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Advanced Course on
Shimura varieties and L-functions

Notes of the Course

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Foreword

This set of notes corresponds to the Advanced Courses on Shimura Varieties and L-functions, held in October 2009 at the Centre de Recerca Matemàtica (CRM) in Bellaterra (Barcelona). The Advanced School was one of the main activities of the Research Programme Arithmetic Geometry, which took place at the CRM from September 2009 to July 2010. The courses were the following:

- *Arithmetic of Shimura curves and the Birch and Swinnerton-Dyer Conjecture*, by Shou-Wu Zhang (Columbia University), and

- *A conjecture of André and Oort*, by Bas Edixhoven (Universiteit Leiden) and Andrei Yafaev (University College London)

The two courses deal with questions which pertain to the flourishing field of number theory and arithmetic geometry.

The aim of the series of lectures delivered by Shou-Wu Zhang is to give a comprehensive description of some recent work of the author and his students on generalisations of the Gross-Zagier formula, Euler systems on Shimura curves and rational points on elliptic curves. More precisely, the course will describe some of the results obtained in the following articles:

1. X. Yuan, S. Zhang, W. Zhang, *Heights of CM-points I: Gross–Zagier formula* (`http://www.math.columbia.edu/ szhang/papers/HCMI.pdf`). This article provides a Gross-Zagier formula in a very general setting.

2. X. Yuan, S. Zhang, W. Zhang, *Heights of CM-points II: Chowla–Selberg formula* (In preparation). This note provides formulae for logarithmic derivatives of Dedekind zeta functions of totally real fields and CM-fields.

3. Y. Tian, S. Zhang, *Euler systems of CM-points on Shimura curves* (In preparation). This article gives a generalization of Kolyvagin's work and some applications to Diophantine equations.

4. X. Yuan, S. Zhang, W. Zhang, *Triple product L-series and Gross–Schoen cycles* (In preparation). This paper contains a formula for the derivative of the triple product L-series and a new construction of rational points on elliptic curves.

The aim of the course delivered by Bas Edixhoven and Andrei Yafaev is to give an introduction to the proof (under the generalised Riemann hypothesis) of the so-called Andre-Oort conjecture by Yafaev, Klingler and Ullmo.

More precisely, the main goal of the lectures will be to describe the results obtained by B. Klingler, E. Ullmo and A. Yafaev in the recent preprints:

1. E. Ullmo, A. Yafaev, *Galois orbits and equidistribution: towards the André-Oort conjecture*, available at
`http://www.math.u-psud.fr/ ullmo/Prebublications/UllmoYafaev2.pdf`

2. B. Klingler, A. Yafaev, *The André-Oort conjecture*, available at
`http://people.math.jussieu.fr/ klingler/papers.html`

This conjecture says that if S is a Shimura variety and Z is any subset of special points of S, then the irreducible components of the Zariski closure of Z are sub-Shimura varieties. Important examples are the moduli spaces of polarised abelian varieties, where the special points are the points corresponding to abelian varieties with (sufficiently many) complex multiplications.

The course will follow the history of the subject, starting with the simplest non-trivial case, and keeping the most technical parts for the end. The main ingredients, Galois orbits, Hecke correspondences and equidistribution, will be introduced. A detailed sketch of the proof mentioned above will be given.

We wish to express our gratitude to the director and the staff of the CRM who helped us in the organization of these courses. We thank the Ingenio-Mathematica programme of the Spanish government and the Catalan Research Funding Agency (AGAUR) for providing financial support for the organization of this Advanced Courses.

The Coordinators

Francesc Bars,
Luis Dieulefait, and
Víctor Rotger

Arithmetic of Shimura curves and the Birch and Swinnerton-Dyer Conjecture

Shou-Wu Zhang

Heights of CM points I
Gross–Zagier formula

Xinyi Yuan, Shou-wu Zhang, Wei Zhang

July 27, 2009

Contents

1 Introduction and statement of main results

In 1984, Gross and Zagier [GZ] proved a formula that relates the Néron–Tate heights of Heegner points to the central derivatives of some Rankin L-series under certain ramification conditions. Since then some generalizations are given in various papers [Zh1, Zh2, Zh3]. The methods of proofs of the Gross–Zagier theorem and all its extensions depend on some newform theories. There are essential difficulties to remove all ramification assumptions in this method. The aim of this paper is a proof of a general formula in which all ramification condition are removed. Such a formula is an analogue of a central value formula of Waldspurger [Wa] and has been more or less formulated by Gross [Gr] in 2002 in term of representation theory. In the following, we want to describe statements of the main results and main idea of proof.

1.1 L-function and root numbers

Let F be a number field with adele ring $\mathbb{A} = \mathbb{A}_F$. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. Let E be a quadratic extension of F, and $\chi : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ a finite character. We assume that

$$\chi|_{\mathbb{A}^\times} \cdot \omega_\pi = 1$$

where ω_π is the central character of π.

Denote by $L(s, \pi, \chi)$ the Rankin-Selberg L-function. Denote by $\epsilon(\frac{1}{2}, \pi_v, \chi_v) = \pm 1$ the local root number at each place v of F, and denote

$$\Sigma = \left\{ v : \epsilon(\frac{1}{2}, \pi_v, \chi_v) \neq \chi_v \eta_v(-1) \right\},$$

where η is the quadratic character of $F^\times\backslash\mathbb{A}^\times$ associated to the extension E/F. Then Σ is a finite set and the global root number is given by

$$\epsilon\left(\frac{1}{2},\pi,\chi\right)=\prod_v \epsilon(\frac{1}{2},\pi_v,\chi_v)=(-1)^{\#\Sigma}.$$

The following is a description of root numbers in terms of linear functionals:

Proposition 1.1.1 (Saito–Tunnell [Tu, Sa]). *Let v be a place of F and B_v a quaternion division algebra over F_v. Let π'_v be the Jacquet–Langlands correspondence of π_v on $B_v^\times$ if π_v is square integrable. Fix embeddings $E_v^\times\subset \mathrm{GL}_2(F_v)$ and $E_v^\times\subset B_v^\times$ as algebraic subgroups. Then $v\in\Sigma$ if and only if*

$$\mathrm{Hom}_{E_v^\times}(\pi_v\otimes\chi_v,\mathbb{C})=0.$$

Moreover

$$\dim \mathrm{Hom}_{E_v^\times}(\pi_v\otimes\chi_v,\mathbb{C})+\mathrm{Hom}_{E_v^\times}(\pi'_v\otimes\chi_v,\mathbb{C})=1.$$

Here the second space is treated as 0 if π'_v is not square integrable.

Arithmetic properties of the L-function depends heavily on the parity of $\#\Sigma$. When $\#\Sigma$ is even, the central value $L(\frac{1}{2},\pi,\chi)$ is related to certain period integral. Explicit formulae have been given by Gross, Waldspurger and S. Zhang. We will recall the treatment of Waldspurger [Wa] in next section.

When $\#\Sigma$ is odd then $L(\frac{1}{2},\pi,\chi)=0$. Under the assumption that E/F is a CM-extension, that π_v is discrete of weight 2 for all infinite place v, and that χ is of finite order, then the central derivative $L'(\frac{1}{2},\pi,\chi)$ is related to the height pairings of some CM divisors on certain Shimura curves. Explicit formulae have been obtained by Gross-Zagier and one of the authors under some unramified assumptions. The goal of this paper is to get a general explicit formula in this odd case without any unramified assumption.

1.2 Waldspurger's formula

Assume that the order of Σ is even in this subsection. We introduce the following notations:

1. B is the unique quaternion algebra over F with ramification set Σ;
2. $B^\times$ is viewed as an algebraic group over F;
3. $T=E^\times$ is a torus of G for a fixed embedding $E\subset B$;
4. $\pi'=\otimes_v\pi'_v$ is the Jacquet–Langlands correspondence of π on $B^\times(\mathbb{A})$.

Define a period integral $\ell(\cdot,\chi):\pi'\to\mathbb{C}$ by

$$\ell(f,\chi)=\int_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} f(t)\chi(t)dt,\quad f\in\pi'.$$

Here the integral uses the Tamagawa measure which is $2L(1,\eta)$.

Assume that ω_π is unitary. Then π' is unitary with Petersson inner product $\langle\cdot,\cdot\rangle$ using Tamagawa measure which has volume 2 on $\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})$. Fix any non-trivial Hermitian form $\langle\cdot,\cdot\rangle_v$ on π'_v so that their product gives $\langle\cdot,\cdot\rangle$. Waldspurger proved the following formula when ω_π is trivial:

Theorem 1.2.1 (Waldspurger when $\omega_\pi = 1$)**.** *Assume that $f = \otimes_v f_v \in \pi'$ is decomposable and nonzero. Then*

$$|\ell(f,\chi)|^2 = \frac{\zeta_F(2)L(\frac{1}{2},\pi,\chi)}{2\ L(1,\pi,\mathrm{ad})}\prod_{v\le\infty}\alpha(f_v,\chi_v),$$

where

$$\alpha(f_v,\chi_v) = \frac{L(1,\eta_v)L(1,\pi_v,\mathrm{ad})}{\zeta_v(2)L(\frac{1}{2},\pi_v,\chi_v)}\int_{F_v^\times\backslash E_v^\times}\langle\pi'_v(t)f_v,f_v\rangle_v\chi_v(t)dt.$$

Moreover, $\alpha(f_v,\chi_v)$ is nonzero and equal to 1 for all but finitely many places v.

We interpret the formula as a result on bilinear functionals. As π' is unitary, the contragredient $\widetilde{\pi}'$ is equal to $\bar{\pi}'$ via integration on $\mathbb{A}^\times B^\times\backslash B_{\mathbb{A}}^\times$ with an invariant measure of volume 2. Thus we have two bilinear functionals on $\pi'\otimes\widetilde{\pi}'$. The first one is

$$\ell(f_1,f_2) = \ell(f_1,\chi)\ell(f_2,\chi^{-1}) = \int_{(Z(\mathbb{A})T(F)\backslash T(\mathbb{A}))^2} f_1(t)f_2(t)\chi(t_1)\bar{\chi}(t_2)dt_1dt_2, \quad f_1\in\pi', f_2\in\widetilde{\pi}'.$$

And the second one is the product of local linear functionals:

$$\alpha(f_1,f_2) = \prod_v \alpha(f_{1v},f_{2v})$$

$$\alpha(f_{1v},f_{2v}) = \frac{L(1,\eta_v)L(1,\pi_v,\mathrm{ad})}{\zeta_v(2)L(\frac{1}{2},\pi_v,\chi_v)}\int_{F_v^\times\backslash E_v^\times}\langle\pi'_v(t)f_{1,v},f_{2,v}\rangle_v\chi_v(t)dt.$$

It is easy to see that both pairings are bilinear and (χ^{-1},χ)-equivariant under the action of $T(\mathbb{A})\times T(\mathbb{A})$. But we know such functionals are unique up to scalar multiples by the uniqueness theorem of the local linear functionals of Saito–Tunnell (Proposition 1.1.1). Therefore, these two functionals must be proportional. Theorem 1.1 says that their ratio is recognized as a combination of special values of L-functions.

1.3 Gross–Zagier formula

Now assume that Σ is odd. We further assume that

1. F is totally real and E is totally imaginary.
2. π_v is discrete of weight 2 at all infinite places v of F.
3. χ is a character of finite order.

In this case, the set Σ must contain all infinite places. We have a totally definite quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ with ramification set Σ which does not have a model over F.

Shimura curves

For each open subgroup U of $\mathbb{B}_f^\times := (\mathbb{B} \otimes_{\mathbb{A}} \mathbb{A}_f)^\times$ which is compact modulo $F^\times \cap U$, we have a (compact) Shimura curve X_U over F. For two open subsets $U_1 \subset U_2$, one has a surjective morphism $\pi_{U_1,U_2} : X_{U_1} \longrightarrow X_{U_2}$ which is bijective if and only if

$$U_1 \cdot F^\times = U_2 \cdot F^\times.$$

Let X be the projective system of X_U. Then X has an action by $\mathbb{B}_f^\times$ given by "*right multiplication*" T_x. More precisely, any $x \in \mathbb{B}_f^\times$ induces a bijection

$$T_x : \quad X_{xUx^{-1}} \longrightarrow X_U.$$

The action T_x is trivial if and only if x is in the closure $\widehat{F^\times}$ of $F^\times$ in $\mathbb{B}_f^\times$. We extend the definition of T_x to whole adeles $\mathbb{B}^\times$ so that the archimedean part $\mathbb{B}_\infty^\times$ acts trivially on X. Then an elements T_x is trivial if and only if $x \in D := \mathbb{B}_\infty^\times \cdot \widehat{F^\times}$. By this reason, it will be more intrinsic to define X_U for U an open and compact subgroup of $\mathbb{B}^\times/D$. The variety X_U can be considered as the quotient of X by U. The induced action of $\mathbb{B}^\times$ on the projective system $\pi_0(X_U)$ of sets of connected components of X_U is factor through the norm maps $\nu : \mathbb{B}^\times \longrightarrow \mathbb{A}^\times$ and makes $\pi_0(X_U)$ a principal homogenous space over $F^\times \backslash \mathbb{A}^\times / \nu(U)$.

For an embedding $\tau : F \longrightarrow \mathbb{R}$, the complex points of X_U at τ forms a Riemann surface $X_{U,\tau}^{\mathrm{an}}$ described as

$$X_{U,\tau}^{\mathrm{an}} = B(\tau)^\times \backslash \mathscr{H}^\pm \times \mathbb{B}_f^\times / U \cup \{\text{cusps}\}$$

where $B(\tau)$ is a quaternion algebra over F with ramification set $\Sigma \setminus \{\tau\}$, $\mathbb{B}_f$ is identified with $B(\tau)_{\mathbb{A}_f}$ as an $\mathbb{A}_f$-algebra, and $B(\tau)^\times$ acts on $\mathscr{H}^\pm$ through an isomorphism $B(\tau)_\tau \simeq M_2(\mathbb{R})$. The isomorphism T_x is realized on X_τ^{an} by right multiplication by x on $\mathbb{B}_f^\times$.

Hodge classes and volumes

On each X_U, there is a Hodge class $\mathscr{L}_U$ which is characterized by the following two properties:

- $\mathscr{L}_U$ is the canonical bundle on X_U if U is sufficiently small;
- $\mathscr{L}_U$ is compatible with pull-back morphisms.

As U varies, $\mathscr{L}_U$ forms an element

$$\mathscr{L} = \varinjlim_U \mathrm{Pic}(X_U).$$

It is known that $\deg(\mathscr{L}_U)$ is also the volume of X_U at an archimedean place τ of F for the measure induced by $dxdy/(2\pi y^2)$ on $\mathscr{H}$. We normalize a measure on $\mathbb{B}^\times/D$ such that for any open compact subgroup U,

$$\mathrm{vol}(U) = \frac{1}{\deg(\mathscr{L}_U)}$$

where $\mathscr{L}_U$ is the Hodge bundle of X_U.

Let $\alpha\in\pi_0(X)$ which is projective limit of elements $\alpha_U\in\pi_0(X_U)$. Let

$$\xi_{\alpha_U}=\deg(\mathscr{L}_U|_{X_{\alpha_U}})^{-1}\mathscr{L}_U|_{X_{\alpha_U}}.$$

Then ξ_{α_U} for a projective limit system and define an element

$$\xi_\alpha\in\varprojlim_U \mathrm{Pic}(X_U)\otimes\mathbb{Q}.$$

CM-divisors

Fix embeddings $E_\mathbb{A}\to\mathbb{B}$ and $E\to\bar F$, and let C be the set of points on $X(\bar F)$ which are fixed by $E^\times$ such that the induced action on the tangent spaces are given by the fixed inclusion $E\longrightarrow\bar F^\times$. Then C is in $X(E^{\mathrm{ab}})$ and is a principal homogenous space of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$. Moreover the projective C_U in $X_U(E^{\mathrm{ab}})$ has an analytic description at each place $\tau_E: E\longrightarrow\mathbb{C}$ over τ as follows: C_{U,τ_E} in $X_{U,\tau}(\mathbb{C})$ is represented by (z_0,t) with $t\in E^\times(\mathbb{A}_f)$ and $z_0\in\mathscr{H}^\pm$ is the unique point fixed by $E^\times$ such that the action of $E^\times$ on the tangent space $T_{\mathscr{H}^\pm,z_0}$ is given by inclusion τ_E.

$$C_U\simeq E^\times\backslash\mathbb{A}_E^\times/U_E \tag{1.3.1}$$

with Galois action of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ given by left multiplication of $\mathbb{A}_E^\times$ and the class field theory.

Fix a point $Y\in C$ and define formally

$$Y_\chi:=\int_{\mathrm{Gal}(E^{\mathrm{ab}}/E)}\chi(\sigma)(Y^\sigma-\xi_Y)d\sigma\in\varprojlim_U \mathrm{Jac}(X_U)=:\mathrm{Alb}(X)\otimes\mathbb{C}$$

where $\xi_Y=\xi_\alpha$ the normalized Hodge class on the connected component containing Y, and $d\sigma$ is a Haar measure on $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ with volume $2L(1,\eta)$ which is the Tamagawa measure on $\mathbb{A}^\times E^\times\backslash\mathbb{A}_E^\times$. The projection $Y_{U,\chi}$ of Y_χ on X_U can be described analytically at a place τ_E of E as follows: we may take Y so that its projection Y_U on X_τ^{an} is represented by $z_0\in\mathscr{H}^\pm$ fixed by $E^\times$ and then $Y_{\chi,U}\neq 0$ only if U_E is contained in $\ker\chi$. In this case

$$Y_\chi=\tau(U)\sum_{t\in E^\times\backslash\mathbb{A}_E^\times/U_E}\chi(t)([t]-\xi_t)$$

where $[t]$ denote the CM-point in C_U via identification (1.3.1) and ξ_t is the Hodge class of degree 1 in the geometric connected component of X_U containing $[z_0,t]$, and $\tau(U)$ is to make

$$\tau(U)|E^\times\backslash\mathbb{A}_E^\times/U_E|=2L(1,\eta).$$

Hecke correspondences

For each $x\in\mathbb{B}^\times$, define a Hecke correspondence $Z(x)_U$ as a cycle in $Z^1(X_U\times X_U)$ as the image of the morphism

$$X_{U\cap xUx^{-1}}\longrightarrow X_U\times X_U,$$

where the first factor is the natural projection, and the second one is the composition of T_x and projection. For any function $\phi \in \mathscr{S}(\mathbb{B}^\times/D)$ bi-invariant under U, we can define

$$\mathrm{T}(\phi)_U = \sum_{x \in U\backslash \mathbb{B}^\times/U} \phi(x) Z(x)_U \in Z^1(X_U^2).$$

The correspondences $\mathrm{T}(\phi)_U$ is compatible with projection of X_U thus forms an element

$$\mathrm{T}(\phi) \in Z^1(X \times X) := \lim_{\overrightarrow{U}} Z^1(X_U^2).$$

It follows that the divisors $\mathrm{T}(\phi)_U Y_\chi$ also form a direct limit system thus defines an element

$$\mathrm{T}(\phi) Y_\chi \in \lim_{\overrightarrow{U}} \mathrm{Jac}(X_U) =: \mathrm{Jac}(X) \otimes \mathbb{C}.$$

The functional

$$\mathscr{S}(\mathbb{B}^\times/D) \longrightarrow \mathrm{Jac}(X) \otimes \mathbb{C}, \quad \phi \mapsto T(\phi) Y_\chi$$

factors through the maximal cuspidal quotient $\widetilde{\mathscr{S}}(\mathbb{B}^\times/D)$ as follows.

For any irreducible representation (V_σ, σ) of $\mathbb{B}_f^\times/D$ (equivalently, an irreducible representation of $\mathbb{B}^\times$ with trivial component at infinity and trivial on $F^\times$), we have a Hecke operator $\rho(\phi)$ acting on σ:

$$v \longrightarrow \rho_\sigma(\phi) := \int_{\mathbb{B}^\times/D} \phi(g) \sigma(g) v dg.$$

Thus we have a well defined $\mathbb{B}^\times \times \mathbb{B}^\times$ equivariant map:

$$\rho_\sigma : \qquad \mathscr{S}(\mathbb{B}^\times/D) \longrightarrow \sigma \otimes \widetilde{\sigma}.$$

Denote

$$\widetilde{\mathscr{S}}(\mathbb{B}^\times/D) = \mathscr{S}(\mathbb{B}^\times/D)/\cap_{\sigma \in S_2(\mathbb{B}^\times)} \ker \rho_\sigma$$

where $S_2(\mathbb{B}^\times)$ runs through the Jacquet–Langlands correspondences of cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ of parallel weight 2 and with trivial central characters at infinities. Then we have an isomorphism of $\mathbb{B}^\times \times \mathbb{B}^\times$-modules:

$$\widetilde{\mathscr{S}}(\mathbb{B}^\times/D) \simeq \oplus_{\sigma \in S_2(\mathbb{B}^\times)} \sigma \otimes \widetilde{\sigma}.$$

We will show that the $T(\phi)Y_\chi$ depends only on the image of ϕ in $\widetilde{\mathscr{S}}(\mathbb{B}^\times/D)$. Thus we can define

$$\mathrm{T}(f \otimes \widetilde{f}) Y_\chi \in \mathrm{Jac}(X) \otimes \mathbb{C}, \qquad f \otimes \widetilde{f} \in \sigma \otimes \widetilde{\sigma}, \quad \sigma \in S_2(\mathbb{B}^\times).$$

Gross–Zagier formula

Notice that the Neron–Tate height paring on $\mathrm{Jac}(X_U)(\bar{F})$ defines a Hermitian pairing on $\mathrm{Jac}(X)(\bar{F}) \otimes \mathbb{C}$ and $\mathrm{Alb}(X) \otimes \mathbb{C}$.

Theorem 1.3.1. *Let π' denote the Jacquet–Langlands correspondence of π on $\mathbb{B}^\times$. Assume that $f = \otimes f_v \in \pi'$ and $\widetilde{f} = \otimes \widetilde{f}_v \in \widetilde{\pi}'$ are decomposable. Then*

$$\langle \mathrm{T}(f \otimes \widetilde{f}) Y_\chi, \quad Y_\chi \rangle_{\mathrm{NT}} = \frac{\zeta_F(2) L'(1/2, \pi, \chi)}{L(1, \pi, ad)} \prod_v \alpha(f_v, \widetilde{f}_v).$$

Remark. In the case that f and $\widetilde{f}$ are new forms, some partial results have been proved in [Zh1, Zh2, Zh3] with more precise formulae under some unramified assumptions.

1.4 Applications

Let π and χ satisfy the same condition as in §1.3. Then we have an abelian variety A defined over E such that

$$L(s, A) = \prod_\sigma L(s + \frac{1}{2}, \pi^\sigma, \chi^\sigma)$$

where $(\pi^\sigma, \chi^\sigma)$ are conjugates of (π, χ) for automorphisms σ of $\mathbb{C}$ in the sense that the Hecke eigenvalues of π^σ and χ^σ are σ-conjugates of those of π and χ. Let $\mathbb{Q}[\pi, \chi]$ denote the subfield generated by Hecke eigenvalues of π and values of χ. Then A has a multiplication by an order in $\mathbb{Q}[\pi, \chi]$. Replace A by an isogenous one, we may assume that A has multiplication by $\mathbb{Z}[\pi, \chi]$.

Theorem 1.4.1 (Tian–Zhang)**.** *Under the assumption above, we have:*

1. *If $\mathrm{ord}_{s=1/2} L(s, \pi, \chi) = 1$, then the Mordell-Weil group $A(E)$ as a $\mathbb{Z}[\pi, \chi]$-module has rank 1 and the Shafarevich-Tate group $Ш(A)$ is finite.*
2. *If $\mathrm{ord}_{s=1/2} L(s, \pi, \chi) = 0$ and A is not of CM-type, then the Mordell-Weil group $A(E)$ is finite. Furthermore, if χ is trivial, and A is geometrically simple, then the Shafarevich-Tate group $Ш(A)$ is also finite.*

Remark. For π as above, there is an abelian variety A_π defined over F with L-series given by

$$L(s, A_\pi) = \prod_\sigma L(s + 1/2, \pi^\sigma).$$

We may assume that A_π has multiplications by the ring $\mathbb{Z}[\pi]$ of the Hecke eigenvalues in π. The variety A up to an isogeny can be obtained by the algebraic tensor product:

$$A := A_\pi \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi, \chi].$$

In terms of algebraic points,

$$A(\bar{E}) = A_\pi(\bar{E}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi, \chi]$$

with usual Galois action by $\mathrm{Gal}(\bar{E}/E)$ on $A(\bar{E})$ and by character χ on $\mathbb{Z}[\pi, \chi]$.

Example. The theorem applies to all modular F-elliptic curves: the elliptic curve A over a Galois extension of F such that A^σ is isogenous to A for all $\sigma \in \mathrm{Gal}(\bar{F}/F)$.

1.5 Idea of proof

Our proof of Theorem 1.3.1 is still based on the idea of Gross–Zagier's paper [GZ], namely, to compare the analytic kernel function representing the central derivative of L-series with the geometric kernel function formed by a generating series of height pairings of CM-points. Many ideas of [Zh1, Zh2] are also used in this paper. The following are some new ingredients:

- Construct the analytic kernel and the geometric kernel systematically using Weil representations. It avoids the use of newform theory and the choice of test vectors. The construction of the analytic kernel is a variation of the idea of Waldspurger [Wa] combined with the *incoherence* philosophy of Kudla [Ku2]. The idea to use the "complete" generating series to construct the geometric kernel is also inspired by Kudla's work.

- Reduce the problem to a special class of *degenerate* Schwartz functions by local representation theoretical reasons. Then many computations are simplified. For example, the constant terms of the Eisenstein series are zero, and the geometric kernel function has no self-intersection involved in that case.

- At bad places, *approximate* the kernel functions by Eisenstein series or theta series on the nearby quaternion algebras, and apply a modularity argument to get an equality. The theta series have no contribution in the integral by the dichotomy result of Tunnell [Tu] and Saito [Sa] we just recalled. It is similar to the idea of geometric pairing in [Zh2].

Now we give more details following the logic order of this paper. First of all, for a given Σ, we will have a quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ with ramification set Σ. The reduced norm on $\mathbb{B}$ defines an orthogonal space $\mathbb{V}$ with group GO of orthogonal similitudes. Then we have a Weil representation of $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GO}(\mathbb{A})$ on the space $\mathscr{S}(\mathbb{V} \times \mathbb{A}^\times)$ of Schwartz functions. For each $\phi \in \mathscr{S}(\mathbb{V} \times \mathbb{A}^\times)$ we will construct an automorphic form $I(s, g, \chi, \phi)$ as a mixed Eisenstein series and theta series to represent the L-series $L(s, \pi, \chi)$.

If Σ is even, then $\mathbb{B} = B_\mathbb{A}$ for some quaternion algebra B over F. Then by Siegel–Weil formulae, $I(0, g, \chi, \phi)$ is equal to a period integral $\theta(g, \chi, \phi)$ of a theta series associate to ϕ. This is essentially the Waldspurger formula.

