23 we may apply the theorem to deduce $THH(tmf;\mathbb{F}_3))$ is Gorenstein of shift 20.

Example 22.8 (*tmf localized at 2*)**.** We apply the theorem to $tmf \longrightarrow \mathbb{F}_3$. First, observe that since $tmf_1(2)$ is small over tmf and $\mathbb{F}_2$ is small over $tmf_1(2)$ then $\mathbb{F}_2$ is small over tmf. Since tmf is Gorenstein of shift -23 we may apply the theorem to deduce $THH(tmf;\mathbb{F}_2))$ is Gorenstein of shift 20.

Example 22.9. We apply the theorem to $e_n \longrightarrow \mathbb{F}_p$, where e_n is the connective Lubin-Tate commutative ring spectrum with homotopy $W(\mathbb{F}_{p^n})[[u_1,\ldots,u_{n-1}]][u]$. From its homotopy we see that it is Gorenstein of shift $-n-3$. From the theorem we conclude $THH(e_n;\mathbb{F}_p)$ is Gorenstein of shift n.

Appendix A. Models of spectra

The purpose of this appendix is to outline the construction of symmetric and orthogonal spectra. The details will not be needed for the lectures, and the point is simply to show that spectra are rather concrete objects.

I. **Naive spectra.** This subsection is designed to explain the idea behind spectra: where they came from and why they were invented. Those already familiar spectra should skip directly to Subsection II, which describes symmetric spectra.

The underlying idea is that spectra are just stabilised spaces and the bonus is that they represent cohomology theories. This definition is perfectly good for additive issues, but it does not have a symmetric monoidal smash product, so is not adequate for commutative algebra.

Definition A.1. A *spectrum* E is a sequence of based spaces E_k for $k \geq 0$ together with structure maps

$$\sigma\colon \Sigma E_k \longrightarrow E_{k+1}.$$

A map of spectra $f\colon E \to F$ is a sequence of maps so that the squares

$$\begin{array}{ccc} \Sigma E_k & \xrightarrow{\Sigma f_k} & \Sigma F_k \\ \downarrow & & \downarrow \\ E_{k+1} & \xrightarrow{f_{k+1}} & F_{k+1} \end{array}$$

commute for all k.

Remark A.2. May and others would call this a 'prespectrum'.

Example A.3. If X is a based space we may define the suspension spectrum $\Sigma^\infty X$ to have kth term $\Sigma^k X$ with the structure maps being the identity.

Remark: It is possible to make a definition of homotopy immediately, but this does not work very well for arbitrary spectra. Nonetheless it will turn out that for finite CW-complexes K, maps out of a suspension spectrum are given by

$$[\Sigma^\infty K, E] = \lim_{\to\ k} [\Sigma^k K, E_k]_{\text{unstable}}.$$

In particular

$$\pi_n(E) := [\Sigma^\infty S^n, E] = \lim_{\to\ k} [S^{n+k}, E_k]_{\text{unstable}}.$$

For example if $E = \Sigma^\infty L$ for a based space L, we obtain the *stable homotopy groups*

$$\pi_n(\Sigma^\infty L) = \lim_{\to\ k} [\Sigma^k S^n, \Sigma^k L]_{\text{unstable}},$$

By the Freudenthal suspension theorem, this is the common stable value of the groups $[\Sigma^k S^n, \Sigma^k L]_{\text{unstable}}$ for large k. Thus spectra have captured stable homotopy groups.

Construction A.4. We can suspend spectra by any integer r, defining $\Sigma^r E$ by

$$(\Sigma^r E)_k = \begin{cases} E_{k-r} & k - r \geq 0 \\ pt & k - r < 0. \end{cases}$$

Notice that if we ignore the first few terms, Σ^r is inverse to Σ^{-r}. Homotopy groups involve a direct limit and therefore do not see these first few terms. Accordingly, once we invert homotopy isomorphisms, the suspension functor becomes an equivalence of categories. Because suspension is an equivalence, we say that we have a *stable* category.

Example A.5. In particular we have sphere spectra. We write $\mathbb{S} = \Sigma^\infty S^0$ for the 0-sphere because of its special role, and then define

$$S^r = \Sigma^r \mathbb{S} \quad \text{for all integers } r.$$

Note that S^r now has meaning for a space and a spectrum for $r \geq 0$, but since we have an isomorphism $S^r \cong \Sigma^\infty S^r$ of spectra for $r \geq 0$ the ambiguity is not important. We extend this ambiguity, by often suppressing Σ^∞.

Example A.6. *Eilenberg-MacLane spectra.* An Eilenberg-MacLane space of type (R, k) for a group R and $k \geq 0$ is a CW-complex $K(R, k)$ with $\pi_k(K(R, k)) = R$ and $\pi_n(K(R, k)) = 0$ for $n \neq k$; any two such spaces are homotopy equivalent. It is well known that in each degree ordinary cohomology is represented by an Eilenberg-MacLane space. Indeed, for any CW-complex X, we have $H^k(X; R) = [X, K(R, k)]_{\text{unstable}}$. In fact, this sequence of spaces, as k varies, assembles to make a spectrum.

To describe this, first note that the suspension functor Σ is defined by smashing with the circle S^1, so it is left adjoint to the loop functor Ω defined by $\Omega X := \mathrm{map}(S^1, X)$ (based loops, with a suitable topology). In fact there is a homeomorphism

$$\mathrm{map}(\Sigma W, X) = \mathrm{map}(W \wedge S^1, X) \cong \mathrm{map}(W, \mathrm{map}(S^1, X)) = \mathrm{map}(W, \Omega X)$$

This passes to homotopy, so looping shifts homotopy in the sense that $\pi_n(\Omega X) = \pi_{n+1}(X)$. We conclude that there is a homotopy equivalence

$$\tilde{\sigma}\colon K(R, k) \xrightarrow{\simeq} \Omega K(R, k+1),$$

and hence we may obtain a spectrum

$$HR = \{K(R, k)\}_{k \geq 0}$$

where the bonding map

$$\sigma\colon \Sigma K(R, k) \longrightarrow K(R, k+1)$$

is adjoint to $\tilde{\sigma}$. Thus we find

$$\begin{aligned}[\Sigma^r \Sigma^\infty X, HR] &= \lim_{\to\, k} [\Sigma^r \Sigma^k X, K(R, k)]_{\text{unstable}} \\ &= \lim_{\to\, k} H^k(\Sigma^r \Sigma^k X; R) = H^{-r}(X; R).\end{aligned}$$

In particular the Eilenberg-MacLane spectrum has homotopy in a single degree like the spaces from which it was built:

$$\pi_k(HR) = \begin{cases} R & k = 0 \\ 0 & k \neq 0. \end{cases}$$

Example A.7. The theory of vector bundles gives rise to topological K-theory. Indeed, the unreduced complex K-theory of an unbased compact space X is given by

$$K(X) = Gr(\mathbb{C}\text{-bundles over } X),$$

where Gr is the Grothendieck group completion. The reduced theory is defined by $K^0(X) = \ker(K(X) \longrightarrow K(pt))$, and represented by the space $BU \times \mathbb{Z}$ in the sense that

$$K^0(X) = [X, BU \times \mathbb{Z}]_{\text{unstable}}.$$

The suspension isomorphism allows one to define $K^{-n}(X)$ for $n \geq 0$, but to give $K^n(X)$ we need Bott periodicity [14, 33]. In terms of the cohomology theory, Bott periodicity states $K^{i+2}(X) \cong K^i(X)$, and in terms of representing spaces it states

$$\Omega^2(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}.$$

Hence we may define the representing spectrum K by giving it $2n$th term $BU \times \mathbb{Z}$ and 2-fold bonding maps adjoint to the Bott periodicity equivalence $BU \times \mathbb{Z} \xrightarrow{\simeq} \Omega^2(BU \times \mathbb{Z})$. We then find

$$[\Sigma^\infty X, K] = \lim_{\to k} [\Sigma^{2k} X, BU \times \mathbb{Z}]_{\text{unstable}} = [X, BU \times \mathbb{Z}]_{\text{unstable}} = K^0(X)$$

Remark A.8. (a) Spectra with the property $\Omega E_{k+1} \simeq E_k$ for all k are called Ω-*spectra* (sometimes pronounced 'loop spectra'). As we saw for K-theory, it is then especially easy to calculate $[\Sigma^\infty X, E]$ since

$$[\Sigma^{k+1} X, E_{k+1}]_{\text{unstable}} \cong [\Sigma^k X, \Omega E_{k+1}]_{\text{unstable}} \cong [\Sigma^k X, E_k]_{\text{unstable}}$$

and all maps in the limit system are isomorphisms.

In particular

$$\pi_n(E) = \pi_n(E_0) \quad \text{for } n \geq 0$$

and in fact more generally

$$\pi_n(E) = \pi_{n+k}(E_k) \quad \text{for } n + k \geq 0.$$

(b) If X is a Ω-spectrum, the 0th term X_0 has the remarkable property that it is equivalent to a k-fold loop space for each k (indeed, $X_0 \simeq \Omega^k X_k$). Spaces with this property are called Ω^∞-*spaces* (sometimes pronounced 'infinite loop spaces'). The space X_0 does not retain information about negative homotopy groups of X, but if $\pi_i(X) = 0$ for $i < 0$ (we say X is *connective*), and we retain information about *how* it is a k-fold loop space for each k we have essentially recovered the spectrum X. The study of Ω^∞-spaces is equivalent to the category of connective spectra in a certain precise sense.

II. **Symmetric spectra** [85, 107]**.** Symmetric spectra give an elementary and combinatorial construction of a symmetric monoidal category of spectra. This is excellent for an immediate access to the formal properties, but to be able to calculate with symmetric spectra and to relate them to the rest of homotopy theory one needs to understand the construction of the *homotopy* category. This is somewhat indirect, and Subsection I was intended as a motivational substitute.

It is usual to give a fully combinatorial construction of symmetric spectra, by basing them on simplicial sets rather than on topological spaces.

Definition A.9. (a) A *symmetric sequence* is a sequence

$$E_0, E_1, E_2, \ldots,$$

of pointed simplicial sets with basepoint preserving action of the symmetric group Σ_n on E_n.

(b) We may define a tensor product $E \otimes F$ of symmetric sequences E and F by

$$(E \otimes F)_n := \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q),$$

where the subscript $+$ indicates the addition of a disjoint basepoint.

It is elementary to check that this has the required formal behaviour.

Lemma A.10. *The product $\otimes$ is symmetric monoidal with unit*

$$(S^0, *, *, *, \dots).$$

Example A.11. The sphere is the symmetric sequence $\mathbb{S} := (S^0, S^1, S^2, \dots)$. Here $S^1 = \Delta^1/\Delta^1$ is the simplicial circle and the higher simplicial spheres are defined by taking smash powers, so that $S^n = (S^1)^{\wedge n}$; this also explains the actions of the symmetric groups.

It is elementary to check that *the sphere is a commutative monoid* in the category of symmetric sequences.

Definition A.12. A *symmetric spectrum* E is a left $\mathbb{S}$-module in symmetric sequences.

Unwrapping the definition, we see that this means E is given by

(1) a sequence $E_0, E_1, E_2, \dots$ of simplicial sets,

(2) maps $\sigma\colon S^1 \wedge X_n \to X_{n+1}$, and

(3) basepoint preserving left actions of Σ_n on X_n which are compatible with the actions in the sense that the composite maps $S^p \wedge X_n \to X_{n+p}$ are $\Sigma_p \times \Sigma_n$ equivariant.

Definition A.13. The smash product of symmetric spectra is

$$E \wedge_{\mathbb{S}} F := \mathrm{coeq}(E \otimes \mathbb{S} \otimes F \rightrightarrows E \otimes F).$$

Proposition A.14. *The tensor product over $\mathbb{S}$ is a symmetric monoidal product on the category of symmetric spectra.*

It is now easy to give the one example most important to commutative algebraists.

Example A.15. For any abelian group M, we define the Eilenberg-MacLane symmetric spectra. For a set T we write $M \otimes T$ for the T-indexed sum of copies of M; this is natural for maps of sets and therefore extends to an operation on simplicial sets. We may then define the Eilenberg-MacLane symmetric spectrum $HM := (M \otimes S^0, M \otimes S^1, M \otimes S^2, \dots)$. It is elementary to check that if R is a commutative ring, then HR is a monoid in the category of $\mathbb{S}$-modules, and if M is an R-module, HM is a module over HR.

The homotopy category is constructed from an appropriate model structure on symmetric spectra in the usual way.

III. **Orthogonal spectra.** The process of passage to homotopy of symmetric spectra can appear complicated. One may use more highly structured objects, and obtain orthogonal spectra [96, 97]. The cost is that examples with so much structure do not often occur naturally, and constructions are not so elementary. However, there are numerous advantages: the passage to homotopy theory is easier to understand, it is easier to construct equivariant and global spectra. In any case all these categories of spectra have equivalent homotopy categories, so examples constructed in one category can be imported into another.

We let $\mathcal{I}$ denote the category of finite dimensional real inner product spaces; the set of morphisms between a pair of objects forms a topological space, and the composition maps are continuous. For example $\mathcal{I}(U, U)$ is the orthogonal group $O(U)$.

Definition A.16. An *$\mathcal{I}$-space* is a continuous functor $X\colon \mathcal{I} \longrightarrow \mathbf{Spaces}_*$ to the category of based spaces.

Notice the large amount of naturality we require: for example $O(U)$ acts on $X(U)$, and an isometry $U \longrightarrow V$ gives a splitting $V = U \oplus V'$ so that $X(U) \longrightarrow X(V) = X(U \oplus V')$ is also $O(U)$-equivariant.

A very important example is the functor $\mathbb{S}$ which takes an inner product space V to its one point compactification S^V.

There is a natural external smash product of $\mathcal{I}$-spaces, so that if X and Y are $\mathcal{I}$-spaces we may form

$$X \overline{\wedge} Y\colon \mathcal{I} \times \mathcal{I} \longrightarrow \mathbf{Spaces}_*$$

by taking $(X \overline{\wedge} Y)(U, V) := X(U) \wedge Y(V)$.

Definition A.17. An *orthogonal spectrum* is an $\mathcal{I}$-space X together with a natural map

$$\sigma\colon X \overline{\wedge} \mathbb{S} \longrightarrow X \circ \oplus$$

so that the evident unit and associativity diagrams commute. Decoding this, we see that the basic structure consists of maps

$$\sigma_{U,V} : X(U) \wedge S^V \longrightarrow X(U \oplus V),$$

and this commutes with the action of $O(U) \times O(V)$.

One may define the objects which play the role of rings without defining the smash product.

Definition A.18. An *$\mathcal{I}$-functor with smash product* (or $\mathcal{I}$-FSP) is an $\mathcal{I}$-space X with a unit $\eta\colon \mathbb{S} \longrightarrow X$ and a natural map $\mu\colon X\overline{\wedge}X \longrightarrow X \circ \oplus$. We require that μ is associative, that η is a unit (and central) in the evident sense. For a commutative $\mathcal{I}$-FSP we impose a commutativity condition on μ.

Note that the unit is given by maps

$$\eta_V : S^V \longrightarrow X(V)$$

and the product μ is given by maps

$$\mu_{U,V} : X(U) \wedge X(V) \longrightarrow X(U \oplus V).$$

Thus, by composition we obtain maps

$$X(U) \wedge S^V \longrightarrow X(U) \wedge X(V) \longrightarrow X(U \oplus V),$$

and one may check that these give an $\mathcal{I}$-FSP the structure of an orthogonal spectrum.

Remark A.19. The notion of $\mathcal{I}$-FSP is closely related to the FSPs introduced by Bökstedt in algebraic K-theory before a symmetric monoidal smash product was available. An FSP is a functor from simplicial sets to simplicial sets with unit and product. The restriction of an FSP to (simplicial) spheres is analogous to a $\mathcal{I}$-FSP and gives rise to a ring in symmetric spectra. □

To define a smash product one first defines the smash product of $\mathcal{I}$-spaces by using a Kan extension to internalize the product $\overline{\wedge}$ described above. Now observe that $\mathbb{S}$ is a monoid for this product and define the smash product of orthogonal spectra to be the coequalizer

$$X \wedge_{\mathbb{S}} Y := \mathrm{coeq}(X \wedge \mathbb{S} \wedge Y \rightrightarrows X \wedge Y).$$

The monoids for this product are essentially the same as $\mathcal{I}$-FSPs.

As for symmetric spectra, a fair amount of model categorical work is necessary to construct the associated homotopy category, but orthogonal spectra have the advantage that the weak equivalences are the stable homotopy isomorphisms

Appendix B. Algebraic definitions: Local and Čech cohomology and homology

Related surveys are given in [67, 68]. The material in this section is based on [56, 66, 75]. Background in commutative algebra can be found in [35, 99].

I. **The functors.** Suppose to begin with that R is a commutative Noetherian ring and that $I = (\alpha_1, \dots, \alpha_n)$ is an ideal in R. We shall be concerned especially with two naturally occurring functors on R-modules: the I-power torsion functor and the I-adic completion functor.

The I-power torsion functor Γ_I is defined by

$$M \longmapsto \Gamma_I(M) = \{x \in M \mid I^k x = 0 \text{ for } k >> 0\}.$$

We say that M is an I-power torsion module if $M = \Gamma_I M$. It is easy to check that the functor Γ_I is left exact.

The I-adic completion functor is defined by

$$M \longmapsto M_I^\wedge = \lim_{\leftarrow k} M/I^k M.$$

The Artin-Rees lemma implies that I-adic completion is exact on finitely generated modules, but it is neither right nor left exact in general.

II. **The stable Koszul complex.** We begin with a sequence $\alpha_1, \dots, \alpha_n$ of elements of R and define various chain complexes. In Subsection III we explain why the chain complexes only depend on the radical of the ideal $I = (\alpha_1, \dots, \alpha_n)$ generated by the sequence, in Subsection IV we define associated homology groups, and in Subsection V we give conceptual interpretations of this homology under Noetherian hypotheses.