If Σ is odd, $I(0, g, \chi, \phi) = 0$ and $I'(0, g, \chi, \phi)$ represents $L'(1/2, \pi, \chi)$. We write $I'(0, g, \chi, \phi)$ as a sum of $I'(0, g, \chi, \phi)(v)$ according to the place of the derivative is taken in the product form of the Whittaker functions of $I(s, g, \chi, \phi)$. Then we compute the holomorphic projection $\mathscr{P}r I'(0, g, \chi, \phi)$. It only changes $I'(0, g, \chi, \phi)(v)$ for archimedean v, but brings some extra term due to the fast growth of the kernel function.

The role of theta series in Waldspurger's work is played by the following generating series of heights of CM-points:

$$Z(g, \chi, \phi) = \langle Z(g, \phi) Y_\chi, Y_\chi \rangle_{\mathrm{NT}}$$

where $Z(g,\phi)$ is a generating series of Hecke operators on X_U introduced by Kudla-Millson extending the classic formula $\sum_n T_n q^n$. The modularity of this generating series is proved in our previous paper [YZZ]. Using Arakelov theory, we may decompose $Z(g,\chi,\phi)$ into a sum of local heights $Z(g,\chi,\phi)(v)$ over all places v of F, with some extra terms involving of the Hodge class.

The problem is reduced to compare the kernels $\mathscr{P}rI'(0,g,\chi,\phi)$ and $2Z(g,\chi,\phi)$. For good v, we show that $I'(0,g,\chi,\phi)(v)$ is exactly equal to $Z(g,\chi,\phi)(v)$ by explicit computations. For bad v, we show both of them can be *approximated* by theta series and Eisenstein series. We also show that the difference of the extra terms can also be *approximated* by Eisenstein series.

To illustrate the idea, we make a general definition. We say a function $\Phi : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ is *approximated* by an automorphic form φ on $\mathrm{GL}_2(\mathbb{A})$ if there exists a finite set S of places of F such that $\Phi(g) = \varphi(g)$ for all $g \in 1_S\mathrm{GL}_2(\mathbb{A}^S)$. Here is a simple fact. If furthermore Φ is automorphic, then $\Phi = \varphi$ identically. It is true since $\mathrm{GL}_2(F)\mathrm{GL}_2(\mathbb{A}^S)$ is dense in $\mathrm{GL}_2(\mathbb{A})$.

Come back to the comparison of the kernel functions. We have shown that $\mathscr{P}rI'(0,g,\chi,\phi) - 2Z(g,\chi,\phi)$ is approximated by a finite sum of theta series and Eisenstein series. By the above simple consequence of modularity, we conclude that it is exactly equal to the sum of these theta series and Eisenstein series. It is not zero, but it is perpendicular to π by simple reasons. These Eisenstein series are automatically perpendicular to π. These theta series are defined on the *nearby* quaternion algebra $B(v)$ over F obtained by changing the invariant of $\mathbb{B}$ at v. They are perpendicular to π by the result of Saito and Tunnel.

In the end, we explain why these local components can be approximated easily. We mainly look at $I'(0,g,\chi,\phi)(v)$. Note that it is a mixed theta series and Eisenstein series with local components of its Fourier coefficients at v replaced by the derivatives. By a local version of the Siegel–Weil formula, it is easy to write it as an integral over $E^\times(\mathbb{A}_f)$ of

$$\mathscr{K}_\phi^{(v)}(g) = \sum_{u\in\mu_K^2\backslash F^\times} \sum_{y\in B(v)} k_{\phi_v}(g,y,u) r(g)\phi^v(y,u).$$

It is a theta series except that at v the function $k_{\phi_v}(g,y,u)$ is not good, so we call it a *pseudo theta series*. The key is to show that $k_{\phi_v}(1,y,u)$ is a Schwartz function of $(y,u) \in B(v)_v \times F_v^\times$ if ϕ_v is degenerate. Then we form the "authentic" theta series

$$\theta(g, k_{\phi_v}\otimes\phi^v) = \sum_{u\in\mu_K^2\backslash F^\times} \sum_{y\in B(v)} r(g)k_{\phi_v}(1,y,u) r(g)\phi^v(y,u).$$

It approximates the original series since they are the same for $g \in 1_v\mathrm{GL}_2(\mathbb{A}^v)$.

As for the local height $Z(g,\chi,\phi)(v)$, we can also write it as a series over $B(v)$. Roughly speaking, the local formal neighborhoods of the integral model of Shimura curve X_U can be uniformized as the quotient of some universal deformation space by the action of $B(v)^\times$. Then the local height pairing on the Shimura curve is a summation of intersections of points in the corresponding orbit indexed by $B(v)^\times$.

1.6 Notation and terminology

Local fields and global fields

We normalize the absolute values, additive characters, and measures following Tate's thesis. Let k be a local field of characteristic zero.

- Normalize the absolute value $|\cdot|$ on k as follows:

  It is the usual one if $k = \mathbb{R}$.

  It is the square of the usual one if $k = \mathbb{C}$.

  It takes the uniformizer to the reciprocal of the cardinality of the residue field if k is non-archimedean.

- Normalize the additive character $\psi : k \to \mathbb{C}^\times$ as follows:

  If $k = \mathbb{R}$, then $\psi(x) = e^{2\pi i x}$.

  If $k = \mathbb{C}$, then $\psi(x) = e^{4\pi i \mathrm{Re}(x)}$.

  If k is non-archimedean, then it is a finite extension of $\mathbb{Q}_p$ for some prime p. Take $\psi = \psi_{\mathbb{Q}_p} \circ \mathrm{tr}_{k/\mathbb{Q}_p}$. Here the additive character $\psi_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is defined by $\psi_{\mathbb{Q}_p}(x) = e^{-2\pi i \iota(x)}$, where $\iota : \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$ is the natural embedding.

- We take the Haar measure dx on k to be self-dual with respect to ψ. More precisely,

  If $k = \mathbb{R}$, then dx is the usual Lebesgue measure.

  If $k = \mathbb{C}$, then dx is twice of the usual Lebesgue measure.

  If k is non-archimedean, then $\mathrm{vol}(O_k) = |d_k|^{\frac{1}{2}}$. Here O_k is the ring of integers and $d_k \in k$ is the different of k over $\mathbb{Q}_p$.

- We take the Haar measure $d^\times x$ on $k^\times$ as follows:

  $$d^\times x = \zeta_k(1)|x|_v^{-1} dx.$$

  Recall that $\zeta_k(s) = (1 - N_v^{-s})^{-1}$ if v is non-archimedean with residue field with N_v-elements, and $\zeta_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\zeta_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. With this normalization, if k is non-archimedean, then $\mathrm{vol}(O_k^\times) = \mathrm{vol}(O_k)$.

Now go back to the totally real field F. For each place v, we choose $|\cdot|_v, \psi_v, dx_v, d^\times x_v$ as above. By tensor products, they induce global $|\cdot|, \psi, dx, d^\times x$. For non-archimedean v, we usually use p_v to denote the corresponding prime ideal, N_v to denote the cardinality of its residue field, and ϖ_v to denote a uniformizer.

Notation on GL_2

We introduce the matrix notation:

$$m(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \quad d(a) = \begin{pmatrix} 1 & \\ & a \end{pmatrix}, \quad d^*(a) = \begin{pmatrix} a & \\ & 1 \end{pmatrix}$$
$$n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

We denote by $P \subset \mathrm{GL}_2$ and $P^1 \subset \mathrm{SL}_2$ the subgroups of upper triangular matrices, and by N the standard unipotent subgroup of them.

For any local field k, the character $\delta : P(k) \to \mathbb{R}^\times$ defined by

$$\delta : \begin{pmatrix} a & b \\ & d \end{pmatrix} \mapsto \left|\frac{a}{d}\right|^{\frac{1}{2}}$$

extends to a function $\delta : \mathrm{GL}_2(k) \to \mathbb{R}^\times$ by Iwasawa decomposition.

For any global field k, the product $\delta = \prod_v \delta_v$ gives a function on $\mathrm{GL}_2(\mathbb{A}_k)$.

Quadratic extensions

Let E/F be a quadratic extension of global fields. We denote by $T = E^\times$ the algebra group over F. We view $V_1 = (E, q = N_{E/F})$ as a two-dimensional vector space over F, which uniquely determines self-dual measures dx on $\mathbb{A}_E$ and E_v for each place v of F. We define the measures on the corresponding multiplicative groups by the same setting as above.

Let $E^1 = \{y \in E^\times : q(y) = 1\}$ act on E by multiplication. It induces an isomorphism $\mathrm{SO}(V_1) \simeq E^1$ of algebraic groups over F. We also have $\mathrm{SO}(V_1) = E^\times/F^\times$ given by

$$E^\times/F^\times \to E^1, \quad t \mapsto t/\bar{t}.$$

It is an isomorphism by Hilbert Theorem 90. We also denote $T^1 = E^1$.

We have the following exact sequence

$$1 \to E^1 \to E^\times \xrightarrow{q} q(E^\times) \to 1.$$

For all places v, we endow E_v^1 the measure such that the quotient measure over $q(E_v^\times)$ is the subgroup measure of $F_v^\times$. On the other hand, we endow $E_v^\times/F_v^\times$ with the quotient measure. It turns out that these two measures induce the same one on $\mathrm{SO}(V_1)$.

Let v be a non-archimedean place of F, denote by $d_v \in O_{F_v}$ the local different of F_v, and by $D_v \in O_{F_v}$ the local discriminant of E_v/F_v. Then $\mathrm{vol}(O_{F_v}) = \mathrm{vol}(O_{F_v}^\times) = |d_v|_v^{\frac{1}{2}}$ and $\mathrm{vol}(O_{E_v}) = \mathrm{vol}(O_{E_v}^\times) = |D_v|^{\frac{1}{2}}|d_v|_v$. Furthermore,

$$\mathrm{vol}(E_v^1) = \begin{cases} 2 & \text{if } v \mid \infty, \\ |d_v|^{\frac{1}{2}} & \text{if } v \nmid \infty \text{ inert in } E, \\ 2|D_v|^{\frac{1}{2}}|d_v|^{\frac{1}{2}} & \text{if } v \nmid \infty \text{ ramified in } E. \end{cases}$$

Notation on quaternion algebra

Fix $F, E, \mathbb{A}, \Sigma$ as introduced at the beginning of the introduction.

Recall that Σ is a finite set of places of F. Denote by $\mathbb{B}$ the unique quaternion algebra over $\mathbb{A}$ such that for every place v of F, the quaternion algebra $\mathbb{B}_v := \mathbb{B} \otimes_{\mathbb{A}} F_v$ over F_v is isomorphic to the matrix algebra if and only if $v \notin \Sigma$. Alternatively, one can define $\mathbb{B}_v$ according to Σ, and $\mathbb{B}$ as a restricted product of $\mathbb{B}_v$.

We say $\mathbb{B}$ is *coherent* if it is a base change of a quaternion algebra over F; otherwise, we say $\mathbb{B}$ is *incoherent*. . It follows that $\mathbb{B}$ is coherent if and only if the cardinality of Σ is even.

The reduce norm q makes $\mathbb{B}$ a quadratic space $\mathbb{V} = (\mathbb{B}, q)$ over $\mathbb{A}$. Fix an embedding $E_{\mathbb{A}} \hookrightarrow \mathbb{B}$ which always exists. It gives an orthogonal decomposition

$$\mathbb{B} = E_{\mathbb{A}} + E_{\mathbb{A}}\mathfrak{j}, \qquad \mathfrak{j}^2 \in \mathbb{A}^\times.$$

Then we get two induced subspaces $\mathbb{V}_1 = (E_{\mathbb{A}}, q)$ and $\mathbb{V}_2 = (E_{\mathbb{A}}\mathfrak{j}, q)$. Apparently $\mathbb{V}_1$ is the base change of the F-space $V_1 = (E, q)$. We usually write $x = x_1 + x_2$ for the corresponding orthogonal decomposition of $x \in \mathbb{V}$.

Assume that the cardinality of Σ is odd. We will keep this assumption throughout this paper except in §1.2 and §2.4. Then $\mathbb{B}$ is incoherent, but we will get a coherent one by increasing or decreasing Σ by one element. For any place v of F, denote by $B(v)$ the quaternion algebra over F obtained from $\mathbb{B}$ by switching the Hasse invariant at v. We call $B(v)$ the nearby quaternion algebra corresponding to v. Throughout this paper, we will fix an identification $B(v) \otimes_F \mathbb{A}^v \cong \mathbb{B}^v$. Fix an embedding $E \hookrightarrow B(v)$ if v is non-split in E. In this case, such an embedding always exists. Then we also have orthogonal decomposition $B(v) = V_1 \oplus V_2(v)$.

For any quaternion algebra B over F_v with a fixed embedding $E_v \hookrightarrow B$, we define

$$\lambda : B^\times \longrightarrow F_v, \qquad x \longmapsto \frac{q(x_2)}{q(x)}$$

where $x = x_1 + x_2$ is the orthogonal decomposition induced by $E_v \hookrightarrow B$. This definition applies to all the quaternion algebras above locally and globally.

Notation on integrations and averages

For a function f on $F^\times \mathbb{A}^\times$ which is invariant under $F_\infty^\times \cdot U$ where U is an open compact subgroup of $\mathbb{A}_f^\times$, we denote the average of f on $\mathbb{A}^\times$ by

$$⨍_{\mathbb{A}^\times} f(z)dz := |F^\times\backslash\mathbb{A}^\times/F_\infty^\times \cdot U|^{-1} \sum_{z \in F^\times\backslash\mathbb{A}^\times/F_\infty^\times \cdot U} f(z).$$

The definition here does not depend on choice of U.

Let G be a reductive group over F with an embedding $F^\times \longrightarrow G$ so that the center of $F^\times\backslash G$ is anisotropic. We denote

$$[G] = \mathbb{A}^\times G(F)\backslash G(\mathbb{A}).$$

Let dg be a measure on $[G]$ and let f be an automorphic function on $G(\mathbb{A})$ with trivial central character at infinity. We define

$$\int_{[G]} f(g)dg := \int_{[G]} dg \fint_{\mathbb{A}^\times} f(zg)dz.$$

$$\fint_{[G]} f(g)dg := \mathrm{vol}([G])^{-1} \int_{[G]} f(g)dg.$$

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2 Weil representation and analytic kernel

In this section, we will review the theory of Weil representation and its applications to integral representations of Rankin–Selberg L-series $L(s,\pi,\chi)$ and to a proof of Waldspurger's central value formula. We will mostly follow Waldspurger's treatment with some modifications including the construction of incoherent Eisenstein series from Weil representation.

We will start with the classical theory of Weil representation of $O(F)\times \mathrm{SL}_2(F)$ on $\mathscr{S}(V)$ for an orthogonal space V over a local field F and its extension to $\mathrm{GO}(F)\times \mathrm{GL}_2(F)$ on $\widetilde{\mathscr{S}}(V\times F^\times)$ by Waldspurger. We then define theta function, state Siegel–Weil formulae, and define normalized local Shimizu lifting. The main result of this section is an integral formula for L-series $L(s,\pi,\chi)$ using a kernel function $I(s,g,\chi,\phi)$. This kernel function is a mixed Eisenstein and theta series attached to each $\phi\in\widetilde{\mathscr{S}}(\mathbb{V}\times\mathbb{A}^\times)$ for $\mathbb{V}$ an orthogonal space obtained form a quaternion algebra over $\mathbb{A}$. The Waldspurger formula is a direct consequence of the Siegel–Weil formula. We conclude the section by proving the vanishing of the kernel function at $s=0$ using information on Whittaker function of incoherent Eisenstein series.

2.1 Weil representations

Let us start with some basic setup on Weil representation. We follow closely from Waldspurger's paper [Wa].

Non-archimedean case

Let F be a non-archimedean local field and (V,q) a quadratic space over F. Let $O=O(V,q)$ denote the orthogonal group of (V,q) and $\widetilde{\mathrm{SL}}_2$ the double cover of SL_2. Then for any nontrivial character ψ of F, the group $\widetilde{\mathrm{SL}}_2(F)\times O(F)$ has an action r on the space $\mathscr{S}(V)$ of locally constant functions with compact support as follows:

- $r(h)\phi(x)=\phi(h^{-1}x), \qquad h\in O(F);$
- $r(m(a))\phi(x)=\chi_V(a)\phi(ax)|a|^{\dim V/2}, \qquad a\in F^\times;$
- $r(n(b))\phi(x)=\phi(x)\psi(bq(x)), \qquad b\in F;$
- $r(w)\phi=\gamma(V,q)\widehat{\phi}.$

Here $\chi_V: k\to\{\pm1\}$ is the corresponding quadratic character, and $\gamma(V,q)$, an 8-th root of unity, is the Weil index.

Archimedean case

When $F\simeq\mathbb{R}$, we may define an analogous representation of the pair $(\mathscr{G},\mathscr{K})$ consisting of the Lie algebra $\mathscr{G}$ and a maximal compact subgroup $\mathscr{K}=\widetilde{\mathrm{SO}}_2(\mathbb{R})\times\mathscr{K}^0$ of $\widetilde{\mathrm{SL}}_2(\mathbb{R})\times O(\mathbb{R})$. More precisely, the maximal compact subgroup $\mathscr{K}^0$ stabilizes a unique orthogonal decomposition

$V = V^+ + V^-$ such that the restrictions of q on $V^\pm$ are positive and negative definite respectively. Then we take $\mathscr{S}(V)$ as to be the space of functions of the form

$$P(x)e^{-2\pi(q(x^+)-q(x^-))}, \qquad x = x^+ + x^-,\, x^\pm \in V^\pm$$

where P is a polynomial function on V. The action of $(\mathscr{G}, \mathscr{K})$ on $\mathscr{S}(V)$ can be deduced formally from the same formulae as above. Notice that the space $\mathscr{S}(V)$ depends on the choice of the maximal compact subgroup $\mathscr{K}^0$ of $O(V)$ and ψ.

Extension to GL_2

Assume that $\dim V$ is even. Following Waldspurger [Wa], we extend this action to an action r of $\mathrm{GL}_2(F) \times \mathrm{GO}(F)$ or more precisely their $(\mathscr{G}, \mathscr{K})$ analogue in real case. In §3.1, we will see that we are actually using a slightly different space of Schwartz functions in the archimedean case.

If F is non-archimedean, let $\widetilde{\mathscr{S}}(V \times F^\times)$ be the space of locally constant and compactly supported functions on $V \times F^\times$. We also write it as $\mathscr{S}(V \times F^\times)$ in the non-archimedean case.

If F is archimedean, let $\widetilde{\mathscr{S}}(V \times F^\times)$ be the space of finite linear combinations of functions of the form

$$H(u)P(x)e^{-2\pi|u|(q(x^+)-q(x^-))}$$

where P is any polynomial function on V, and H is any compactly supported smooth function on $\mathbb{R}^\times$.

The Weil representation is extended by the following formulae:

- $r(h)\phi(x, u) = \phi(h^{-1}x, \nu(h)u), \qquad h \in \mathrm{GO}(F);$
- $r(g)\phi(x, u) = r_u(g)\phi(x, u), \qquad g \in \mathrm{SL}_2(F);$
- $r(d(a))\phi(x, u) = \phi(x, a^{-1}u)|a|^{-\dim V/4}, \qquad a \in F^\times.$

Here $\nu : \mathrm{GO}(F) \to F^\times$ denotes the similitude map. In the second formula $\phi(x, u)$ is viewed a function of x, and r_u is the Weil representation on V with new norm uq.

By the action of $\mathrm{GO}(F)$ above, we introduce an action of $\mathrm{GO}(F)$ on $V \times F^\times$ given by

$$h \circ (x, u) := (hx, \nu(h)^{-1}u).$$

This action stabilizes the subset

$$(V \times F^\times)_a := \{(x, u) \in V \times F^\times : uq(x) = a\}.$$

Global case

Now we assume that F is a number field and that (V,q) is an orthogonal space over F. Then we can define a Weil representation r on $\mathscr{S}(V_{\mathbb{A}})$ (which actually depends on ψ) of $\widetilde{\mathrm{SL}}_2(\mathbb{A})\times O(V_{\mathbb{A}})$. When $\dim V$ is even, we can define an action r of $\mathrm{GL}_2(\mathbb{A})\times \mathrm{GO}(V_{\mathbb{A}})$ on $\widetilde{\mathscr{S}}(V_{\mathbb{A}}\times\mathbb{A}^\times)$ which is the restricted tensor product of $\widetilde{\mathscr{S}}(V_v\times F_v)$ with spherical element as characteristic function of $V_{O_{F_v}}\times O_{F_v}^\times$ once a global lattice is chosen.

Notice that the representation r depends only on the quadratic space $(V_{\mathbb{A}},q)$ over $\mathbb{A}$. We may define representations directly for a pair $(\mathbb{V},q)$ of a free $\mathbb{A}$-module $\mathbb{V}$ with non-degenerate quadratic form q. It still makes sense to define $\widetilde{\mathscr{S}}(\mathbb{V}\times\mathbb{A}^\times)$ to be the restricted tensor product of $\widetilde{\mathscr{S}}(\mathbb{V}_v\times F_v^\times)$. The Weil representation extends in this case.

If $(\mathbb{V},q)$ is a base change of an orthogonal space over F, then we call this Weil representation is *coherent*; otherwise it is called *incoherent*.

Siegel–Weil formula

Let F be a number field, and (V,q) a quadratic space over F. Then for any $\phi\in\mathscr{S}(V_{\mathbb{A}})$, we can form a theta series as a function on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}}_2(\mathbb{A})\times O(F)\backslash O(\mathbb{A})$:

$$\theta(g,h,\phi)=\sum_{x\in V} r(g,h)\phi(x),\qquad (g,h)\in\widetilde{\mathrm{SL}}_2(\mathbb{A})\times O(\mathbb{A}).$$

Similarly, when V has even dimension we can define theta series for $\phi\in\widetilde{\mathscr{S}}(V_{\mathbb{A}}\times\mathbb{A}^\times)$ as an automorphic form on $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})\times\mathrm{GO}(F)\backslash\mathrm{GO}(\mathbb{A})$:

$$\theta(g,h,\phi)=\sum_{(x,u)\in V\times F^\times} r(g,h)\phi(x,u).$$

Now we recall the Siegel–Weil formulae. For $\phi\in\mathscr{S}(V_{\mathbb{A}})$, $s\in\mathbb{C}$, we have a section

$$g\mapsto \delta(g)^s r(g)\phi(0)$$

in $\mathrm{Ind}_{P^1}^{\mathrm{SL}_2}(\chi_V|\cdot|^{\dim V/2+s})$ where δ is the modulo function explained in the introduction. Thus we can form an Eisenstein series

$$E(s,g,\phi)=\sum_{\gamma\in P^1(F)\backslash \mathrm{SL}_2(F)}\delta(\gamma g)^s r(\gamma g)\phi(0).$$

We are interested in the Siegel–Weil formula in the following cases:

(1) $(V,q)=(E,\mathrm{N}_{E/F})$ with E a quadratic extension of F;

(2) $(V,q)=(B,q)$ with B a quaternion algebra over F and q the reduced norm;

(3) $(V,q)=(B_0,q)$ where B_0 is the subspace of trace free elements of a quaternion algebra B over F and q is induced from the reduced norm on B.

The Tamagawa number of $\mathrm{SO}(\mathbb{A})$ in the above cases are respectively $2L(1,\eta),2,2$. Set $m = L(1,\eta),2,2$ respectively in these cases.

Theorem 2.1.1 (Siegel–Weil formula). *Let V, m be as above, and $\phi \in \mathscr{S}(V(\mathbb{A}))$. Then*

$$\int_{\mathrm{SO}(F)\backslash \mathrm{SO}(\mathbb{A})} \theta(g,h,\phi)dh = mE(0,g,\phi)$$

where dh is the Tamagawa measure on $\mathrm{SO}(\mathbb{A})$.

Notice that the Eisenstein series can be defined for any quadratic space $(\mathbb{V},q)$ over $\mathbb{A}$ with rational $\det V$. Kudla has proposed a connection between the derivative of Eisenstein series and arithmetic intersection of certain arithmetic cycles on Shimura varieties.

Local Siegel–Weil formula

For any $a \in F^\times$, the above Siegel–Weil formula yields an identity of the a-th Fourier coefficients (also called Whittaker function) of the two sides as follows:

$$\mathrm{vol}(\mathrm{SO}_{x_a}(F)\backslash \mathrm{SO}_{x_a}(\mathbb{A})) \int_{\mathrm{SO}_{x_a}(\mathbb{A})\backslash \mathrm{SO}(\mathbb{A})} r(g,h)\phi(x_a)dh = m\int_{\mathbb{A}} r(wn(b)g)\phi(0)\ \psi(-ab)db.$$

Here $x_a \in V$ is any fixed element of norm a, and SO_{x_a} denotes the stabilizer of x_a in SO. If such an x_a does not exist, the left-hand side is considered to be zero.

Note that both integrals above are products of local integrals. It follows that the identity induces an identity at every place, and vice versa. Actually Weil [We] proved the Siegel–Weil formula by first showing the local version below.

We state with a quadratic space (V,q) over a local field k. For any $a \in F$, denote by $V(a)$ the set of elements of V with norm a. If it is non-empty, then any $x_a \in V(a)$ gives a bijection $V(a) \cong \mathrm{SO}_{x_a}(k)\backslash \mathrm{SO}(k)$. Under this identity, $\mathrm{SO}(k)$-invariant measures of $V(a)$ correspond to Haar measures of $\mathrm{SO}_{x_a}(k)\backslash \mathrm{SO}(k)$. They are unique up to scalar mutiples.

Theorem 2.1.2 ([We], local Siegel–Weil). *Let (V,q) be a quadratic space over a local field k. Then the following are true:*

(1) There is a unique $\mathrm{SO}(k)$-invariant measure $d_\psi x$ of $V(a)$ for every $a \in k^\times$ such that

$$\Phi(a) := \int_{V(a)} \phi(x)d_\psi x$$

gives a continuous function for $a \in k^\times$, and such that

$$\int_k \Phi(a)da = \int_V \phi(x)dx.$$

Here da, dx are the self-dual measures on k, V with respect to ψ.

(2) With the above measure,

$$\int_k r(wn(b)g)\phi(0)\ \psi(-ab)db = \gamma(V,q)\int_{V(a)} r(g)\phi(x)d_\psi x, \quad \forall a \in k^\times, \phi \in \mathscr{S}(V).$$

The right-hand side is consided to be zero if $V(a)$ is empty.

The measure $d\mu_a$ is very easy to determine for small groups in practice. In the case $a = 0$, there is a similar result with some complication caused by analytic continuation. We omit it here.

2.2 Shimizu lifting

Let F be a local field and B a quaternion algebra over F. Write $V = B$ as an orthogonal space with quadratic form q defined by the reduced norm on B. Let $B^\times \times B^\times$ act on V by

$$x \mapsto h_1 x h_2^{-1}, \qquad x \in V = B, \quad h_i \in B^\times.$$

Then we have an exact sequence:

$$1 \longrightarrow F^\times \longrightarrow (B^\times \times B^\times) \rtimes \{1, \iota\} \longrightarrow \mathrm{GO}(F) \longrightarrow 1.$$

Here ι acts on $V = B$ by the canonical involution, and on $B^\times \times B^\times$ by $(h_1, h_2) \mapsto (h_2^{\iota-1}, h_1^{\iota-1})$, and here $F^\times$ is embedded into the middle group by $x \mapsto (x, x) \rtimes 1$. The theta lifting of any representation π of $\mathrm{GL}_2(F)$ on GO is induced by the representation $\mathrm{JL}(\widetilde{\pi}) \otimes \mathrm{JL}(\pi)$ on $B^\times \times B^\times$, where $\mathrm{JL}(\pi)$ is the Jacquet–Langlands correspondence of π. Recall that $\mathrm{JL}(\pi) \neq 0$ only if $B = M_2(F)$ or π is discrete, and that if $\mathrm{JL}(\pi) \neq 0$,

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F) \times B^\times \times B^\times}(\widetilde{\mathscr{S}}(V \times F^\times) \otimes \pi, \mathrm{JL}(\pi) \otimes \mathrm{JL}(\widetilde{\pi})) = 1.$$

This space has a normalized form θ as follows: for any $\varphi \in \pi$ realized as $W_{-1}(g)$ in a Whittaker model for the additive character ψ^{-1}, $\phi \in \widetilde{\mathscr{S}}(V \times F^\times)$, and for $\mathscr{F}$ the canonical form on $\mathrm{JL}(\pi) \otimes \mathrm{JL}(\widetilde{\pi})$,

$$\mathscr{F}\theta(\phi \otimes \varphi) = \frac{\zeta(2)}{L(1, \pi, \mathrm{ad})} \int_{N(F)\backslash \mathrm{GL}_2(F)} W_{-1}(g) r(g) \phi(1,1) dg \tag{2.2.1}$$

The constant before the integral is used to normalize the form so that $\mathscr{F}\theta(\phi \otimes \varphi) = 1$ when every thing is unramified.

Let F be a number field and B a quaternion algebra over F. Then the Shimizu lifting can be realized as a global theta lifting:

$$\theta(\phi \otimes \varphi)(h) = \frac{\zeta(2)}{2L(1, \pi, \mathrm{ad})} \int_{\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g) \theta(g, h, \phi) dg, \qquad h \in B_\mathbb{A}^\times \times B_\mathbb{A}^\times. \tag{2.2.2}$$

Here is the relation between global theta lifting and normalized local theta lifting:

Proposition 2.2.1. *We have a decomposition $\theta = \otimes\theta_v$ in*

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A})\times B_{\mathbb{A}}^\times\times B_{\mathbb{A}}^\times}(\widetilde{\mathscr{S}}(V_{\mathbb{A}}\times\mathbb{A}^\times)\otimes\pi, \mathrm{JL}(\widetilde{\pi})\otimes\mathrm{JL}(\pi)).$$

Proof. It suffices to show that for decomposable $\phi = \otimes\phi_v$ and $\varphi = \otimes\varphi_v$,

$$\mathscr{F}\theta(\phi\otimes\varphi) = \prod \mathscr{F}\theta_v(\phi_v\otimes\varphi_v).$$

The inner product between $\mathrm{JL}(\pi)$ and $\mathrm{JL}(\widetilde{\pi})$ is given by integration on the diagonal of $(\mathbb{A}^\times B^\times\backslash B_{\mathbb{A}}^\times)^2$. Let $V = V_0\oplus V_1$ correspond to the decomposition $B = B_0\oplus F$ with B_0 the subspace of trace free elements. Then the diagonal can be identified with $\mathrm{SO}' = \mathrm{SO}(V_0)$. Thus we have globally

$$\mathscr{F}\theta(\phi\otimes\varphi) = \int_{\mathrm{SO}'(F)\backslash\mathrm{SO}'(\mathbb{A})} dh \int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})} \theta(g,h,\phi)\varphi(g)dg.$$

To compute this integral, we interchange the order of integrals and apply the Siegel–Weil formula:

$$\int_{\mathrm{SO}'(F)\backslash\mathrm{SO}'(\mathbb{A})} \theta(g,h,\phi)dh = 2\left(\sum_{\gamma\in P(F)\backslash\mathrm{GL}_2(F)} \delta(\gamma g)^s \sum_{(x,u)\in V_1\times F} r(\gamma g)\phi(x,u)\right)_{s=0}.$$

Thus we have

$$\begin{aligned}\mathscr{F}\theta(\phi\otimes\varphi) =& 2\int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})} \varphi(g) \sum_{\gamma\in P(F)\backslash\mathrm{GL}_2(F)} \sum_{(x,u)\in V_1\times F} r(\gamma g)\phi(x,u)dg \\ =& 2\int_{P(F)\backslash\mathrm{GL}_2(\mathbb{A})} \varphi(g) \sum_{(x,u)\in V_1\times F^\times} r(g)\phi(x,u)dg\end{aligned}$$

The sum here can be written as a sum of two parts $I_1 + I_2$ invariant under $P(F)$ where I_1 is the sum over $x\neq 0$ and I_2 over $x = 0$. It is easy to see that I_2 is invariant under $N(\mathbb{A})$ which contributes 0 to the integral as φ is cuspidal. The sum I_1 is a single orbit over diagonal group. Thus we have

$$\begin{aligned}&2\int_{N(F)\backslash\mathrm{GL}_2(\mathbb{A})} \varphi(g)r(g)\phi(1,1)dg \\ =&2\int_{N(\mathbb{A})\backslash\mathrm{GL}_2(\mathbb{A})} W_{-1}(g)r(g)\phi(1,1)dg.\end{aligned}$$

□

2.3 Integral representations of *L*-series

In the following we want to describe an integral representation of the L-series $L(s,\pi,\chi)$. Let F be a number field with ring of adeles $\mathbb{A}$. Let $\mathbb{B}$ be a quaternion algebra with ramification set Σ. Fix an embedding $E_{\mathbb{A}} \hookrightarrow \mathbb{B}$. We have an orthogonal decomposition

$$\mathbb{B} = E_{\mathbb{A}} + E_{\mathbb{A}}\mathfrak{j}, \qquad \mathfrak{j}^2 \in \mathbb{A}^\times.$$

Write $\mathbb{V}$ for the orthogonal space $\mathbb{B}$ with reduced norm q, and $\mathbb{V}_1 = E_{\mathbb{A}}$ and $\mathbb{V}_2 = E_{\mathbb{A}}\mathfrak{j}$ as subspaces of $V_{\mathbb{A}}$. Then $\mathbb{V}_1$ is coherent and is the base change of F-space $V_1 := E$, and $\mathbb{V}_2$ is coherent if and only if Σ is even.

For $\phi \in \widetilde{\mathscr{S}}(\mathbb{V}\times\mathbb{A}^\times)$, we can form a mixed Eisenstein–Theta series

$$I(s,g,\phi) = \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s \sum_{(x_1,u)\in V_1\times F^\times} r(\gamma g)\phi(x_1,u)$$

Define its χ-component:

$$I(s,g,\chi,\phi) = \int_{T(F)\backslash T(\mathbb{A})} \chi(t) I(s,g,r(t,1)\phi)dt.$$

For $\varphi \in \pi$, we want to compute the integral

$$P(s,\chi,\phi,\varphi) = \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g) I(s,g,\chi,\phi)dg.$$

Proposition 2.3.1 (Waldspurger)**.** *If $\phi = \otimes\phi_v$ and $\varphi = \otimes\varphi_v$ are decomposable, then*

$$P(s,\chi,\phi,\varphi) = \prod_v P_v(s,\chi_v,\phi_v,\varphi_v)$$

where

$$P_v(s,\chi_v,\phi_v,\varphi_v) = \int_{Z(F_v)\backslash T(F_v)} \chi(t)dt \int_{N(F_v)\backslash \mathrm{GL}_2(F_v)} \delta(g)^s W_{-1,v}(g) r(g)\phi_v(t^{-1}, q(t))dg.$$

Here W denotes the Whittaker function of ϕ.

Proof. Bring the definition formula of $I(s,g,\chi,\phi)$ to obtain an expression for $P(s,\chi,\phi,\varphi)$:

$$\int_{Z(\mathbb{A})P(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g)\delta(g)^s \int_{T(F)\backslash T(\mathbb{A})} \chi(t) \sum_{(x,u)\in V_1\times F^\times} r(g,(t,1))\phi(x,u)dtdg.$$

We decompose the first integral as a double integral:

$$\int_{Z(\mathbb{A})P(F)\backslash \mathrm{GL}_2(\mathbb{A})} dg = \int_{Z(\mathbb{A})N(\mathbb{A})P(F)\backslash \mathrm{GL}_2(\mathbb{A})} \int_{N(F)\backslash N(\mathbb{A})} dndg$$

and perform the integral on $N(F)\backslash N(\mathbb{A})$ to obtain:

$$\int_{Z(\mathbb{A})N(\mathbb{A})P(F)\backslash \mathrm{GL}_2(\mathbb{A})} \delta(g)^s dg \int_{T(F)\backslash T(\mathbb{A})} \chi(t) \sum_{(x,u)\in V_1\times F^\times} W_{-q(x)u}(g) r(g,(t,1))\phi(x,u) dt.$$

Here as φ is cuspidal, the term $x = 0$ has no contribution to the integral. In this way, we may change variable $(x,u) \mapsto (x, q(x^{-1})u)$ to obtain the following expression of the sum:

$$\begin{aligned} &\sum_{(x,u)\in E^\times \times F^\times} W_{-u}(g) r(g,(t,1))\phi(x, q(x^{-1})u) \\ =& \sum_{(x,u)\in E^\times \times F^\times} W_{-u}(g) r(g,(tx,1))\phi(1,u). \end{aligned}$$

The sum over $x \in E^\times$ collapses with quotient $T(F) = E^\times$. Thus the integral becomes

$$\int_{Z(\mathbb{A})N(\mathbb{A})P(F)\backslash \mathrm{GL}_2(\mathbb{A})} \delta(g)^s dg \int_{T(\mathbb{A})} \chi(t) \sum_{u\in F^\times} W_{-u}(g) r(g,(t,1))\phi(1,u) dt.$$

The expression does not change if we make the substitution $(g, au) \mapsto (gd(a)^{-1}, u)$. Thus we have

$$\int_{Z(\mathbb{A})N(\mathbb{A})P(F)\backslash \mathrm{GL}_2(\mathbb{A})} \delta(g)^s dg \int_{T(\mathbb{A})} \chi(t) \sum_{u\in F^\times} W_{-u}(d(u^{-1})g) r(d(u^{-1})g,(t,1))\phi(1,u) dt.$$

The sum over $u \in F^\times$ collapses with quotient $P(F)$, thus we obtain the following expression:

$$P(s,\chi,\phi,\varphi) = \int_{Z(\mathbb{A})N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} \delta(g)^s dg \int_{T(\mathbb{A})} \chi(t) W_{-1}(g) r(g)\phi(t^{-1}, q(t)) dt.$$

We may decompose the inside integral as

$$\int_{Z(\mathbb{A})\backslash T(\mathbb{A})} \int_{Z(\mathbb{A})}$$

and move the integral $\int_{Z(\mathbb{A})\backslash T(\mathbb{A})}$ to the outside. Then we use the fact that $\omega_\pi \cdot \chi|_{\mathbb{A}^\times} = 1$ to obtain

$$P(s,\chi,\phi,\varphi) = \int_{Z(\mathbb{A})\backslash T(\mathbb{A})} \chi(t) dt \int_{N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} \delta(g)^s W_{-1}(g) r(g)\phi(t^{-1}, q(t)) dg.$$

□

When everything is unramified, Waldspurger has computed these integrals:

$$P_v(s,\chi_v,\phi_v,\varphi_v) = \frac{L((s+1)/2, \pi_v, \chi_v)}{L(s+1, \eta_v)}.$$

Thus we may define a normalized integral P_v° by

$$P_v(s,\chi_v,\phi_v,\varphi_v)=\frac{L((s+1)/2,\pi_v,\chi_v)}{L(s+1,\eta_v)}P_v^\circ(s,\chi_v,\phi_v,\varphi_v).$$

This normalized local integral P_v° will be regular at $s=0$ and equal to

$$\frac{L(1,\eta_v)L(1,\pi_v,\mathrm{ad})}{\zeta_v(2)L(1/2,\pi_v,\chi_v)}\int_{Z(F_v)\backslash T(F_v)}\chi_v(t)\mathscr{F}\left(\mathrm{JL}(\pi)(t)\theta(\phi_v\otimes\varphi_v)\right)dt.$$

We may write this as $\alpha_v(\theta(\phi_v\otimes\varphi_v))$ with

$$\alpha_v\in\mathrm{Hom}(\mathrm{JL}(\pi)\otimes\mathrm{JL}(\widetilde{\pi}),\mathbb{C})$$

given by the integration of matrix coefficients:

$$\alpha_v(f\otimes\widetilde{f})=\frac{L(1,\eta_v)L(1,\pi_v,\mathrm{ad})}{\zeta_v(2)L(1/2,\pi_v,\chi_v)}\int_{Z(F_v)\backslash T(F_v)}\chi_v(t)(\pi(t)f,\widetilde{f})dt. \tag{2.3.1}$$

And we define an element $\alpha:=\otimes_v\alpha_v$ in $\mathrm{Hom}(\mathrm{JL}(\pi)\otimes\mathrm{JL}(\widetilde{\pi}),\mathbb{C})$.

We now take value or derivative at $s=0$ to obtain

Proposition 2.3.2.

$$P(0,\chi,\phi,\varphi)=\frac{L(1/2,\pi,\chi)}{L(1,\eta)}\prod_v\alpha_v(\theta(\phi_v\otimes\varphi_v)).$$

If Σ is odd, then $L(1/2,\pi,\chi)=0$, and

$$P'(0,\chi,\phi,\varphi)=\frac{L'(1/2,\pi,\chi)}{2L(1,\eta)}\prod_v\alpha_v(\theta(\phi_v\otimes\varphi_v)).$$

Remark. Let $\mathscr{A}_\Sigma(\mathrm{GL}_2,\chi)$ denote the direct sum of cusp forms π on $\mathrm{GL}_2(\mathbb{A})$ such that $\Sigma(\pi,\chi)=\Sigma$. If Σ is even, let $\mathscr{I}(g,\chi,\phi)$ be the projection of $I(0,g,\chi,\phi)$ on $\mathscr{A}_\Sigma(\mathrm{GL}_2,\chi^{-1})$. If Σ is odd, let $\mathscr{I}'(g,\chi,\phi)$ denote the projection of $I'(0,g,\chi,\phi)$ on $\mathscr{A}_\Sigma(\mathrm{GL}_2,\chi^{-1})$. Then have shown that $\mathscr{I}(g,\chi,\phi)$ and $\mathscr{I}'(g,\chi,\phi)$ represent the functionals

$$\varphi\mapsto\frac{L(1/2,\pi,\chi)}{L(1,\eta)}\alpha(\theta(\phi\otimes\varphi))\qquad\text{if }\Sigma\text{ is even}$$

$$\varphi\mapsto\frac{L'(1/2,\pi,\chi)}{2L(1,\eta)}\alpha(\theta(\phi\otimes\varphi))\qquad\text{if }\Sigma\text{ is odd}$$

on π respectively.

2.4 Proof of Waldspurger formula

Assume that Σ is even and we recall Waldspurger's proof of his central value formula. Now the space $\mathbb{V} = V(\mathbb{A})$ is coming from a global $V = B$ over F. For $\phi \in \widetilde{\mathscr{S}}(\mathbb{V} \times \mathbb{A}^\times)$ and $\varphi \in \pi$, we want to compute the double period integral of $\theta(\phi \otimes \varphi)$:

$$\int_{(T(F)Z(\mathbb{A})\backslash T(\mathbb{A}))^2} \theta(\phi \otimes \varphi)(t_1, t_2)\chi(t_1)\chi^{-1}(t_2)dt_1 dt_2.$$

Using definition, this equals

$$\frac{\zeta(2)}{2L(1, \pi, \mathrm{ad})} \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g)\theta(g, \phi, \chi)dg$$

where

$$\theta(g, \chi, \phi) = \int_{Z^\Delta(\mathbb{A})T(F)^2\backslash T(\mathbb{A})^2} \theta(g, (t_1, t_2), \phi)\chi(t_1 t_2^{-1})dt_1 dt_2,$$

where Z^Δ is the image of the diagonal embedding $F^\times \to (B^\times)^2$. We change variable $t_1 = tt_2$ to get a double integral

$$\theta(g, \chi, \phi) = \int_{T(F)\backslash T(\mathbb{A})} \chi(t)dt \int_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} \theta(g, (tt_2, t_2), \phi)dt_2,$$

Notice that the diagonal embedding $Z\backslash T$ can be realized as $\mathrm{SO}(Ej)$ in the decomposition $V = B = E + Ej$. Thus we can apply Siegel–Weil formula 2.1.1 to obtain

$$\theta(g, \chi, \phi) = L(1, \eta)I(0, g, \chi, \phi).$$

Here recall

$$I(s, g, \phi) = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s \sum_{(x,u)\in E\times F^\times} r(\gamma g, (t, 1)\phi)(x, u).$$

Combining with Proposition 2.3.2, we have

$$\int_{(T(F)Z(\mathbb{A})\backslash T(\mathbb{A}))^2} \theta(\phi \otimes \varphi)(t_1, t_2)\chi(t_1)\chi^{-1}(t_2)dt_1 dt_2 = \frac{\zeta(2)L(1/2, \pi, \chi)}{2L(1, \pi, \mathrm{ad})}\alpha(\theta(\phi \otimes \varphi)).$$

This is certainly an identity as functionals on $\mathrm{JL}(\pi) \otimes \mathrm{JL}(\widetilde{\pi})$:

$$\ell_\chi \otimes \ell_{\chi^{-1}} = \frac{\zeta(2)L(1/2, \pi, \chi)}{2L(1, \pi, \mathrm{ad})} \cdot \alpha. \tag{2.4.1}$$

2.5 Vanishing of the kernel function

We want to compute the Fourier expansion of $I(s, g, \phi)$ in the case that Σ is odd, and prove that they vanish at $s = 0$. We will mainly work on $\phi \in \widetilde{\mathscr{S}}(\mathbb{V} \times \mathbb{A}^\times)$.

Recall that we have the orthogonal decomposition $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$, where $\mathbb{V}_1 = E_\mathbb{A}$ and $\mathbb{V}_2 = E_\mathbb{A}\mathrm{j}$. It yields a decomposition

$$\widetilde{\mathscr{S}}(\mathbb{V} \times \mathbb{A}^\times) = \widetilde{\mathscr{S}}(\mathbb{V}_1 \times \mathbb{A}^\times) \otimes \widetilde{\mathscr{S}}(\mathbb{V}_2 \times \mathbb{A}^\times).$$

More precisely, any $\phi_1 \in \widetilde{\mathscr{S}}(\mathbb{V}_1 \times \mathbb{A}^\times)$ and $\phi_2 \in \widetilde{\mathscr{S}}(\mathbb{V}_2 \times \mathbb{A}^\times)$ gives $\phi_1 \otimes \phi_2 \in \widetilde{\mathscr{S}}(\mathbb{V} \times \mathbb{A}^\times)$ defined by

$$(\phi_1 \otimes \phi_2)(x_1 + x_2, u) := \phi_1(x_1, u)\phi_2(x_2, u).$$

And any element of $\widetilde{\mathscr{S}}(\mathbb{V} \times \mathbb{A}^\times)$ is a finite linear combination of functions of the form $\phi_1 \otimes \phi_2$. More importantly, the decomposition preserves Weil representation in the sense that

$$r(g, (t_1, t_2))(\phi_1 \otimes \phi_2)(x, u) = r_1(g, (t_1, t_2))\phi_1(x_1, u)\ r_2(g, (t_1, t_2))\phi_2(x_2, u)$$

for any $(g, (t_1, t_2)) \in \mathrm{GL}_2(\mathbb{A}) \times E^\times(\mathbb{A}) \times E^\times(\mathbb{A})$. Here we write r_1, r_2 for the Weil representation associated to the vector spaces $\mathbb{V}_1, \mathbb{V}_2$. The group $E^\times(\mathbb{A}) \times E^\times(\mathbb{A})$ acts on $\mathbb{V}_\ell$ by $(t_1, t_2) \circ x_\ell = t_1 x_\ell t_2^{-1}$. It is compatible with the action on $\mathbb{V}$.

By linearity, we may reduce the computation to the decomposable case $\phi = \phi_1 \otimes \phi_2$. Then we have

$$I(s, g, \phi) = \sum_{u \in F^\times} \theta(g, u, \phi_1) E(s, g, u, \phi_2),$$

where

$$\theta(g, u, \phi_1) = \sum_{x \in E} r_1(g)\phi_1(x, u),$$
$$E(s, g, u, \phi_2) = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s r_2(\gamma g)\phi_2(0, u).$$

It suffices to study the behavior of Eisenstein series at $s = 0$. The work of Kudla-Rallis in more general setting shows that the incoherent Eisenstein series $E(s, g, u, \phi_2)$ vanishes at $s = 0$ by local reasons. For reader's convenience, we will show this fact by detailed analysis of the local Whittaker functions. We will omit ϕ_1 and ϕ_2 in the notation.

We start with the Fourier expansion of Eisenstein series:

$$E(s, g, u) = E_0(s, g, u) + \sum_{a \in F^\times} E_a(s, g, u)$$

where the a-th Fourier coefficient is

$$E_a(s, g, u) = \int_{F\backslash \mathbb{A}} E\left(s, \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} g, u\right) \psi(-ab)db, \quad a \in F.$$

An easy calculation gives the following result.

Lemma 2.5.1.

$$\begin{aligned} E_0(s,g,u) &= \delta(g)^s r_2(g)\phi_2(0,u) - \int_{\mathbb{A}} \delta(wn(b)g)^s \int_{\mathbb{V}_2} r_2(g)\phi_2(x_2,u)\psi(buq(x_2))d_ux_2db, \\ E_a(s,g,u) &= -\int_{\mathbb{A}} \delta(wn(b)g)^s \int_{\mathbb{V}_2} r_2(g)\phi_2(x_2,u)\psi(b(uq(x_2)-a))d_ux_2db, \quad a\in F^\times. \end{aligned}$$

Here the measure d_ux_2 on $\mathbb{V}_2$ is the self-dual Haar measure with respect to uq.

Proof. It is a standard result that the constant term is given by

$$E_0(s,g,u) = \delta(g)^s r_2(g)\phi_2(0,u) + \int_{N(\mathbb{A})} \delta(wng)^s r_2(wng)\phi_2(0,u)dn.$$

By definition, we have

$$\begin{aligned} r_2(wn(b)g)\phi_2(0,u) &= \gamma(\mathbb{V}_2)\int_{\mathbb{V}_2} r_2(n(b)g)\phi_2(x_2,u)d_ux_2 \\ &= \gamma(\mathbb{V}_2)\int_{\mathbb{V}_2} r_2(g)\phi_2(x_2,u)\psi(buq(x_2))d_ux_2. \end{aligned}$$

Here $\gamma(\mathbb{V}_2)$ is the Weil index of the quadratic space $(\mathbb{V}_2, uq)$, which is apparently independent of $u\in F^\times$. By the orthogonal decomposition $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$, we have $\gamma(\mathbb{V}) = \gamma(\mathbb{V}_1)\gamma(\mathbb{V}_2)$. Here $\gamma(\mathbb{V}_1) = 1$ since $\mathbb{V}_1 = E(\mathbb{A}_f)$ is coherent, and $\gamma(\mathbb{V}) = \prod_v \gamma(\mathbb{V}_v, q) = (-1)^{\#\Sigma} = -1$ since Σ is assumed to be odd. It follows that $\gamma(\mathbb{V}_2) = -1$. So we get the result for $E_0(s,g,u)$. The formula for $E_a(s,g,u)$ is computed similarly. □

Notation. We introduce the following notations:

$$\begin{aligned} W_a(s,g,u,\phi_2) &= \int_{\mathbb{A}} \delta(wn(b)g)^s \int_{\mathbb{V}_2} r_2(g)\phi_2(x_2,u)\psi(b(uq(x_2)-a))d_ux_2db, \\ W_{a,v}(s,g,u,\phi_2) &= \int_{F_v} \delta(wn(b)g)^s \int_{\mathbb{V}_{2,v}} r_2(g)\phi_{2,v}(x_2,u)\psi_v(b(uq(x_2)-a))d_ux_2db, \\ W^\circ_{0,v}(s,g,u,\phi_2) &= \frac{L(s+1,\eta_v)}{L(s,\eta_v)}|D_v|^{-\frac12}|d_v|^{-\frac12}W_{0,v}(s,g,u), \\ W^\circ_0(s,g,u,\phi_2) &= \prod_v W^\circ_{0,v}(s,g,u,\phi_2). \end{aligned}$$

We usually omit the dependence on ϕ_2 if no confusion occurs. Notice that the normalizing factor $\dfrac{L(s+1,\eta_v)}{L(s,\eta_v)}$ has a zero at $s=0$ when E_v is split, and is equal to π^{-1} at $s=0$ when v is archimedean. Note that we use the convention that $|D_v| = |d_v| = 1$ if v is archimedean.

Now we list the precise values of these local Whittaker functions when $s=0$. They follow from the local Siegel–Weil formula. In the incoherent case, they will lead to the vanishing of our kernel function at $s=0$.

Proposition 2.5.2. *(1) In the sense of analytic continuation for s,*

$$W^{\circ}_{0,v}(0,g,u) = r_2(g)\phi_{2,v}(0,u), \qquad W_0(0,g,u) = r_2(g)\phi_2(0,u).$$

Furthermore, for almost all places v,

$$W^{\circ}_{0,v}(s,g,u) = \delta_v(g)^{-s} r_2(g)\phi_{2,v}(0,u).$$

(2) Assume $a \in F_v^{\times}$.

(a) If au^{-1} is not represented by $(\mathbb{V}_{2,v}, q_{2,v})$, then $W_{a,v}(0,g,u) = 0$.

(b) Assume that there exists $\xi \in \mathbb{V}_{2,v}$ satisfying $q(\xi) = au^{-1}$. Then

$$W_{a,v}(0,g,u) = \frac{1}{L(1,\eta_v)} \int_{E_v^1} r_2(g)\phi_{2,v}(\xi\tau,u)d\tau.$$

Proof. By the action of $d(F_v^{\times})$, it suffices show all the identities for $g \in \mathrm{SL}_2(F_v)$.

All the equalities at $s=0$ are consequences of the local Siegel–Weil formula in Theorem 2.1.2. We immediately know that they are true up to constant multiples independent of g and ϕ_v. Many cases are in the literature. See [KRY1] for example.

As for $W^{\circ}_{0,v}(s,g,u)$, it is the image of the normalized intertwining operator of $\delta(g)^s r_2(g)\phi_{2,v}(0$ for $g \in \mathrm{SL}_2(F_v)$. Hence we know the equality for almost all places. □

Proposition 2.5.3. *$E(0,g,u) = 0$, and thus $I(0,g,\phi) = 0$.*

Proof. It is a direct corollary of Proposition 2.5.2. The Eisenstein series has Fourier expansion

$$E(0,g,u) = \sum_{a \in F} E_a(0,g,u).$$

The vanishing of the constant term easily follows Proposition 2.5.2. For $a \in F^{\times}$,

$$E_a(0,g,u) = -\prod_v W_{a,v}(0,g,u)$$

is nonzero only if au^{-1} is represented by $(E_v, q_{2,v})$ at each v.