We begin with a single element $\alpha \in R$, and an integer $s \geq 0$, and define the sth unstable Koszul complex by

$$K_s^\bullet(\alpha) = (\alpha^s \colon R \longrightarrow R)$$

where the non-zero modules are in cohomological degrees 0 and 1. These complexes form a direct system as s varies,

$$\begin{array}{cccccc}
K_1^\bullet(\alpha) & = (& R & \xrightarrow{\alpha} & R &) \\
\downarrow & = & \downarrow & & \downarrow\alpha & \\
K_2^\bullet(\alpha) & = (& R & \xrightarrow{\alpha^2} & R &) \\
\downarrow & = & \downarrow & & \downarrow\alpha & \\
K_3^\bullet(\alpha) & = (& R & \xrightarrow{\alpha^3} & R &) \\
\downarrow & = & \downarrow & & \downarrow\alpha &
\end{array}$$

and the direct limit is the *flat stable* Koszul complex

$$K_\infty^\bullet(\alpha) = (R \longrightarrow R[1/\alpha]) .$$

When defining local cohomology, it is usual to use the complex $K_\infty^\bullet(\alpha)$ of flat modules. However, we shall need a complex of projective R-modules to define the dual local homology modules. Accordingly, we take a particularly convenient projective approximation $PK_\infty^\bullet(\alpha)$ to $K_\infty^\bullet(\alpha)$. Instead of taking the direct limit of the $K_s^\bullet(\alpha)$, we take their homotopy direct limit. This makes the translation to the topological context straightforward. More concretely, our model for $PK^{\bullet}_{\infty}(\alpha)$ is displayed as the upper row in the homology isomorphism

$$\begin{array}{ccccc} PK^{\bullet}(\alpha) & = (& R \oplus R[x] & \xrightarrow{\langle 1, \alpha x - 1\rangle} & R[x] \ ) \\ \downarrow & & \downarrow{\scriptstyle \langle 1,0\rangle} & & \downarrow{\scriptstyle g} \\ K^{\bullet}(\alpha) & = (& R & \longrightarrow & R[1/\alpha] \ ), \end{array}$$

where $g(x^i) = 1/\alpha^i$. Like $K^{\bullet}_{\infty}(\alpha)$, this choice of $PK^{\bullet}_{\infty}(\alpha)$ is non-zero only in cohomological degrees 0 and 1.

The stable Koszul cochain complex for a sequence $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is obtained by tensoring together the complexes for the elements, so that

$$K^{\bullet}_{\infty}(\boldsymbol{\alpha}) = K^{\bullet}_{\infty}(\alpha_1) \otimes_R \cdots \otimes_R K^{\bullet}_{\infty}(\alpha_n),$$

and similarly for the projective complex $PK^{\bullet}_{\infty}(\boldsymbol{\alpha})$.

III. **Invariance statements.** We list some basic properties of the stable Koszul complex, leaving proofs to the reader.

Lemma B.1. *If β is in the ideal $I = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, then $K^{\bullet}_{\infty}(\boldsymbol{\alpha})[1/\beta]$ is exact.*

Note that, by construction, we have an augmentation map

$$\varepsilon \colon K^{\bullet}_{\infty}(\boldsymbol{\alpha}) \longrightarrow R.$$

Corollary B.2. *Up to quasi-isomorphism, the complex $K^{\bullet}_{\infty}(\boldsymbol{\alpha})$ depends only on the radical of the ideal I.*

In view of Corollary B.2 it is reasonable to write $K^{\bullet}_{\infty}(I)$ for $K^{\bullet}_{\infty}(\boldsymbol{\alpha})$. Since $PK^{\bullet}_{\infty}(\boldsymbol{\alpha})$ is a projective approximation to $K^{\bullet}_{\infty}(\boldsymbol{\alpha})$, it too depends only on the radical of I. We also write $K^{\bullet}_{s}(I) = K^{\bullet}_{s}(\alpha_1) \otimes \cdots \otimes K^{\bullet}_{s}(\alpha_n)$, but this is an abuse of notation since even its homology groups do depend on the choice of generators.

IV. **Local homology and cohomology.** The local cohomology and homology of an R-module M are then defined by

$$H^*_I(R; M) = H^*(PK^{\bullet}_{\infty}(I) \otimes M)$$

and

$$H^I_*(R; M) = H_*(\mathrm{Hom}(PK^{\bullet}_{\infty}(I), M).$$

Note that we could equally well use the flat stable Koszul complex in the definition of local cohomology, as is more usual. Lemma B.1 shows that $H^*_I(M)[1/\beta] = 0$ if $\beta \in I$, so $H^*_I(M)$ is an I-power torsion module and supported over I.

It is immediate from the definitions that local cohomology and local homology are related by a third quadrant universal coefficient spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_R(H_I^{-t}(R), M) \Longrightarrow H^I_{-t-s}(R; M), \tag{1}$$

with differentials $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$.

V. **Derived functors.** We gave our definitions in terms of specific chain complexes. The meaning of the definitions appears in the following two theorems.

Theorem B.3 (Grothendieck [75])**.** *If R is Noetherian, then the local cohomology groups calculate the right derived functors of the left exact functor $M \longmapsto \Gamma_I(M)$. In symbols,*

$$H_I^n(R; M) = (R^n\Gamma_I)(M). \qquad \square$$

This result may be used to give an explicit expression for local cohomology in familiar terms. Indeed, since $\Gamma_I(M) = \lim_{\rightarrow r} \operatorname{Hom}(R/I^r, M)$, and the right derived functors of the right-hand side are obvious, we have

$$(R^n\Gamma_I)(M) \cong \lim_{\rightarrow r} \operatorname{Ext}_R^n(R/I^r, M).$$

The description in terms of the stable Koszul complex is usually more practical.

Theorem B.4 (Greenlees-May [66])**.** *If R is Noetherian, then the local homology groups calculate the left derived functors of the (not usually right exact) I-adic completion functor $M \longmapsto M_I^{\wedge}$. Writing L_n^I for the left derived functors of I-adic completion, this gives*

$$H_n^I(R; M) = L_n^I(M). \qquad \square$$

The conclusions of Theorem B.3 and B.4 are true under much weaker hypotheses [66, 105].

VI. **The shape of local cohomology.** One is used to the idea that I-adic completion is often exact, so that L_0^I is the most significant of the left derived functors. However, it is the top non-vanishing right derived functor of Γ_I that is the most significant. Some idea of the shape of these derived functors can be obtained from the following result. Observe that the complex $PK_\infty^\bullet(\boldsymbol{\alpha})$ is non-zero only in cohomological degrees between 0 and n, so that local homology and cohomology are zero above dimension n. A result of Grothendieck usually gives a much better bound. We write $\dim(R)$ for the Krull dimension of R and $\operatorname{depth}_I(M)$ for the I-depth of a module M (the length of the longest M-regular sequence from I).

Theorem B.5 (Grothendieck [74])**.** *If R is Noetherian of Krull dimension d, then*

$$H_I^i(M) = 0 \quad \textit{and} \quad H_i^I(M) = 0 \ \textit{ if } i > d.$$

If $e = \operatorname{depth}_I(M)$ then

$$H_I^i(M) = 0 \textit{ if } i < e.$$

If R is Noetherian, M is finitely generated, and $IM \neq M$, then

$$H_I^e(M) \neq 0. \qquad \square$$

Grothendieck's proof of vanishing begins by noting that local cohomology is sheaf cohomology with support. It then proceeds by induction on the Krull dimension and reduction to the irreducible case. The statement about depth is elementary, and proved by induction on the length of the I-sequence (see [99, 16.8]).

The Universal Coefficient Theorem gives a useful consequence for local homology.

Corollary B.6. *If R is Noetherian and $\mathrm{depth}_I(R) = \dim(R) = d$, then*

$$L^I_s M = \mathrm{Ext}^{d-s}_R(H^d_I(R), M). \qquad \square$$

For example if $R = \mathbb{Z}$ and $I = (p)$, then $H^*_{(p)}(\mathbb{Z}) = H^1_{(p)}(\mathbb{Z}) = \mathbb{Z}/p^\infty$. Therefore the corollary states that

$$L^{(p)}_0 M = \mathrm{Ext}(\mathbb{Z}/p^\infty, M) \quad \text{and} \quad L^{(p)}_1 M = \mathrm{Hom}(\mathbb{Z}/p^\infty, M),$$

as was observed in Bousfield-Kan [32, VI.2.1].

VII. **Čech homology and cohomology.** We have motivated local cohomology in terms of I-power torsion, and it is natural to consider the difference between the torsion and the original module. In geometry this difference would be more fundamental than the torsion itself, and local cohomology would then arise by considering functions with support.

To construct a good model for this difference, observe that $\varepsilon\colon K^\bullet_\infty(\boldsymbol{\alpha}) \longrightarrow R$ is an isomorphism in degree zero and define the flat Čech complex $\check{C}^\bullet(I)$ to be the complex $\Sigma(\ker \varepsilon)$. Thus, if $i \geq 0$, then $\check{C}^i(I) = K^{i+1}(I)$. For example, if $I = (\alpha, \beta)$, then

$$\check{C}^\bullet(I) = (\ R[1/\alpha] \oplus R[1/\beta] \longrightarrow R[1/(\alpha\beta)]\ ).$$

The differential $K^0(I) \longrightarrow K^1(I)$ specifies a chain map $R \longrightarrow \check{C}^\bullet(I)$ whose fibre is exactly $K^\bullet_\infty(I)$. Thus we have a fibre sequence

$$\boxed{K^\bullet_\infty(I) \longrightarrow R \longrightarrow \check{C}^\bullet(I).}$$

We define the projective version $P\check{C}^\bullet(I)$ similarly, using the kernel of the composite of ε and the quasi-isomorphism $PK^\bullet_\infty(I) \longrightarrow K^\bullet_\infty(I)$; note that $P\check{C}^\bullet(I)$ is non-zero in cohomological degree -1.

The Čech cohomology of an R-module M is then defined by

$$\check{C}H^*_I(R; M) = H^*(\check{C}^\bullet(I) \otimes M).$$

VIII. **Čech cohomology and Čech covers.** To explain why $\check{C}^\bullet(I)$ is called the Čech complex, we describe how it arises by using the Čech construction to calculate cohomology from a suitable open cover. More precisely, let Y be the closed subscheme of $X = \mathit{Spec}(R)$ determined by I. The space $V(I) = \{\wp | \wp \supset I\}$ decomposes as $V(I) = V(\alpha_1) \cap \cdots \cap V(\alpha_n)$, and there results an open cover of the open subscheme $X - Y$ as the union of the complements $X - Y_i$ of the closed subschemes

Y_i determined by the principal ideals (α_i). However, $X - Y_i$ is isomorphic to the affine scheme $Spec(R[1/\alpha_i])$. Since affine schemes have no higher cohomology,

$$H^*(\mathrm{Spec}(R[1/\alpha_i]); \tilde{M}) = H^0(\mathrm{Spec}(R[1/\alpha_i]); \tilde{M}) = M[1/\alpha_i],$$

where $\tilde{M}$ is the sheaf associated to the R-module M. Thus the E_1 term of the Mayer-Vietoris spectral sequence for this cover collapses to the chain complex $\check{C}^\bullet(I)$, and

$$H^*(X - Y; \tilde{M}) \cong \check{C}H^*_I(M).$$

Appendix C. Homotopical analogues of the algebraic definitions

We now transpose the algebra from Appendix B into the homotopy theoretic context. It is convenient to note that it is routine to extend the algebra to graded rings, and we will use this without further comment below. We assume the reader is already comfortable working with ring spectra, but there is an introduction with full references in [60].

We replace the standing assumption that R is a commutative $\mathbb{Z}$-algebra by the assumption that it is a commutative $\mathbb{S}$-algebra, where $\mathbb{S}$ is the sphere spectrum. The category of R-modules is now the category of R-module *spectra*. Since the derived category of a ring R is equivalent to the derived category of the associated Eilenberg-MacLane spectrum [110], the work of Appendix B can be reinterpreted in the new context. To emphasize the algebraic analogy, we write $\otimes_R$ and Hom_R for the smash product over R and function spectrum of R-maps and 0 for the trivial module. In particular $X \otimes_{\mathbb{S}} Y = X \wedge Y$ and $\mathrm{Hom}_{\mathbb{S}}(X, Y) = F(X, Y)$.

This section is based on [55, 56].

I. Koszul spectra. For $\alpha \in \pi_* R$, we define the stable Koszul spectrum $K(\alpha)$ by the fibre sequence

$$K(\alpha) \longrightarrow R \longrightarrow R[1/\alpha],$$

where $R[1/\alpha] = \mathrm{holim}_{\rightarrow}(R \xrightarrow{\alpha} R \xrightarrow{\alpha} \cdots)$. Analogous to the filtration by degree in chain complexes, we obtain a filtration of the R-module $K(\alpha)$ by viewing it as

$$\Sigma^{-1}(R[1/\alpha] \cup CR).$$

Next we define the stable Koszul spectrum for the sequence $\alpha_1, \ldots, \alpha_n$ by

$$K(\alpha_1, \ldots, \alpha_n) = K(\alpha_1) \otimes_R \cdots \otimes_R K(\alpha_n),$$

and give it the tensor product filtration.

The topological analogue of Lemma B.1 states that if $\beta \in I$ then

$$K(\alpha_1, \ldots, \alpha_n)[1/\beta] \simeq 0;$$

this follows from Lemma B.1 and the spectral sequence (3) below. We may now use precisely the same proof as in the algebraic case to conclude that the homotopy type of $K(\alpha_1, \ldots, \alpha_n)$ depends only on the radical of the ideal $I = (\alpha_1, \ldots, \alpha_n)$. We therefore write $K(I)$ for $K(\alpha_1, \ldots, \alpha_n)$.

II. **Localization and completion.** With motivation from Theorems B.3 and B.4, we define the homotopical I-power torsion (or local cohomology) and homotopical completion (or local homology) modules associated to an R-module M by

$$\Gamma_I(M) = K(I) \otimes_R M \quad \text{and} \quad \Lambda_I(M) = M_I^\wedge = \mathrm{Hom}_R(K(I), M). \tag{2}$$

In particular, $\Gamma_I(R) = K(I)$.

Because the construction follows the algebra so precisely, it is easy give methods of calculation for the homotopy groups of these R-modules. We use the product of the filtrations of the $K(\alpha_i)$ given above and obtain spectral sequences

$$E^2_{s,t} = H_I^{-s,-t}(R_*; M_*) \Longrightarrow \pi_{s+t}(\Gamma_I M) \tag{3}$$

with differentials $d^r \colon E^r_{s,t} \to E^r_{s-r,t+r-1}$ and

$$E_2^{s,t} = H^I_{-s,-t}(R^*; M^*) \Longrightarrow \pi_{-(s+t)}(M_I^\wedge) \tag{4}$$

with differentials $d_r \colon E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$.

III. **The Čech spectra.** Similarly, we define the Čech spectrum by the cofibre sequence

$$\boxed{K(I) \longrightarrow R \longrightarrow \check{C}(I).} \tag{5}$$

We define the homotopical localization (or Čech cohomology) and Čech homology modules associated to an R-module M by

$$M[I^{-1}] = \check{C}(I) \otimes_R M \quad \text{and} \quad \Delta^I(M) = \mathrm{Hom}_R(\check{C}(I), M). \tag{6}$$

In particular, $R[I^{-1}] = \check{C}(I)$. Once again, we have spectral sequences for calculating their homotopy groups from the analogous algebraic constructions.

IV. **Basic properties.** We can now give topological analogues of some basic pieces of algebra that we used in Section B. Recall that the algebraic Koszul complex $K^\bullet_\infty(I)$ is a direct limit of unstable complexes $K^\bullet_s(I)$ that are finite complexes of free modules with homology annihilated by a power of I. We say that an R-module M is a I-power torsion module if its R_*-module M_* of homotopy groups is a I-power torsion module; equivalently, M_* must have support over I.

Lemma C.1. *The R-module $K^\bullet_\infty(I)$ is a homotopy direct limit of finite R-modules $K_s(I)$, each of which has homotopy groups annihilated by some power of I. Therefore $K^\bullet_\infty(I)$ is a I-power torsion module.*

The following lemma is an analogue of the fact that $\check{C}^\bullet(I)$ is a chain complex which is a finite sum of modules $R[1/\alpha]$ for $\alpha \in I$.

Lemma C.2. *The R-module $\check{C}(I)$ has a finite filtration by R-submodules with subquotients that are suspensions of modules of the form $R[1/\alpha]$ with $\alpha \in I$.*

These lemmas are useful in combination.

Corollary C.3. *If M is a I-power torsion module then $M \otimes_R \check{C}(I) \simeq 0$; in particular $K(I) \otimes_R \check{C}(I) \simeq 0$.*

Proof: Since $M[1/\alpha] \simeq 0$ for $\alpha \in I$, Lemma C.2 gives the conclusion for M. □

Appendix D. Completion at ideals and Bousfield localization

Bousfield localizations include both completions at ideals and localizations at multiplicatively closed sets, but one may view these Bousfield localizations as falling into the types typified by completion at p and localization away from p. Thinking in terms of $Spec(R_*)$, this is best viewed as the distinction between localization at a closed set and localization at the complementary open subset. In this section we deal with the closed sets and with the open sets in Section E. This appendix is based on [65, 66, 56].

I. **Homotopical completion.** As observed in the proof of Lemma C.1, we have $K(\alpha) = \underset{\rightarrow}{\mathrm{holim}}_s \Sigma^{-1} R/\alpha^s$ and therefore

$$M^\wedge_{(\alpha)} = \mathrm{Hom}_R(\underset{\rightarrow}{\mathrm{holim}}_s \Sigma^{-1} R/\alpha^s, M) \simeq \underset{\leftarrow}{\mathrm{holim}}_s M/\alpha^s.$$

If $I = (\alpha, \beta)$, then

$$\begin{aligned} M^\wedge_I &= \mathrm{Hom}_R(K(\alpha) \otimes_R K(\beta), M) \\ &= \mathrm{Hom}_R(K(\alpha), \mathrm{Hom}_R(K(\beta), M)) \\ &= (M^\wedge_{(\beta)})^\wedge_{(\alpha)}, \end{aligned}$$

and so on inductively. This should help justify the notation $M^\wedge_I = \mathrm{Hom}_R(K(I), M)$.