In fact, $\mathbb{B}_v$ is split if and only if there exists a non-zero element $\xi \in \mathbb{V}_{2,v}$ (equivalently, for all nonzero $\xi \in \mathbb{V}_{2,v}$) such that $-q(\xi) \in q(E_v^{\times})$. In another word, the following identities hold:

$$\varepsilon(\mathbb{B}_v) = \eta_v(-q(\xi)),$$

where $\varepsilon(\mathbb{B}_v)$ is the Hasse invariant of $\mathbb{B}_v$ and η_v is the quadratic character induced by E_v.

Now the existence of ξ at v such that $q_{2,v}(\xi) = au^{-1}$ is equivalent to

$$\varepsilon(\mathbb{B}_v) = \eta_v(-au^{-1}).$$

If this is true for all v, then

$$\prod_v \varepsilon(\mathbb{B}_v) = \prod_v \eta_v(-au^{-1}) = 1.$$

It contradicts to the incoherence condition that the order of Σ is odd. □

3 Derivative of the kernel function

In this section, we want to study the derivative of the kernel function for L-series when Σ is odd and π has discrete components of weight 2 at archimedean places. The main content of this section is various local formulae.

In §3.1, we extend the result in the last section to functions in $\phi \in \mathscr{S}(\mathbb{V} \times \mathbb{A}^\times)$, the maximal quotient of $\widetilde{\mathscr{S}}(\mathbb{V} \times \mathbb{A}^\times)$ with trivial action by $\mathrm{GO}(F_\infty)$.

In §3.2, we introduce the space $\mathscr{S}^0(\mathbb{V}_v \times F_v^\times)$ of degenerate Schwartz functions for any non-archimedean place v. The main result ensures that this space is sufficient in our proof.

In §3.3, we decompose the kernel function $I'(0,g)$ into a sum of infinite many local terms $I'(0,g)(v)$ indexed by places v of F non-split in E. Each local term is a period integral of some kernel function $\mathscr{K}^{(v)}(g,(t_1,t_2))$.

In §3.4, we deal with the v-part $I'(0,g)(v)$ for non-archimedean v. An explicit formula is given in the unramified case, and an approximation is given in the ramified case assuming the Schwartz function is degenerate.

In §3.5, we show an explicit result of the v-part $I'(0,g)(v)$ for archimedean v.

In §3.6, we review a general formula of holomorphic projection, and estimate the growth of the kernel function.

In §3.7, we compute the holomorphic projection of the kernel $I'(0,g,\chi)$. It has the main part, and the part coming from the fast growth of the kernel function.

3.1 Discrete series of weight two at infinite places

In §2, we reviewed the result of Waldspurger, especially his extension of Weil representations to the Schwartz function space $\widetilde{\mathscr{S}}(V \times F^\times)$. In this paper, we will use the same spaces for non-archimedean places, but different spaces for archimedean spaces.

In the case that (V,F) is non-archimedean, we still consider Weil representations over the space $\mathscr{S}(V \times F^\times) = \widetilde{\mathscr{S}}(V \times F^\times)$ consisting of locally constant and compactly supported functions on $V \times F^\times$.

In the archimedean case, our space $\mathscr{S}(V \times F^\times)$ is different from $\widetilde{\mathscr{S}}(V \times F^\times)$. We make this change mainly because the generating series to be introduced in next section uses the standard Schwartz function which is not in the original space.

Discrete series of weight two at infinity

Let V be a positive definite quadratic space of even dimension $2d$ over a local field $F \simeq \mathbb{R}$. Let $\mathscr{S}(V \times F^\times)$ denote the space of functions on $V \times F^\times$ of the form

$$(P_1(uq(x)) + \mathrm{sgn}(u)P_2(uq(x)))\, e^{-2\pi|u|q(x)}$$

with polynomials P_i on $\mathbb{R}$. Here $\mathrm{sgn}(u) = u/|u|$ denotes the sign of $u \in F^\times$.

The Weil representation r acts on $\mathscr{S}(V \times F^\times)$ by the same method in §2. Apparently, all functions in $\mathscr{S}(V \times F^\times)$ are invariant under $\mathrm{GO}(F)$. The space actually gives the discrete series of weight d of $\mathrm{GL}_2(F)$.

We can also find the connection between $\mathscr{S}(V\times F^\times)$ and $\widetilde{\mathscr{S}}(V\times F^\times)$. For example, $\mathscr{S}(V\times F^\times)$ can be a quotient of $\widetilde{\mathscr{S}}(V\times F^\times)$ (resp. $\widetilde{\mathscr{S}}(V\times F^\times)^{O(F)}$) via integration over $\mathrm{GO}(F)$ (resp. $Z(F)$):

$$\phi\mapsto\int_{\mathrm{GO}(V)} r(h)\phi dh,\qquad (\text{resp.} \int_{Z(F)} r(h)\phi dh).$$

Here $\widetilde{\mathscr{S}}(V\times F^\times)^{O(F)}$ denotes the subspace of functions in $\widetilde{\mathscr{S}}(V\times F^\times)$ which are invariant under $O(F)$. It can be obtained by integrals over $O(F)$ in a similar way.

The the *standard Schwartz function* $\phi\in\mathscr{S}(V\times F^\times)$ is the Gaussian

$$\phi(x,u)=\frac{1}{2}(1+\mathrm{sgn}(u))e^{-2\pi|u|q(x)}.$$

Then one verifies that if $q(x)\neq 0$, then

$$r(g)\phi(x,u)=W^{(d)}_{uq(x)}(g).$$

Here $W^{(d)}_a(g)$ is the *standard Whittaker function* of weight d for character $e^{2\pi i a x}$:

$$W^{(d)}_a(g)=\begin{cases}|y_0|^{\frac{d}{2}}e^{di\theta} & \text{if } a=0\\ |y_0|^{\frac{d}{2}}e^{2\pi i a(x_0+iy_0)}e^{di\theta} & \text{if } ay_0>0\\ 0 & \text{if } ay_0<0\end{cases}$$

for any $a\in\mathbb{R}$ and

$$g=\begin{pmatrix} z_0 & \\ & z_0\end{pmatrix}\begin{pmatrix} y_0 & x_0\\ & 1\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\in\mathrm{GL}_2(\mathbb{R})$$

in the form of Iwasawa decomposition. Here we normalize $z_0>0$.

In the end, we introduce some global notations. For any (coherent or incoherent) quadratic space $\mathbb{V}$ over $\mathbb{A}$ which is positive definite of dimension $2d$ at infinity, denote restricted tensor product

$$\mathscr{S}(\mathbb{V}\times\mathbb{A}^\times):=\otimes_v\mathscr{S}(\mathbb{V}_v\times F_v^\times).$$

For any $a\in\mathbb{A}^\times$ and $x\in\mathbb{V}$ with $q(x)\neq 0$, denote

$$r(g,h)\phi(x)_a:=r(g,h)\phi(x,aq(x)^{-1}).$$

Use similar notations in the local case. Note that if ϕ_∞ is standard, the archimedean part

$$r(g,h)\phi_\infty(x,aq(x)^{-1})=W^{(d)}_{a,\infty}(g_\infty)$$

is independent of x_∞,h_∞. Thus we use the notation

$$r(g,h)\phi(x)_a:=r(g_f,h_f)\phi_f(x,aq(x)^{-1})W^{(d)}_{a,\infty}(g_\infty)$$

even for $h\in\mathrm{GO}(\mathbb{V}_f)$ and $x\in\mathbb{V}_f$ as long as a and g have infinite components.

Theta series

First let us consider the definition of theta series. Let V be a positive definite quadratic space over a totally real field F. Let $\phi \in \mathscr{S}(V(\mathbb{A}) \times \mathbb{A}^\times)$. There is an open compact subgroup $K \subset \mathrm{GO}(\mathbb{A}_f)$ such that ϕ_f is invariant under the action of K by Weil representation. Denote $K_Z = K \cap \mathbb{A}^\times$ and $\mu_K = F^\times \cap K$. Then μ_K is a subgroup of the unit group $O_F^\times$, and thus is a finitely generated abelian group. Our theta series is of the following form:

$$\theta(g,h,\phi) = \sum_{(x,u)\in\mu_K\backslash(V\times F^\times)} r(g,h)\phi(x,u).$$

The definition here depends on the choice of K. A normalization is given by

$$\widetilde{\theta}(g,h,\phi) = [\mathbb{A}_f^\times : F^\times K_Z]\mathrm{vol}(K_Z)\theta(g,h,\phi)$$

By choosing a different fundamental domain, we can rewrite the sum as

$$\varepsilon_K^{-1} \sum_{u\in\mu_K^2\backslash F^\times} \sum_{x\in V} r(g,h)\phi(x,u).$$

Here $\varepsilon_K = |\{1,-1\}\cap K| \in \{1,2\}$. In particular, $\varepsilon_K = 1$ for K small enough. The summation over u is well-defined since $\phi(x,u) = r(\alpha)\phi(x,u) = \phi(\alpha x, \alpha^{-2}u)$ for any $\alpha \in \mu_K$. It is an automorphic form on $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GO}(\mathbb{A})$, provided the absolute convergence.

To show the convergence, we claim that the summation over u is actually a finite sum depending on (g,h). For fixed (g,h), there is a compact subset $K' \subset \mathbb{A}_f^\times$ such that $r(g,h)\phi_f(x,u) \neq 0$ only if $u \in K'$. Thus the summation is taken over $u \in \mu_K^2\backslash(F^\times \cap K')$, which is a finite set since both μ_K^2 and $F^\times \cap K'$ are finite-index subgroups of the unit group $O_F^\times$.

Alternatively, we may construct theta series for above $\phi \in \mathscr{S}(V(\mathbb{A}) \times \mathbb{A}^\times)$ by some function $\widetilde{\phi} = \widetilde{\phi}_\infty \otimes \phi_f \in \widetilde{\mathscr{S}}(\mathbb{V} \times \mathbb{A}^\times)^{O(F_\infty)}$ such that

$$\int_{Z(F_\infty)} r(h)\widetilde{\phi} dh = \phi.$$

In this case,

$$\begin{aligned}\widetilde{\theta}(g,h,\phi) =& [\mathbb{A}_f^\times : F^\times K_Z]\mathrm{vol}(K_Z) \int_{Z(F_\infty)/\mu_K} \theta(g,zh,\widetilde{\phi})dz \\ =& [\mathbb{A}^\times : F^\times(F_\infty^\times \cdot K_Z)] \int_{F^\times\backslash F^\times(F_\infty^\times\cdot K_Z)} \theta(g,zh,\widetilde{\phi})dz.\end{aligned}$$

Kernel functions

For $\phi \in \mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)$, we may define the mixed Eisenstein-theta series and their normalizations as

$$I(s,g,\phi) = \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s \sum_{(x_1,u)\in\mu_K\backslash V_1\times F^\times} r(\gamma g)\phi(x_1,u),$$

$$\widetilde{I}(s, g, \phi) = [\mathbb{A}_f^\times : F^\times K_Z]\mathrm{vol}(K_Z)I(s, g, \phi).$$

If χ has trivial component at infinity, then we may define

$$I(s, g, \chi, \phi) = [\mathbb{A}_f^\times : F^\times \cdot K_Z]^{-1} \int_{T(F)\backslash T(\mathbb{A})/Z(F_\infty)K_Z} \chi(t)\widetilde{I}(s, g, r(t, 1)\phi)dt.$$

Here $K_Z = Z(\mathbb{A}) \cap K$. It is clear that $I(s, g, \phi)$ is a finite linear combination of $I(s, g, \chi, \phi)$. In terms of integration $\int_{[G]}$ introduced in Introduction, the above definition is written as

$$I(s, g, \chi, \phi) = \int_{[T]} \chi(t)I(s, g, r(t, 1)\phi)dt.$$

Now we explain our reason for this normalization. We claim that if

$$\widetilde{\phi} = \widetilde{\phi}_\infty \otimes \phi_f \in \widetilde{\mathscr{S}}(\mathbb{V} \times \mathbb{A}^\times)^{O(F_\infty)}$$

such that

$$\phi_\infty = \int_{Z(F_\infty)} r(z)\widetilde{\phi}_\infty dz,$$

then

$$\widetilde{I}(s, g, \phi) = [\mathbb{A}_f^\times : F^\times \cdot K_Z]\mathrm{vol}(K_Z) \int_{Z(F_\infty^\times)/\mu_F} I(s, g, r(z)\widetilde{\phi})dz$$

$$I(s, g, \chi, \phi) = I(s, g, \chi, \widetilde{\phi}).$$

Indeed, in the definition of $I(s, g, \chi, \phi)$ in section 2.4, we may decompose the integral over $T(F)\backslash T(\mathbb{A})$ into double integrals over $T(F)\backslash T(\mathbb{A})/Z(F_\infty)K_Z$ and integrals over $Z(F_\infty)K_Z T(F)/T(F)$ to obtain

$$\begin{aligned} I(s, g, \chi, \widetilde{\phi}) &= \int_{T(F)\backslash T(\mathbb{A})/Z(F_\infty)K_Z} \chi(t)dt \int_{T(F)\backslash T(F)Z(F_\infty)K_Z} I(s, g, r(tz, 1)\widetilde{\phi})dz \\ &= \int_{[T]} \chi(t)dt \cdot [\mathbb{A}_f^\times : F^\times K_Z] \int_{T(F)\backslash T(F)Z(F_\infty)K_Z} I(s, g, r(tz, 1)\widetilde{\phi})dz \end{aligned}$$

The inside integral domain can be identified with

$$T(F)\backslash T(F)Z(F_\infty)K_Z \simeq \mu_K \backslash Z(F_\infty)K_Z$$

with has a fundamental domain $Z(F_\infty)/\mu_K \times K_Z$. Thus the second integral can be written as

$$\mathrm{vol}(K_Z) \int_{Z(F_\infty)/\mu_K} I(s, g, r(tz, 1)\widetilde{\phi})dz.$$

For this last integral, we write the sum over $V_1 \times F^\times$ in the definition of $I(s, g, \widetilde{\phi})$ as a double sum over $\mu_K \backslash V_1 \times F^\times$ and a sum of μ_K. The first sum commutes with integral over $Z(F_\infty)/\mu_K$ while the second the sum collapse with quotient $Z(F_\infty)/\mu_K$ to get a simple integral over $Z(F_\infty)$ which changes $\widetilde{\phi}$ to ϕ.

3.2 Degenerate Schwartz functions at finite places

In this subsection we introduce a class of "degenerate" Schwartz functions at a non-archimedean place. It is generally very difficult to obtain an explicit formula of $I'(0,g)(v)$ for a ramified finite prime v. When we choose a degenerate Schwartz function at v, the function $I'(0,g)(v)$ turns out to be easier to control. The same phenomena happens in the geometric side. The main result roughly says these degenerate functions generate the space of all functions in some sense.

Let v be a finite place of F. Recall that d_v is the local different of F at v. If v is split in E, define

$$\begin{aligned}\mathscr{S}^0(\mathbb{B}_v \times F_v^\times) = \{\phi_v \in \mathscr{S}(\mathbb{B}_v \times F_v^\times) :\\ \phi_v(x,u) = 0 \text{ if } v(uq(x)) \geq -v(d_v)\}.\end{aligned}$$

If v is non-split in E, define

$$\begin{aligned}\mathscr{S}^0(\mathbb{B}_v \times F_v^\times) = \{\phi_v \in \mathscr{S}(\mathbb{B}_v \times F_v^\times) :\\ \phi_v(x,u) = 0 \text{ if } v(uq(x)) \geq -v(d_v) \text{ or } v(uq(x_2)) \geq -v(d_v)\}.\end{aligned}$$

Here x_2 denotes the orthogonal projection of x in $\mathbb{V}_{2,v} = E_v \mathfrak{j}_v$. We say such Schwartz functions are *degenerate*.

A global $\phi \in \mathscr{S}(\mathbb{B} \times \mathbb{A}^\times)$ is *degenerate* at v if ϕ_v is degenerate. This is an assumption we usually need to deal with singularities. The following is another assumption we will make in §3-5 to simplify computation. It will kill the constant terms of Eisenstein series of weight one and weight two, self-intersections in the geometric kernel, and local heights at ordinary points.

Global Degeneracy Assumption. *Fix two different non-archimedean places v_1, v_2 of F which are non-split in E. Assume that $g \in P(F_{v_1,v_2})\mathrm{GL}_2(\mathbb{A}^{v_1,v_2})$ and $\phi \in \mathscr{S}(\mathbb{V} \times \mathbb{A}^\times)$ is degenerate at v_1, v_2.*

The rest of this subsection is dedicated to prove the following result, which allows us to extend our final result from degenerate Schwartz functions to general Schwartz functions in the last section.

Proposition 3.2.1. *Let v be a non-archimedean place and π_v an infinite dimensional irreducible representation of $\mathrm{GL}_2(F_v)$. Then for any nonzero $\mathrm{GL}_2(F_v)$-equivariant homomorphism $\mathscr{S}(\mathbb{B}_v \times F_v^\times) \to \pi_v$, the image of $\mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$ in π_v is nonzero.*

For convenience, we introduce

$$\mathscr{S}^0_{\mathrm{weak}}(\mathbb{B}_v \times F_v^\times) = \begin{cases} \{\phi_v \in \mathscr{S}(\mathbb{B}_v \times F_v^\times) : \phi_v|_{(\mathbb{B}_v{}^{\mathrm{sing}} \cup E_v) \times F_v^\times} = 0\}, & \text{if } E_v/F_v \text{ is nonsplit;} \\ \{\phi_v \in \mathscr{S}(\mathbb{B}_v \times F_v^\times) : \phi_v|_{\mathbb{B}_v{}^{\mathrm{sing}} \times F_v^\times} = 0\}, & \text{if } E_v/F_v \text{ is split.}\end{cases}$$

Here $\mathbb{B}_v{}^{\mathrm{sing}} = \{x \in \mathbb{B}_v : q(x) = 0\}$. By compactness, for any $\phi_v \in \mathscr{S}^0_{\mathrm{weak}}(\mathbb{B}_v \times F_v^\times)$, there exists a constant c such that $\phi_v(x,u) = 0$ if $v(uq(x)) > c$. The some result holds for $uq(x_2)$ in the non-split case. Then it is easy to see that $\mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$ generates $\mathscr{S}^0_{\mathrm{weak}}(\mathbb{B}_v \times F_v^\times)$ under the action of the group $m(F_v^\times) \subset \mathrm{GL}_2(F_v)$ of elements $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$. Thus the former can be viewed as an effective version of the later. In particular, it suffices to prove the proposition for $\mathscr{S}^0_{\mathrm{weak}}(\mathbb{B}_v \times F_v^\times)$.

We will prove a more general result. For simplicity, let F be a non-archimedean local field and let (V, q) be a non-degenerate quadratic space over F of even dimension. Then we have the Weil representation of $\mathrm{GL}_2(F)$ on $\mathscr{S}(V \times F^\times)$, the space of Bruhat-Schwartz functions on $V \times F^\times$. Let

$$\alpha : \mathscr{S}(V \times F^\times) \to \sigma$$

be a surjective morphism to an irreducible and admissible representation σ of $\mathrm{GL}_2(F)$. We will prove the following result which obviously implies Proposition 3.2.1.

Proposition 3.2.2. *Let W be a proper subspace of V of even dimension. Assume that σ is not one-dimensional, and that in the case $W \neq 0$, W is non-degenerate, and that its orthogonal complement W' is anisotropic. Then there is a function $\phi \in \mathscr{S}(V \times F^\times)$ with a nonzero image in σ such that the support* $\mathrm{supp}(\phi)$ *of ϕ contains only elements (x, u) such that $q(x) \neq 0$ and that*

$$W(x) := W + Fx$$

is non-degenerate of dimension $\dim W + 1$.

Let us start with the following Proposition which allows us to modify any test function to a function with support at points $(x, u) \in V \times F^\times$ with components x of nonzero norm $q(x) \neq 0$.

Proposition 3.2.3. *Let $\phi \in \mathscr{S}(V \times F^\times)$ be an element with a nonzero image in σ. Then there is a function $\widetilde{\phi} \in \mathscr{S}(V \times F^\times)$ with a nonzero image in σ such that*

$$\mathrm{supp}(\widetilde{\phi}) \subset \mathrm{supp}(\phi) \cap \left(V_{q \neq 0} \times F^\times\right).$$

The key to prove this proposition is the following lemma. It is well-known but we give a proof for readers' convenience.

Lemma 3.2.4. *Let σ be an irreducible admissible representation of* $\mathrm{GL}_2(F)$ *whose dimension is greater than one. Then the only vector in σ invariant under the action of the unipotent group $N(F)$ is zero.*

Proof. Let v be such an invariant vector. By smoothness, it is also fixed by some compact open subgroup U of $\mathrm{SL}_2(F)$. Then v is invariant under the subgroup generated by $N(F)$ and U. It is easy to see that $U - P^1(F)$ is non-empty, and let $\gamma \in U - P^1(F)$ be one element. A basic fact asserts that $\mathrm{SL}_2(F)$ is generated by $N(F)$ and γ as long as γ is not in $P^1(F)$. It follows that v is invariant under $\mathrm{SL}_2(F)$.

If $v \neq 0$, then by irreducibility σ is generated by v under the action of $\mathrm{GL}_2(F)$. It follows that all elements of σ are invariant under $\mathrm{SL}_2(F)$. Thus the representation σ factors through the determinant map, which implies it must be one-dimensional. Contradiction! □

Proof of Proposition 3.2.3. Applying the lemma above, we obtain an element $b \in F$ such that

$$\sigma(n(b))\alpha(\phi) - \alpha(\phi) \neq 0.$$

The left hand side is equal to $\alpha(\widetilde{\phi})$ with

$$\widetilde{\phi} = r(n(b))\phi - \phi.$$

By definition, we have

$$\widetilde{\phi}(x,u) = (\psi(buq(x)) - 1)\phi(x).$$

Thus such a $\widetilde{\phi}$ has support

$$\mathrm{supp}(\widetilde{\phi}) \subset \mathrm{supp}(\phi) \cap \left(V_{q\neq 0} \times F^\times\right).$$

□

Proof of Proposition 3.2.2. If $W = 0$, then the result is implied by Proposition 3.2.3. Thus we assume that $W \neq 0$. We have an orthogonal decomposition $V = W \oplus W'$, and an identification

$$\mathscr{S}(V \times F^\times) = \mathscr{S}(W \times F^\times) \otimes \mathscr{S}(W' \times F^\times).$$

The action of $\mathrm{GL}_2(F)$ is given by actions on $\mathscr{S}(W \times F^\times)$ and $\mathscr{S}(W' \times F^\times)$ respectively. Choose any $\phi \in \mathscr{S}(V \times F^\times)$ such that $\alpha(\phi) \neq 0$. We may assume that $\phi = f \otimes f'$ is a pure tensor.

Since W' is anisotropic, $\mathscr{S}(W'_{q\neq 0} \times F^\times)$ is a subspace in $\mathscr{S}(W' \times F^\times)$ with quotient $\mathscr{S}(F^\times)$. The quotient map is given by evaluation at $(0,u)$. Thus

$$\mathscr{S}(W' \times F^\times) = \mathscr{S}(W'_{q\neq 0} \times F^\times) + r(w)\mathscr{S}(W'_{q\neq 0} \times F^\times)$$

as w acts as the Fourier transform up to a scale multiple. In this way, we may write

$$f' = f'_1 + r(w)f'_2, \qquad f'_i \in \mathscr{S}(W'_{q\neq 0} \times F^\times).$$

Then we have a decomposition

$$\phi = \phi_1 + r(w)\phi_2, \quad \phi_1 := f \otimes f'_1, \ \phi_2 := r(w^{-1})f \otimes f'_2.$$

One of $\alpha(\phi_i) \neq 0$, and the support of this ϕ_i consists of points (x,u) such that $W(x)$ is non-degenerate. Applying Proposition 3.2.3 to this ϕ_i, we get the function we want. □

3.3 Decomposition of the kernel function

We now compute the derivative of the kernel function $I(s,g,\phi)$ for $\phi \in \mathscr{S}(\mathbb{V} \times \mathbb{A}^\times)$. We will decompose it to a sum of local parts $I(s,g,\phi)(v)$ under the global degeneracy assumption introduced in the last subsection.

It suffices to assume $\phi = \phi_1 \otimes \phi_2$ with $\phi_i \in \mathscr{S}(\mathbb{V}_i \times \mathbb{A}^\times)$ and ϕ_1, ϕ_2 are pure tensors of local Schwartz functions. All definitions and results here can be extended by linearity. In this case,

$$I(s, g, \phi) = \sum_{u \in \mu_K^2 \backslash F^\times} \theta(g, u, \phi_1) E(s, g, u, \phi_2).$$

It amounts to compute the derivative of the Eisenstein series $E(s, g, u, \phi_2)$. We may further assume that both ϕ_i have standard components at infinity. As before we will suppress the dependence on ϕ.

Let us start with the Fourier expansion:

$$E(s, g, u) = E_0(s, g, u) - \sum_{a \in F^\times} W_a(s, g, u),$$

where the constant term

$$E_0(s, g, u) = \delta(g)^s r_2(g)\phi_2(0, u) - W_0(s, g, u).$$

Denote by $F(v)$ the set of $a \in F^\times$ that is represented by $(E(\mathbb{A}^v), uq_2^v)$ but not by $(E_v, uq_{2,v})$. Then $F(v)$ is non-empty only if E is non-split at v. By Proposition 2.5.2, $W_{a,v}(0, g, u) = 0$ for any $a \in F(v)$. Then taking the derivative yields

$$W'_a(0, g, u) = W'_{a,v}(0, g, u) W_a^v(0, g, u).$$

It follows that

$$E'(0, g, u) = E'_0(0, g, u) - \sum_{v \text{ nonsplit}} \sum_{a \in F(v)} W'_{a,v}(0, g, u) W_a^v(0, g, u).$$

Notation. *For any non-split place v, denote the v-part by*

$$E'(0, g, u, \phi_2)(v) := \sum_{a \in F(v)} W'_{a,v}(0, g, u, \phi_2) W_a^v(0, g, u, \phi_2).$$

$$I'(0, g, \phi)(v) := \sum_{u \in \mu_K^2 \backslash F^\times} \theta(g, u, \phi_1) E'(0, g, u, \phi_2)(v).$$

For simplicity, we usually write $I'(0, g)(v)$ for $I'(0, g, \phi)(v)$. By definition, we have a decomposition

$$I'(0, g) = -\sum_{v \text{ nonsplit}} I'(0, g)(v) + \sum_{u \in \mu_K^2 \backslash F^\times} \theta(g, u) E'_0(0, g, u).$$

We first take care of $I'(0, g)(v)$ for any fixed non-split v. Denote by $B = B(v)$ the nearby quaternion algebra, the unique quaternion algebra over F obtained from $\mathbb{B}$ by changing the Hasse invariant at v. Then we have a splitting $B = E + Ej$. Let $V = (B, q)$ be the corresponding quadratic space with the reduced norm q, and $V = V_1 + V_2$ be the corresponding orthogonal decomposition. We identify the quadratic spaces $V_{2,w} = \mathbb{V}_{2,w}$ unless $w = v$. It follows that for $a \in F^\times$, $a \in F(v)$ if and only if a is represented by (V_2, uq).