When $R = \mathbb{S}$ is the sphere spectrum and $p \in \mathbb{Z} \cong \pi_0(\mathbb{S})$, $K(p)$ is a Moore spectrum for $\mathbb{Z}/p^\infty$ in degree -1 and we recover the usual definition

$$X^\wedge_p = F(S^{-1}/p^\infty, X)$$

of p-completions of spectra as a special case, where $F(A, B) = \mathrm{Hom}_{\mathbb{S}}(A, B)$ is the function spectrum. The standard short exact sequence for the calculation of the homotopy groups of $X^\wedge_p$ in terms of 'Ext completion' and 'Hom completion' follows directly from Corollary B.6.

Since p-completion has long been understood to be an example of a Bousfield localization, our next task is to show that completion at I is a Bousfield localization in general.

II. **Bousfield's terminology.** Fix an R-module E. A spectrum A is *E-acyclic* if $A \otimes_R E \simeq 0$; a map $f\colon X \longrightarrow Y$ is an *E-equivalence* if its cofibre is E-acyclic. An R-module M is *E-local* if $E \otimes_R T \simeq 0$ implies $\mathrm{Hom}_R(T, M) \simeq 0$. A map $Y \longrightarrow L_E Y$ is a *Bousfield E-localization* of Y if it is an E-equivalence and $L_E Y$ is E-local. This means that $Y \longrightarrow L_E Y$ is terminal among E-equivalences with domain Y, and the Bousfield localization is therefore unique if it exists. Similarly, we may replace the single spectrum E by a class $\mathcal{E}$ of objects E, and require the conditions hold for all such E

The following is a specialization of a change of rings result to the ring map $\mathbb{S} \longrightarrow R$.

Lemma D.1. *Let $\mathcal{E}$ be a class of R-modules. If an R-module N is $\mathcal{E}$-local as an R-module, then it is $\mathcal{E}$-local as an $\mathbb{S}$-module.*

Proof: If $E \wedge T = E \otimes_{\mathbb{S}} T \simeq *$ for all E, then $E \otimes_R (R \otimes_{\mathbb{S}} T) \simeq 0$ for all E and therefore $F(T, N) = \mathrm{Hom}_{\mathbb{S}}(T, N) \simeq \mathrm{Hom}_R(R \otimes_{\mathbb{S}} T, N) \simeq 0$. □

III. **Homotopical completion is a Bousfield localization.** The class that will concern us most is the class I-**Tors** of finite I-power torsion R-modules M. Thus M must be a finite cell R-module, and its R_*-module M_* of homotopy groups must be a I-power torsion module.

Theorem D.2. *For any finitely generated ideal I of R_* the map $M \longrightarrow M_I^\wedge$ is Bousfield localization in the category of R-modules in each of the following equivalent senses:*

(i) *with respect to the R-module $\Gamma_I(R) = K(I)$,*
(ii) *with respect to the class I-**Tors** of finite I-power torsion R-modules,*
(iii) *with respect to the R-module $K_s(I)$ for any $s \geq 1$.*

Furthermore, the homotopy groups of the completion are related to local homology groups by a spectral sequence

$$E^2_{s,t} = H^I_{s,t}(M_*) \Longrightarrow \pi_{s+t}(M_I^\wedge).$$

If R_* is Noetherian, the E^2 term consists of the left derived functors of I-adic completion: $H^I_s(M_*) = L^I_s(M_*)$.

Proof: We begin with (i). Since

$$\mathrm{Hom}_R(T, M_I^\wedge) \simeq \mathrm{Hom}_R(T \otimes_R K(I), M),$$

it is immediate that $M_I^\wedge$ is $K(I)$-local. We must prove that the map $M \longrightarrow M_I^\wedge$ is a $K(I)$-equivalence. The fibre of this map is $\mathrm{Hom}_R(\check{C}(I), M)$, so we must show that

$$\mathrm{Hom}_R(\check{C}(I), M) \otimes_R K(I) \simeq 0.$$

By Lemma C.1, $K(I)$ is a homotopy direct limit of terms $K_s(I)$. Each $K_s(I)$ is in I-**Tors**, and we see by their definition in terms of cofibre sequences and smash products that their duals $DK_s(I)$ are also in I-**Tors**, where $DM = \mathrm{Hom}_R(M, R)$. Since $K_s(I)$ is a finite cell R-module,

$$\mathrm{Hom}_R(\check{C}(I), M) \otimes_R K_s(I) = \mathrm{Hom}_R(\check{C}(I) \otimes_R DK_s(I), M),$$

and $\check{C}(I) \otimes_R DK_s(I) \simeq 0$ by Corollary C.3. Parts (ii) and (iii) are similar but simpler. For (iii), observe that we have a cofibre sequence $R/\alpha^s \longrightarrow R/\alpha^{2s} \longrightarrow R/\alpha^s$, so that all of the $K_{js}(I)$ may be constructed from $K_s(I)$ using a finite number of cofibre sequences. □

Appendix E. Localization away from ideals and Bousfield localization

In this section we turn to localization away from the closed set defined by an ideal I. First, observe that, when $I = (\alpha)$, $M[I^{-1}]$ is just $R[\alpha^{-1}] \otimes_R M = M[\alpha^{-1}]$. However, the higher Čech cohomology groups give the construction for general finitely generated ideals a quite different algebraic flavour, and $M[I^{-1}]$ is rarely a localization of M at a multiplicatively closed subset of R_*. This appendix is based on [56].

I. **The Čech complex as a Bousfield localization.** To characterize this construction as a Bousfield localization, we consider the class I-**Inv** of R-modules M for which there is an element $\alpha \in I$ such that $\alpha\colon M \longrightarrow M$ is an equivalence.

Theorem E.1. *For any finitely generated ideal $I = (\alpha_1, \ldots, \alpha_n)$ of R_*, the map $M \longrightarrow M[I^{-1}]$ is Bousfield localization in the category of R-modules in each of the following equivalent senses:*

(i) *with respect to the R-module $R[I^{-1}] = \check{C}(I)$,*
(ii) *with respect to the class I-**Inv**,*
(iii) *with respect to the set $\{R[1/\alpha_1], \ldots, R[1/\alpha_n]\}$.*

Furthermore, the homotopy groups of the localization are related to Čech cohomology groups by a spectral sequence

$$E^2_{s,t} = \check{C}H_I^{-s,-t}(M_*) \Longrightarrow \pi_{s+t}(M[I^{-1}]).$$

If R_* is Noetherian, the E^2 term can be viewed as the cohomology of $Spec(R_*)\backslash V(I)$ with coefficients in the sheaf associated to M_*.

Remark E.2. One may also characterize the map $\Gamma_I(M) \longrightarrow M$ by a universal property analogous to that of the cellular approximation in spaces.

On the one hand, $\Gamma_I(M)$ is constructed from $K_1(I)$ by C.1, and on the other hand, the map induces an equivalence of $\mathrm{Hom}_R(K_1(I), \cdot)$ since, by Lemma C.2, $\mathrm{Hom}_R(K_1(I), \check{C}(I)) \simeq 0$.

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Support theory for triangulated categories

SUPPORT THEORY FOR TRIANGULATED CATEGORIES NOTES FOR THE ADVANCED SCHOOL "BUILDING BRIDGES BETWEEN ALGEBRA AND TOPOLOGY"

GREG STEVENSON

Contents

Introduction

The aim of this lecture series is to give both an introduction and an overview of the developing field of tensor triangular geometry and support varieties. We seek to understand the coarse structure, i.e., lattices of suitable subcategories, of triangulated categories. Put another way, given a triangulated category T and objects $X, Y \in \mathsf{T}$, we study the question of when one can obtain Y from X by taking cones, suspensions, and (possibly infinite) coproducts.

The lectures are structured as follows. In the first lecture we review work of Paul Balmer in the case of essentially small (rigid) tensor triangulated categories. Here one can introduce a certain topological space, the spectrum, which solves the classification problem for thick tensor ideals. The spectrum is an example of a so-called "support variety" and provides a conceptual framework which unites earlier results in various examples such as perfect complexes over schemes, stable categories of modular representations, and the finite stable homotopy category. In this section we first meet the notion of supports and lay the foundations for defining supports in more general settings.

In the second lecture we turn to the infinite case, i.e., compactly generated tensor triangulated categories. Following work of Balmer and Favi we use the compact objects and the Balmer spectrum to define a notion of support for objects of such categories. Along the way we discuss smashing localisations and the associated generalised Rickard idempotents which are the key to the definition of the support given in [BF11].

The third lecture serves as an introduction to actions of tensor triangulated categories. This allows us to define a relative version of the supports introduced in the second lecture; in this way we can, at least somewhat, escape the tyranny of monoidal structures. After introducing actions, the associated support theory, and outlining some of the fundamental lemmas concerning supports and actions, we come to the main abstract result of this course - the local-to-global principle. This theorem, which already provides new insight in the situation of Lecture 2, reduces the study of lattices of localising subcategories to the computation of these lattices in smaller (and hopefully simpler) subcategories.

Finally, the fourth lecture focusses on illustrating some applications of the abstract machinery from the preceding lectures; there will be, of course, examples along the way but here we concentrate on a pair of examples where the machinery of actions has been successfully applied. First we discuss singularity categories of affine hypersurfaces and the corresponding classification problem for localising subcategories. We will also, time allowing, briefly discuss applications to studying derived categories of representations of quivers over arbitrary commutative noetherian rings. The details on this latter application don't appear in this set of notes.

1. Lecture 1: The Balmer spectrum

The main reference for this section is the paper [Bal05] by Paul Balmer. We follow his exposition fairly closely, albeit with two major differences: firstly, in order to simplify the discussion and since we will not require it later we do not work in full generality, and secondly we omit many of the technical details in the interest of time. The interested reader should consult [Bal05], [Bal10] and the references within for further details.

This lecture mainly consists of definitions, but we provide several examples on the way and ultimately get to Balmer's classification of thick tensor ideals in terms of the spectrum.

1.1. **Rigid tensor triangulated categories.**

Throughout this lecture K will denote an essentially small triangulated category. We use lowercase letters k, l, m for objects of K and denote its suspension functor by Σ.

We begin by introducing the main player in this lecture.

Definition 1.1. An *essentially small tensor triangulated category* is a triple $(\mathsf{K}, \otimes, \mathbf{1})$, where K is an essentially small triangulated category and $(\otimes, \mathbf{1})$ is a symmetric monoidal structure on K such that $\otimes$ is an exact functor in each variable. Slightly more explicitly,

$$- \otimes - \colon \mathsf{K} \times \mathsf{K} \longrightarrow \mathsf{K}$$

is a symmetric monoidal structure on K with unit $\mathbf{1}$ and with the property that, for all $k \in \mathsf{K}$, the endofunctors $k \otimes -$ and $- \otimes k$ are exact.

Remark 1.2. Throughout we shall not generally make explicit the associativity, symmetry, and unit constraints for the symmetric monoidal structure on a tensor triangulated category. By standard coherence results for monoidal structures this will not get us into any trouble.

Definition 1.3. Let K be an essentially small tensor triangulated category. Assume that K is closed symmetric monoidal, i.e., for each $k \in \mathsf{K}$ the functor $k \otimes -$ has a right adjoint which we denote $\hom(k, -)$. These functors can be assembled into a bifunctor $\hom(-,-)$ which we call the *internal hom* of K. By definition one has, for all $k, l, m \in \mathsf{K}$, the tensor-hom adjunction

$$\mathsf{K}(k \otimes l, m) \cong \mathsf{K}(l, \hom(k, m)),$$

with corresponding units and counits

$$\eta_{k,l} \colon l \longrightarrow \hom(k, k \otimes l) \quad \text{and} \quad \epsilon_{k,l} \colon \hom(k,l) \otimes k \longrightarrow l.$$

The *dual* of $k \in \mathsf{K}$ is the object

$$k^\vee = \hom(k, \mathbf{1}).$$

Given $k, l \in \mathsf{K}$ there is a natural *evaluation map*

$$k^\vee \otimes l \longrightarrow \hom(k, l),$$

which is defined by following the identity map on l through the composite

$$\mathsf{K}(l,l) \xrightarrow{\sim} \mathsf{K}(l \otimes \mathbf{1}, l) \xrightarrow{\mathsf{K}(l \otimes \epsilon_{k,\mathbf{1}}, l)} \mathsf{K}(k \otimes k^\vee \otimes l, l) \xrightarrow{\sim} \mathsf{K}(k^\vee \otimes l, \hom(k,l)).$$

We say that K is *rigid* if for all $k, l \in \mathsf{K}$ this natural evaluation map is an isomorphism

$$k^\vee \otimes l \xrightarrow{\sim} \hom(k, l).$$

Remark 1.4. If K is rigid then, given $k \in \mathsf{K}$, there is a natural isomorphism

$$(k^\vee)^\vee \cong k,$$

and the functor $k^\vee \otimes -$ is both a left and a right adjoint to $k \otimes -$.

Example 1.5. Let us provide some standard examples of essentially small rigid tensor triangulated categories:

(1) Given a commutative ring R the category $\mathsf{D}^{\mathrm{perf}}(R)$ of *perfect complexes* i.e., those complexes in the derived category which are quasi-isomorphic to a bounded complex of finitely generated projectives, is symmetric monoidal via the left derived tensor product $\otimes^{\mathbf{L}}_R$ (which we will usually denote just as $\otimes_R$ or $\otimes$ if the ring is clear) with unit the stalk complex R sitting in degree 0. The category $\mathsf{D}^{\mathrm{perf}}(R)$ is easily checked to be rigid.

(2) Let G be a finite group and k a field whose characteristic divides the order of G (this is not necessary but rules out trivial cases). We write $\operatorname{mod} kG$ for the category of finite dimensional kG-modules. This is a Frobenius category, so its stable category $\underline{\operatorname{mod}}kG$, which is obtained by factoring out maps factoring through projectives, is triangulated. Moreover, it is a rigid tensor triangulated category via the usual tensor product $\otimes_k$ with the diagonal action and unit object the trivial representation k.

(3) The finite stable homotopy category $\mathrm{SH}^{\mathrm{fin}}$ together with the smash product of spectra is a rigid tensor triangulated category with unit object the sphere spectrum S^0.

1.2. **Ideals and the spectrum.** From this point onward K denotes an essentially small rigid tensor triangulated category with tensor product $\otimes$ and unit $\mathbf{1}$. For much of what follows the rigidity assumption is overkill, but it simplifies the discussion in several places and will be a necessary hypotheses in the future lectures.

We begin by recalling the subcategories of K with which we will be concerned. All subcategories of K are assumed to be full and replete (i.e., closed under isomorphisms). The simple, but beautiful, idea is to view K as a very strange sort of ring. One takes thick subcategories to be the analogue of additive subgroups and then defines ideals and prime ideals in the naive way.

Definition 1.6. A triangulated subcategory I of K is *thick* if it is closed under taking direct summands, i.e., if $k \oplus k' \in \mathsf{I}$ then both k and k' lie in I.

A thick subcategory I of K is a (thick) *tensor-ideal* if given any $k \in \mathsf{K}$ and $l \in \mathsf{I}$ the tensor product $k \otimes l$ lies in I. Put another way we require the functor

$$\mathsf{K} \times \mathsf{I} \xrightarrow{\otimes} \mathsf{K}$$

to factor through I.

Finally, a proper thick tensor-ideal P of K is *prime* if given $k, l \in \mathsf{K}$ such that $k \otimes l \in \mathsf{P}$ then at least one of k or l lies in P.

Remark 1.7. Since we have assumed K is essentially small the collections of thick subcategories, thick tensor-ideals, and prime tensor-ideals each form a set.

Remark 1.8. Rigidity of K provides the following simplification when dealing with tensor-ideals. Given $k \in \mathsf{K}$ the adjunctions between $k \otimes -$ and $k^\vee \otimes -$ imply that k is a summand of $k \otimes k \otimes k^\vee$ and that $k^\vee$ is a summand of $k \otimes k^\vee \otimes k^\vee$. The former implies that every tensor-ideal is radical i.e., if I is a tensor-ideal and $k^{\otimes n} \in \mathsf{I}$ then $k \in \mathsf{I}$. The latter implies that every tensor-ideal is closed under taking duals.

An important special case of the above discussion is the following: if $k \in \mathsf{K}$ is such that $k^{\otimes n} \cong 0$ then $k \cong 0$.

Given a collection of objects $S \subseteq \mathsf{K}$ we denote by $\mathrm{thick}(S)$ (resp. $\mathrm{thick}^{\otimes}(S)$) the smallest thick subcategory (resp. thick tensor-ideal) containing S. We use $\mathrm{Thick}(\mathsf{K})$ and $\mathrm{Thick}^{\otimes}(K)$ respectively to denote the sets of thick subcategories and thick tensor-ideals of K. Both of these sets of subcategories are naturally ordered by inclusion and form complete lattices whose meet is given by intersection.

Lemma 1.9. *Suppose that K is generated by the tensor unit, i.e.* $\mathrm{thick}(\mathbf{1}) = \mathsf{K}$. *Then every thick subcategory is a tensor-ideal:*

$$\mathrm{Thick}(\mathsf{K}) = \mathrm{Thick}^{\otimes}(K).$$

Proof. Exercise. $\square$

Example 1.10. The lemma applies to both $\mathsf{D}^{\mathrm{perf}}(R)$ and $\mathrm{SH}^{\mathrm{fin}}$ which are generated by their respective tensor units R and S^0. It is not necessarily the case that $\underline{\mathrm{mod}}kG$ is generated by its tensor unit k.

The prime ideals, somewhat unsurprisingly, receive special attention (and notation).

Definition 1.11. The *spectrum of* K is the set

$$\operatorname{Spc}\mathsf{K} = \{\mathsf{P} \subseteq \mathsf{K} \mid \mathsf{P} \text{ is prime}\}$$

of prime ideals of K.

We next record some elementary but crucial facts about prime ideals and $\operatorname{Spc}\mathsf{K}$.