Proposition 3.3.1. *For non-split v,*

$$I'(0,g,\phi)(v) = 2\fint_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} \mathscr{K}_\phi^{(v)}(g,(t,t))dt,$$

where the integral $\fint$ *employs the Haar measure of total volume one, and*

$$\mathscr{K}_\phi^{(v)}(g,(t_1,t_2)) = \mathscr{K}_{r(t_1,t_2)\phi}^{(v)}(g) = \sum_{u\in\mu_K^2\backslash F^\times}\sum_{y\in V-V_1} k_{r(t_1,t_2)\phi_v}(g,y,u)r(g,(t_1,t_2))\phi^v(y,u).$$

Here for any $y = y_1 + y_2 \in V_v - V_{1v}$,

$$k_{\phi_v}(g,y,u) = \frac{L(1,\eta_v)}{\mathrm{vol}(E_v^1)} r(g)\phi_{1,v}(y_1,u)W'_{uq(y_2),v}(0,g,u).$$

Proof. By Proposition 2.5.2,

$$\begin{aligned}
E'(0,g,u)(v) &= \sum_{y_2\in E^1\backslash(V_2-\{0\})} W'_{uq(y_2),v}(0,g,u)W^v_{uq(y_2)}(0,g,u)\\
&= \frac{1}{L^v(1,\eta)}\sum_{y_2\in E^1\backslash(V_2-\{0\})} W'_{uq(y_2),v}(0,g,u)\int_{E^1(\mathbb{A}^v)} r(g)\phi_2^v(y_2\tau,u)d\tau\\
&= \frac{1}{\mathrm{vol}(E_v^1)L^v(1,\eta)}\sum_{y_2\in E^1\backslash(V_2-\{0\})}\int_{E^1(\mathbb{A})} W'_{uq(y_2\tau),v}(0,g,u)r(g)\phi_2^v(y_2\tau,u)d\tau\\
&= \frac{1}{\mathrm{vol}(E_v^1)L^v(1,\eta)}\int_{E^1\backslash E^1(\mathbb{A})}\sum_{y_2\in V_2-\{0\}} W'_{uq(y_2\tau),v}(0,g,u)r(g)\phi_2^v(y_2\tau,u)d\tau.
\end{aligned}$$

Therefore, we have the following expression for $I'(0,g)(v)$:

$$\begin{aligned}
I'(0,g)(v) =&\frac{1}{\mathrm{vol}(E_v^1)L^v(1,\eta)}\sum_{u\in\mu_K^2\backslash F^\times}\sum_{y_1\in V_1} r(g)\phi_1(y_1,u)\cdot\\
&\cdot\int_{E^1\backslash E^1(\mathbb{A})}\sum_{y_2\in V_2-\{0\}} W'_{uq(y_2\tau),v}(0,g,u)r(g)\phi_2^v(y_2\tau,u)d\tau
\end{aligned}$$

Move two sums inside integral to obtain:

$$\int_{\mathbb{A}^\times E^\times\backslash\mathbb{A}_E^\times}\sum_{u\in\mu_K^2\backslash F^\times}\sum_{\substack{y=y_1+y_2\in V\\ y_2\neq 0}} r(g)\phi_{1,v}(y_1,u)W'_{uq(y_2t^{-1}\bar{t}),v}(0,g,u)r(g)\phi^v(t^{-1}yt,u)dt$$

By definition of k_{ϕ_v} and $\mathscr{K}_\phi^{(v)}$, we have

$$\begin{aligned}
I'(0,g)(v) =&\frac{1}{L^v(1,\eta)}\int_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})}\sum_{u\in\mu_K^2\backslash F^\times}\sum_{x\in V-V_1} k_{\phi_v}(g,t^{-1}yt,u)r(g)\phi^v(t^{-1}yt,u)dt\\
=&\frac{1}{L(1,\eta)}\int_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})}\mathscr{K}_\phi^{(v)}(g,(t,t))dt.
\end{aligned}$$

Since $\mathrm{vol}(Z(\mathbb{A})T(F)\backslash T(\mathbb{A})) = 2L(1,\eta)$, we get the result. Here we have used the relation $k_{\phi_v}(g, t^{-1}yt, u) = k_{r(t,t)\phi_v}(g, y, u)$ in the lemma below. □

Lemma 3.3.2. *The function $k_{\phi_v}(g, y, u)$ behaves like Weil representation under the action of $P(F_v)$ and $E_v^\times \times E_v^\times$. Namely,*

$$\begin{aligned} k_{\phi_v}(m(a)g, y, u) &= |a|^2 k_{\phi_v}(g, ay, u), \quad a \in F_v^\times \\ k_{\phi_v}(n(b)g, y, u) &= \psi(buq(y)) k_{\phi_v}(g, y, u), \quad b \in F_v \\ k_{\phi_v}(d(c)g, y, u) &= |c|^{-1} k_{\phi_v}(g, y, c^{-1}u), \quad c \in F_v^\times \\ k_{r(t_1,t_2)\phi_v}(g, y, u) &= k_{\phi_v}(g, t_1^{-1} y t_2, q(t_1 t_2^{-1})u), \quad (t_1, t_2) \in E_v^\times \times E_v^\times \end{aligned}$$

Proof. These identities follow from the definition of Weil representation and some transformation of integrals. We will only verify the first identity. By definition, we can compute the transformation of $m(a)$ on Whittaker function directly:

$$\begin{aligned} W_{a_0,v}(s, m(a)g, u) &= \int_{F_v} \delta(wn(b)m(a)g)^s r_2(wn(b)m(a)g)\phi_{2,v}(0,u)\psi_v(-a_0 b)db \\ &= \int_{F_v} \delta(m(a^{-1})wn(ba^{-2})g)^s r_2(m(a^{-1})wn(ba^{-2})g)\phi_{2,v}(0,u)\psi_v(-a_0 b)db \\ &= |a|^{-s} \int_{F_v} \delta(wn(ba^{-2})g)^s r_2(wn(ba^{-2})g)\phi_{2,v}(0,u)|a|^{-1}\eta_v(a)\psi_v(-a_0 b)db \\ &= |a|^{-s-1}\eta_v(a) \int_{F_v} \delta(wn(b)g)^s r_2(wn(b)g)\phi_{2,v}(0,u)\psi_v(-a_0 a^2 b)|a|^2 db \\ &= |a|^{-s+1}\eta_v(a) W_{a^2 a_0, v}(s, g, u). \end{aligned}$$

It follows that

$$W'_{a_0,v}(0, m(a)g, u) = |a|\eta_v(a) W'_{a^2 a_0,v}(0, g, u).$$

This implies the result by combining

$$r(m(a)g)\phi_{1,v}(y_1, u) = |a|\eta_v(a)\phi_{1,v}(ay_1, u).$$

□

Now we take care of the contribution from the constant term $E_0'(0, g, u)$. It will simply vanish in some degenerate case.

Proposition 3.3.3. *Under the global degeneracy assumption in §3.2,*

$$I'(0, g, \phi) = -\sum_{v \text{ nonsplit}} I'(0, g, \phi)(v), \quad \forall g \in P(F_{v_1,v_2})\mathrm{GL}_2(\mathbb{A}^{v_1,v_2}).$$

Proof. We want to check that

$$E_0'(0, g, u) = \log \delta(g) r_2(g)\phi_2(0, u) - W_0'(0, g, u)$$

vanishes if $g_{v_\ell} \in P(F_\ell)$ for $\ell = 1, 2$.

Recall that the degeneration condition gives $\phi_{v_\ell}(E_v, F_v^\times) = 0$ for $\ell = 1, 2$. By linearity we can assume that $\phi_{v_\ell} = \phi_{1,v_\ell} \otimes \phi_{2,v_\ell}$ with $\phi_{2,v_\ell}(0, F_v^\times) = 0$. Then it is immediate that $r_2(g)\phi_2(0, u) = 0$ since $r(g_{v_1})\phi_{2,v_1}(0, u) = 0$ by our degeneration assumption.

Take derivative on

$$W_0(s, g, u) = \frac{L(s, \eta)}{L(s+1, \eta)} W_0^\circ(s, g, u) \prod_v |D_v|^{\frac{1}{2}} |d_v|^{\frac{1}{2}} = \frac{L(s, \eta)/L(0, \eta)}{L(s+1, \eta)/L(1, \eta)} \prod_v W_{0,v}^\circ(s, g, u).$$

We obtain

$$W_0'(0, g, u) = \frac{d}{ds}|_{s=0} \left(\log \frac{L(s, \eta)}{L(s+1, \eta)} \right) W_0^\circ(0, g, u) + \sum_v W_{0,v}^\circ{}'(0, g, u) \prod_{v' \neq v} W_{0,v'}^\circ(0, g, u).$$

By Proposition 2.5.2, we get

$$W_0'(0, g, u) = \frac{d}{ds}|_{s=0} \left(\log \frac{L(s, \eta)}{L(s+1, \eta)} \right) r(g)\phi_2(0, u) + \sum_v W_{0,v}^\circ{}'(0, g, u) r(g^v)\phi_2^v(0, u).$$

Then $r(g)\phi_2(0, u) = 0$ as above, and $r(g^v)\phi_2^v(0, u) = 0$ for any v since it has $r(g_{v_\ell})\phi_{2,v_\ell}(0, u)$ as a factor for at least one ℓ.

□

3.4 Non-archimedean components

Assume that v is a non-archimedean place non-split in E. Resume the notations in the last section. We now consider the local kernel function $k_{\phi_v}(g, y, u)$, which has the expression

$$k_{\phi_v}(g, y, u) = \frac{L(1, \eta_v)}{\mathrm{vol}(E_v^1)} r(g)\phi_{1,v}(y_1, u) W'_{uq(y_2),v}(0, g, u, \phi_{2,v}), \quad y = y_1 + y_2 \in V_v - V_{1v}$$

if $\phi_v = \phi_{1,v} \otimes \phi_{2,v}$.

Main results

Let v be a non-archimedean place non-split in E, and B_v be the quaternion division algebra over F_v non-isomorphic to $\mathbb{B}_v$.

Proposition 3.4.1. *The following results under different assumptions are true:*

(1) *Assume that v is unramified in E, unramified in $\mathbb{B}$, and unramified over $\mathbb{Q}$. Assume further that ϕ_v is the characteristic function of $O_{\mathbb{B}_v} \times O_{F_v}^\times$. Then*

$$k_{\phi_v}(1, y, u) = 1_{O_{B_v}}(y) 1_{O_{F_v}^\times}(u) \frac{v(q(y_2)) + 1}{2} \log N_v.$$

(2) *Assume that* $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$. *Then* $k_{\phi_v}(1, y, u)$ *extends to a Schwartz function of* $(y, u) \in B_v \times F_v^\times$.

We consider its consequences. We first look at (1). In that unramified case, it is easy to see that

$$k_{\phi_v}(1, y, u) = k_{\phi_v}(g, y, u), \quad g \in \mathrm{GL}_2(O_{F_v}).$$

Then by Iwasawa decomposition and Lemma 3.3.2, we know $k_{r(t_1,t_2)\phi_v}(g, y, u)$ explicitly for all $(g, (t_1, t_2))$. It will cancel the local height of CM points at v.

Now we consider a place v that does not satisfy the conditions in (1). Then the computation of k_{ϕ_v} may be very complicated or useless. It is better to consider the whole series

$$\mathscr{K}_\phi^{(v)}(g, (t_1, t_2)) = \sum_{u \in \mu_K^2 \backslash F^\times} \sum_{y \in V - V_1} k_{r(t_1,t_2)\phi_v}(g, y, u)\ r(g, (t_1, t_2))\phi^v(y, u).$$

It looks like a theta series. We call it a *pseudo-theta series*. It has a strong connection with the usual theta series.

In the proposition, we have shown that $k_{\phi_v}(y, u) = k_{\phi_v}(1, y, u)$ extends to a Schwartz function for $(y, u) \in V_v \times F_v^\times$ if ϕ_v is degenerate. We did this because we want to compare the above pseudo-theta series with the usual theta series

$$\theta(g, (t_1, t_2), k_{\phi_v} \otimes \phi^v) = \sum_{u \in \mu_K^2 \backslash F^\times} \sum_{y \in V} r(g, (t_1, t_2))k_{\phi_v}(y, u)\ r(g, (t_1, t_2))\phi^v(y, u).$$

It seems that these two series have a good chance to equal if $g_v = 1$. In fact, it is supported by the equality

$$r(t_1, t_2)k_{\phi_v}(y, u) = k_{r(t_1,t_2)\phi_v}(1, y, u)$$

shown in Lemma 3.3.2.

Another difficulty for them to be equal is that the summations of y are over different spaces. This problem is solved by the global degeneracy assumption in §3.2. In fact, if ϕ is degenerate at v_1, v_2. Assume that $v_1 \neq v$ and keep in mind that $g_{v_1} = 1$. Then $r(t_1, t_2)\phi_{v_1}(y, u) = 0$ for all $y \in E_v$. In particular, it forces $r(g, (t_1, t_2))\phi^v(y, u) = 0$ for $y \in V_1$. Therefore, the two series are equal if $g_{v,v_1,v_2} = 1$. We can also extend the equality to $P(F_{v,v_1,v_2})\mathrm{GL}_2(\mathbb{A}^{v,v_1,v_2})$ by Lemma 3.3.2.

Corollary 3.4.2. *Let* v *be a non-archimedean place non-split in* E. *Assume that* ϕ *is degenerate at* v. *Assume the global degeneracy assumption. Then*

$$\mathscr{K}_\phi^{(v)}(g, (t_1, t_2)) = \theta(g, (t_1, t_2), k_{\phi_v} \otimes \phi^v)$$

for all

$$(g, (t_1, t_2)) \in P(F_{v,v_1,v_2})\mathrm{GL}_2(\mathbb{A}^{v,v_1,v_2}) \times T(\mathbb{A}) \times T(\mathbb{A}).$$

In that situation, we say $\mathscr{K}_\phi^{(v)}$ is *approximated* by $\theta(k_{\phi_v} \otimes \phi^v)$. They are usually not equal, unless we know the modularity of the pseudo-theta series.

The computation

To prove Proposition 3.4.1, we first show a formula for the Whittaker function $W_{a,v}(s,1,u)$ in the most general case.

Proposition 3.4.3. *Let v be any non-archimedean place of F.*

(1) For any $a \in F_v$,

$$W_{a,v}(s,1,u) \quad = \quad |d_v|^{\frac{1}{2}}(1-N_v^{-s})\sum_{n=0}^{\infty} N_v^{-ns+n}\int_{D_n(a)} \phi_{2,v}(x_2,u)d_u x_2,$$

where $d_u x_2$ is the self-dual measure of $(\mathbb{V}_{2,v}, uq)$ and

$$D_n(a) = \{x_2 \in \mathbb{V}_{2,v} : uq(x_2) \in a + p_v^n d_v^{-1}\}.$$

(2) Assume that $\phi_{2,v}(x_2,u) = 0$ if $v(uq(x_2)) > -v(d_v)$. Then there is a constant $c > 0$ such that $W_{a,v}(s,1,u) = 0$ identically for all $a \in F_v$ satisfying $v(a) > c$ or $v(a) < -c$.

Proof. We first compute (1). Recall that

$$W_{a,v}(s,1,u) = \int_{F_v} \delta(wn(b))^s \int_{\mathbb{V}_{2,v}} \phi_{2,v}(x_2,u)\psi_v(b(uq(x_2)-a))d_u x_2 db.$$

It suffices to verify the formulae for Whittaker functions under the condition that $u = 1$. The general case is obtained by replacing q by uq and $\phi_{2,v}(x_2)$ by $\phi_{2,v}(x_2,u)$. We will drop the dependence on u to simplify the notation. Then we write

$$W_{a,v}(s,1) = \int_{F_v} \delta(wn(b))^s \int_{\mathbb{V}_{2,v}} \phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2 db.$$

By

$$\delta(wn(b)) = \begin{cases} 1 & \text{if } b \in O_{F_v}, \\ |b|^{-1} & \text{otherwise,} \end{cases}$$

we will split the integral over F_v into the sum of an integral over O_{F_v} and an integral over $F_v - O_{F_v}$. Then

$$\begin{aligned} W_{a,v}(s,1) = & \int_{O_{F_v}} \int_{\mathbb{V}_{2,v}} \phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2 db \\ & + \int_{F_v - O_{F_v}} |b|^{-s} \int_{\mathbb{V}_{2,v}} \phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2 db \end{aligned}$$

The second integral can be decomposed as

$$\begin{aligned}
&\sum_{n=1}^{\infty}\int_{p_v^{-n}-p_v^{-n+1}} N_v^{-ns}\int_{\mathbb{V}_{2,v}}\phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db\\
=&\sum_{n=1}^{\infty}\int_{p_v^{-n}} N_v^{-ns}\int_{\mathbb{V}_{2,v}}\phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db\\
&-\sum_{n=1}^{\infty}\int_{p_v^{-(n-1)}} N_v^{-ns}\int_{\mathbb{V}_{2,v}}\phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db.
\end{aligned}$$

Combine with the first integral to obtain

$$\begin{aligned}
W_{a,v}(s,1)=&\sum_{n=0}^{\infty}\int_{p_v^{-n}} N_v^{-ns}\int_{\mathbb{V}_{2,v}}\phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db\\
&-\sum_{n=0}^{\infty}\int_{p_v^{-n}} N_v^{-(n+1)s}\int_{\mathbb{V}_{2,v}}\phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db\\
=&(1-N_v^{-s})\sum_{n=0}^{\infty}N_v^{-ns}\int_{p_v^{-n}}\int_{\mathbb{V}_{2,v}}\phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db.
\end{aligned}$$

As for the last double integral, change the order of the integration. The integral on b is nonzero if and only if $q(x_2)-a\in p_v^n d_v^{-1}$. Here d_v is the local different of F over $\mathbb{Q}$, and also the conductor of ψ_v. Then we have

$$\begin{aligned}
W_{a,v}(s,1) &= (1-N_v^{-s})\sum_{n=0}^{\infty}N_v^{-ns}\mathrm{vol}(p_v^{-n})\int_{D_n(a)}\phi_{2,v}(x_2)dx_2\\
&= |d_v|^{\frac{1}{2}}(1-N_v^{-s})\sum_{n=0}^{\infty}N_v^{-ns+n}\int_{D_n(a)}\phi_{2,v}(x_2)dx_2.
\end{aligned}$$

It proves (1).

Now we show (2) using (1). The key is that only those $D_n(a)$ with $n\geq 0$ are involved in the formula. Recall that

$$D_n(a)=\{x_2\in\mathbb{V}_{2,v}: uq(x_2)\in a+p_v^n d_v^{-1}\}.$$

If $v(a)<-v(d_v)$, then for every $x_2\in D_n(a)$, we have $v(uq(x_2))=v(a)$. Then $\phi_{2,v}(x_2,u)=0$ if $v(a)$ is too small. It follows that $W_{a,v}(s,1,u)=0$ if $v(a)$ is too small. This is apparently true for all Schwartz function ϕ_v.

If $v(a)\geq -v(d_v)$, then for every $x_2\in D_n(a)$, we have $v(uq(x_2))\geq -v(d_v)$. By the assumption, $\phi_{2,v}(\cdot,u)$ is zero on $D_n(a)$. In that case, $W_{a,v}(s,1,u)=0$ identically. It proves the result.

□

Proof of Proposition 3.4.1. Both results are obtained as consequence of Proposition 3.4.3. We first look at (2). It suffices to consider the case that $\phi_v = \phi_{1,v} \otimes \phi_{2,v}$ with $\phi_{2,v}$ satisfies the condition of Proposition 3.4.3 (2). In deed, any degenerate ϕ_v is a finite linear combination of such $\phi_{1,v} \otimes \phi_{2,v}$. Set $k_{\phi_v}(1, y, u)$ to be zero if $y_2 = 0$. It is easy to see that it gives a Schwartz function by Proposition 3.4.3 (2).

Now we consider (1). It suffices to show that for any $a \in F(v)$,

$$W'_{a,v}(0, 1, u) = 1_{O_{F_v}}(a) 1_{O_{F_v}^\times}(u) \frac{v(a) + 1}{2} (1 + N_v^{-1}) \log N_v.$$

Use the formula in Proposition 3.4.3. We need to simplify

$$D_n(a) = \{x_2 \in \mathbb{V}_{2,v} : uq(x_2) - a \in p_v^n\}.$$

We first have $v(a) \neq v(q(x_2))$ because a is not represented by $uq(x_2)$. Actually $v(q(x_2))$ is always even and $v(a)$ must be odd. Then

$$v(q(x_2) - a) = \min\{v(a), v(q(x_2))\}, \quad \forall\ x_2 \in \mathbb{V}_{2,v}.$$

We see that $D_n(a)$ is empty if $v(a) < n$. Otherwise, it is equal to

$$D_n := \{x_2 \in \mathbb{V}_{2,v} : uq(x_2) \in p_v^n\}.$$

It follows that

$$W_{a,v}(s, 1, u) \quad = \quad (1 - N_v^{-s}) \sum_{n=0}^{v(a)} N_v^{-ns+n} \int_{D_n} \phi_{2,v}(x_2, u) d_u x_2.$$

It is a finite sum and we don't have any convergence problem. Then

$$W'_{a,v}(0, 1, u) \quad = \quad \log N_v \sum_{n=0}^{v(a)} N_v^n \int_{D_n} \phi_{2,v}(x_2, u) d_u x_2.$$

It is nonzero only if $u \in O_{F_v}^\times$ and $a \in O_{F_v}$. Identify $\mathbb{V}_{2,v}$ with E_v. Then

$$D_n = \{x_2 \in E_v : q(x_2) \in p_v^n\} = p_v^{[\frac{n+1}{2}]} O_{E_v}.$$

And $\mathrm{vol}(D_n) = N_v^{-2[\frac{n+1}{2}]}$. Note that $v(a)$ is odd since it is not represented by q_2. Then it is easy to have

$$W'_{a,v}(0, 1, u) = \log N_v \sum_{n=0}^{v(a)} N_v^{n-2[\frac{n+1}{2}]} = \frac{v(a) + 1}{2} (1 + N_v^{-1}).$$

□

3.5 Archimedean places

For an archimedean place v, the quaternion algebra $\mathbb{B}_v$ is isomorphic to the Hamiltonian quaternion. We will compute $k_{\phi_v}(g,y,u)$ for standard ϕ_v introduced in §3.1. The computation here is done by [KRY1].

The result involves the exponential integral Ei defined by

$$\mathrm{Ei}(z) = \int_{-\infty}^{z} \frac{e^t}{t} dt, \quad z \in \mathbb{C}.$$

Another expression is

$$\mathrm{Ei}(z) = \gamma + \log(-z) + \int_0^z \frac{e^t - 1}{t} dt,$$

where γ is the Euler constant. It follows that it has a logarithmic singularity near 0. This fact is useful when we compare the result here with the archimedean local height, since we know that Green's functions have a logarithmic singularity.

Proposition 3.5.1.

$$k_{\phi_v}(g,y,u) = \begin{cases} -\dfrac{1}{2}\mathrm{Ei}(4\pi u q(y_2) y_0)\ |y_0| e^{2\pi i u q(y)(x_0 + i y_0)} e^{2i\theta} & \text{if } uy_0 > 0 \\ 0 & \text{if } uy_0 < 0 \end{cases}$$

for any

$$g = \begin{pmatrix} z_0 & \\ & z_0 \end{pmatrix} \begin{pmatrix} y_0 & x_0 \\ & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{GL}_2(F_v)$$

in the form of the Iwasawa decomposition.

Proof. It suffices to show the formula in the case $g = 1$. The general case is obtained by Proposition 3.3.2 and the fact that $r(k_\theta)\phi_v = e^{2i\theta}\phi_v$.

Now we show that

$$k_{\phi_v}(1,y,u) = \begin{cases} -\frac{1}{2}\mathrm{Ei}(4\pi u q(y_2)) e^{-2\pi u q(y)} & \text{if } u > 0; \\ 0 & \text{if } u < 0. \end{cases}$$

It amounts to show that, for any $a \in F(v)$,

$$W'_{a,v}(0,1,u) = \begin{cases} -\pi e^{-2\pi a}\mathrm{Ei}(4\pi a) & \text{if } u > 0; \\ 0 & \text{if } u < 0. \end{cases}$$

Assume $u > 0$ since the case $u < 0$ is trivial. Then $a < 0$ by the condition $a \in F(v)$. We compute explicitly as follows. First of all, it is easy to check $\delta(wn(b)) = 1/(1+b^2)$. It follows that

$$\begin{aligned} W_{a,v}(s,1,u) &= \int_{F_v} \delta(wn(b))^s \int_{V_{2v}} \phi_{2,v}(x_2,u)\psi_v(b(uq(x_2)-a)) d_u x_2 db \\ &= \int_{\mathbb{R}} \left(\frac{1}{1+b^2}\right)^{\frac{s}{2}} \int_{\mathbb{C}} e^{-2\pi u q(x_2) + 2\pi i b(uq(x_2)-a)} d_u x_2 db \end{aligned}$$

Take an isometry of quadratic space $(V_{2v}, uq) \simeq (\mathbb{C}, |\cdot|^2)$ to obtain

$$\begin{aligned}&\int_{\mathbb{R}} \left(\frac{1}{1+b^2}\right)^{\frac{s}{2}} \int_{\mathbb{C}} e^{-2\pi|x_2|^2} e^{2\pi ib(|x_2|^2-a)} d_u x_2 db \\ =&\int_{\mathbb{R}} \left(\frac{1}{1+b^2}\right)^{\frac{s}{2}} \int_{\mathbb{C}} e^{-2\pi(1-ib)|x_2|^2} e^{-2\pi iab} d_u x_2 db = \int_{\mathbb{R}} \left(\frac{1}{1+b^2}\right)^{\frac{s}{2}} \frac{1}{1-ib} e^{-2\pi iab} db \\ =&\int_{\mathbb{R}} (1+ib)^{-\frac{s}{2}} (1-ib)^{-\frac{s}{2}-1} e^{-2\pi iab} db.\end{aligned}$$

By the computation in [KRY1], page 19,

$$\frac{d}{ds}|_{s=0} \int_{\mathbb{R}} (1+ib)^{-\frac{s}{2}} (1-ib)^{-\frac{s}{2}-1} e^{-2\pi iab} db = -\pi e^{-2\pi a} Ei(4\pi a).$$

Thus

$$W'_{a,v}(0,1,u) = -\pi e^{-2\pi a} \mathrm{Ei}(4\pi a).$$

□

3.6 Holomorphic projection

In this subsection we consider the general theory of holomorphic projection which we will apply to the form $I'(0,g,\chi)$ in the next subsection. Denote by $\mathscr{A}(\mathrm{GL}_2(\mathbb{A}),\omega)$ the space of automorphic forms of central character ω, by $\mathscr{A}_0(\mathrm{GL}_2(\mathbb{A}),\omega)$ the subspace of cusp forms, and by $\mathscr{A}_0^{(2)}(\mathrm{GL}_2(\mathbb{A}),\omega)$ the subspace of holomorphic cusp forms of parallel weight two.

The usual Petersson inner product is just

$$(f_1, f_2)_{\mathrm{pet}} = \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} f_1(g)\overline{f_2(g)} dg, \quad f_1, f_2 \in \mathscr{A}(\mathrm{GL}_2(\mathbb{A}),\omega).$$

Denote by $\mathscr{P}r : \mathscr{A}(\mathrm{GL}_2(\mathbb{A}),\omega) \to \mathscr{A}_0^{(2)}(\mathrm{GL}_2(\mathbb{A}),\omega)$ the orthogonal projection. Namely, for any $f \in \mathscr{A}(\mathrm{GL}_2(\mathbb{A}),\omega)$, the image $\mathscr{P}r(f)$ is the unique form in $\mathscr{A}_0^{(2)}(\mathrm{GL}_2(\mathbb{A}),\omega)$ such that

$$(\mathscr{P}r(f), \varphi)_{\mathrm{pet}} = (f, \varphi)_{\mathrm{pet}}, \quad \forall \varphi \in \mathscr{A}_0^{(2)}(\mathrm{GL}_2(\mathbb{A}),\omega).$$

We simply call $\mathscr{P}r(f)$ the holomorphic projection of f. Apparently $\mathscr{P}r(f) = 0$ if f is an Eisenstein series.

A General Formula

For any automorphic form f for $\mathrm{GL}_2(\mathbb{A})$ we define a Whittaker function

$$f_{\psi,s}(g) = (4\pi)^{\deg F} W^{(2)}(g_\infty) \int_{Z(F_\infty)N(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \delta(g)^s f_\psi(g_f h) \overline{W^{(2)}(h)} dh.$$

Here $W^{(2)}$ is the standard holomorphic Whittaker function of weight two at infinity, and f_ψ denotes the Whittaker function of f. As $s \to 0$, the limit of the integral is holomorphic at $s = 0$.

Proposition 3.6.1. *Let $f \in \mathscr{A}(\mathrm{GL}_2(\mathbb{A}), \omega)$ be a form with asymptotic behavior*

$$f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) = O_g(|a|^{1-\epsilon})$$

as $a \in \mathbb{A}^\times$, $|a| \to \infty$ for some $\epsilon > 0$. Then the holomorphic projection $\mathscr{P}r(f)$ has Whittaker function

$$\mathscr{P}r(f)_\psi(g_\infty g_f) = \lim_{s\to 0} f_{\psi,s}(g).$$

Proof. For any Whittaker function W of $\mathrm{GL}_2(\mathbb{A})$ with decomposition $W(g) = W^{(2)}(g_\infty)W_f(g_f)$ such that $W^{(2)}(g_\infty)$ is standard holomorphic of weight 2 and that $W_f(g_f)$ is compactly supported modulo $Z(\mathbb{A}_f)N(\mathbb{A}_f)$, the Poincaré series is defined as

$$\varphi_W(g) := \lim_{s\to 0+} \sum_{\gamma\in Z(F)N(F)\backslash G(F)} W(\gamma g)\delta(\gamma g)^s,$$

where

$$\delta(g) = |a_\infty/d_\infty|^{\frac{1}{2}}, \qquad g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k, \qquad k \in U$$

where U is the standard maximal compact subgroup of $\mathrm{GL}_2(\mathbb{A})$. Assume that W and f have the same central character. Since f has asymptotic behavior as in the proposition, their inner product can be computed as follows:

$$\begin{aligned}(f, \varphi_W)_{\mathrm{pet}} &= \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} f(g)\overline{\varphi_W(g)}dg \\ &= \lim_{s\to 0} \int_{Z(\mathbb{A})N(F)\backslash \mathrm{GL}_2(\mathbb{A})} f(g)\overline{W(g)}\delta(g)^s dg \\ &= \lim_{s\to 0} \int_{Z(\mathbb{A})N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} f_\psi(g)\overline{W(g)}\delta(g)^s dg. \end{aligned} \tag{3.6.1}$$

We may apply this formula to $\mathscr{P}r(f)$ which has the same inner product with φ_W as f. Write

$$\mathscr{P}r(f)_\psi(g) = W^{(2)}(g_\infty)\mathscr{P}r(f)_\psi(g_f).$$

Then the above integral is a product of integrals over finite places and integrals at infinite places:

$$\int_{Z(\mathbb{R})N(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R})} |W^{(2)}(g)|^2 dg = \int_0^\infty y^2 e^{-4\pi y} dy/y^2 = (4\pi)^{-1}.$$

In other words, we have

$$(f, \varphi_W)_{\mathrm{pet}} = (4\pi)^{-\deg F} \int_{Z(\mathbb{A}_f)N(\mathbb{A}_f)\backslash \mathrm{GL}_2(\mathbb{A}_f)} \mathscr{P}r(f)_\psi(g_f)\overline{W(g_f)}dg_f. \tag{3.6.2}$$

As $\overline{W}$ can be any Whittaker function with compact support modulo $Z(\mathbb{A}_f)N(\mathbb{A}_f)$, the combinations of the above formulae give the proposition. □

We introduce an operator $\mathscr{P}r'$ formally defined on the function space of $N(F)\backslash \mathrm{GL}_2(\mathbb{A})$. For any function $f : N(F)\backslash \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$, denote as above

$$f_{\psi,s}(g) = (4\pi)^{\deg F} W^{(2)}(g_\infty) \int_{Z(F_\infty)N(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \delta(g)^s f_\psi(g_f h) \overline{W^{(2)}(h)} dh$$

if it has meromorphic continuation around $s = 0$. Here f_ψ denotes the first Fourier coefficient of f. Denote

$$\mathscr{P}r'(f)_\psi(g_f g_\infty) = \widetilde{\lim}_{s\to 0} f_{\psi,s}(g),$$

where the "quasi-limit" $\widetilde{\lim}_{s\to 0}$ denotes the constant term of the Laurent expansion at $s = 0$. Finally, we write

$$\mathscr{P}r'(f)(g) = \sum_{a\in F^\times} \mathscr{P}r'(f)_\psi(d^*(a)g).$$

The the above result is just $\mathscr{P}r(f) = \mathscr{P}r'(f)$ under the growth condition. In general, $\mathscr{P}r'(f)$ is not automorphic when f is automorphic but fails the growth condition of Proposition 3.6.1.

Growth of the kernel function

Now we want to consider the growth of $I'(0, g, \chi)$ so that we can apply the formula. Recall that

$$I(s, g, \chi) = \int_{[T]} I(s, g, r(t, 1)\phi)\chi(t)dt = \mathrm{vol}(K_Z) \int_{T(F)\backslash T(\mathbb{A})/Z(F_\infty)K_Z} \chi(t)\widetilde{I}(s, g, r(t, 1)\phi)dt$$

It has central character $\chi|_{\mathbb{A}_F^\times} = \omega_\pi^{-1}$. It is equal to a finite linear combination of some $I(s, g, r(t, 1)\phi)$. The growth does not satisfy the condition of Proposition 3.6.1, but can be canceled by several Eisenstein series. We will write down those Eisenstein series explicitly.

Recall that

$$I(s, g, \phi) = \sum_{u\in \mu_K^2\backslash F^\times} \theta(g, u, \phi_1)E(s, g, u, \phi_2).$$

In their explicit Fourier expansions, the non-constant terms of $\theta(g, u, \phi_1)$ and $E(s, g, u, \phi_2)$ decay exponentially. So the growth of $I(s, g, \phi)$ and its derivative is determined by the "absolute constant term"

$$I_{00}(s, g, \phi) = \sum_{u\in \mu_U^2\backslash F^\times} I_{00}(s, g, u, \phi)$$

where

$$I_{00}(s, g, u, \phi) = \theta_0(g, u, \phi_1)E_0(s, g, u, \phi_2)$$

is the product of the constant terms of the theta series and the Eisenstein series. Note that we use the notation I_{00} since it is only a part of the constant term I_0 of I.

Let $\mathscr{J}(s,g,u,\phi)$ be the Eisenstein series formed by $I_{00}(s,g,u,\phi)$:

$$\mathscr{J}(s,g,u,\phi)=\sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} I_{00}(s,\gamma g,u,\phi).$$

Then we denote

$$\mathscr{J}(s,g,\phi)=\sum_{u\in\mu_U^2\backslash F^\times}\mathscr{J}(s,g,u,\phi),$$

$$\mathscr{J}(s,g,\chi,\phi)=\int_{[T]}\mathscr{J}(s,g,r(t,1)\phi)\chi(t)dt.$$

We will see that $\mathscr{J}'(0,g,\chi)$ cancels the growth of $I'(0,g,\chi)$. But let us first look at its structure.

By definition

$$I_{00}(s,g,u,\phi)=\delta(g)^s r(g)\phi(0,u)-\delta(g)^s r_1(g)\phi_1(0,u)W_0(s,g,u,\phi_2).$$

Recall that the computation in Proposition 3.3.3 gives

$$W_0'(0,g,u)=c_0 r(g)\phi_2(0,u)+W_0^{\circ}{}'(0,g,u)$$

with the constant

$$c_0=\frac{d}{ds}|_{s=0}\left(\log\frac{L(s,\eta)}{L(s+1,\eta)}\right).$$

Thus the absolute constant term $I_{00}'(0,g,u,\phi)$ has three parts:

$$I_{00}'(0,g,u,\phi)=\log\delta(g)r(g)\phi(0,u)-c_0 r(g)\phi(0,u)-r_1(g)\phi_1(0,u)W_0^{\circ}{}'(0,g,u).$$

Hence we introduce

$$J(s,g,u,\phi)=\sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)}\delta(\gamma g)^s r(\gamma g)\phi(0,u),$$

$$\widetilde{J}(s,g,u,\phi)=\sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} r_1(\gamma g)\phi_1(0,u)W_0^{\circ}(s,\gamma g,u,\phi_2).$$

By the above manners of summation and integration for $\mathscr{J}$, we introduce the series

$$J(s,g,\phi),\ \ J(s,g,\chi),\ \ \widetilde{J}(s,g,\phi),\ \ \widetilde{J}(s,g,\chi).$$

Finally, we have

$$\mathscr{J}'(0,g,\chi)=J'(0,g,\chi)-\widetilde{J}'(0,g,\chi)-c_0 J(0,g,\chi).$$

We remark that the behavior of $\widetilde{J}$ is not that different from J. For example, it is easy to have $\widetilde{J}(0,g,u)=J(0,g,u)$ by Proposition 2.5.2. The proposition also implies that the sections defining $\widetilde{J}(s,g,u)$ and $J(-s,g,u)$ are equal at almost all places. Thus their Whittaker functions match at almost all places. It follows that $\widetilde{J}'(0,g,u)$ is very close to $-J'(0,g,u)$. This property will be used later.

Proposition 3.6.2. *(1) The growth of*

$$I'\left(0, \begin{pmatrix} a & \\ & 1 \end{pmatrix} g, \chi\right) - \mathscr{J}'\left(0, \begin{pmatrix} a & \\ & 1 \end{pmatrix} g, \chi\right)$$

has asymptotic behavior $O(|a|^{1-\epsilon})$*.*

(2) If χ *does not factor through the norm map* $q : \mathbb{A}_E^\times \to \mathbb{A}^\times$*, then* $\mathscr{J}(s, g, \chi) = 0$ *identically.*

Otherwise, χ *can be induced from two different finite characters* μ_i *on* $F^\times \backslash \mathbb{A}^\times$*, i.e.,*

$$\chi = \mu_1 \circ q = \mu_2 \circ q.$$

Then the Eisenstein series $J(s, g, \chi)$ *is equal to the sum of two Einstein series associated to principal series with characters*

$$\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \mapsto \left|\frac{a}{b}\right|^{1+\frac{s}{2}} \mu_i(ab), \quad i = 1, 2.$$

The result is true for $\widetilde{J}(s, g, \chi)$ *if we replace the above exponent* $1 + \dfrac{s}{2}$ *by* $1 - \dfrac{s}{2}$*.*

Proof. We first look at the characters of $J(s, g, \chi)$. Equivalently, we consider

$$f(s, g, \chi) = \int_{[T]} \chi(t) \sum_{u \in \mu_U^2 \backslash F^\times} \delta(g)^s r(g) \phi(0, uq(t)) dt.$$

It apparently vanishes if χ is non-trivial on $T^1(\mathbb{A}) = \{t \in T(\mathbb{A}) : q(t) = 1\}$.

Assume that χ is trivial on $T^1(\mathbb{A})$, so the integral is invariant on $T^1(\mathbb{A})$. It is essentially an integration on

$$T(F) \backslash T(\mathbb{A}) / Z(F_\infty) K_Z T^1(\mathbb{A}) \xrightarrow{q} q(E^\times) \backslash q(\mathbb{A}_E^\times) / F_{\infty,+}^\times K_Z^2,$$

where the norm map above is an isomorphism. Thus we have

$$f(s, g, \chi) = \mathrm{vol}(K_Z) \int_{q(E^\times) \backslash q(\mathbb{A}_E^\times) / F_{\infty,+}^\times K_Z^2} \mu_i(\alpha) \sum_{u \in \mu_U^2 \backslash F^\times} \delta(g)^s r(g) \phi(0, u\alpha) d\alpha.$$

Note that the image of $F^\times q(\mathbb{A}_E^\times)$ has index two in $\mathbb{A}^\times$ by class field theory, so there are exactly two μ_1, μ_2 extending χ. Furthermore,

$$\mu_i|_{F^\times q(\mathbb{A}_E^\times)} = \frac{1}{2}(\mu_1 + \mu_2).$$

Thus we can write

$$f(s, g, \chi) = \frac{1}{2}(f_1(s, g, \chi) + f_2(s, g, \chi))$$

where

$$f_i(s,g,\chi) = \int_{F_{\infty,+}^\times\backslash\mathbb{A}^\times} \mu_i(\alpha)\delta(g)^s r(g)\phi(0,\alpha)d\alpha.$$

Note that the measure on $F_{\infty,+}^\times\backslash\mathbb{A}^\times$ to make the above equality is different from the natural one. It is clear that

$$f_i\left(s, \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} g, \chi\right) = \left|\frac{a}{b}\right|^{1+\frac{s}{2}} \mu_i(ab) f_i(s,g,\chi).$$

They define Einsenstein series $J(s,g,\mu_i)$ and give the decomposition

$$J(s,g,\chi) = \frac{1}{2}(J(s,g,\mu_1) + J(s,g,\mu_2)).$$

The vanishing of $\widetilde{J}(s,g,\chi)$ when χ does not factor through the norm map follows the same reason by noting that $W_0^\circ(s,g,u,\phi_2)$ actually transfers similarly under the action of T. Otherwise, by the character μ_1,μ_2, we can similarly decompose

$$\widetilde{J}(s,g,\chi) = \frac{1}{2}(\widetilde{J}(s,g,\mu_1) + \widetilde{J}(s,g,\mu_2))$$

where $\widetilde{J}(s,g,\mu_i)$ is the Eisenstein series associated to some principal series with the character

$$\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \longmapsto \left|\frac{a}{b}\right|^{1-\frac{s}{2}} \mu_i(ab).$$

Now we look at the growth of $I'(0,g,\chi) - \mathscr{J}'(0,g,\chi)$. Since the non-constant parts always have exponential decay, we only need to consider the growth of $I'_{00}(0,g,\chi) - \mathscr{J}'_0(0,g,\chi)$. By definition, this difference is equal to the part coming form the intertwining operators in those four Eisenstein series $J(s,g,\mu_i), \widetilde{J}(s,g,\mu_i), i = 1,2$.

We look that the intertwining part $M(s)f_i$ in $J(s,g,\mu_i)$ for example. We have

$$(M(s)f_i)\left(s, \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} g, \chi\right) = \left|\frac{a}{b}\right|^{-\frac{s}{2}} \mu_i^{-1}(ab)\widetilde{f}_i(s,g,\chi).$$

The derivative

$$(M(s)f_i)'\left(0, \begin{pmatrix} a & * \\ 0 & b \end{pmatrix} g, \chi\right) = -\frac{1}{2}\log\left|\frac{a}{b}\right| \mu_i^{-1}(ab)\widetilde{f}_i(0,g,\chi) + \mu_i^{-1}(ab)\widetilde{f}'_i(0,g,\chi) = O(\log\left|\frac{a}{b}\right|).$$

Similarly, we can take care of $\widetilde{J}(s,g,\mu_i)$.

□

3.7 Holomorphic kernel function

By Proposition 3.6.2, we can apply Proposition 3.6.1 to the form $I'(0,g,\chi) - \mathcal{J}'(0,g,\chi)$. The result gives

$$\mathscr{P}r(I'(0,g,\chi)) = \mathscr{P}r'(I'(0,g,\chi) - \mathcal{J}'(0,g,\chi)) = \mathscr{P}r'(I'(0,g,\chi)) - \mathscr{P}r'(\mathcal{J}'(0,g,\chi))$$

where the operator $\mathscr{P}r'$ is defined after Proposition 3.6.1. It is reduced to compute $\mathscr{P}r'(I'(0,g,\chi))$ and $\mathscr{P}r'(\mathcal{J}'(0,g,\chi))$. Of course, it suffices to compute $\mathscr{P}r'(I'(0,g))$ and $\mathscr{P}r'(\mathcal{J}'(0,g))$. They are the goal below.

The main part

We now apply the holomorphic projection formula to $I'(0,g)$. Recall that in Proposition 3.3.3, we have the simple decomposition

$$I'(0,g) = -\sum_{v \text{ nonsplit}} I'(0,g)(v)$$

under the global degeneracy assumption in §3.2.

Proposition 3.7.1. *Assume the global degeneracy assumption. Then*

$$\mathscr{P}r'(I'(0,g,\phi)) = -\sum_{v|\infty} \overline{I}'(0,g,\phi)(v) - \sum_{v\nmid\infty \text{ nonsplit}} I'(0,g,\phi)(v), \qquad \forall g \in P(F_{v_1,v_2})\mathrm{GL}_2(\mathbb{A}^{v_1,v_2}).$$

Here $I'(0,g,\phi)(v)$ is the same as in Proposition 3.3.1, and for any archimedean v,

$$\overline{I}'(0,g,\phi)(v) = 2\fint_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} \overline{\mathscr{K}}_{\phi}^{(v)}(g,(t,t))dt,$$

$$\overline{\mathscr{K}}_{\phi}^{(v)}(g,(t_1,t_2)) = \sum_{a\in F^\times} \widetilde{\lim}_{s\to 0} \sum_{y\in\mu_K\backslash(B(v)_+^\times - E^\times)} r(g,(t_1,t_2))\phi(y)_a\ k_{v,s}(y),$$

$$k_{v,s}(y) = \frac{\Gamma(s+1)}{2(4\pi)^s}\int_1^\infty \frac{1}{t(1-\xi_v(y)t)^{s+1}}dt.$$

Proof. It suffices to check that $\mathscr{P}r'(I'(0,g)(v)) = I'(0,g)(v)$ for finite v, and $\mathscr{P}r'(I'(0,g)(v)) = \overline{I}'(0,g)(v)$ for infinite v. They actually hold for all $g \in \mathrm{GL}_2(\mathbb{A})$.

By Proposition 3.3.1,

$$I'(0,g)(v) = 2\fint_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} \mathscr{K}_{\phi}^{(v)}(g,(t,t))dt$$

with

$$\mathscr{K}_{\phi}^{(v)}(g,(t_1,t_2)) = \sum_{u\in\mu_K^2\backslash F^\times}\sum_{y\in B(v)-E} k_{r(t_1,t_2)\phi_v}(g,y,u)r(g,(t_1,t_2))\phi^v(y,u).$$

Note that the integral above is just a finite sum.

We have a simple rule

$$r(n(b)g,(t_1,t_2))\phi^v(y,u)=\psi(uq(y)b)\ r(g,(t_1,t_2))\phi^v(y,u),$$

and its analogue

$$k_{r(t_1,t_2)\phi_v}(n(b)g,y,u)=\psi(uq(y)b)k_{r(t_1,t_2)\phi_v}(g,y,u)$$

showed in Proposition 3.3.2. By these rules it is easy to see that the first Fourier coefficient is given by

$$\mathscr{K}_\phi^{(v)}(g,(t_1,t_2))_\psi=\sum_{(y,u)\in\mu_K\backslash((B(v)-E)\times F^\times)_1}k_{r(t_1,t_2)\phi_v}(g_v,y_v,u_v)r(g,(t_1,t_2))\phi^v(y,u).$$

If v is non-archimedean, all the infinite components are already holomorphic of weight two. So the operator $\mathscr{P}r'$ doesn't change $\mathscr{K}_\phi^{(v)}(g,(t_1,t_2))_\psi$ at all. Thus

$$\mathscr{P}r'(\mathscr{K}_\phi^{(v)}(g,(t_1,t_2)))=\sum_{a\in F^\times}\mathscr{K}_\phi^{(v)}(d^*(a)g,(t_1,t_2))_\psi.$$

It is easy to check that it is exactly equal to $\mathscr{K}_\phi^{(v)}(g,(t_1,t_2))$ by Proposition 3.3.2 that $k_{r(t_1,t_2)\phi_v}$ transforms according to the Weil representation under upper triangular matrices. We conclude that $\mathscr{P}r'$ doesn't change $\mathscr{K}_\phi^{(v)}(g,(t_1,t_2))$, and thus we have $\mathscr{P}r'(I'(0,g)(v))=I'(0,g)(v)$.

Now we look at the case that v is archimedean. The only difference is that we need to replace $k_{\phi_v}(g,y,u)$ by some $\widetilde{k}_{\phi_v,s}(g,y,u)$, and then take a "quasi-limit" $\widetilde{\lim}$. It suffices to consider the case that $uq(y)=1$, it is given by

$$\widetilde{k}_{\phi_v,s}(g,y,u)=4\pi W^{(2)}(g_v)\int_{F_{v,+}}y_0^se^{-2\pi y_0}k_{\phi_v}(d^*(y_0),y,u)\frac{dy_0}{y_0}.$$

Then $\widetilde{k}_{\phi_v,s}(g,y,u)\neq 0$ only if $u>0$, since $k_{\phi_v}(d^*(y_0),y,u)\neq 0$ only if $u>0$.

Assume that $u>0$, which is equivalent to $q(y)>0$ since we assume $uq(y)=1$ for the moment. By Proposition 3.5.1,

$$\begin{aligned}
&\int_{F_{v,+}}y_0^se^{-2\pi y_0}k_{\phi_v}(d^*(y_0),y,u)\frac{dy_0}{y_0}=-\frac{1}{2}\int_{F_{v,+}}y_0^se^{-2\pi y_0}\ \mathrm{Ei}(4\pi uq(y_2)y_0)\ y_0e^{-2\pi y_0}\frac{dy_0}{y_0}\\
=&-\frac{1}{2}\int_0^\infty y_0^{s+1}e^{-4\pi y_0}\ \mathrm{Ei}(-4\pi\alpha y_0)\frac{dy_0}{y_0}\qquad (\alpha=-uq(y_2)=-\frac{q(y_2)}{q(y)}>0)\\
=&\frac{1}{2}\int_0^\infty y_0^{s+1}e^{-4\pi y_0}\int_1^\infty t^{-1}e^{-4\pi\alpha y_0t}dt\frac{dy_0}{y_0}=\frac{1}{2}\int_1^\infty t^{-1}\int_0^\infty y_0^{s+1}e^{-4\pi(1+\alpha t)y_0}\frac{dy_0}{y_0}dt\\
=&\frac{\Gamma(s+1)}{2(4\pi)^{s+1}}\int_1^\infty\frac{1}{t(1+\alpha t)^{s+1}}dt.
\end{aligned}$$

Hence,

$$\widetilde{k}_{\phi_v,s}(g,y,u) = W^{(2)}(g_v)\frac{\Gamma(s+1)}{2(4\pi)^s}\int_1^\infty \frac{1}{t(1-\frac{q(y_2)}{q(y)}t)^{s+1}}dt = W^{(2)}(g_v)k_{v,s}(y).$$

This matches the result in the proposition. Since $k_{v,s}(y)$ is invariant under the multiplication action of $F^\times$ on y, it is easy to get

$$\mathscr{P}r'(\mathscr{K}_\phi^{(v)}(g,(t_1,t_2))) = \overline{\mathscr{K}}_\phi^{(v)}(g,(t_1,t_2)), \qquad \mathscr{P}r'(I'(0,g)(v)) = \overline{I'}(0,g)(v).$$

□

Holomorphic projection of the Einsenstein series

Now we compute $\mathscr{P}r'(\mathscr{J}'(0,g,\phi))$. Recall that in last section we have obtained

$$\mathscr{J}'(0,g,\phi) = J'(0,g,\phi) - \widetilde{J}'(0,g,\phi) - c_0 J(0,g,\phi).$$

Here

$$J(s,g,\phi) = \sum_{u\in\mu_U^2\backslash F^\times} J(s,g,u,\phi),$$
$$J(s,g,u,\phi) = \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s r(\gamma g)\phi(0,u).$$

And $\widetilde{J}(s,g,\phi)$ has a similar formula. We have Fourier expansion

$$J(s,g,u) = \sum_{a\in F} J_a(s,g,u)$$

where

$$J_a(s,g,u) = \int_{\mathbb{A}} \delta(wn(b)g)^s\ r(wn(b)g)\phi(0,u)\ \psi(-ab)db, \quad a\in F^\times;$$
$$J_{a,v}(s,g,u) = \int_{F_v} \delta_v(wn(b)g)^s\ r(wn(b)g)\phi_v(0,u)\ \psi_v(-ab)db, \quad a\in F_v^\times.$$

We also introduce the normalization:

$$J_{a,v}^\circ(s,g,u) = \zeta_v(s+2)\ J_{a,v}(s,g,u), \quad a\in F_v^\times.$$

We will obtain a precise formula for $J_{a,v}(s,g,u)$ at unramified v later. Then we can see that $J_{a,v}^\circ(s,g,u) = 1$ for almost all v. It explains the reason to introduce the normalization.