Proposition 1.12 ([Bal05, Proposition 2.3])**.** *Let* K *be as above.*

(a) *Let S be a set of objects of* K *containing* $\mathbf{1}$ *and such that if $k, l \in S$ then $k \otimes l \in S$. If S does not contain* 0 *then there exists a* $\mathsf{P} \in \operatorname{Spc}\mathsf{K}$ *such that* $\mathsf{P} \cap S = \varnothing$.

(b) *For any proper thick tensor-ideal* $\mathsf{I} \subsetneq \mathsf{K}$ *there exists a maximal proper thick tensor-ideal* M *with* $\mathsf{I} \subseteq \mathsf{M}$.

(c) *Maximal proper thick tensor-ideals are prime.*

(d) *The spectrum is not empty:* $\operatorname{Spc}\mathsf{K} \neq \varnothing$.

1.3. **Supports and the Zariski topology.** We now define, for each object $k \in \mathsf{K}$, a subset of prime ideals at which k is "non-zero". This is the central construction of the lecture, allowing us to both put a topology on $\operatorname{Spc}\mathsf{K}$ and to understand $\operatorname{Thick}^{\otimes}(\mathsf{K})$ in terms of the resulting topological space.

Definition 1.13. Let k be an object of K. The *support* of k is the subset

$$\operatorname{supp} k = \{\mathsf{P} \in \operatorname{Spc}\mathsf{K} \mid k \notin \mathsf{P}\}.$$

We denote by

$$U(k) = \{\mathsf{P} \in \operatorname{Spc}\mathsf{K} \mid k \in \mathsf{P}\}$$

the complement of $\operatorname{supp} k$.

Let us give some initial intuition for the way in which one can think of k as being supported at $\mathsf{P} \in \operatorname{supp} k$. We can form the Verdier quotient K/P of K by P, which comes with a canonical projection $\pi\colon \mathsf{K} \longrightarrow \mathsf{K}/\mathsf{P}$. Since P is thick the kernel of π is precisely P and so, as $k \notin \mathsf{P}$, the object $\pi(k)$ is non-zero in the quotient.

We now list the main properties of the support.

Lemma 1.14 ([Bal05, Lemma 2.6])**.** *The assignment $k \mapsto \operatorname{supp} k$ given by the support satisfies the following properties:*

(a) $\operatorname{supp}\mathbf{1} = \operatorname{Spc}\mathsf{K}$ *and* $\operatorname{supp} 0 = \varnothing$;

(b) $\operatorname{supp}(k \oplus l) = \operatorname{supp}(k) \cup \operatorname{supp}(l)$;

(c) $\operatorname{supp}(\Sigma k) = \operatorname{supp} k$;

(d) *for any distinguished triangle*

$$k \longrightarrow l \longrightarrow m \longrightarrow \Sigma k$$

in K *there is a containment*

$$\operatorname{supp} l \subseteq (\operatorname{supp} k \cup \operatorname{supp} m);$$

(e) $\operatorname{supp}(k \otimes l) = \operatorname{supp}(k) \cap \operatorname{supp}(l)$.

Proof. We refer to Balmer's paper for the details, where the corresponding results are proved for the subsets $U(k)$. These properties, as stated above, appear in [Bal05, Definition 3.1]. However, we suggest proving these statements as they provide an instructive exercise for beginners. □

Another very important property of the support is that it can detect whether or not an object is zero.

Lemma 1.15. *Given $k \in \mathsf{K}$ we have* $\operatorname{supp} k = \varnothing$ *if and only if* $k \cong 0$.

Proof. By [Bal05, Corollary 2.4] the support of k is empty if and only if k is tensor-nilpotent i.e., $k^{\otimes n} \cong 0$. As K is rigid there are no non-zero tensor-nilpotent objects by Remark 1.8. □

A consequence of the above lemma is that the family of subsets $\{\operatorname{supp} k \mid k \in \mathsf{K}\}$ form a basis of closed subsets for a topology on $\operatorname{Spc} \mathsf{K}$.

Definition 1.16. The *Zariski topology* on $\operatorname{Spc} \mathsf{K}$ is the topology defined by the basis of closed subsets

$$\{\operatorname{supp} k \mid k \in \mathsf{K}\}.$$

The closed subsets of $\operatorname{Spc} \mathsf{K}$ are of the form

$$\begin{aligned} Z(S) &= \{\mathsf{P} \in \operatorname{Spc} \mathsf{K} \mid S \cap \mathsf{P} = \varnothing\} \\ &= \bigcap_{k \in S} \operatorname{supp}(k) \end{aligned}$$

where $S \subseteq \mathsf{K}$ is an arbitrary family of objects.

Before continuing to the universal property of $\operatorname{Spc} \mathsf{K}$ and the classification theorem we discuss some topological properties of $\operatorname{Spc} \mathsf{K}$.

Proposition 1.17 ([Bal05, Proposition 2.9])**.** *Let* $\mathsf{P} \in \operatorname{Spc} \mathsf{K}$. *The closure of* P *is*

$$\overline{\{\mathsf{P}\}} = \{\mathsf{Q} \in \operatorname{Spc} \mathsf{K} \mid \mathsf{Q} \subseteq \mathsf{P}\}.$$

In particular, $\operatorname{Spc} \mathsf{K}$ *is* T_0 *i.e., for* $\mathsf{P}_1, \mathsf{P}_2 \in \operatorname{Spc} \mathsf{K}$

$$\overline{\{\mathsf{P}_1\}} = \overline{\{\mathsf{P}_2\}} \;\Rightarrow\; \mathsf{P}_1 = \mathsf{P}_2.$$

Proof. Let $S_0 = \mathsf{K} \setminus \mathsf{P}$ denote the complement of P. It is immediate that $\mathsf{P} \in Z(S_0)$ and one easily checks that if $\mathsf{P} \in Z(S)$ then $S \subseteq S_0$. Thus for any such S we have $Z(S_0) \subseteq Z(S)$ i.e., $Z(S_0)$ is the smallest closed subset containing P. This shows

$$\overline{\{\mathsf{P}\}} = Z(S_0) = \{\mathsf{Q} \in \operatorname{Spc} \mathsf{K} \mid \mathsf{Q} \subseteq \mathsf{P}\}$$

as claimed. The assertion that $\operatorname{Spc} \mathsf{K}$ is T_0 follows immediately. □

Remark 1.18. This proposition is the first indication of the mental gymnastics that occur when dealing with $\operatorname{Spc} \mathsf{K}$ versus $\operatorname{Spec} R$ for a commutative ring R. A wealth of further information on the relationship between prime ideals in R and prime tensor-ideals in $\operatorname{Spc} \mathsf{D}^{\mathrm{perf}}(R)$ can be found in [Bal10].

In fact $\operatorname{Spc} \mathsf{K}$ is much more than just a T_0 space.

Definition 1.19. A topological space X is a *spectral space* if it verifies the following properties:

1. (1) X is T_0;
2. (2) X is quasi-compact;
3. (3) the quasi-compact open subsets of X are closed under finite intersections and form an open basis of X;
4. (4) every non-empty irreducible closed subset of X has a generic point.

Given spectral spaces X and Y, a *spectral map* $f\colon X \longrightarrow Y$ is a continuous map such that for any quasi-compact open $U \subseteq Y$ the preimage $f^{-1}(U)$ is quasi-compact.

Typical examples of spectral spaces are given by $\operatorname{Spec} R$ where R is a commutative ring. In fact Hochster has shown in [Hoc69] that any spectral topological space is of this form. Further examples include the topological space underlying any quasi-compact and quasi-separated scheme. Another class of examples is provided by the following result.

Theorem 1.20 ([BKS07]). *Let* K *be as above. The space* $\operatorname{Spc}\mathsf{K}$ *is spectral.*

Thus the spaces we produce by taking spectra of tensor triangulated categories are particularly nice and enjoy many desirable properties. As a complement to this theorem let us record a few related facts which appear in Balmer's work.

Proposition 1.21. *For any rigid tensor triangulated category* K *the following assertions hold.*

(a) *Any non-empty closed subset of* $\operatorname{Spc}\mathsf{K}$ *contains at least one closed point.*
(b) *For any* $k \in \mathsf{K}$ *the open subset* $U(k) = \operatorname{Spc}\mathsf{K} \setminus \operatorname{supp}(k)$ *is quasi-compact.*
(c) *Any quasi-compact open subset of* $\operatorname{Spc}\mathsf{K}$ *is of the form* $U(k)$ *for some* $k \in \mathsf{K}$.

Proof. Both results can be found in [Bal05]: the first is Corollary 2.12, and the second two are the content of Proposition 2.14. ☐

In particular, it follows from the above proposition and Lemma 1.14 that the $U(k)$ for $k \in \mathsf{K}$ give a basis of quasi-compact open subsets for $\operatorname{Spc}\mathsf{K}$ that is closed under finite intersections as is required in the definition of a spectral space.

Remark 1.22. Recall, or prove as an exercise, that a space is noetherian (i.e., satisfies the descending chain condition for closed subsets) if and only if every open subset is quasi-compact. Combining this with (c) above we see that if $\operatorname{Spc}\mathsf{K}$ is noetherian then every open subset is of the form $U(k)$ for some $k \in \mathsf{K}$ and hence every closed subset is the support of some object.

1.4. **The classification theorem.** We now come to the first main result of this lecture series, namely the abstract classification of thick tensor-ideals of K in terms of $\operatorname{Spc}\mathsf{K}$. The starting point to this story is an abstract axiomatisation of the properties of the support, based on Lemma 1.14.

Definition 1.23. A *support data* on $(\mathsf{K}, \otimes, \mathbf{1})$ is a pair (X, σ) where X is a topological space and σ is an assignment associating to each object k of K a closed subset $\sigma(k)$ of X such that:

(a) $\sigma(\mathbf{1}) = X$ and $\sigma(0) = \varnothing$;
(b) $\sigma(k \oplus l) = \sigma(k) \cup \sigma(l)$;
(c) $\sigma(\Sigma k) = \sigma(k)$;
(d) for any distinguished triangle
$$k \longrightarrow l \longrightarrow m \longrightarrow \Sigma k$$
in K there is a containment
$$\sigma(l) \subseteq (\sigma(k) \cup \sigma(m));$$
(*e*) $\sigma(k \otimes l) = \sigma(k) \cap \sigma(l)$.

A *morphism of support data* $f : (X, \sigma) \longrightarrow (Y, \tau)$ on K is a continuous map $f : X \longrightarrow Y$ such that for every $k \in \mathsf{K}$ the equality
$$\sigma(k) = f^{-1}(\tau(k))$$

is satisfied. An isomorphism of support data is a morphism f of support data where f is a homeomorphism.

It should, given the definition of the support, come as no surprise that the pair $(\operatorname{Spc}\mathsf{K}, \operatorname{supp})$ is a universal support data for K.

Theorem 1.24 ([Bal05, Theorem 3.2])**.** *Let K be as throughout. The pair $(\operatorname{Spc}\mathsf{K}, \operatorname{supp})$ is the terminal support data on K. Explicitly, given any support data (X,σ) on K there is a unique morphism of support data $f\colon (X,\sigma) \longrightarrow (\operatorname{Spc}\mathsf{K}, \operatorname{supp})$ i.e., a unique continuous map f such that*

$$\sigma(k) = f^{-1}\operatorname{supp}(k)$$

for all $k \in \mathsf{K}$. This unique morphism is given by sending $x \in X$ to

$$f(x) = \{k \in \mathsf{K} \mid x \notin \sigma(k)\}.$$

Sketch of proof. We give a brief sketch of the argument. We have seen in Lemma 1.14 that the pair $(\operatorname{Spc}\mathsf{K}, \operatorname{supp})$ is in fact a support data for K. By the same lemma it is clear that for $x \in X$ the full subcategory $f(x)$, defined as in the statement, is a proper thick tensor-ideal: it is proper by (a), closed under summands by (b), closed under Σ by (c), closed under extensions by (d), and a tensor-ideal by (e). To see that it is prime suppose $k \otimes l$ lies in $f(x)$. By axiom (e) for a support data we have

$$x \notin \sigma(k \otimes l) = \sigma(k) \cap \sigma(l).$$

and hence x must not lie in at least one of $\sigma(k), \sigma(l)$ implying that one of k or l lies in $f(x)$.

The map f is a map of support data as $f(x) \in \operatorname{supp} k$ if and only if $k \notin f(x)$ if and only if $x \in \sigma(k)$ i.e.,

$$f^{-1}\operatorname{supp}(k) = \{x \in X \mid f(x) \in \operatorname{supp} k\} = \{x \in X \mid x \in \sigma(k)\} = \sigma(k).$$

This also proves continuity of f by definition of the Zariski topology on $\operatorname{Spc}\mathsf{K}$.

We leave the unicity of f to the reader (or it can be found as [Bal05, Lemma 3.3]. $\square$

Before stating the promised classification theorem we need one more definition (we also provide two bonus definitions which will be useful both here and later).

Definition 1.25. Let X be a spectral space and let $W \subseteq X$ be a subset of X. We say W is *specialisation closed* if for any $w \in W$ and $w' \in \overline{\{w\}}$ we have $w' \in W$. That is, W is specialisation closed if it is the union of the closures of its elements. Given w, w' as above we call w' a *specialisation* of w. Dually, we say W is *generisation closed* if given any $w' \in W$ and a $w \in X$ such that $w' \in \overline{\{w\}}$ then we have $w \in W$. In this situation we call w a *generisation* of w'.

A *Thomason subset* of X is a subset of the form

$$\bigcup_{\lambda \in \Lambda} V_\lambda$$

where each V_λ is a closed subset of X with quasi-compact complement. Note that any Thomason subset is specialisation closed. We denote by $\operatorname{Thom}(X)$ the collection of Thomason subsets of X. It is a poset with respect to inclusion and in fact also carries the structure of a complete lattice where the join is given by taking unions.

Remark 1.26. As observed in Remark 1.22 if X is noetherian then every open subset is quasi-compact. Thus the Thomason subsets of X are precisely the specialisation closed subsets.

Remark 1.27. By Proposition 1.21 for any $k \in \mathsf{K}$ the subset $\operatorname{supp}(k)$ is a Thomason subset of $\operatorname{Spc}\mathsf{K}$, as is any union of supports of objects. Part (c) of the same proposition implies the converse, namely that any Thomason subset V can be written as a union of supports of objects of k. This observation explains the role of Thomason subsets in the following theorem.

Theorem 1.28 ([Bal05, Theorem 4.10])**.** *The assignments*

$$\sigma\colon \operatorname{Thick}^{\otimes}(\mathsf{K}) \longrightarrow \operatorname{Thom}(\operatorname{Spc}\mathsf{K}), \quad \sigma(\mathsf{I}) = \bigcup_{k\in\mathsf{I}} \operatorname{supp}(k)$$

and

$$\tau\colon \operatorname{Thom}(\operatorname{Spc}\mathsf{K}) \longrightarrow \operatorname{Thick}^{\otimes}(\mathsf{K}), \quad \tau(V) = \{k \in \mathsf{K} \mid \operatorname{supp}(k) \subseteq V\},$$

for $\mathsf{I} \in \operatorname{Thick}^{\otimes}(\mathsf{K})$ *and* $V \in \operatorname{Thom}(\mathsf{K})$ *give an isomorphism of lattices*

$$\operatorname{Thick}^{\otimes}(\mathsf{K}) \cong \operatorname{Thom}(\mathsf{K}).$$

Sketch of proof. By the previous remark and the properties of the support (Lemma 1.14) both assignments are well defined, i.e., $\sigma(\mathsf{I})$ is a Thomason subset and $\tau(V)$ is a thick tensor-ideal. It is clear that both morphisms are inclusion preserving, so we just need to check that they are inverse to one another.

Consider the thick tensor-ideal $\tau\sigma(\mathsf{I})$. It is clear that $\mathsf{I} \subseteq \tau\sigma(\mathsf{I})$. The equality is proved by showing that

$$\mathsf{I} = \bigcap_{\substack{\mathsf{P}\in\operatorname{Spc}\mathsf{K} \\ \mathsf{I}\subseteq\mathsf{P}}} \mathsf{P} = \tau\sigma(\mathsf{I}).$$

We refer to [Bal05] for the details.

On the other hand consider the Thomason subset $\sigma\tau(V)$. In this case it is clear that $\sigma\tau(V) \subseteq V$. By the remark preceding the theorem we know V can be written as a union of supports of objects and so this containment is in fact an equality. □

Remark 1.29. By Stone duality the lattice $\operatorname{Thom}(\operatorname{Spc}\mathsf{K})$ actually determines $\operatorname{Spc}\mathsf{K}$. So by the theorem if we know $\operatorname{Thick}^{\otimes}(\mathsf{K})$ we can recover $\operatorname{Spc}\mathsf{K}$.

To conclude this section of the notes we briefly explain what the spectrum is in each of our current running examples.

Example 1.30. Let R be a commutative ring. Then

$$\operatorname{Spc}\mathsf{D}^{\mathrm{perf}}(R) \cong \operatorname{Spec} R.$$

This computation is due to Neeman [Nee92] (and Hopkins) in the case that R is noetherian and Thomason [Tho97] in general. Applying the above theorem, in combination with Lemma 1.9, tells us that thick subcategories of $\mathsf{D}^{\mathrm{perf}}(R)$ are in bijection with Thomason subsets of $\operatorname{Spec} R$.

Example 1.31. As previously let $\mathrm{SH}^{\mathrm{fin}}$ denote the finite stable homotopy category. The description of the thick subcategories of $\mathrm{SH}^{\mathrm{fin}}$ by Devinatz, Hopkins, and Smith [DHS88] allows one to compute $\operatorname{Spc}\mathrm{SH}^{\mathrm{fin}}$. A nice picture of this space can be found in [Bal10, Corollary 9.5].

Example 1.32. Let G be a finite group and k a field whose characteristic divides the order of G. By a result of Benson, Carlson, and Rickard [BCR97] we have

$$\operatorname{Spc} \underline{\mathrm{mod}} kG \cong \operatorname{Proj} H^{\bullet}(G;k)$$

i.e., the spectrum of the stable category is the space underlying the projective scheme associated to the group cohomology ring.

2. Lecture 2: Generalised Rickard idempotents and supports for rigidly-compactly generated tensor triangulated categories

In this lecture we use the Balmer spectrum to explain a theory of supports for triangulated categories admitting arbitrary coproducts (so being interesting and essentially small become mutually exclusive). Our main reference is [BF11] and again, we follow it relatively closely.

2.1. Rigidly-compactly generated tensor triangulated categories.

In this section we introduce the main players of this section, rigidly-compactly generated tensor triangulated categories. Let us begin by recalling the notion of a compactly generated triangulated category.

Definition 2.1. Let T be a triangulated category admitting all set-indexed coproducts. We say an object $t \in \mathsf{T}$ is *compact* if $\mathsf{T}(t,-)$ preserves arbitrary coproducts, i.e., if for any family $\{X_\lambda \mid \lambda \in \Lambda\}$ of objects in T the natural morphism

$$\bigoplus_{\lambda\in\Lambda} \mathsf{T}(t, X_\lambda) \longrightarrow \mathsf{T}(t, \coprod_{\lambda\in\Lambda} X_\lambda)$$

is an isomorphism.

We say T is *compactly generated* if there is a set G of compact objects of T such that an object $X \in \mathsf{T}$ is zero if and only if

$$\mathsf{T}(g, \Sigma^i X) = 0 \text{ for every } g \in G \text{ and } i \in \mathbb{Z}$$

We denote by T^c the full subcategory of compact objects of T and note that it is an essentially small thick subcategory of T.

Example 2.2. Let us give some examples of compactly generated triangulated categories which will be used as illustrations throughout.

(1) Let R be a ring and denote by $\mathsf{D}(R)$ the unbounded derived category of R. Then $\mathsf{D}(R)$ is compactly generated and R is a compact generator for $\mathsf{D}(R)$.
(2) Let SH denote the stable homotopy category. The sphere spectrum S^0 is a compact generator for SH and hence it is compactly generated.
(3) Let G be a finite group and let k be a field whose characteristic divides the order of G. Them $\underline{\mathrm{Mod}}kG$, the stable category of aribtrary kG-modules is a compactly generated triangulated category.

Definition 2.3. A *compactly generated tensor triangulated category* is a triple $(\mathsf{T}, \otimes \mathbf{1})$ where T is a compactly generated tensor triangulated category, and $(\otimes, \mathbf{1})$ is a symmetric monoidal structure on T such that the tensor product $\otimes$ is a coproduct preserving exact functor in each variable and the compact objects T^c form a tensor subcategory. In particular, we require that the unit $\mathbf{1}$ is compact.

Remark 2.4. We will frequently supress the tensor product and unit and just refer to T as a compactly generated tensor triangulated category.

Remark 2.5. By Brown representability [Nee96, Theorem 3.1] and the assumption that the tensor product is coproduct preserving it is automatic that T is closed symmetric monoidal - for each $X \in \mathsf{T}$ the functor $X \otimes -$ has a right adjoint which we denote $\hom(X, -)$.

Finally, we come to the combination of hypotheses that we can get the most mileage out of.

Definition 2.6. A *rigidly-compactly generated tensor triangulated category* T is a compactly generated tensor triangulated category such that the full subcategory T^c of compact objects is rigid. Explicitly, not only is T^c a tensor subcategory of T, but it is closed under the internal hom (which exists by the previous remark) and is rigid in the sense of Definition 1.3.

Example 2.7. Each of the triangulated categories covered in Example 2.2 (with the proviso in (1) that the ring is commutative) is a rigidly-compactly generated tensor triangulated category via the left derived tensor product, smash product, and tensor product over the ground field with the diagonal action respectively.

2.2. **Localising sequences and smashing localisations.** We now review some fundamental definitions concerning the particular subcategories of a rigidly-compactly generated tensor triangulated category that we will wish to consider. Of course some of the definitions we make are valid more generally, but we will restrict ourselves to what we need. Throughout we fix a rigidly-compactly generated tensor triangulated category T.

Definition 2.8. Let L be a triangulated subcategory of T. The subcategory L is *localising* if it is closed under arbitrary coproducts in T. Dually, it is *colocalising* if it is closed under arbitrary products in T.

Let I be a localising subcategory of T. We call I a *localising tensor-ideal* if for any $X \in \mathsf{T}$ and $Y \in \mathsf{I}$ the object $X \otimes Y$ lies in I.

Given a family of objects $S \subseteq \mathsf{T}$ we denote by $\mathrm{loc}(S)$ (resp. $\mathrm{loc}^{\otimes}(S)$) the smallest localising subcategory (resp. localising tensor-ideal) of T containing S and by $\mathrm{Loc}(\mathsf{T})$ (resp. $\mathrm{Loc}^{\otimes}(\mathsf{T})$) the collection of all localising subcategories (resp. localising tensor-ideal) of T.

Remark 2.9. There is, *a priori*, no reason that either $\mathrm{Loc}(\mathsf{T})$ or $\mathrm{Loc}^{\otimes}(\mathsf{T})$ should be sets. In fact, whether or not these collections can form proper classes in the given situation is a longstanding open problem.

Remark 2.10. By [Nee01, Proposition 1.6.8] every localising and colocalising subcategory of T is automatically closed under direct summands i.e., (co)localising implies thick.

Remark 2.11. One can show, see for instance [Nee96, Lemma 3.2], that G is a set of compact generators for T if and only if $\mathrm{loc}(G) = \mathsf{T}$.

The fact from the above remark is very important in practice. For instance, it is a key point in proving the following elementary but useful lemma. Perhaps more importantly, the proof illustrates an argument which appears frequently in this subject.

Lemma 2.12. *Let T be a rigidly-compactly generated tensor triangulated category. Then, for $t \in \mathsf{T}^c$ the isomorphism of endofunctors of T^c*

$$t^{\vee} \otimes (-) \xrightarrow{\alpha} \hom(t, -)$$

extends to an isomorphism of the corresponding endofunctors of T*. That is, for any* $X \in \mathsf{T}$ *we have* $t^\vee \otimes X \cong \hom(t, X)$*. In particular,* $\hom(t, -)$ *preserves coproducts and* $t \otimes (-)$ *is left and right adjoint to* $t^\vee \otimes (-)$ *when viewed as endofunctors of* T*.*

Proof. As in Definition 1.3 we can define a natural transformation

$$\alpha \colon t^\vee \otimes (-) \longrightarrow \hom(t, -)$$

of functors $\mathsf{T} \longrightarrow \mathsf{T}$. Define a full subcategory of T as follows

$$\mathsf{S} = \{X \in \mathsf{T} \mid \alpha_X \colon t^\vee \otimes X \longrightarrow \hom(t, X) \text{ is an isomorphism}\}.$$

As both functors involved preserve suspensions and coproducts and α is also compatible with suspensions and coproducts we see that S is closed under coproducts and suspension. Given a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ with $X, Y \in \mathsf{S}$ the naturality of α guarantees the commutativity of the following diagram

$$\begin{tikzcd}
t^\vee \otimes X \arrow[r] \arrow[d, "\alpha_X"] & t^\vee \otimes Y \arrow[r] \arrow[d, "\alpha_Y"] & t^\vee \otimes Z \arrow[r] \arrow[d, "\alpha_Z"] & {} \\
\hom(t, X) \arrow[r] & \hom(t, Y) \arrow[r] & \hom(t, Z) \arrow[r] & {}
\end{tikzcd}$$

As both functors are exact the rows are triangles and so as α_X and α_Y are isomorphisms it follows that α_Z must also be an isomorphism and hence $Z \in \mathsf{S}$. Thus S is a localising subcategory and, by rigidity of T^c, we have $\mathsf{T}^c \subseteq \mathsf{S}$. It then follows from [Nee96, Lemma 3.2] that $\mathsf{S} = \mathsf{T}$ which proves the first assertion of the Lemma; the rest of the statements are immediate. □

Another illustration is given by the following analogue of Lemma 1.9.

Lemma 2.13. *Let* T *be a rigidly-compactly generated tensor triangulated category. If* $\mathbf{1}$ *is a compact generator for* T*, i.e., if* $\mathsf{T} = \operatorname{loc}(\mathbf{1})$*, then every localising subcategory of* T *is a localising tensor-ideal.*

Proof. Exercise. □

Localising subcategories are so-called because they are the kernels of localisation functors. We will only give a brief outline of the formalism that we need and do not go into the details of forming Verdier quotients. There are many excellent sources for further information on this topic such as [Nee01], [BN93], and [Kra].

Definition 2.14. A *localisation sequence* is a diagram

$$\begin{tikzcd}
\mathsf{L} \arrow[r, shift left, "i_*"] & \mathsf{T} \arrow[l, shift left, "i^!"] \arrow[r, shift left, "j^*"] & \mathsf{C} \arrow[l, shift left, "j_*"]
\end{tikzcd}$$

where $i^!$ is right adjoint to i_* and j_* is right adjoint to j^*, i_* and j_* are fully faithful and hence embed L and C as a localising and a colocalising subcategory respectively, and we have equalities

$$(i_*\mathsf{L})^\perp = j_*\mathsf{C} \quad \text{and} \quad {}^\perp(j_*\mathsf{C}) = i_*\mathsf{L},$$

where

$$(i_*\mathsf{L})^\perp = \{Y \in \mathsf{T} \mid \mathsf{T}(i_*L, Y) = 0 \text{ for all } L \in \mathsf{L}\}$$

and

$${}^\perp(j_*\mathsf{C}) = \{Y \in \mathsf{T} \mid \mathsf{T}(Y, j_*C) = 0 \text{ for all } C \in \mathsf{C}\}.$$

We refer to the composite $i_*i^!$ as the *acyclisation* functor and j_*j^* as the *localisation* functor corresponding to this localisation sequence.

We will frequently abuse the notation is such situations and identify L and C with their images under the fully faithful functors i_* and j_*.

The existence of a localisation sequence provides a great deal of information and has many consequences. We list a few of the ones we will need in the following proposition and suggest perusing the references given before the definition for more details as well as proofs of the statements made below (there are also relevant references in [BF11, Theorem 2.6] where these statements also appear).

Proposition 2.15. *If we have a localisation sequence as in the definition then the following statements hold.*

(a) *The composites j^*i_* and $i^!j_*$ are zero. Moreover, the kernel of j^* is precisely L.*

(b) *The composite*
$$\mathsf{C} \xrightarrow{j_*} \mathsf{T} \longrightarrow \mathsf{T}/\mathsf{L}$$
is an equivalence. In particular, the Verdier quotient T/L is locally small and the canonical projection $\mathsf{T} \longrightarrow \mathsf{T}/\mathsf{L}$ has a right adjoint.

(c) *For every $X \in \mathsf{T}$ there is a distinguished triangle*
$$i_*i^!X \longrightarrow X \longrightarrow j_*j^*X \longrightarrow \Sigma i_*i^!X.$$
These triangles are functorial and unique in the sense that given any distinguished triangle
$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \Sigma X \quad \text{with } X' \in \mathsf{L} \text{ and } X'' \in \mathsf{C}$$
there are unique isomorphisms $X' \cong i_*i^!X$ and $X'' \cong j_*j^*X$.

(d) *A localisation sequence is completely determined by either of the pairs of adjoint functors $(i_*, i^!)$ or (j^*, j_*).*

In nature localising sequences are actually rather easy to come by. This is a consequence of the following form of Brown representability.

Theorem 2.16. *Let T be a compactly generated triangulated category, $L \subseteq \mathsf{T}$ a set of objects, and set $\mathsf{L} = \mathrm{loc}(S)$. Then the inclusion $i_* \colon \mathsf{L} \longrightarrow \mathsf{T}$ admits a right adjoint. In particular, L fits into a localisation sequence*
$$\begin{tikzcd} \mathsf{L} \arrow[r, shift left, "i_*"] & \mathsf{T} \arrow[l, shift left, "i^!"] \arrow[r, shift left, "j^*"] & \mathsf{T}/\mathsf{L} \arrow[l, shift left, "j_*"] \end{tikzcd}$$

Proof. This follows from Section 7.2 of [Kra] combined with [Nee01, Theorem 8.4.4]. □

By definition, given any localisation sequence as above the functors i_* and j^*, by virtue of being left adjoints, preserve coproducts.

Definition 2.17. A localisation sequence as in Definition 2.14 is *smashing* if $i^!$ (or equivalently j_*) preserves coproducts. In this case we call L a *smashing subcategory* of T. If L is moreover a tensor-ideal we call it a *smashing tensor-ideal.*

In a smashing localisation sequence C is also, under the embedding j_*, a localising subcategory of T. The standard source of smashing localisation sequences is the following well known lemma.

Lemma 2.18. *Let* T *be a compactly generated triangulated category and* $S \subseteq \mathsf{T}^c$ *a set of compact objects. Then* $\mathsf{S} = \mathrm{loc}(S)$ *is a smashing subcategory i.e., the inclusion* $i_*\colon \mathsf{S} \longrightarrow \mathsf{T}$ *admits a coproduct preserving right adjoint* $i^!$.

Proof. We know from Theorem 2.16 that i_* admits a right adjoint $i^!$. It follows from [Nee96, Theorem 5.1] that $i^!$ preserves coproducts. □

2.3. **Generalised Rickard idempotents and supports.** Throughout this section we will assume T is a rigidly-compactly generated tensor triangulated category such that $\mathrm{Spc}\,\mathsf{T}^c$, the spectrum of the compact objects, if a noetherian topological space. Recall from Remark 1.26 that in this case the Thomason subsets of $\mathrm{Spc}\,\mathsf{T}^c$ are precisely the specialisation closed subsets.

We now introduce the objects which allow us to extend the support defined in the first lecture for objects of T^c to arbitrary objects of T. This is achieved via the following key construction based on certain smashing localisations. This result is standard, but we include some details. Further information and references can be found in [BF11].

Theorem 2.19. *Let* T *be a compactly generated triangulated category,* $S \subseteq \mathsf{T}^c$ *a set of compact objects, and set* $\mathsf{S} = \mathrm{loc}^{\otimes}(S)$. *Consider the corresponding smashing localisation sequence*

$$\begin{tikzcd} \mathsf{S} \arrow[r, shift left, "i_*"] & \mathsf{T} \arrow[l, shift left, "i^!"] \arrow[r, shift left, "j^*"] & \mathsf{S}^{\perp}. \arrow[l, shift left, "j_*"] \end{tikzcd}$$

Then:

(a) $\mathsf{S}^{\perp}$ *is a localising tensor-ideal;*

(b) *there are isomorphisms of functors*

$$i_*i^!\mathbf{1} \otimes (-) \cong i_*i^! \quad \text{and} \quad j_*j^*\mathbf{1} \otimes (-) \cong j_*j^*;$$

(c) *the objects* $i_*i^!\mathbf{1}$ *and* $j_*j^*\mathbf{1}$ *satisfy*

$$i_*i^!\mathbf{1} \otimes i_*i^!\mathbf{1} \cong i_*i^!\mathbf{1}, \quad j_*j^*\mathbf{1} \otimes j_*j^*\mathbf{1} \cong j_*j^*\mathbf{1}, \quad \text{and} \quad i_*i^!\mathbf{1} \otimes j_*j^*\mathbf{1} \cong 0,$$

i.e., they are tensor idempotent *and tensor to* 0.

Proof. The claimed smashing localisation sequence exists by Lemma 2.18. The only hitch is that one needs to know S is in fact generated by objects of T^c; in fact $\mathrm{loc}^{\otimes}(S) = \mathrm{loc}(\mathrm{thick}^{\otimes}(S))$, this can be proved directly or deduced from [Ste13, Lemma 3.8] and is left as an exercise. We begin by proving (a); this is the main point and the other statements follow in a straightforward manner from abstract properties of localisations.

Consider the following full subcategory of T

$$\mathsf{M} = \{X \in \mathsf{T} \mid X \otimes \mathsf{S}^{\perp} \subseteq \mathsf{S}^{\perp}\}.$$

As $\otimes$ is exact and coproduct preserving in each variable and $\mathsf{S}^{\perp}$ is localising it is straightforward to verify that M is a localising subcategory. Let t be a compact object of T and $Y \in \mathsf{S}^{\perp}$. We have isomorphisms for any $Z \in \mathsf{S}$

$$\begin{aligned} \mathsf{T}(Z, t \otimes Y) &\cong \mathsf{T}(Z, \hom(t^{\vee}, Y)) \\ &\cong \mathsf{T}(Z \otimes t^{\vee}, Y) \\ &= 0 \end{aligned}$$

where the first isomorphism is via Lemma 2.12, the second is by adjunction, and the final equality holds as S is a tensor ideal so $Z \otimes t^{\vee} \in \mathsf{S}$ and $Y \in \mathsf{S}^{\perp}$. As $Z \in \mathsf{S}$ was arbitrary this shows $t \otimes Y \in \mathsf{S}^{\perp}$ i.e., $\mathsf{T}^c \subseteq \mathsf{M}$. It follows from Remark 2.11 that $\mathsf{M} = \mathsf{T}$ which says precisely that $\mathsf{S}^{\perp}$ is a tensor-ideal.