Proposition 3.7.2. *(1) The Whittaker function of $J'(0,g,\phi) + \widetilde{J}'(0,g,\phi)$ is holomorphic of weight two.*

(2) There exists a constant c such that

$$\mathscr{P}r'(\mathscr{J}'(0,g,\phi)) = c\ J_*(0,g,\phi) - (J'_*(0,g,\phi) + \widetilde{J}'_*(0,g,\phi)) + 2\sum_{v\nmid\infty} J'(0,g,\phi)(v).$$

Here J_* (resp. $\widetilde{J}_*$) denotes the non-constant part of J (resp. $\widetilde{J}$), and

$$J'(0,g,\phi)(v) = \frac{1}{\zeta_v(2)} \sum_{u\in\mu_U^2\backslash F^\times} \sum_{a\in F^\times} J^{\circ\prime}_{1,v}(0,d^*(a)g,au,\phi) J^v_a(0,g,u,\phi).$$

Proof. We first show that $J'(0,g,\phi) + \widetilde{J}'(0,g,\phi)$ is holomorphic. It suffices to show the Whittaker function of $J'(0,g,u)+\widetilde{J}'(0,g,u)$ is holomorphic. By linearity, assume $\phi = \phi_1\otimes\phi_2$. By definition, $J'(0,g,u) + \widetilde{J}'(0,g,u)$ is formed by the section

$$\Phi(g,u) := \log\delta(g)\ r(g)\phi(0,u) + r(g)\phi_1(0,u) W_0^{\circ\prime}(0,g,u,\phi_2)$$

by summing over $P(F)\backslash \mathrm{GL}_2(F)$. By the product formula,

$$W_0^{\circ\prime}(0,g,u) = \sum_v W^{\circ\prime}_{0,v}(0,g_v,u) W_0^{v,\circ}(0,g^v,u) = \sum_v W^{\circ\prime}_{0,v}(0,g_v,u) r(g^v)\phi_2^v(0,u).$$

Here the second identity follows from Proposition 2.5.2. Thus

$$\Phi(g,u) = r(g^v)\phi^v(0,u)\ r(g_v)\phi_{1,v}(0,u) \sum_v \Big(\log\delta(g_v)\ r(g_v)\phi_{2,v}(0,u) + W_0^{\circ\prime}(0,g_v,u,\phi_{2,v})\Big).$$

Proposition 2.5.2 asserts that

$$\log\delta(g)\ r(g)\phi_2(0,u) + W_0^{\circ\prime}(0,g,u,\phi_2) = 0$$

for almost all places v. More importantly, it is true for all archimedean places v since we are assuming ϕ is standard at infinity. Therefore, the summation above for $\Phi(g,u)$ is actually for finitely many non-archimedean places v. In particular, the archimedean part

$$\Phi_\infty(g,u) = r(g_\infty)\phi_\infty(0,u)$$

is holomorphic. It proves that $J'(0,g,\phi) + \widetilde{J}'(0,g,\phi)$ is holomorphic.

Now we consider the holomorphic projection. We regroup the expression as

$$\mathscr{J}'(0,g,\phi) = 2J'(0,g,\phi) - (J'(0,g,\phi) + \widetilde{J}'(0,g,\phi)) - c_0 J(0,g,\phi).$$

Since the Whittaker functions of $J'(0,g,\phi) + \widetilde{J}'(0,g,\phi)$ and $J(0,g,\phi)$ are already holomorphic, the holomorphic projection doesn't change them except dropping their constant terms.

It remains to obtain $\mathscr{P}r' J'(0,g,\phi)$. Look at the Whittaker function

$$J_1'(0,g,\phi) = \sum_{u\in\mu_U^2\backslash F^\times} J_1'(0,g,u).$$

By

$$J_1(s,g,u) = \frac{1}{\zeta_F(s+2)} \prod_v J^\circ_{1,v}(s,g,u),$$

we have

$$\begin{aligned} J_1'(0,g,u) &= -\frac{\zeta_F'(2)}{\zeta_F(2)^2} \prod_v J^\circ_{1,v}(0,g,u) + \frac{1}{\zeta_F(2)} \sum_v J^{\circ\prime}_{1,v}(0,g,u) \prod_{w\neq v} J^\circ_{1,w}(0,g,u) \\ &= -\frac{\zeta_F'(2)}{\zeta_F(2)} J_1(0,g,u) + \sum_v \frac{1}{\zeta_v(2)} J^{\circ\prime}_{1,v}(0,g,u) J_1^v(0,g,u). \end{aligned}$$

The holomorphic projection doesn't change the first term since it is already holomorphic. Similarly, it doesn't change $J^{\circ\prime}_{1,v}(0,g,u)J_1^v(0,g,u)$ for all non-archimedean v. If v is archimedean, then the holomorphic projection formula turns $J^{\circ\prime}_{1,v}(0,g,u)$ into some multiple of the standard $W_v^{(2)}(g_v)$. Since $W_v^{(2)}(g_v)$ is proportional to $J_{1,v}(0,g,u)$, we obtain

$$\mathscr{P}r' \left(J^{\circ\prime}_{1,v}(0,g,u) J_1^v(0,g,u) \right) = (*) J^{\circ\prime}_{1,v}(0,g,u) J_1^v(0,g,u) = (*) J_1(0,g,u).$$

Here $(*)$ denotes some constant. In summary, we have obtained

$$\mathscr{P}r' J_1'(0,g,u) = (*) J_1(0,g,u) + \sum_{v\nmid\infty} \frac{1}{\zeta_v(2)} J^{\circ\prime}_{1,v}(0,g,u) J_1^v(0,g,u).$$

It follows that

$$\mathscr{P}r' J'(0,g,u) = (*) J_*(0,g,u) + \sum_{v\nmid\infty} \sum_{a\in F^\times} \frac{1}{\zeta_v(2)} J^{\circ\prime}_{1,v}(0,d^*(a)g,u) J_1^v(0,d^*(a)g,u).$$

By the relation

$$J_{1,v}(0,d^*(a)g,u) = J_{a,v}(0,g,a^{-1}u),$$

we see that the sum over u gives

$$\mathscr{P}r' J'(0,g,\phi) = (*) J_*(0,g,\phi) + \sum_{v\nmid\infty} J'(0,g,\phi)(v).$$

□

Remark. (1) The behavior of $J'(0,g,\phi) + \widetilde{J}'(0,g,\phi)$ is quite similar to $J(0,g,\phi)$. Their Whittaker functions are the same at all archimedean place and almost all non-archimedean places.

(2) After integration against χ, the infinite sum $\sum_{v\nmid\infty} J'(0,g,\phi)(v)$ gives a finite linear combinations of Eisenstein series and their derivations in the sense of [Zh1]. Another derivation is the infinite sum in Proposition 4.6.5 coming from height pairings of Hodge bundles. Lemma 4.5.2 of [Zh1] asserts that these derivations cancel each other for almost all v. In this paper, we would rather show the cancellation by explicit calculation. See Proposition 4.6.6 (1).

In the end, we show a result that will be used in the comparison with the geometric kernel related to the Hodge class. It computes the unramified case, and controls the singularities of the ramified case for some degenerate Schwartz functions.

Lemma 3.7.3. *Let v be a non-archimedean place of F. Then the following results under different assumptions are true:*

(1) Assume that v is unramified in $\mathbb{B}$ and unramified over $\mathbb{Q}$, and assume that ϕ_v is standard. Then for any $a, u \in F_v^\times$,

$$J_{a,v}^\circ(s, 1, u) = 1_{O_{F_v}}(a) 1_{O_{F_v}^\times}(u) \sum_{n=0}^{v(a)} N_v^{-n(s+1)}.$$

It follows that

$$J_{1,v}^{\circ\prime}(0, d^*(a), au) = 1_{O_{F_v}}(a) 1_{O_{F_v}^\times}(u) \left(\frac{v(a)}{2} \sum_{n=0}^{v(a)} N_v^{-n} - \sum_{n=0}^{v(a)} n N_v^{-n} \right) \log N_v.$$

(2) Assume that ϕ_v is degenerate. Then there is a constant $c > 0$ such that $J_{a,v}(s, 1, u) = 0$ identically for all $a \in F_v^\times$ satisfying $v(a) > c$ or $v(a) < -c$.

Proof. It is analogous to Proposition 3.4.1 and Proposition 3.4.3. We first prove (2). The formula in Proposition 3.4.3 (1) is actually applicable for any quadratic space. In the case here, we have

$$J_{a,v}(s, 1, u) = \gamma(\mathbb{B}_v) |d_v|^{\frac{1}{2}} (1 - N_v^{-s}) \sum_{n=0}^{\infty} N_v^{-ns+n} \int_{D_n(a)} \phi_v(x, u) dx,$$

where

$$D_n(a) = \{x \in \mathbb{B}_v : uq(x) - a \in p_v^n d_v^{-1}\}.$$

The only difference is the appearance of the Weil index $\gamma(\mathbb{B}_v) = \pm 1$ due to a different normalization. Then proof is exactly the same.

Now we consider (1). It is easy to see

$$J_{1,v}(s, d^*(a) g, u) = |a|^{-\frac{s}{2}} J_{a,v}(s, g, a^{-1} u).$$

Then

$$J_{1,v}^{\circ\prime}(0, d^*(a), au) = J_{a,v}^{\circ\prime}(0, 1, u) - \frac{1}{2} \log |a|_v J_{a,v}^\circ(0, 1, u).$$

So the second equality is a consequence of the first one.

The first identity in (1) is essentially a standard result. The computation is very simple since everything is unramified. We still include it here. Since ϕ_v is standard, it is invariant under the action of $\mathrm{GL}_2(O_{F_v})$. By Iwasawa decomposition, it is very easy to obtain

$$r(g)\phi_v(0, u) = \delta(g)^2 \ \phi_v(0, \det(g)^{-1} u), \quad g \in \mathrm{GL}_2(F_v).$$

Then $J_{a,v}(s,1,u)$ is nonzero only if $u \in O_{F_v}^{\times}$. In that case,

$$J_{a,v}(s,1,u) = \int_{F_v} \delta(wn(b))^{s+2}\ \psi_v(-ab)db.$$

The computation is similar to the first part of Proposition 3.4.3. Note that

$$\delta(wn(b)) = \begin{cases} 1 & \text{if } b \in O_{F_v}, \\ |b|^{-1} & \text{otherwise,} \end{cases}$$

Then

$$J_{a,v}(s,1,u) = \int_{O_{F_v}} \psi_v(-ab)db + \int_{F_v - O_{F_v}} |b|^{-(s+2)}\psi_v(-ab)db.$$

The second integral is equal to

$$\begin{aligned} &\sum_{n=1}^{\infty} \int_{p_v^{-n} - p_v^{-(n-1)}} N_v^{-n(s+2)}\psi_v(-ab)db \\ =&\sum_{n=1}^{\infty} N_v^{-n(s+2)} \int_{p_v^{-n}} \psi_v(-ab)db - N_v^{-(s+2)} \sum_{n=0}^{\infty} N_v^{-n(s+2)} \int_{p_v^{-n}} \psi_v(-ab)db. \end{aligned}$$

Combine with the first integral to obtain

$$J_{a,v}(s,1,u) = (1 - N_v^{-(s+2)}) \sum_{n=0}^{\infty} N_v^{-n(s+2)} \int_{p_v^{-n}} \psi_v(-ab)db.$$

The last integral is nonzero only if $a \in p_v^n$. In this case it is equal to N_v^n. Hence,

$$J_{a,v}(s,1,u) = (1 - N_v^{-(s+2)}) \sum_{n=0}^{v(a)} N_v^{-n(s+1)}.$$

It gives the result.

□

4 Shimura curves, Hecke operators, CM-points

In this section, we will review the theory of Shimura curves, generating series of Hecke operators, and a preliminary decomposition of height pairing of CM-points using Arakelov theory.

In §4.1, we will describe a projective system of Shimura curves X_U for a totally definite incoherent quaternion algebra $\mathbb{B}$ over a totally real field F indexed by compact open subgroups U of $\mathbb{B}_f^\times$.

In §4.2, we will define a generating function $Z(g,\phi)$ with coefficients in $\mathrm{Pic}(X_U \times X_U)$ for each $\phi \in \mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)$. And we will show that it is automorphic in $g \in \mathrm{GL}_2(\mathbb{A})$ if ϕ_∞ is standard, which we will assume in the rest of this paper. This series is an extension of Kudla's generating series for Shimura varieties of orthogonal type [Ku1]. In this case, the modularity of its cohomology class is proved by Kudla-Millson [KM1, KM2, KM3]. The modularity as Chow cycles are proved in our previous work [YZZ].

In §4.3, for E a CM extension of F, we define a set of points with CM by E bijective to $E^\times\backslash\mathbb{B}_f^\times$. For two CM-points represented by $\beta_i \in \mathbb{B}_f^\times$, we define a function $Z(g,(\beta_1,\beta_2),\phi)$ for $g \in \mathrm{GL}_2(\mathbb{A})$ by means of height pairing. It can be viewed as a function in $g \in \mathrm{GL}_2(\mathbb{A})$ and $(\beta_1,\beta_2) \in \mathrm{GO}(\mathbb{A})$ compatible with Weil representation on $\mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)$. This function is automorphic for g and left invariant under the diagonal action of $T(\mathbb{A})$ on (β_1,β_2).

In §4.4 and §4.5, using Arakelov theory, we will decompose $Z(g,(t_1,t_2),\phi)$ into the sum of its main part $\langle Z_*(g,\phi)t_1,t_2\rangle$ and some height pairings with arithmetic Hodge classes. Then $\langle Z_*(g,\phi)t_1,t_2\rangle$ is decomposed an infinite sum of local pairings:

$$-\sum_v i_v(Z_*(g,\phi)t_1,t_2)\log N_v - \sum_v j_v(Z_*(g,\phi)t_1,t_2)\log N_v.$$

The decomposition is valid under the global degeneracy assumption in §3.2.

In §4.6, we consider the height pairings with arithmetic Hodge classes. We show that its difference with the holomorphic projection of the Eisenstein series in Proposition 3.7.2 can be approximated by a finite sum of Eisenstein series. In particular, we decompose the height pairing into a sum of certain local terms over places v of F, and show the match of these local terms with $J'(0,g,\phi)(v)$ for unramified v.

The computation of the local pairing $i_v(Z_*(g,\phi)t_1,t_2)$ and $j_v(Z_*(g,\phi)t_1,t_2)$ is the content of next section.

4.1 Shimura curves

In the following, we will review the theory of Shimura curves following our previous paper [Zh2]. Let F be a totally real number field. Let $\mathbb{B}$ be a quaternion algebra over $\mathbb{A}$ with odd ramification set Σ including all archimedean places. Then for each open subset U of $\mathbb{B}_f^\times$ we have a Shimura curve X_U. The curve is not geometrically connected; its set of connected components can be parameterized by $F_+^\times\backslash\mathbb{A}_f^\times/q(U)$. For each archimedean place τ of F, the set of complex points at τ forms a Riemann surface as follows:

$$X_{U,\tau}^{\mathrm{an}} = B(\tau)_+^\times\backslash\mathscr{H}\times\mathbb{B}_f^\times/U \cup \{\mathrm{Cusps}\}$$

where $B(\tau)_+^\times$ is the group of totally positive elements in a quaternion algebra $B(\tau)$ over F with ramification set $\Sigma \setminus \{\tau\}$ with an action on $\mathscr{H}^\pm$ by some fixed isomorphisms

$$B(\tau) \otimes_\tau \mathbb{R} = M_2(\mathbb{R})$$

$$B(\tau) \otimes \mathbb{A}_f \simeq \mathbb{B}_f,$$

and where $\{\mathrm{Cusp}\}$ is the set of cusps which is non-empty only when $F = \mathbb{Q}$ and $\mathbb{B}_f = M_2(\mathbb{A}_f)$.

For two open compact subgroups $U_1 \subset U_2$ of $\mathbb{B}_f^\times$, one has a canonical morphism $\pi_{U_1,U_2} : X_{U_1} \to X_{U_2}$ which satisfies the composition property. Thus we have a projective system X of curves X_U. For any $x \in \mathbb{B}_f$, we also have isomorphism $T_x : X_U \to X_{x^{-1}Ux}$ which induces an automorphism on the projective system X. It is compatible with multiplication on $\mathbb{B}_f^\times$ in the sense that $T_{xy} = T_x \cdot T_y$. All of these morphisms on X_U's has obvious description on the complex manifolds $X_{U,\tau}(\mathbb{C})$. The induced actions are the obvious one on the set of connected components after taking norm of U_i and x.

An important tool to study Shimura curves is to use modular interpretation. For a fixed archimedean place τ, the space $\mathscr{H}^\pm$ parameterizes Hodge structures on $V_0 := B(\tau)$ which has type $(-1,0) + (0,-1)$ (resp $(0,0)$) on $V_0 \otimes_\tau \mathbb{R}$ (resp. $V_0 \otimes_\sigma \mathbb{R}$ for other archimedean places $\sigma \neq \tau$). The non-cuspidal part of $X_{U,\tau}(\mathbb{C})$ parameterizes Hodge structure and level structures on a $B(\tau)$-module V of rank 1.

Due to the appearance of type $(0,0)$, the curve X_U does not parameterize abelian varieties unless $F = \mathbb{Q}$. To get a modular interpretation, we use an auxiliary imaginary quadratic extension K over F with complex embeddings $\sigma_K : K \to \mathbb{C}$ for each archimedean places σ of F other than τ. These σ_K's induce a Hodge structure on K which has type $(0,0)$ on $K \otimes_\tau \mathbb{R}$ and type $(-1,0)+(0,-1)$ on $K \otimes_\sigma \mathbb{R}$ for all $\sigma \neq \tau$. Now the tensor product of Hodge structures on $V_K := V \otimes_F K$ is of type $(-1,0)+(0,-1)$. In this way, X_U parameterizes some abelian varieties with homology group H_1 isomorphic to V_K. The construction makes X_U a curve over the reflex field for σ_K's:

$$K^\sharp = \mathbb{Q}\left(\sum_{\sigma\neq\tau} \sigma(x), \quad x \in K\right).$$

See our paper [Zh1] for a construction following Carayol in the case $K = F(\sqrt{d})$ with $d \in \mathbb{Q}$ where $\sigma_K(\sqrt{d})$ is chosen independent of σ.

The curve X_U has a *Hodge class* $\mathscr{L}_U \in \mathrm{Pic}(X_U) \otimes \mathbb{Q}$ which is compatible with pull-back morphism and isomorphic to the canonical class $\omega_{X_U/F}$ when U is sufficiently small. We also define a *normalized Hodge class* $\xi_U \in \mathrm{Pic}(X_U) \otimes \mathbb{Q}$ which has degree 1 on each connected component and is proportional to $\mathscr{L}_U$. One can use ξ_U to define a projection $\mathrm{Div}(X_U) \to \mathrm{Div}^0(X_U)$ by sending D to $D - \deg(D)\xi_U$. Here $\deg(D)$ is understood to be a function on the set $\{X_i\}$ of geometrically connected components of X_U given by $\deg D(X_i) := \deg(D|_{X_i})$. The class $\deg D \cdot \xi_U$ has restriction $\deg(D|_{X_i})\xi_{X_i}$ on X_i.

The Jacobian variety J_U of X_U is defined to be the variety over F to represent line bundles on X_U (or base change) of degree 0 on every connected components X_i. Over an extension of F, J_U is the product of Jacobian varieties J_i of the connected components X_i. The Néron–Tate height $\langle \cdot, \cdot \rangle$ on $J_U(\bar{F})$ is defined using the Poincare divisor on $J_i \times J_i$.

4.2 Hecke correspondences and generating series

We want to define some correspondences on X_U, i.e., some divisor classes on $X_U \times X_U$. The projective system of surfaces $X_U \times X_U$ has an action by $\mathbb{B}_f^\times \times \mathbb{B}_f^\times$. Let K denote the open compact subgroup $K = U \times U$, and write $M_K = X_U \times X_U$.

Hecke operators

For any double coset UxU of $U\backslash\mathbb{B}_f^\times/U$, we have a Hecke correspondence

$$Z(x)_U \in \mathrm{Div}(X_U \times X_U)$$

defined as the image of the morphism

$$(\pi_{U\cap xUx^{-1},U}, \pi_{U\cap x^{-1}Ux,U} \circ T_x) : \qquad X_{U\cap xUx^{-1}} \longrightarrow X_U^2.$$

It is defined over F.

In terms of complex points at a place of F as above, the Hecke correspondence $Z(x)_U$ takes

$$[z, \beta] \longmapsto \sum_i [z, \beta x_i]$$

for points on $X_{U,\tau}(\mathbb{C})$ represented by $[z, \beta] \in \mathscr{H}^\pm \times \mathbb{B}_f^\times$ where x_i are representatives of UxU/U. We usually abbreviate $Z(x)_U$ as $Z(x)$.

Hodge classes

On $M_K := X_U \times X_U$, one has a *Hodge class* $\mathscr{L}_K \in \mathrm{Pic}(M_K) \otimes \mathbb{Q}$ defined as

$$\mathscr{L}_K = \frac{1}{2}(p_1^*\mathscr{L}_U + p_2^*\mathscr{L}_U).$$

Next we introduce some notations for components of $\mathscr{L}_K$.

The geometrically connected components of X_U are indexed by $F_+^\times\backslash\mathbb{A}_f^\times/q(U)$, and we use $X_{U,\alpha}$ to denote the corresponding component for $\alpha \in F_+^\times\backslash\mathbb{A}_f^\times/q(U)$. Then the geometrically connected components of $M_K = X_U \times X_U$ are naturally indexed by $(\alpha_1, \alpha_2) \in (F_+^\times\backslash\mathbb{A}_f^\times/q(U))^2$. For any $\alpha \in F_+^\times\backslash\mathbb{A}_f^\times/q(U)$, denote

$$M_{K,\alpha} = \coprod_{\beta \in F_+^\times\backslash\mathbb{A}_f^\times/q(U)} X_{U,\beta} \times X_{U,\alpha\beta}$$

as a subvariety of M_K. It is still defined over F. Then

$$M_K = \coprod_{\alpha \in F_+^\times\backslash\mathbb{A}_f^\times/q(U)} M_{K,\alpha}.$$

View the Hodge bundle

$$\mathscr{L}_{K,\alpha} = \mathscr{L}_K|_{M_{K,\alpha}}$$

of $M_{K,\alpha}$ as a line bundle or divisor of M_K by trivial extension outside $M_{K,\alpha}$.

Generating Function

Let $\mathbb{V}$ denote the orthogonal space $\mathbb{B}$ with quadratic form q. Let $\mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)$ denote the space $\mathscr{S}(V_\infty, F_\infty^\times)\otimes\mathscr{S}(V_{\mathbb{A}_f}\times\mathbb{A}_f^\times)$ which is isomorphic to the maximal quotient of $\widetilde{\mathscr{S}}(\mathbb{V}\times\mathbb{A}^\times)$ via integration on $\mathrm{GO}(F_\infty)$.

For any $(x,u)\in\mathbb{V}\times\mathbb{A}^\times$, let us define a cycle $Z(x,u)_K$ on $X_U\times X_U$ as follows. This cycle is non-vanishing only if $q(x)u\in F^\times$ or $x=0$. If $q(x)u\in F^\times$, then we define $Z(x,u)_K$ to be the Hecke operator $Z(x)_U=UxU$ defined in last subsection. If $x=0$, then we define

$$Z(0,u)_K=-\frac{1}{[\mu_U':\mu_U^2]}\sum_{\alpha\in F_+^\times\backslash F^\times}\mathscr{L}_{K,\alpha u^{-1}}.$$

Here $\mu_U=F^\times\cap U$ like μ_K and $\mu_U'=F_+^\times\cap q(U)$. The index $[\mu_U':\mu_U^2]$ is always finite, and it is equal to 1 for sufficiently small U.

For $\phi\in\mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)$ which is invariant under $K\cdot\mathrm{GO}(F_\infty)$, we can form a generating series and its normalization

$$Z(g,\phi)=\sum_{(x,u)\in(K\cdot\mathrm{GO}(F_\infty))\backslash\mathbb{V}\times\mathbb{A}^\times} r(g)\phi(x,u)Z(x,u)_K,\qquad g\in\mathrm{GL}_2(\mathbb{A}).$$

$$\widetilde{Z}(g,\phi)=[\mathbb{A}_f^\times:F^\times\cdot K_Z]\mathrm{vol}(Z_K)Z(g,\phi).$$

The factor in the normalization is taken so that the definition is compatible with pull-back maps in Chow groups in the projection $M_{K_1}\to M_{K_2}$ with $K_i=U_i\times U_i$ and $U_1\subset U_2$. Thus it defines an element in the direct limit $\mathrm{Pic}(M):=\lim_K\mathrm{Pic}(M_K)$. For any $h\in(\mathbb{B}_f^\times)^2$, let $\rho(h)$ denote the pull-back morphism on $\mathrm{Pic}(M)$ by right translation of h. Then it is easy to verify

$$Z(g,r(h)\phi)=\rho(h)Z(g,\phi).$$

A similar but easier property is $Z(g,r(g')\phi)=Z(gg',\phi)$ for any $g'\in\mathrm{GL}_2(\mathbb{A})$.

It is always convenient to write

$$Z(g,\phi)=Z_0(g,\phi)+Z_*(g,\phi),$$

where $Z_0(g,\phi)$ denotes the constant term and $Z_*(g,\phi)$ denotes the non-constant part. It is easy to see that they also have the following expressions:

$$\begin{aligned}Z_0(g,\phi)&=-\sum_{\alpha\in F_+^\times\backslash\mathbb{A}_f^\times/q(U)}\sum_{u\in\mu_U^2\backslash F^\times} r(g)\phi(0,\alpha^{-1}u)\mathscr{L}_{K,\alpha},\\ Z_*(g,\phi)&=\sum_{a\in F^\times}\sum_{x\in K\backslash\mathbb{B}_f^\times} r(g)\phi(x)_aZ(x)_U=\sum_{a\in F^\times}\sum_{x\in U\backslash\mathbb{B}_f^\times/U} r(g)\phi(x)_aZ(x)_U.\end{aligned}$$

Here the symbol

$$\phi(x)_a=\phi(x,aq(x)^{-1})$$

is introduced in §3.1.

Proposition 4.2.1. *The series $Z(g,\phi)$ is absolutely convergent and defines an automorphic form on $\mathrm{GL}_2(\mathbb{A})$ with coefficients in $\mathrm{Pic}(X_U\times X_U)_{\mathbb{C}}$.*

We will reduce the proposition to the modularity proved in [YZZ].

We first recall the related result in [YZZ]. Let $M'_{K'}$ be the Shimura variety of orthogonal type associated to the subgroup

$$\mathrm{GSpin}(\mathbb{V})=\{(g_1,g_2)\in\mathbb{B}^\times\times\mathbb{B}^\times: q(g_1)=q(g_2)\}$$

for any open compact subgroup $K'\subset\mathrm{GSpin}(\mathbb{V}_f)$. For any Schwartz function $\phi'\in\mathscr{S}(\mathbb{V})^{O(F_\infty)}$, one can define a generating series

$$Z(g,\phi')=\sum_{y\in K'\cdot O(F_\infty)\backslash\mathbb{V}} r(g)\phi'(y)Z(y)_{K'},\quad g\in\mathrm{SL}_2(\mathbb{A}).$$

Here $Z(y)_{K'}$ is non-zero only if $q(y)\in F_+^\times$ or $y=0$. If $y=0$, then $Z(y)_{K'}=-\mathscr{L}_{K'}$. If $q(y)\in F^\times$, then $Z(y)_{K'}$ is defined as some special divisor of $M'_{K'}$ whose image in M_K is just $Z(y)_U$ under the natural map $M'_{K'}\to M_K$. In [YZZ], we have shown that $Z(g,\phi')$ is absolutely convergent and defines an automorphic form on $\mathrm{SL}_2(\mathbb{A})$.