Now we show (b). Consider the localisation triangle

$$i_* i^! \mathbf{1} \longrightarrow \mathbf{1} \longrightarrow j_* j^* \mathbf{1} \longrightarrow$$

from Proposition 2.15 (c). Given $X \in \mathsf{T}$ we can tensor this triangle with X to obtain the distinguished triangle

$$i_* i^! \mathbf{1} \otimes X \longrightarrow X \longrightarrow j_* j^* \mathbf{1} \otimes X \longrightarrow$$

As both S and $\mathsf{S}^{\perp}$ are tensor-ideals the leftmost and rightmost terms of this latter triangle lie in S and $\mathsf{S}^{\perp}$ respectively. Uniqueness and functoriality of localisation triangles, also observed in (c) of the aforementioned proposition, then guarantees unique isomorphisms

$$i_* i^! X \cong i_* i^! \mathbf{1} \otimes X \quad \text{and} \quad j_* j^* X \cong j_* j^* \mathbf{1} \otimes X$$

which can be assembled to the desired isomorphisms of functors. We leave (c) as an exercise so the interested reader can familiarise themselves with the properties of the localisation and acyclisation functors ($j_* j^*$ and $i_* i^!$ respectively) associated to localisation sequences. ☐

We now apply this proposition to certain subcategories arising from Theorem 1.28. Let $\mathcal{V}$ be a specialisation closed subset of $\operatorname{Spc} \mathsf{T}^c$. Recall that $\tau(\mathcal{V})$ denotes the associated thick tensor-ideal

$$\tau(\mathcal{V}) = \{t \in \mathsf{T}^c \mid \operatorname{supp} t \subseteq \mathcal{V}\}.$$

We set

$$\varGamma_{\mathcal{V}} \mathsf{T} = \operatorname{loc}(\tau(\mathcal{V})).$$

By Lemma 2.18 there is an associated smashing localisation sequence

$$\varGamma_{\mathcal{V}} \mathsf{T} \rightleftarrows \mathsf{T} \rightleftarrows L_{\mathcal{V}} \mathsf{T}$$

and we denote the corresponding acyclisation and localisation functors by $\varGamma_{\mathcal{V}}$ and $L_{\mathcal{V}}$ respectively. By [BF11, Theorem 4.1] $\varGamma_{\mathcal{V}} \mathsf{T}$ is not only a smashing subcategory but a smashing tensor-ideal. Thus applying the preceding theorem yields tensor idempotents $\varGamma_{\mathcal{V}} \mathbf{1}$ and $L_{\mathcal{V}} \mathbf{1}$ which give rise to the acyclisation and localisation functors by tensoring. It is these idempotents that are used to define the support; the intuition is that, for $X \in \mathsf{T}$, $\varGamma_{\mathcal{V}} \mathbf{1} \otimes X$ is the "piece of X supported on $\mathcal{V}$" and $L_{\mathcal{V}} \mathbf{1} \otimes X$ is the "piece of X supported on the complement of $\mathcal{V}$".

Let us make a quick comment on notation before continuing. From this point forward we will generally denote points of $\operatorname{Spc} \mathsf{T}^c$ by $x, y, \ldots$ rather than in a notation suggestive of prime tensor-ideals as in the last section. This is due to the fact that we will work with $\operatorname{Spc} \mathsf{T}^c$ as an abstract topological space which, together with supp, forms a support data rather than explicitly with prime tensor-ideals.

Definition 2.20. For every $x \in \operatorname{Spc} \mathsf{T}^c$ we define subsets of the spectrum

$$\mathcal{V}(x) = \overline{\{x\}}$$

and

$$\mathcal{Z}(x) = \{y \in \operatorname{Spc} \mathsf{T}^c \mid x \notin \mathcal{V}(y)\}.$$

Both of these subsets are specialization closed and hence Thomason as we have assumed $\operatorname{Spc} \mathsf{T}^c$ is noetherian. We note that

$$\mathcal{V}(x) \setminus (\mathcal{Z}(x) \cap \mathcal{V}(x)) = \{x\}$$

i.e., these two Thomason subsets let us pick out the point x.

Definition 2.21. Let x be a point of $\operatorname{Spc} \mathsf{T}^c$. We define a tensor-idempotent

$$\varGamma_x \mathbf{1} = (\varGamma_{\mathcal{V}(x)} \mathbf{1} \otimes L_{\mathcal{Z}(x)} \mathbf{1}).$$

Following the intuition above, for an object $X \in \mathsf{T}$, the object $\varGamma_x \mathbf{1} \otimes X$ is supposed to be the "piece of X which lives only over the point $x \in \operatorname{Spc} \mathsf{T}^c$.

Remark 2.22. Given any Thomason subsets $\mathcal{V}$ and $\mathcal{W}$ of $\operatorname{Spc} \mathsf{T}^c$ such that $\mathcal{V} \setminus (\mathcal{V} \cap \mathcal{W}) = \{x\}$ we can define a similar object by forming the tensor product $\varGamma_{\mathcal{V}} \mathbf{1} \otimes L_{\mathcal{W}} \mathbf{1}$. By [BF11, Corollary 7.5] any such object is uniquely isomorphic to $\varGamma_x \mathbf{1}$.

Example 2.23. As a quick respite from the abstraction let us provide a detailed example describing what these idempotents look like in $\mathsf{D}(R)$ the derived category of a noetherian commutative ring.

Given an element $f \in R$ we define the *stable Koszul complex* $K_\infty(f)$ to be the complex concentrated in degrees 0 and 1

$$\cdots \longrightarrow 0 \longrightarrow R \longrightarrow R_f \longrightarrow 0 \longrightarrow \cdots$$

where the only non-zero morphism is the canonical map to the localisation. Given a sequence of elements $\mathbf{f} = \{f_1, \ldots, f_n\}$ of R we set

$$K_\infty(\mathbf{f}) = K_\infty(f_1) \otimes \cdots \otimes K_\infty(f_n).$$

We define the *Ĉech complex* of $\mathbf{f}$ to be the suspension of the kernel of the canonical morphism $K(\mathbf{f}) \longrightarrow R$. This is a degreewise split epimorphism and so we get a triangle in $K(A)$

$$K_\infty(\mathbf{f}) \longrightarrow R \longrightarrow \check{C}(\mathbf{f}) \longrightarrow \Sigma K_\infty(\mathbf{f}).$$

Explicitly we have

$$\check{C}(\mathbf{f})^t = \bigoplus_{i_0 < \cdots < i_t} R_{f_{i_0} \cdots f_{i_t}}$$

for $0 \leq t \leq n-1$ and $K_\infty(\mathbf{f})$ is degreewise the same complex desuspended and with R in degree 0. For an ideal I of R we define $K(I)$ and $\check{C}(I)$ by choosing a set of generators for I; the complex obtained is independent of the choice of generators up to quasi-isomorphism in $D(R)$. We note that these complexes are K-flat.

Using these complexes we can give the following well known explicit descriptions of the Rickard idempotents corresponding to some specialization closed subsets. For an ideal $I \subseteq R$ and $\mathfrak{p} \in \operatorname{Spec} R$ a prime ideal there are natural isomorphisms in $D(R)$:

(1) $\varGamma_{\mathcal{V}(I)} R \cong K_\infty(I)$;
(2) $L_{\mathcal{V}(I)} R \cong \check{C}(I)$;
(3) $L_{\mathcal{Z}(\mathfrak{p})} R \cong R_{\mathfrak{p}}$.

In particular the objects $\varGamma_{\mathfrak{p}} R = \varGamma_{\mathcal{V}(\mathfrak{p})} R \otimes L_{\mathcal{Z}(\mathfrak{p})} R$ giving rise to supports on $D(R)$ and $S(R)$ are naturally isomorphic to $K_\infty(\mathfrak{p}) \otimes R_{\mathfrak{p}}$.

Proof. Statements (1) and (2) are already essentially present in [Har67]; in the form stated here they can be found as special cases of [Gre01] Lemma 5.8. For the third statement simply observe that the full subcategory of complexes with homological support in $\mathcal{U}(\mathfrak{p})$ is the essential image of the inclusion of $D(R_{\mathfrak{p}})$. □

We now come to the main definition of this section.

Definition 2.24. For $X \in \mathsf{T}$ we define the *support* of X to be

$$\operatorname{supp} X = \{x \in \operatorname{Spc} \mathsf{T}^c \mid \varGamma_x \mathbf{1} \otimes X \neq 0\}.$$

We do not introduce notation to distinguish this notion of support, applied to compact objects, from the one for the first lecture as they agree. The following proposition records this fact as well as several other important properties of the support and the tensor-idempotents we have defined.

Proposition 2.25 ([BF11, 7.17, 7.18])**.** *The support defined above satisfies the following properties.*

(*a*) *The two notions of support coincide for any compact object of* T.

(*b*) *For any set-indexed family of objects* $\{X_\lambda \mid \lambda \in \Lambda\}$ *we have*

$$\operatorname{supp}(\coprod_{\lambda \in \Lambda} X_\lambda) = \bigcup_{\lambda \in \Lambda} \operatorname{supp} X_\lambda.$$

(*c*) *For every* $X \in \mathsf{T}$ $\operatorname{supp}(\Sigma X) = \operatorname{supp}(X)$.

(*d*) *Given a distinguished triangle* $X \longrightarrow Y \longrightarrow Z \longrightarrow$ *in* T *we have*

$$\operatorname{supp}(Y) \subseteq \operatorname{supp}(X) \cup \operatorname{supp}(Z).$$

(*e*) *For any* $X, Y \in \mathsf{T}$ *we have* $\operatorname{supp}(X \otimes Y) \subseteq \operatorname{supp}(X) \cap \operatorname{supp}(Y)$.

(*f*) *For any* $X \in \mathsf{T}$ *and Thomason subset* $\mathcal{V} \subseteq \operatorname{Spc} \mathsf{T}^c$ *we have*

$$\operatorname{supp}(\varGamma_{\mathcal{V}} \mathbf{1} \otimes X) = (\operatorname{supp} X) \cap \mathcal{V}$$
$$\operatorname{supp}(L_{\mathcal{V}} \mathbf{1} \otimes X) = (\operatorname{supp} X) \cap (\operatorname{Spc} \mathsf{T}^c \setminus \mathcal{V}).$$

Remark 2.26. Note that (e) is weaker than the corresponding statement given in Lemma 1.14. It is also worth noting that we have not mentioned that an object $X \in \mathsf{T}$ is 0 if and only if $\operatorname{supp} X = \varnothing$. In fact this latter statement is much more subtle for rigidly-compactly generated tensor triangulated categories. We will show, with an additional hypothesis, that it does indeed hold though when we discuss the local-to-global principle in the next lecture.

3. Lecture 3: Tensor actions and the local-to-global principle

In the last lecture we saw how to define a notion of support for rigidly-compactly generated tensor triangulated categories using the spectrum of the compacts. In this lecture we will explain a relative version, where one defines supports in terms of the action of a rigidly-compactly generated tensor triangulated category. What we have seen is then the special case when such a category acts on itself in the obvious way. The main reference for this section is [Ste13].

Continuing our analogy from the first lecture of a tensor triangulated category to a strange sort of ring, we will now study the corresponding module theory.

3.1. **Actions and submodules.** We define here the notion of left action and express a sinistral bias by only considering left actions and referring to them simply as actions.

We fix throughout this lecture a rigidly-compactly generated tensor triangulated category T and a compactly generated triangulated category K. We will moreover assume throughout that $\operatorname{Spc}\mathsf{T}^c$ is noetherian; a lot of the theory can be developed without this hypothesis but we avoid the related technicalities for the sake of simplicity.

Definition 3.1. A *left action* of T on K is a functor

$$*\colon \mathsf{T}\times\mathsf{K}\longrightarrow\mathsf{K}$$

which is exact in each variable, i.e., for all $X\in\mathsf{T}$ and $A\in\mathsf{K}$ the functors $X*(-)$ and $(-)*A$ are exact, together with natural isomorphisms

$$a_{X,Y,A}\colon (X\otimes Y)*A\xrightarrow{\sim} X*(Y*A)$$

and

$$l_A\colon \mathbf{1}*A\xrightarrow{\sim} A$$

for all $X,Y\in\mathsf{T}$, $A\in\mathsf{K}$, compatible with the biexactness of $(-)*(-)$ and satisfying the following conditions:

(1) The associator a satisfies the pentagon condition which asserts that the following diagram commutes for all X,Y,Z in T and A in K

$$\begin{tikzcd}
 & X*(Y*(Z*A)) & \\
X*((Y\otimes Z)*A) \arrow[ur, "X*a_{Y,Z,A}"] & & (X\otimes Y)*(Z*A) \arrow[ul, "a_{X,Y,Z*A}"'] \\
(X\otimes(Y\otimes Z))*A \arrow[u, "a_{X,Y\otimes Z,A}"] & & ((X\otimes Y)\otimes Z)*A \arrow[ll] \arrow[u, "a_{X\otimes Y,Z,A}"']
\end{tikzcd}$$

where the bottom arrow is the associator of $(\mathsf{T},\otimes,\mathbf{1})$.

(2) The unitor l makes the following squares commute for every X in T and A in K

$$\begin{tikzcd}
X*(\mathbf{1}*A) \arrow[r, "X*l_A"] & X*A \arrow[d, "1_{X*A}"] \\
(X\otimes\mathbf{1})*A \arrow[u, "a_{X,\mathbf{1},A}"] \arrow[r] & X*A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1}*(X*A) \arrow[r, "l_{X*A}"] & X*A \arrow[d, "1_{X*A}"] \\
(\mathbf{1}\otimes X)*A \arrow[u, "a_{\mathbf{1},X,A}"] \arrow[r] & X*A
\end{tikzcd}$$

where the bottom arrows are the right and left unitors of $(\mathsf{T},\otimes,\mathbf{1})$.

(3) For every A in K and $r,s\in\int$ the diagram

$$\begin{tikzcd}
\Sigma^r\mathbf{1}*\Sigma^s A \arrow[r, "\sim"] \arrow[d, "\wr"'] & \Sigma^{r+s}A \arrow[d, "(-1)^{rs}"] \\
\Sigma^r(\mathbf{1}*\Sigma^s A) \arrow[r, "\sim"'] & \Sigma^{r+s}A
\end{tikzcd}$$

is commutative, where the left vertical map comes from exactness in the first variable of the action, the bottom horizontal map is the unitor, and the top map is given by the composite

$$\Sigma^r \mathbf{1} * \Sigma^s A \longrightarrow \Sigma^s(\Sigma^r \mathbf{1} * A) \longrightarrow \Sigma^{r+s}(\mathbf{1} * A) \xrightarrow{l} \Sigma^{r+s} A$$

whose first two maps use exactness in both variables of the action.

(4) The functor $*$ distributes over coproducts whenever they exist i.e., for families of objects $\{X_i\}_{i\in I}$ in T and $\{A_j\}_{j\in J}$ in K, and X in T, A in K the canonical maps

$$\coprod_i (X_i * A) \xrightarrow{\sim} (\coprod_i X_i) * A$$

and

$$\coprod_j (X * A_j) \xrightarrow{\sim} X * (\coprod_j A_j)$$

are isomorphisms whenever the coproducts concerned, on both the left and the right of each isomorphism, exist.

Remark 3.2. Given composable morphisms f, f' in T and g, g' in K one has

$$(f' * g')(f * g) = (f'f * g'g)$$

by functoriality of $\mathsf{T} \times \mathsf{K} \xrightarrow{*} \mathsf{K}$.

We also note it follows easily from the definition that both $0_{\mathsf{T}} * (-)$ and $(-) * 0_{\mathsf{K}}$ are isomorphic to the zero functor.

We view K as a module over T and from now on we will use the terms module and action interchangeably. There are of course, depending on the context, natural notions of T-submodule.

Definition 3.3. Let $\mathsf{L} \subseteq \mathsf{K}$ be a localising (thick) subcategory. We say L is a localising (thick) T-*submodule* of K if the functor

$$\mathsf{T} \times \mathsf{L} \xrightarrow{*} \mathsf{K}$$

factors via L i.e., L is closed under the action of T. We note that in the case $\mathsf{K} = \mathsf{T}$ acts on itself by $\otimes$ this gives the notion of a localising (thick) tensor-ideal of T. By a smashing or compactly generated (by compact objects in the ambient category) submodule we mean the obvious things.

Notation 3.4. For a collection of objects $\mathcal{A}$ in K we denote, as before, by $\mathrm{loc}(\mathcal{A})$ the smallest localising subcategory containing $\mathcal{A}$ and by $\mathrm{loc}^*(\mathcal{A})$ the smallest localising T-submodule of K containing $\mathcal{A}$. Following earlier notation we let $\mathrm{Loc}^*_{\mathsf{T}}(\mathsf{K})$ denote the lattice of localising submodules of K with respect to the action of T. Usually the action in question is clear and we omit the T from the notation.

Given also a collection of objects $\mathcal{X}$ of T we denote by

$$\mathcal{X} * \mathcal{A} = \mathrm{loc}^*(X * A \mid X \in \mathcal{X}, A \in \mathcal{A})$$

the localising submodule generated by products of the objects from $\mathcal{X}$ and $\mathcal{A}$.

We have the following useful technical result for working with tensor-ideals and submodules generated by prescribed sets of objects.

Lemma 3.5 ([Ste13, Lemma 3.12])**.** *Formation of localising* T*-submodules commutes with the action i.e., given a collection of objects* $\mathcal{X}$ *of* T *and a collection of objects* $\mathcal{A}$ *of* K *we have*

$$\begin{aligned}\operatorname{loc}^{\otimes}(\mathcal{X}) * \operatorname{loc}(\mathcal{A}) &= \operatorname{loc}(\mathcal{X}) * \operatorname{loc}(\mathcal{A}) \\ &= \mathcal{X} * \mathcal{A} \\ &= \operatorname{loc}(Z * (X * A) \mid Z \in \mathsf{T}, X \in \mathcal{X}, A \in \mathcal{A}).\end{aligned}$$

Another useful lemma in examples, which frequently removes the need to keep track of submodules versus localising subcategories is the following analogue of Lemmas 1.9 and 2.13.

Lemma 3.6 ([Ste13, Lemma 3.13])**.** *If* T *is generated as a localising subcategory by the tensor unit* $\mathbf{1}$ *then every localising subcategory of* K *is a* T*-submodule.*

3.2. **Supports.** We now introduce supports for K relative to the action of T. This is done in the naive way, extending what we have seen in the last lecture. Recall from Definition 2.21 the objects

$$\varGamma_x\mathbf{1} = \varGamma_{\mathcal{V}(x)}\mathbf{1} \otimes L_{\mathcal{Z}(x)}\mathbf{1}.$$

Notation 3.7. We use $\varGamma_x\mathsf{K}$, for $x \in \operatorname{Spc}\mathsf{T}^c$, to denote the essential image of $\varGamma_x\mathbf{1} * (-)$. It is a T-submodule as for any $X \in \mathsf{T}$ and $A \in \varGamma_x\mathsf{K}$

$$X * A \cong X * (\varGamma_x\mathbf{1} * A') \cong \varGamma_x\mathbf{1} * (X * A')$$

for some $A' \in \mathsf{K}$ as, by virtue of being in the essential image of $\varGamma_x\mathbf{1} \otimes (-)$ we must have such an A' and an isomorphism

$$A \cong \varGamma_x\mathbf{1} \otimes A'.$$

Similarly we will often write $\varGamma_x A$ as shorthand for $\varGamma_x\mathbf{1} * A$.

Definition 3.8. Given A in K we define the support of A to be the set

$$\operatorname{supp}_{(\mathsf{T},*)} A = \{x \in \operatorname{Spc}\mathsf{T}^c \mid \varGamma_x A \neq 0\}.$$

When the action in question is clear we will omit the subscript from the notation.

We have analogous properties for this notion of support as for the support when T acts on itself as in Proposition 2.25.

Proposition 3.9. *The support assignment* $\operatorname{supp}_{(\mathsf{T},*)}$ *satisfies the following properties:*

(a) *given a triangle*

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

in K *we have* $\operatorname{supp} B \subseteq \operatorname{supp} A \cup \operatorname{supp} C$*;*

(b) *for any* A *in* K *and* $i \in \int$

$$\operatorname{supp} A = \operatorname{supp} \Sigma^i A;$$

(c) *given a set-indexed family* $\{A_\lambda\}_{\lambda\in\Lambda}$ *of objects of* K *there is an equality*

$$\operatorname{supp} \coprod_\lambda A_\lambda = \bigcup_\lambda \operatorname{supp} A_\lambda;$$

(d) *the support satisfies the separation axiom i.e., for every specialization closed subset* $\mathcal{V} \subseteq \operatorname{Spc} \mathsf{T}^c$ *and every object* A *of* K

$$\begin{aligned} \operatorname{supp} \mathit{\Gamma}_{\mathcal{V}} \mathbf{1} * A &= (\operatorname{supp} A) \cap \mathcal{V} \\ \operatorname{supp} L_{\mathcal{V}} \mathbf{1} * A &= (\operatorname{supp} A) \cap (\operatorname{Spc} \mathsf{T}^c \setminus \mathcal{V}). \end{aligned}$$

3.3. **The local-to-global principle and parametrising submodules.** An action allows us to understand, at least to some extent, the lattice $\operatorname{Loc}^*(\mathsf{K})$. In this section we will define assignments relating supports to localising submodules. We then introduce the local-to-global principle, a version of which was first considered in [BIK11], and explain some of its consequences with respect to these assignments. We then give a general result on the validity of the local-to-global principle together with an essentially complete proof. This powerful technical result allows us to reduce classification problems into smaller (but usually difficult) pieces. Along the way we will finally show that the support, in the sense we have defined above, is sufficiently refined to determine whether or not an object is 0.

Definition 3.10. We define order preserving assignments

$$\left\{ \text{ subsets of } \operatorname{Spc} \mathsf{T}^c \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \text{ localising submodules of } \mathsf{K} \right\}$$

where both collections are ordered by inclusion. For a localising submodule L we set

$$\sigma(\mathsf{L}) = \operatorname{supp} \mathsf{L} = \{x \in \operatorname{Spc} \mathsf{T}^c \mid \mathit{\Gamma}_x \mathsf{L} \neq 0\}$$

and for a subset $W \subseteq \operatorname{Spc} \mathsf{T}^c$ we set

$$\tau(W) = \{A \in \mathsf{K} \mid \operatorname{supp} A \subseteq W\}.$$

Both of these are well defined; this is clear for σ and for τ it follows from Proposition 3.9.

Definition 3.11. We say $\mathsf{T} \times \mathsf{K} \xrightarrow{*} \mathsf{K}$ satisfies the *local-to-global principle* if for each A in K

$$\operatorname{loc}^*(A) = \operatorname{loc}^*(\mathit{\Gamma}_x A \mid x \in \operatorname{Spc} \mathsf{T}^c).$$

The local-to-global principle has the following rather pleasing consequences for the assignments σ and τ of Definition 3.10.

Lemma 3.12. *Suppose the local-to-global principle holds for the action of* T *on* K *and let* W *be a subset of* $\operatorname{Spc} \mathsf{T}^c$*. Then*

$$\tau(W) = \operatorname{loc}^*(\mathit{\Gamma}_x \mathsf{K} \mid x \in W \cap \sigma \mathsf{K}).$$

Proof. By the local-to-global principle we have for every object A of K an equality

$$\operatorname{loc}^*(A) = \operatorname{loc}^*(\mathit{\Gamma}_x A \mid x \in \operatorname{Spc} \mathsf{T}^c).$$

Thus

$$\begin{aligned} \tau(W) &= \operatorname{loc}^*(A \mid \operatorname{supp} A \subseteq W) \\ &= \operatorname{loc}^*(\mathit{\Gamma}_x A \mid A \in \mathsf{K}, x \in W) \\ &= \operatorname{loc}^*(\mathit{\Gamma}_x A \mid A \in \mathsf{K}, x \in W \cap \sigma \mathsf{K}) \\ &= \operatorname{loc}^*(\mathit{\Gamma}_x \mathsf{K} \mid x \in W \cap \sigma \mathsf{K}). \end{aligned}$$

□

Proposition 3.13. *Suppose the local-to-global principle holds for the action of* T *on* K *and let* W *be a subset of* $\operatorname{Spc}\mathsf{T}^c$. *Then there is an equality of subsets*

$$\sigma\tau(W) = W \cap \sigma\mathsf{K}.$$

In particular, τ *is injective when restricted to subsets of* $\sigma\mathsf{K}$.

Proof. With $W \subseteq \operatorname{Spc}\mathsf{T}^c$ as in the statement we have

$$\begin{aligned}\sigma\tau(W) &= \operatorname{supp}\tau(W)\\ &= \operatorname{supp}\operatorname{loc}^*(\mathit{\Gamma}_x\mathsf{K} \mid x \in W \cap \sigma\mathsf{K}),\end{aligned}$$

the first equality by definition and the second by the last lemma. Thus $\sigma\tau(W) = W \cap \sigma\mathsf{K}$ as claimed: by the properties of the support (Proposition 3.9) we have $\sigma\tau(W) \subseteq W \cap \sigma\mathsf{K}$ and it must in fact be all of $W \cap \sigma\mathsf{K}$ as $x \in \sigma\mathsf{K}$ if and only if $\mathit{\Gamma}_x\mathsf{K}$ contains a non-zero object. ☐

Now we go about showing that we have access to these results in significant generality.

Definition 3.14. We will say the tensor triangulated category T *has a model* if it occurs as the homotopy category of a monoidal model category.

Our main interest in such categories is that the existence of a monoidal model provides a good theory of homotopy colimits compatible with the tensor product.

Remark 3.15. Of course instead of requiring that T arose from a monoidal model category we could, for instance, ask that T was the underlying category of a stable monoidal derivator. In fact we will only use directed homotopy colimits so one could use a weaker notion of a stable monoidal "derivator" only having homotopy left and right Kan extensions for certain diagrams; to be slightly more precise one could just ask for homotopy left and right Kan extensions along the smallest full 2-subcategory of the category of small categories satisfying certain natural closure conditions and containing the ordinals (one can see the discussion before [Gro13] Definition 4.21 for further details).

We begin by showing that, when T has a model, taking the union of a chain of specialisation closed subsets is compatible with taking the homotopy colimit of the associated idempotents.

Lemma 3.16. *Suppose* T *has a model. Then for any chain* $\{\mathcal{V}_i\}_{i\in I}$ *of specialisation closed subsets of* $\operatorname{Spc}\mathsf{T}^c$ *with union* $\mathcal{V}$ *there is an isomorphism*

$$\mathit{\Gamma}_{\mathcal{V}}\mathbf{1} \cong \operatorname{hocolim}\mathit{\Gamma}_{\mathcal{V}_i}\mathbf{1}$$

where the structure maps are the canonical ones.

Proof. As each $\mathcal{V}_i$ is contained in $\mathcal{V}$ there are corresponding inclusions for $i < j$

$$\mathit{\Gamma}_{\mathcal{V}_i}\mathsf{T} \subseteq \mathit{\Gamma}_{\mathcal{V}_j}\mathsf{T} \subseteq \mathit{\Gamma}_{\mathcal{V}}\mathsf{T}$$

which give rise to commuting triangles of canonical morphisms

$$\begin{tikzcd} \mathit{\Gamma}_{\mathcal{V}_i}\mathbf{1} \arrow[rr] \arrow[dr] & & \mathit{\Gamma}_{\mathcal{V}}\mathbf{1} \\ & \mathit{\Gamma}_{\mathcal{V}_j}\mathbf{1} \arrow[ur] & \end{tikzcd}$$

We remind the reader that $\Gamma_{\mathcal{V}_i}\mathsf{T}$ denotes the localising tensor-ideal generated by those compact objects with support in $\mathcal{V}_i$ and $\Gamma_{\mathcal{V}_i}\mathbf{1}$ is the associated idempotent for the acyclisation (see the discussion before Definition 2.20).

From these morphisms we get an induced morphism from the homotopy colimit of the $\Gamma_{\mathcal{V}_i}\mathbf{1}$ to $\Gamma_{\mathcal{V}}\mathbf{1}$ which we complete to a triangle

$$\operatorname{hocolim}_I \Gamma_{\mathcal{V}_i}\mathbf{1} \longrightarrow \Gamma_{\mathcal{V}}\mathbf{1} \longrightarrow Z \longrightarrow \Sigma \operatorname{hocolim}_I \Gamma_{\mathcal{V}_i}\mathbf{1}.$$

In order to prove the lemma it is sufficient to show that Z is isomorphic to the zero object in T.

The argument in [Bou83] extends to show localising subcategories are closed under homotopy colimits so this triangle consists of objects of $\Gamma_{\mathcal{V}}\mathsf{T}$. By definition $\Gamma_{\mathcal{V}}\mathsf{T}$ is the full subcategory of T generated by those objects of T^c whose support is contained in $\mathcal{V}$. Thus $Z \cong 0$ if for each compact object t with $\operatorname{supp} t \subseteq \mathcal{V}$ we have $\operatorname{Hom}(t, Z) = 0$; we remark that there is no ambiguity here as by Proposition 2.25 the two notions of support, that of [Bal05] and [BF11], agree for compact objects. In particular the support of any compact object is closed.

Recall from Theorem 1.20 that by [BKS07] $\operatorname{Spc}\mathsf{T}^c$ is spectral (see Definition 1.19) and we have assumed it is also noetherian. Thus $\operatorname{supp} t$, by virtue of being closed, is a finite union of irreducible closed subsets. We can certainly find a $j \in I$ so that $\mathcal{V}_j$ contains the generic points of these finitely many irreducible components which implies $\operatorname{supp} t \subseteq \mathcal{V}_j$ by specialisation closure. Therefore, by adjunction, it is enough to show

$$\begin{aligned}\operatorname{Hom}(t, Z) &\cong \operatorname{Hom}(\Gamma_{\mathcal{V}_j} t, Z)\\ &\cong \operatorname{Hom}(t, \Gamma_{\mathcal{V}_j} Z)\end{aligned}$$

is zero, as this implies $Z \cong 0$ and we get the claimed isomorphism.

In order to show the claimed hom-set vanishes let us demonstrate that $\Gamma_{\mathcal{V}_j} Z$ is zero. Observe that tensoring the structure morphisms $\Gamma_{\mathcal{V}_{i_1}}\mathbf{1} \longrightarrow \Gamma_{\mathcal{V}_{i_2}}\mathbf{1}$ for $i_2 \geq i_1 \geq j$ with $\Gamma_{\mathcal{V}_j}\mathbf{1}$ yields canonical isomorphisms

$$\Gamma_{\mathcal{V}_j}\mathbf{1} \cong \Gamma_{\mathcal{V}_j}\mathbf{1} \otimes \Gamma_{\mathcal{V}_{i_1}}\mathbf{1} \xrightarrow{\sim} \Gamma_{\mathcal{V}_j}\mathbf{1} \otimes \Gamma_{\mathcal{V}_{i_2}}\mathbf{1} \cong \Gamma_{\mathcal{V}_j}\mathbf{1}.$$

Thus applying $\Gamma_{\mathcal{V}_j}$ to the sequence $\{\Gamma_{\mathcal{V}_i}\mathbf{1}\}_{i\in I}$ gives a diagram whose homotopy colimit is $\Gamma_{\mathcal{V}_j}\mathbf{1}$. From this we deduce that the first morphism in the resulting triangle

$$\Gamma_{\mathcal{V}_j}\mathbf{1} \otimes \operatorname{hocolim}_I \Gamma_{\mathcal{V}_i}\mathbf{1} \longrightarrow \Gamma_{\mathcal{V}_j}\mathbf{1} \longrightarrow \Gamma_{\mathcal{V}_j} Z$$

is an isomorphism as T is the homotopy category of a monoidal model category so the tensor product commutes with homotopy colimits. This forces $\Gamma_{\mathcal{V}_j} Z \cong 0$ completing the proof. ☐

Lemma 3.17. *Let $P \subseteq \operatorname{Spc}\mathsf{T}^c$ be given and suppose A is an object of K such that $\Gamma_x A \cong 0$ for all $x \in (\operatorname{Spc}\mathsf{T}^c \setminus P)$. If T has a model then A is an object of the localising subcategory*

$$\mathsf{L} = \operatorname{loc}(\Gamma_y \mathsf{K} \mid y \in P).$$

Proof. Let $\Lambda \subseteq \mathcal{P}(\operatorname{Spc}\mathsf{T}^c)$ be the set of specialisation closed subsets $\mathcal{W}$ such that $\Gamma_{\mathcal{W}} A$ is in $\mathsf{L} = \operatorname{loc}(\Gamma_y \mathsf{K} \mid y \in P)$. We first note that Λ is not empty. Indeed, as T^c is rigid the only compact objects with empty support are the zero objects by [Bal07, Corollary 2.5] so

$$\Gamma_{\varnothing}\mathsf{T} = \operatorname{loc}(t \in \mathsf{T}^c \mid \operatorname{supp}_{(\mathsf{T},\otimes)} t = \varnothing) = \operatorname{loc}(0)$$

giving $\mathit{\Gamma}_\varnothing A = 0$ and hence $\varnothing \in \Lambda$.

Since L is localising, Lemma 3.16 shows the set Λ is closed under taking increasing unions: as mentioned above the argument in [Bou83] extends to show that localising subcategories are closed under directed homotopy colimits in our situation. Thus Λ contains a maximal element Y by Zorn's lemma. We claim that $Y = \operatorname{Spc}\mathsf{T}^c$.

Suppose $Y \neq \operatorname{Spc}\mathsf{T}^c$. Then since $\operatorname{Spc}\mathsf{T}^c$ is noetherian $\operatorname{Spc}\mathsf{T}^c \setminus Y$ contains a maximal element z with respect to specialisation. We have

$$L_Y\mathbf{1} \otimes \mathit{\Gamma}_{Y\cup\{z\}}\mathbf{1} \cong \mathit{\Gamma}_z\mathbf{1}$$

as $Y \cup \{z\}$ is specialisation closed by maximality of z and Remark 2.22 tells us that we can use any suitable pair of Thomason subsets to define $\mathit{\Gamma}_z\mathbf{1}$. So $L_Y\mathit{\Gamma}_{Y\cup\{z\}}A \cong \mathit{\Gamma}_z A$ and by our hypothesis on vanishing either $\mathit{\Gamma}_z\mathsf{K} \subseteq \mathsf{L}$ or $\mathit{\Gamma}_z A = 0$. Considering the triangle

$$\begin{tikzcd}
\mathit{\Gamma}_Y\mathit{\Gamma}_{Y\cup\{z\}}A \arrow[r] \arrow[d, "\wr"'] & \mathit{\Gamma}_{Y\cup\{z\}}A \arrow[r] & L_Y\mathit{\Gamma}_{Y\cup\{z\}}A \arrow[d, "\wr"] \\
\mathit{\Gamma}_Y A & & \mathit{\Gamma}_z A
\end{tikzcd}$$

we see that in either case, since $\mathit{\Gamma}_Y A$ is in L, that $Y \cup \{z\} \in \Lambda$ contradicting maximality of Y. Hence $Y = \operatorname{Spc}\mathsf{T}^c$ and so A is in L. $\square$

Proposition 3.18. *Suppose* T *has a model. Then the local-to-global principle holds for the action of* T *on* K. *Explicitly for any* A *in* K *there is an equality of* T-*submodules*

$$\operatorname{loc}^*(A) = \operatorname{loc}^*(\mathit{\Gamma}_x A \mid x \in \operatorname{supp} A).$$

Proof. By Lemma 3.17 applied to the action

$$\mathsf{T} \times \mathsf{T} \xrightarrow{\otimes} \mathsf{T}$$

we see $\mathsf{T} = \operatorname{loc}(\mathit{\Gamma}_x\mathsf{T} \mid x \in \operatorname{Spc}\mathsf{T}^c)$. Since $\mathit{\Gamma}_x\mathsf{T} = \operatorname{loc}^\otimes(\mathit{\Gamma}_x\mathbf{1})$ it follows that the set of objects $\{\mathit{\Gamma}_x\mathbf{1} \mid x \in \operatorname{Spc}\mathsf{T}^c\}$ generates T as a localising tensor-ideal. By Lemma 3.5 given an object $A \in \mathsf{K}$ we get a generating set for $\mathsf{T} * \operatorname{loc}(A)$:

$$\mathsf{T} * \operatorname{loc}(A) = \operatorname{loc}^\otimes(\mathit{\Gamma}_x\mathbf{1} \mid x \in \operatorname{Spc}\mathsf{T}) * \operatorname{loc}(A) = \operatorname{loc}^*(\mathit{\Gamma}_x A \mid x \in \operatorname{supp} A).$$

But it is also clear that $\mathsf{T} = \operatorname{loc}^\otimes(\mathbf{1})$ so, by Lemma 3.5 again,

$$\mathsf{T} * \operatorname{loc}(A) = \operatorname{loc}^\otimes(\mathbf{1}) * \operatorname{loc}(A) = \operatorname{loc}^*(A)$$

and combining this with the other string of equalities gives

$$\operatorname{loc}^*(A) = \mathsf{T} * \operatorname{loc}(A) = \operatorname{loc}^*(\mathit{\Gamma}_x A \mid x \in \operatorname{supp} A)$$

which completes the proof. $\square$

We thus have the following theorem concerning the local-to-global principle for actions of rigidly-compactly generated tensor triangulated categories.

Theorem 3.19. *Suppose* T *is a rigidly-compactly generated tensor triangulated category with a model and that* $\operatorname{Spc}\mathsf{T}^c$ *is noetherian. Then* T *satisfies the following properties:*

(i) *The local-to-global principle holds for the action of* T *on itself;*
(ii) *The associated support theory detects vanishing of objects i.e.,* $X \in \mathsf{T}$ *is zero if and only if* $\operatorname{supp} X = \varnothing$*;*

(iii) *For any chain* $\{\mathcal{V}_i\}_{i\in I}$ *of specialisation closed subsets of* $\operatorname{Spc}\mathsf{T}^c$ *with union* $\mathcal{V}$ *there is an isomorphism*

$$\Gamma_{\mathcal{V}}\mathbf{1} \cong \operatorname{hocolim} \Gamma_{\mathcal{V}_i}\mathbf{1}$$

where the structure maps are the canonical ones.

Furthermore, the relative versions of (i) and (ii) hold for any action of T *on a compactly generated triangulated category* K.

Proof. That (iii) always holds is the content of Lemma 3.16 and we have proved in Proposition 3.18 that (i) holds. To see (i) implies (ii) observe that if $\operatorname{supp} X = \varnothing$ for an object X of T then the local-to-global principle yields

$$\operatorname{loc}^{\otimes}(X) = \operatorname{loc}^{\otimes}(\Gamma_x X \mid x \in \operatorname{Spc}\mathsf{T}^c) = \operatorname{loc}^{\otimes}(0)$$

so $X \cong 0$.

Finally, we saw in Proposition 3.18 that the relative version of (i) holds. This in turn implies (ii) for supports with values in $\operatorname{Spc}\mathsf{T}^c$ by the same argument as we have used in the proof of (i)⇒(ii) above. □

4. Lecture 4: Applications and examples

In this final lecture we concentrate on giving examples to illustrate how one can use the local-to-global principle in practice.

4.1. **Singularity categories of hypersurfaces.** In this section we outline the solution to the classification problem for localising subcategories of singularity categories of hypersurface rings. The intention is to indicate the manner in which these computations usually proceed. In particular, we highlight the manner in which the local-to-global principle is used. We begin by defining the main players and explaining the action we will use. Further details, including the analogous results in the non-affine case, can be found in [Ste12].

We can, for the time being, work quite generally. Suppose R is a arbitrary commutative noetherian ring. Then one defines a category

$$\mathsf{D}_{\mathrm{Sg}}(R) := \mathsf{D}^b(\operatorname{mod} R)/\mathsf{D}^{\mathrm{perf}}(R),$$

where $D^b(\operatorname{mod} R)$ is the bounded derived category of finitely generated R-modules and $\mathsf{D}^{\mathrm{perf}}(R)$ is the full subcategory of complexes locally isomorphic to bounded complexes of finitely generated projectives. This category measures the singularities of R. In particular, $\mathsf{D}_{\mathrm{Sg}}(R)$ vanishes if and only if R is regular, it is related to other measures of the singularities of R for example maximal Cohen-Macaulay modules (see [Buc87]), and its properties reflect the severity of the singularities of R. The particular category which will concern us is the stable derived category of Krause [Kra05], namely

$$\mathsf{S}(R) := \mathsf{K}_{\mathrm{ac}}(\operatorname{Inj} R)$$

the homotopy category of acyclic complexes of injective R-modules. We slightly abuse standard terminology by calling $\Sigma(R)$ the *singularity category* of R. We will also need to consider the following categories

$$\mathsf{D}(R) := \mathsf{D}(\operatorname{Mod} R), \quad \text{and} \quad \mathsf{K}(R) := K(\operatorname{Mod} R)$$

the unbounded derived category of R and the homotopy category of R-modules. As indicated above we denote by $\operatorname{Inj} R$ the category of injective R-modules and we write $\operatorname{Flat} R$ for the category of flat R-modules.

The main facts about $\mathsf{S}(R)$ that we will need are summarised in the following theorem of Krause.

Theorem 4.1 ([Kra05] Theorem 1.1)**.** *Let R be a commutative noetherian ring.*

(1) *There is a recollement*

$$\begin{tikzcd} \mathsf{S}(R) \arrow[r, "I"] & \mathsf{K}(\operatorname{Inj} R) \arrow[l, bend right, "I_\lambda"'] \arrow[l, bend left, "I_\rho"] \arrow[r, "Q"] & \mathsf{D}(R) \arrow[l, bend right, "Q_\lambda"'] \arrow[l, bend left, "Q_\rho"] \end{tikzcd}$$

where each functor is right adjoint to the one above it.

(2) *The triangulated category $\mathsf{K}(\operatorname{Inj} R)$ is compactly generated, and Q induces an equivalence*

$$\mathsf{K}(\operatorname{Inj} R)^{\mathrm{c}} \longrightarrow \mathsf{D}^b(\operatorname{mod} R).$$

(3) *The sequence*

$$\mathsf{D}(R) \xrightarrow{Q_\lambda} \mathsf{K}(\operatorname{Inj} R) \xrightarrow{I_\lambda} \mathsf{S}(R)$$

is a localisation sequence. Therefore $\mathsf{S}(R)$ is compactly generated, and $I_\lambda \circ Q_\rho$ induces (up to direct factors) an equivalence

$$\mathsf{D}_{\mathrm{Sg}}(R) \longrightarrow \mathsf{S}(R)^{\mathrm{c}}.$$

Now let us outline the proof that there is an action

$$\mathsf{D}(R) \times \mathsf{S}(R) \xrightarrow{\odot} \mathsf{S}(R)$$

in the sense of Definition 3.1. This comes down to showing one can take K-flat resolutions (see Definition 4.4) of objects of $\mathsf{D}(R)$ in a way which, although not necessarily initially functorial, becomes functorial after tensoring with an acyclic complex of injectives. Our starting point is work of Neeman [Nee08] and Murfet [Mur07] which provides a category that naturally acts on $\mathsf{K}(\operatorname{Inj} R)$.

There is an obvious action of $\mathsf{K}(\operatorname{Flat} R)$ on $\mathsf{K}(\operatorname{Inj} R)$, given by simply taking the tensor product of complexes. We recall this is in fact defined as, when R is noetherian, the tensor product of a complex of flats with a complex of injectives is again a complex of injectives. However, $\mathsf{K}(\operatorname{Flat} R)$ may contain many objects which act trivially. Thus we will now pass to a more manageable quotient.

In order to do so we remind the reader of the notion of pure acyclicity. In [MS] a complex F in $\mathsf{K}(\operatorname{Flat} R)$ is defined to be *pure acyclic* if it is exact and has flat syzygies. Such complexes form a triangulated subcategory of $\mathsf{K}(\operatorname{Flat} R)$ which we denote by $\mathsf{K}_{\mathrm{pac}}(\operatorname{Flat} R)$ and we say that a morphism with pure acyclic mapping cone is a *pure quasi-isomorphism.* The point is that tensoring a pure acyclic complex of flats with a complex of injectives yields a contractible complex by [Nee08, Corollary 9.7].

Following Neeman and Murfet we consider $\mathsf{N}(\operatorname{Flat} R)$, which is defined to be the quotient $\mathsf{K}(\operatorname{Flat} R)/\mathsf{K}_{\mathrm{pac}}(\operatorname{Flat} R)$. What we have observed above shows the action of $\mathsf{K}(\operatorname{Flat} R)$ on $\mathsf{K}(\operatorname{Inj} R)$ factors via the projection $\mathsf{K}(\operatorname{Flat} R) \longrightarrow \mathsf{N}(\operatorname{Flat} R)$. Next we describe a suitable subcategory of $\mathsf{N}(\operatorname{Flat} R)$ which will act on $\mathsf{S}(R)$.

Recall from above that $\mathsf{S}(R)$ is a compactly generated triangulated category. Consider $E = \coprod_\lambda E_\lambda$ where E_λ runs through a set of representatives for the isomorphism classes of compact objects in $\mathsf{S}(R)$. We define a homological functor

$H\colon \mathsf{K}(\mathrm{Flat}\, R) \longrightarrow \mathrm{Ab}$ by setting, for F an object of $\mathsf{K}(\mathrm{Flat}\, R)$,

$$H(F) = H^0(F \otimes_R E)$$

where the tensor product is taken in $\mathsf{K}(R)$. This is a coproduct preserving homological functor since we are merely composing the exact coproduct preserving functor $(-) \otimes_R E$ with the coproduct preserving homological functor H^0.

In particular every pure acyclic complex lies in the kernel of H.

Definition 4.2. With notation as above we denote by $A_\otimes(\mathrm{Inj}\, R)$ the quotient $\ker(H)/\mathsf{K}_{\mathrm{pac}}(\mathrm{Flat}\, R)$, where

$$\ker(H) = \{F \in \mathsf{K}(\mathrm{Flat}\, R) \mid H(\Sigma^i F) = 0 \ \forall i \in \mathbb{Z}\}.$$

The next lemma shows this is the desired subcategory of $\mathsf{N}(\mathrm{Flat}\, R)$.

Lemma 4.3 ([Ste12, Lemma 4.3])**.** *An object F of $K(\mathrm{Flat}\, X)$ lies in $\ker(H)$ if and only if the exact functor*

$$F \otimes_{\mathcal{O}_X} (-)\colon K(\mathrm{Inj}\, X) \longrightarrow K(\mathrm{Inj}\, X)$$

restricts to

$$F \otimes_{\mathcal{O}_X} (-)\colon S(X) \longrightarrow S(X).$$

In particular, $A_\otimes(\mathrm{Inj}\, X)$ consists of the pure quasi-isomorphism classes of objects which act on $S(X)$.

Before continuing we recall the notion of K-flatness. By taking K-flat resolutions we get an action of $\mathsf{D}(R)$ via the above category.

Definition 4.4. We say that a complex of flat R-modules F is *K-flat* provided $F \otimes_R (-)$ sends quasi-isomorphisms to quasi-isomorphisms (or equivalently if $F \otimes_R E$ is an exact complex for any exact complex of R-modules E).

Lemma 4.5. *There is a fully faithful, exact, coproduct preserving functor*

$$\mathsf{D}(R) \longrightarrow A_\otimes(\mathrm{Inj}\, R).$$

Proof. There is, by the proof of Theorem 5.5 of [Mur07], a fully faithful, exact, coproduct preserving functor $\mathsf{D}(R) \longrightarrow \mathsf{N}(\mathrm{Flat}\, R)$ given by taking K-flat resolutions and inducing an equivalence

$$\mathsf{D}(R) \cong {}^{\perp}\mathsf{N}_{\mathrm{ac}}(\mathrm{Flat}\, R).$$

This functor given by taking resolutions factors via $A_\otimes(\mathrm{Inj}\, R)$ since K-flat complexes send acyclics to acyclics under the tensor product. $\square$

Taking K-flat resolutions and then tensoring gives the desired action

$$(-) \odot (-)\colon \mathsf{D}(R) \times \mathsf{S}(R) \longrightarrow \mathsf{S}(R)$$

by an easy argument: K-flat resolutions are well behaved with respect to the tensor product so the necessary compatibilities follow from those of the tensor product of complexes.

Remark 4.6. Recall that every complex in $\mathsf{K}^-(\mathrm{Flat}\, R)$, the homotopy category of bounded above complexes of flat R-modules, is K-flat. Thus when acting by the subcategory $\mathsf{K}^-(\mathrm{Flat}\, R)$ there is an equality $\odot = \otimes_R$.

Thus we can apply all of the machinery we have introduced for actions of rigidly-compactly generated triangulated categories. Recall from Example 1.30 that $\operatorname{Spc} \mathsf{D}(R)^c = \operatorname{Spc} \mathsf{D}^{\mathrm{perf}}(R) \cong \operatorname{Spec} R$; henceforth we will identify these spaces. As in Definition 3.8 we get a notion of support on $\mathsf{S}(R)$ with values in $\operatorname{Spec} R$; we will denote the support of an object A of $\mathsf{S}(R)$ simply by $\operatorname{supp} A$. As the category $\mathsf{D}(R)$ has a model the local-to-global principle (Theorem 3.19) holds. In particular, not only do we have assignments as in Definition 3.10

$$\left\{ \text{ subsets of } \operatorname{Spec} R \ \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \text{ localising subcategories of } \mathsf{S}(R) \ \right\} \tag{4.1}$$

where for a localising subcategory L we set

$$\sigma(\mathsf{L}) = \operatorname{supp} \mathsf{L} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \varGamma_{\mathfrak{p}} \mathsf{L} \neq 0\}$$

and for a subset $W \subseteq \operatorname{Spec} R$ we set

$$\tau(W) = \{A \in \mathsf{S}(R) \mid \operatorname{supp} A \subseteq W\}.$$

we have the following additional information.

Proposition 4.7. *Given a subset $W \subseteq \operatorname{Spec} R$ there is an equality of subcategories*

$$\tau(W) = \operatorname{loc}(\varGamma_{\mathfrak{p}} \mathsf{S}(R) \mid \mathfrak{p} \in W)$$

Proof. This is just a restatement of Lemma 3.12 combined with Lemma 2.13. $\square$

We now describe $\sigma \mathsf{S}(R)$, the support of the singularity category. In order to describe it we need to recall the following definition.

Definition 4.8. The *singular locus* of R is the set

$$\operatorname{Sing}(R) = \{\mathfrak{p} \in \operatorname{Spec} R \mid R_{\mathfrak{p}} \text{ is not regular}\},$$

i.e., the collection of those prime ideals $\mathfrak{p}$ such that $R_{\mathfrak{p}}$ has infinite global dimension. Recall that this is equivalent to the residue field $k(\mathfrak{p})$ having infinite projective dimension over $R_{\mathfrak{p}}$.

Proposition 4.9 ([Ste12, Proposition 5.7])**.** *For any $\mathfrak{p} \in \operatorname{Sing} R$ the object $\varGamma_{\mathfrak{p}} I_{\lambda} Q_{\rho} k(\mathfrak{p})$ is non-zero in $\mathsf{S}(R)$. Thus $\varGamma_{\mathfrak{p}} \mathsf{S}(R)$ is non-trivial for all such $\mathfrak{p}$ yielding the equality*

$$\sigma \mathsf{S}(R) = \operatorname{Sing} R.$$

We now restrict ourselves to hypersurfaces, which we define below, so we can state the classification theorem, and provide a rough sketch of the proof.

Definition 4.10. Let $(R, \mathfrak{m}, k)$ be a noetherian local ring. We say R is a *hypersurface* if its $\mathfrak{m}$-adic completion $\hat{R}$ can be written as the quotient of a regular ring by a regular element (i.e., a non-zero divisor).

Given a not necessarily local commutative noetherian ring R if, when localized at each prime ideal $\mathfrak{p}$ in $\operatorname{Spec} R$, $R_{\mathfrak{p}}$ is a hypersurface we say that R is *locally a hypersurface.*

Theorem 4.11 ([Ste12, Theorem 6.13])**.** *If R is a noetherian ring which is locally a hypersurface then there is an isomorphism of lattices*

$$\left\{ \textit{ subsets of } \operatorname{Sing} R \ \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \textit{ localising subcategories of } \mathsf{S}(R) \ \right\}$$

given by the assignments of (4.1). This restricts to the equivalent lattice isomorphisms

$$\left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \operatorname{Sing} R \end{array} \right\} \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{localising subcategories of } \mathsf{S}(R) \\ \textit{generated by objects of } \mathsf{S}(R)^c \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} \textit{specialization closed} \\ \textit{subsets of } \operatorname{Sing} R \end{array} \right\} \rightleftarrows \left\{ \textit{thick subcategories of } \mathsf{D}_{\mathrm{Sg}}(R) \right\}.$$

Sketch of proof. We only sketch the proof of the first bijection, the others follow from it but the proof is somewhat more involved.

By Proposition 3.13, combined with the computation of $\sigma\mathsf{S}(R)$ given in Proposition 4.9, we know that τ is injective with left inverse σ. Suppose then that L is a localising subcategory and consider

$$\tau\sigma\mathsf{L} = \operatorname{loc}(\varGamma_{\mathfrak{p}}\mathsf{S}(R) \mid \varGamma_{\mathfrak{p}}\mathsf{L} \neq 0)$$

where we have this equality by Proposition 4.7. By [Ste12, Theorem 6.12] (together with a short reduction to the local case) each of the localising subcategories $\varGamma_{\mathfrak{p}}\mathsf{S}(R)$ is *minimal* i.e., it has no proper non-zero localising subcategories. Thus, if $\varGamma_{\mathfrak{p}}\mathsf{L}$ is non-zero it must, by this minimality, be equal to $\varGamma_{\mathfrak{p}}\mathsf{S}(R)$. By the local-to-global principle we know

$$\mathsf{L} = \operatorname{loc}(\varGamma_{\mathfrak{p}}\mathsf{L} \mid \mathfrak{p} \in \operatorname{Sing} R\}$$

and so putting all this together yields

$$\mathsf{L} = \operatorname{loc}(\varGamma_{\mathfrak{p}}\mathsf{L} \mid \mathfrak{p} \in \operatorname{Sing} R\} = \operatorname{loc}(\varGamma_{\mathfrak{p}}\mathsf{S}(R) \mid \varGamma_{\mathfrak{p}}\mathsf{L} \neq 0) = \tau\sigma\mathsf{L}.$$

This proves we have the claimed bijection. □

Remark 4.12. The strategy evident in the above proof sketch is really the thing to remember. Given the local-to-global principle for an action of T on K it is sufficient to check that each $\varGamma_x\mathsf{K}$ is either zero or has no non-trivial, proper localising subcategories; if this is the case one has an isomorphism of lattices

$$\operatorname{Loc}(\mathsf{K}) \cong \{\text{subsets of } \sigma\mathsf{K}\}.$$

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