For any $h\in\mathbb{B}_f^\times\times\mathbb{B}_f^\times$, let i_h denote the composition of the embedding $M'\to M$ and the translation T_h of right multiplication by h on M. On the level K, it gives a finite map $i_h: M'_{K^h}\to M_K$ whose image in M_K is exactly $M_{K,\nu(h)}$. Here $K^h=\mathrm{GSpin}(\mathbb{V}_f)\cap hKh^{-1}$ is an open compact subgroup of $\mathrm{GSpin}(\mathbb{V}_f)$. One can verify that the degree of i_h onto its image is exactly equal to $[\mu'_U:\mu_U^2]$. Hence i_h is an isomorphism for sufficiently small U.

To prove the modularity of $Z(g,\phi)$, it suffices to prove that for the restriction

$$Z(g,\phi)_\alpha:=Z(g,\phi)|_{M_{K,\alpha}}$$

for all $\alpha\in F_+^\times\backslash\mathbb{A}_f^\times/q(U)$. Fix one $h\in\mathbb{B}_f^\times\times\mathbb{B}_f^\times$ such that $\nu(h)\in\alpha^{-1}F_+^\times q(U)$. Consider the finite map $i_h: M'_{K^h}\to M_{K,\alpha}$, and we want to express $Z(g,\phi)_\alpha$ as the push-forward of some generating function on M'_{K^h}.

It is easy to see that for any $x\in\mathbb{B}_f^\times$, the Hecke operator $Z(x)_U=UxU$ is completely contained in $Z(g,\phi)_{q(x)}$. It has contribution to $Z(g,\phi)_\alpha$ if and only if $q(x)\in\alpha F_+^\times q(U)$, in which case we can find $y\in Khx$ with norm in $F_+^\times$. It follows that

$$Z(g,\phi)_\alpha=-\sum_{u\in\mu_U^2\backslash F^\times} r(g,h)\phi(0,u)\mathscr{L}_{K,\alpha}+\sum_{u\in\mu'_U\backslash F^\times}\sum_{\substack{y\in K^hO(F_\infty)\backslash\mathbb{V}\\ q(y)\in F_+^\times}} r(g,h)\phi(y,u)Z(h^{-1}y)_U.$$

Note that $i_{h*}\mathscr{L}_{K^h}=\deg(i_h)\mathscr{L}_{K,\alpha}$ and $i_{h*}Z(y)_{K^h}=Z(h^{-1}y)_U$. We see that

$$Z(g,\phi)_\alpha=i_{h*}\sum_{u\in\mu'_U\backslash F^\times}\sum_{y\in K^hO(F_\infty)\backslash\mathbb{V}} r(g,h)\phi(y,u)Z(y)_{K^h}=i_{h*}\sum_{u\in\mu'_K\backslash F^\times} Z(1,r(g,h)\phi(\cdot,u)).$$

We have thus shown that $Z(g,\phi)_\alpha$ is the push forward of a finite sum of generating series for M'. It follows that $Z(g,\phi)_\alpha$ is invariant under the left translation by elements in $\mathrm{SL}_2(F)$

and is absolutely convergent. By definition, it is clear that $Z(g,\phi)$ is also invariant under left translation by elements of the form $d(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$. Thus $Z(g,\phi)_\alpha$ is invariant under the left translation by $\mathrm{GL}_2(F)$.

Action of the generating function

It is easy to write down the action of the non-constant part

$$Z_*(g,\phi) = \sum_{a\in F^\times} \sum_{x\in U\backslash \mathbb{B}_f^\times/U} r(g)\phi(x)_a Z(x)_U$$

on a point $[z,\beta]$ in X_U. By definition,

$$Z(x)_U[z,\beta] = \sum_j [z,\beta\alpha_j], \quad \text{if} \ \ UxU = \coprod_j \alpha_j U.$$

It follows that

$$Z_*(g,\phi)[z,\beta] \ = \ \sum_{a\in F^\times} \sum_{x\in \mathbb{B}_f^\times/U} r(g)\phi(x)_a [z,\beta x].$$

For any $\alpha \in F_+^\times\backslash \mathbb{A}_f^\times/q(U)$, we have considered generating function

$$Z(g,\phi)_\alpha = Z(g,\phi)|_{M_{K,\alpha}}.$$

Its non-constant part is given by

$$Z_*(g,\phi)_\alpha = \sum_{u\in \mu_U'\backslash F^\times} \sum_{y\in K^h\backslash \mathbb{B}_f^{\mathrm{ad}}} r(g,(h,1))\phi(y,u) Z(h^{-1}y)_U.$$

Here h is any element of $\mathbb{B}_f^\times$ such that $q(h) \in \alpha^{-1}F_+^\times q(U)$, and $K^h = \mathrm{GSpin}(\mathbb{V}_f) \cap hKh^{-1}$. Here we introduce the following notations:

$$\begin{aligned} \mathbb{B}_f^a &= \mathbb{B}_f(a) = \{x \in \mathbb{B}_f : q(x) = a\}, \quad a \in \mathbb{A}_f^\times; \\ \mathbb{B}_f^{\mathrm{ad}} &= \{x \in \mathbb{B}_f : q(x) \in F_+^\times\} = \bigcup_{a\in F_+^\times} \mathbb{B}_f(a). \end{aligned}$$

Note that the first notation is also valid in the local case. And the infinite component of $r(g,(h,1))\phi(y,u)$ is understood to be $W^{(2)}_{uq(y)}(g_\infty)$, which makes sense for $q(y) \in F_+^\times$.

We are going to write down the action of $Z_*(g,\phi)_\alpha$ on X_U. Assume that h is an element of $\mathbb{B}_f^\times$ for simplicity. That is, the second component is trivial. The action of $Z(h^{-1}y)_U$ is given by the coset $Uh^{-1}yU/U$. We have identities

$$Uh^{-1}yU/U = Kh^{-1}y/U = h^{-1}(hKh^{-1}y/U) = h^{-1}(K^h y/U^1).$$

Here $U^1 = U \cap \mathbb{B}_f^1 = \{b \in U : q(b) = 1\}$. By this it is easy to see that

$$Z_*(g,\phi)_\alpha \ [z,\beta] = \sum_{u\in \mu_U'\backslash F^\times} \sum_{y\in \mathbb{B}_f^{\mathrm{ad}}/U^1} r(g,(h,1))\phi(y,u) \ [z,\beta h^{-1}y].$$

Degree of the generating function

Now we compute the degree of the generating function $Z(g,\phi)_\alpha = Z(g,\phi)|_{M_{K,\alpha}}$ for any $\alpha \in F_+^\times\backslash\mathbb{A}_f^\times/q(U)$. It is a correspondence from X_β to $X_{\alpha\beta}$ for any $\beta \in F_+^\times\backslash\mathbb{A}_f^\times/q(U)$. The degree of this correspondence is just the degree of $Z(g,\phi)_\alpha D$ for any degree-one divisor D on X_β. By definition, we see that $\deg Z(x)_U = [UxU : U]$.

Recall that right before Proposition 3.6.2, we have defined

$$J(s,g,u,\phi) = \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s r(\gamma g)\phi(0,u),$$
$$J(s,g,\phi) = \sum_{u\in\mu_U^2\backslash F^\times} J(s,g,u,\phi).$$

The following result can be viewed as an arithmetic variant of the Siegel–Weil formula.

Proposition 4.2.2. *Denote by κ_U the degree of the Hodge bundle $\mathscr{L}_U$ on any geometrically connected component of X_U. Then*

(1)

$$J(0,g,\phi) = \sum_{u\in\mu_U^2\backslash F^\times} r(g)\phi(0,u) - \frac{2}{\kappa_U}\sum_{u\in\mu_U'\backslash F^\times}\sum_{y\in\mathbb{B}_f^{\mathrm{ad}}/U^1} r(g)\phi(y,u).$$

(2) For any $\alpha \in F_+^\times\backslash\mathbb{A}_f^\times/q(U)$,

$$\deg Z(g,\phi)_\alpha = -\frac{1}{2}\kappa_U J(0,g,r(h)\phi).$$

Here h is any element of $\mathbb{B}_f^\times\times\mathbb{B}_f^\times$ such that $\nu(h)\in\alpha^{-1}F_+^\times q(U)$.

Proof. It is easy to see that (2) can be reduced to (1). In fact, the right-hand side of (2) is independent of the choice of h, so we assume that the second component of h is 1. Write $Z(g,\phi)_\alpha = Z_0(g,\phi)_\alpha + Z_*(g,\phi)_\alpha$. Recall that

$$Z_0(g,\phi)_\alpha \;=\; -\sum_{u\in\mu_U^2\backslash F^\times} r(g,h)\phi(0,u)\mathscr{L}_{K,\alpha}.$$

By $\mathscr{L}_K = \frac{1}{2}(p_1^*\mathscr{L}_U + p_2^*\mathscr{L}_U)$, we get

$$\deg Z_0(g,\phi)_\alpha = -\frac{1}{2}\kappa_U\sum_{u\in\mu_U^2\backslash F^\times} r(g,h)\phi(0,u).$$

As for $J_0(0,g,u,\phi)$, the intertwining operator vanishes at $s=0$ due to a pole of the Dedekind zeta function. Hence the constant terms

$$J_0(0,g,u,\phi) = r(g)\phi(0,u), \qquad J_0(0,g,\phi) = \sum_{u\in\mu_U^2\backslash F^\times} r(g)\phi(0,u).$$

Thus we have shown the identity for constant terms:

$$\deg Z_0(g,\phi)_\alpha = -\frac{1}{2}\kappa_U J_0(0,g,r(h)\phi).$$

As for the non-constant part, we have the formula

$$Z_*(g,\phi)_\alpha\ [z,\beta] = \sum_{u\in\mu_U'\backslash F^\times}\ \sum_{y\in\mathbb{B}_f^{\mathrm{ad}}/U^1} r(g,h)\phi(y,u)\ [z,\beta h^{-1}y].$$

Then we simply have

$$\deg Z_*(g,\phi)_\alpha = \sum_{u\in\mu_U'\backslash F^\times}\ \sum_{y\in\mathbb{B}_f^{\mathrm{ad}}/U^1} r(g,h)\phi(y,u).$$

It shows that (1) implies (2).

The truth of (1) follows from a precise case of the local Siegel–Weil formula (Theorem 2.1.2). Note that both sides of (1) are automorphic, and their constant terms are the same. By modularity, it suffices to show the non-constant part

$$A(g):\ = \sum_{u\in\mu_U'\backslash F^\times}\ \sum_{a\in F_+^\times}\ \sum_{y\in\mathbb{B}_f(a)/U^1} r(g)\phi(y,u)$$

is a scalar multiple of $J_*(0,g,\phi)$. Let y_a be any fixed element in $\mathbb{B}_f^\times$ with norm a. Then the last summation

$$\sum_{y\in\mathbb{B}_f(a)/U^1} r(g)\phi(y,u) = \frac{1}{\mathrm{vol}(U^1)\mathrm{vol}(\mathbb{B}^1_\infty)}\int_{\mathbb{B}^1} r(g)\phi(by_a,u)db.$$

By the local Siegel–Weil theorem, the integral is exactly equal to $-J_{au}(0,g,u,\phi)$. Here we use the "Tamagawa measure" on $\mathbb{B}^1$. The negative sign comes out because the product of the Weil indexes at all places is -1 in the incoherence case. Thus we have

$$A(g) = -\frac{1}{[\mu_U':\mu_U^2]\mathrm{vol}(U^1)\mathrm{vol}(\mathbb{B}^1_\infty)}J_*(0,g,\phi).$$

It proves the result. □

Remark. The above proof actually implies a formula

$$\frac{1}{2}\kappa_U = \frac{1}{[\mu_U':\mu_U^2]\mathrm{vol}(U^1)\mathrm{vol}(\mathbb{B}^1_\infty)}.$$

We can interpret $\frac{1}{2}\kappa_U$ as the volume of the invariant measure $\dfrac{dxdy}{4\pi y^2}$ on any connected component of $X_{U,\sigma}(\mathbb{C})$ for any archimedean place σ. Then the formula about $12\kappa_U$ is equivalent to the fact that the Tamagawa number of B^1 is 1 where $B = B(\sigma)$ is the nearby quaternion algebra. The factor $[\mu'_U : \mu_U^2]$ is just the degree of the natural map

$$(B^1 \cap U^1)\backslash \mathscr{H} \to (B_+^\times \cap U)\backslash \mathscr{H}.$$

In particular, if U is maximal, we have an explicit formula

$$\kappa_U = \frac{2d_F^{\frac{3}{2}}\zeta_F(2)}{[O_{F,+}^\times : (O_F^\times)^2](4\pi^2)^{[F:\mathbb{Q}]}} \prod_{v\in\Sigma_f}(N_v - 1).$$

Here ζ_F denotes the finite part of the Dedekind zeta function of F. See also Vignéras [Vi].

On the other hand, recall from [Zh1] that the Hodge bundle on each connected component X_i has degree

$$2g(X_i) - 2 + \sum_{p\in X_i}\left(1 - \frac{1}{u_p}\right)$$

here u_p is the local index of p.

4.3 CM-points and height series

In this subsection we define our geometric kernel function $Z(g, \chi, \phi)$ by means of height pairing of CM-points. It is automatically cuspidal though the original generating function $Z(g, \phi)$ does not need to be.

CM-points

Let E be an imaginary quadratic extension of F with an embedding $E(\mathbb{A}_f) \subset \mathbb{B}_f$. Then $X_U(\bar{F})$ has a set of CM-points defined over E^{ab}, the maximal abelian extension of E. The set CM_U is stable under action of $\mathrm{Gal}(E^{\mathrm{ab}}/F)$ and Hecke operators. More precisely, we have a projective system of bijections

$$\mathrm{CM}_U \simeq E^\times\backslash\mathbb{B}_f^\times/U \tag{4.3.1}$$

which is compatible with action of Hecke operators and such that the Galois action is given as follows: the action of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ acts by right multiplication of elements of $E^\times(\mathbb{A}_f)$ via the reciprocity law in class field theory:

$$E^\times\backslash E^\times(\mathbb{A}_f) \longrightarrow \mathrm{Gal}(E^{\mathrm{ab}}/E).$$

Such a bijection is unique up to left multiplication by elements in $E^\times(\mathbb{A}_f)$.

If τ is a real place of F, then the set CM_U can be described as a subset of

$$X_{U,\tau}(\mathbb{C}) = B(\tau)^\times\backslash\mathscr{H}^\pm \times \mathbb{B}_f^\times/U \cup \{\mathrm{cusps}\}$$

as

$$\mathrm{CM}_U = B(\tau)^\times\backslash B(\tau)^\times z_0 \times \mathbb{B}_f^\times/U \cup \{\mathrm{cusps}\} \simeq E^\times\backslash\mathbb{B}_f^\times/U$$

where E is embedded into $B(\tau)$ compatible with isomorphism $B(\tau)_{\mathbb{A}_f} = \mathbb{B}_f$ and $z_0 \in \mathscr{H}$ is unique fixed point of $E^\times$.

Any $\beta \in \mathbb{B}_f^\times$ gives a CM-point in CM_U which is denoted by $[\beta]_U$ or just $[\beta]$ or β if U is clear. Let T denote $E^\times$ as an algebraic group over F. We are particularly interested in the case that $\beta \in T(\mathbb{A}_f)$, i.e. CM-points that are in the image in X_U of the zero-dimensional Shimura variety

$$C_U = T(F)\backslash T(\mathbb{A}_f)/U_T$$

associated to T. Here $U_T = U \cap T(\mathbb{A}_f)$. Notice that the set C_U does not depend on the choice of bijection 4.3.1.

Height series

For any $\phi \in \mathscr{S}(\mathbb{B} \times \mathbb{A}^\times)^{U\times U}$, we define the Néron–Tate height pairing

$$Z(g,(\beta_1,\beta_2),\phi) = \langle Z(g,\phi)([\beta_1]_U - \deg([\beta_1]_U)\xi_U), \quad [\beta_2]_U - \deg([\beta_2]_U)\xi_U\rangle_{\mathrm{NT}}, \qquad \beta_1,\beta_2 \in \mathbb{B}_f^\times.$$

Using the projection formula of height pairing, we see that this definition does not depend on the choice of U. Also this definition does not depend on the choice of the bijection 4.3.1 since the height pairing is invariant under Galois action. It follows that

$$Z(g,(t\beta_1,t\beta_2),\phi) = Z(g,(\beta_1,\beta_2),\phi).$$

In this way, we may view this as a function on $\mathrm{GO}(\mathbb{V})$ through projection

$$\mathrm{GO}(\mathbb{V}) = \Delta(\mathbb{A})\backslash\mathbb{B}^\times \times \mathbb{B}^\times \longrightarrow \mathrm{GO}(\mathbb{V}_f) = \Delta(\mathbb{A}_f)\backslash\mathbb{B}_f^\times \times \mathbb{B}_f^\times$$

Using the projection formula and the formula $Z(g, r(h)\phi) = \rho(h)Z(g,\phi)$ where $\rho(h)$ is the pull-back morphism of right translation by $h \in \mathbb{B}_f^\times \times \mathbb{B}_f^\times$, one can show that the resulting function $Z(g,h,\phi)$ is equivariant under Weil representation:

$$Z(g_2,h_2,r(g_1,h_1)\phi) = Z(g_2g_1,h_2h_1,\phi), \qquad g_i \in \mathrm{GL}_2(\mathbb{A}), \quad h_i \in \mathrm{GO}(\mathbb{V}).$$

This function is automorphic for the first variable and invariant for the second variable under left diagonal multiplication by $\mathbb{A}_E^\times$.

For a finite character χ of $T(F)\backslash T(\mathbb{A})$, we can define

$$Z(g,\chi,\phi) := \int_{[T]} \widetilde{Z}(g,(t,1),\phi)\chi(t)dt = [\mathbb{A}_f^\times : F^\times {\cdot} U_Z]^{-1}\int_{T(F)\backslash T(\mathbb{A})/Z(F_\infty)U_Z} Z(g,(t,1),\phi)\chi(t)dt.$$

Here $U_Z = U \cap \mathbb{A}^\times$ is the equal to K_Z since $K = U \times U$.

Using definition of $Z(g,(t_1,t_2),\phi)$, and Heegner divisor

$$Y_\chi = \int_{[T]} \chi(t)([t] - \xi_t)dt$$

we obtain

$$Z(g,\chi,\phi)=\langle Z(g,\phi)Y_\chi,[1]\rangle_N T.$$

By Galois invariance of the Neron–Tate height pairing, we obtain

$$Z(g,\chi,\phi)=\frac{1}{2L(1,\eta)}\langle Z(g,\phi)Y_\chi,Y_\chi\rangle_N T.$$

The integral is essentially a summation on the finite set $C_U=T(F)\backslash T(\mathbb{A}_f)/U_T$:

$$Z(g,\chi,\phi)=\frac{2L(1,\eta)}{|C_U|^2}\langle Z(g,\phi)C_U,C_U\rangle_{\mathrm{NT}}.$$

Cuspidality of the height series

Our first observation is that the above $Z(g,(\beta_1,\beta_2),\phi)$ is a cusp form. It follows from the cuspidality of $Z(g,\phi)(\beta_1-\deg(\beta_1)\xi)$. More precisely, the action

$$Z_0(g,\phi)(\beta_1-\deg(\beta_1)\xi)=0$$

where $Z_0(g,\phi)$ denotes the constant term of $Z(g,\phi)$. In fact, as a correspondence, $Z_0(g,\phi)$ is a linear combination of hodge classes $\mathscr{L}_{U,\alpha,\beta}$, the (α,β)-component of the total hodge bundle

$$\mathscr{L}_K=\frac{1}{2}(p_1^*\mathscr{L}_U+p_2^*\mathscr{L}_U).$$

But it is very easy to see that the degree-zero cycle $\beta_1-\deg(\beta_1)\xi$ has trivial image under these correspondences.

Therefore,

$$Z(g,(\beta_1,\beta_2),\phi)=\langle Z_*(g,\phi)(\beta_1-\deg(\beta_1)\xi),\beta_2-\deg(\beta_2)\xi\rangle_{\mathrm{NT}}$$

where the non-constant part

$$Z_*(g,\phi)=\sum_{a\in F^\times}\sum_{x\in K\backslash\mathbb{B}_f^\times}r(g)\phi(x)_a Z(x)_U.$$

Its action is simply given by

$$Z_*(g,\phi)[\beta_1]\ =\ \sum_{a\in F^\times}\sum_{x\in\mathbb{B}_f^\times/U}r(g)\phi(x)_a[\beta_1 x].$$

Simplification of the notation

We usually drop the dependence of these series on ϕ. For example, we will abbreviate

$$Z(g,\phi),\ Z_0(g,\phi),\ Z_*(g,\phi),\ Z_*(g,\phi)_\alpha,\ Z(g,(\beta_1,\beta_2),\phi),\ Z(g,\chi,\phi)$$

respectively as

$$Z(g),\quad Z_0(g),\quad Z_*(g),\quad Z_*(g)_\alpha,\quad Z(g,(\beta_1,\beta_2)),\quad Z(g,\chi).$$

4.4 Arithmetic intersection pairing

We recall the notion of admissible arithmetic extension in [Zh2]. It gives a nice decomposition of the Neron-Tate height by the arithmetic Hodge index theorem of Faltings [Fa].

Hodge index theorem

Let F be any number field, and X be a complete smooth curve over F. Note that X may not be geometrically connected. Denote by $\mathrm{Div}^0(X_{\bar{F}})$ the group of divisors on $X_{\bar{F}}$ that has degree zero on every connected component of $X_{\bar{F}}$, and by $\mathrm{Pic}^0(X_{\bar{F}})$ the rational equivalence classes in $\mathrm{Div}^0(X_{\bar{F}})$. The Neron-Tate height gives a bilinear pairing $\langle \cdot, \cdot \rangle_{\mathrm{NT}}$ on $\mathrm{Pic}^0(X_{\bar{F}})$.

Let $D_1, D_2 \in \mathrm{Div}^0(X_{\bar{F}})$ be any two divisors. Let L be any field extension of F such that D_1 and D_2 are defined over L. Take a regular integral model $\mathscr{Y}$ of X_L over O_L. Let $\widehat{D}_i$ be a *flat* arithmetic extension of D_i on $\mathscr{Y}$, i.e. a flat arithmetic divisor on $\mathscr{Y}$ with generic fibre D_i. Here we say an arithmetic divisor is *flat* if its curvature form is zero at every archimedean place and its intersection with any finite vertical divisor on $\mathscr{Y}$ is zero. Then the Hodge index theorem asserts that

$$\langle D_1, D_2 \rangle_{\mathrm{NT}} = -\frac{1}{[L:F]} \widehat{D}_1 \cdot \widehat{D}_2.$$

The flat extensions $\widehat{D}_i$ are unique up to linear combinations of special fibres. The normalized intersection number on the right-hand side depends only on the rational equivalence classes of D_1, D_2.

Admissible extension

The Néron–Tate height is a canonical pairing on $\mathrm{Pic}^0(X_{\bar{F}})$. We can extend it to a pairing on $\mathrm{Div}(X_{\bar{F}})$, which does not preserve rational equivalence and depends on the choice of a reference arithmetic class. For the rest of this section, fix an arithmetic class $\hat{\xi} \in \varprojlim \widehat{\mathrm{Pic}}(\mathscr{Y})$ whose generic fibre has degree one on any geometrically connected component. Here the inverse limit is taking over all integral models $\mathscr{Y}$ of X_L over O_L for all extensions L/F, and the arrows between different models are the pull-back maps on arithmetic divisors.

Let $D_1, D_2 \in \mathrm{Div}(X_{\bar{F}})$. Let $L, \mathscr{Y}$ be as above. Assume further that $\hat{\xi}$ is represented by an arithmetic divisor on $\mathscr{Y}$. We still denote it by $\hat{\xi}$. Any arithmetic extension of D_i on $\mathscr{Y}$ is of the form $\widehat{D}_i = (\overline{D}_i + V_i, g_i)$, where $\overline{D}_i$ is the Zariski closure of D_i, V_i is some finite vertical divisor, and g_i is some green's function of $\overline{D}_i$ at infinity. We $\widehat{D}_i$ a *$\hat{\xi}$-admissible* extension if the following conditions hold on each connected component of $\mathscr{Y}$:

- The difference $\widehat{D}_i - \deg D_i \cdot \hat{\xi}$ is flat;
- The integral $\int g_i c_1(\hat{\xi}) = 0$ at any archimedean place;
- The intersection $(V_i \cdot \hat{\xi})_v = 0$ at any non-archimedean place v.

With these extensions, define a pairing

$$\langle D_1, D_2\rangle := -\frac{1}{[L:F]}\widehat{D}_1\cdot\widehat{D}_2.$$

Once L and $\mathscr{Y}$ are given, the $\widehat{\xi}$-admissible extension is unique. When varying L and $\mathscr{Y}$, the $\widehat{\xi}$-admissibility is preserved with pull-back. Hence the pairing depends only on D_1, D_2 and the choice of $\widehat{\xi}$.

Decomposition of the pairing

There is a non-canonical decomposition $\langle\cdot,\cdot\rangle = -i-j$ depending on the choice of a model. Let $X, \widehat{\xi}$ be as above. Fix a regular integral model $\mathscr{Y}_0$ of X_{L_0} over O_{L_0} for some field extension L_0 of F with a fixed embedding $L_0 \hookrightarrow \bar{K}$. Assume that $\widehat{\xi}$ is realized as a divisor on $\mathscr{Y}_0$.

Let $D_1, D_2 \in \mathrm{Div}(X_{\bar{F}})$ be two divisors. Assume that D_2 is defined over L_0. Let L be any field extension of L_0 such that D_1 is defined over L. Then we can decompose $\langle D_1, D_2\rangle$ according to the model $\mathscr{Y}_{0,O_L}$.

We first consider the case that $\mathscr{Y}_{0,O_L}$ is regular. Let $\widehat{D}_i = (\overline{D}_i + V_i, g_i)$ be the $\widehat{\xi}$-admissible extensions on the model. Note that $V_1\cdot\widehat{D}_2 = 0$ since V_1 is orthogonal to both $\widehat{D}_2 - \deg D_2\cdot\hat{\xi}$ and $\widehat{\xi}$. Similarly, the integral of g_1 on the curvature of g_2 is zero. It follows that

$$\langle D_1, D_2\rangle = -\frac{1}{[L:F]}\overline{D}_1\cdot\widehat{D}_2.$$

Define

$$i(D_1, D_2) = \frac{1}{[L:F]}\overline{D}_1\cdot(\overline{D}_2, g_2), \qquad j(D_1, D_2) = \frac{1}{[L:F]}\overline{D}_1\cdot V_2.$$

Then we have a decomposition

$$\langle D_1, D_2\rangle = -i(D_1, D_2) - j(D_1, D_2),$$

The decomposition is still valid even if $\mathscr{Y}_{0,O_L}$ is not regular. We have the $\widehat{\xi}$-admissible extension of D_2 on the regular model $\mathscr{Y}_0$. Pull it back to $\mathscr{Y}_{0,O_L}$. We get the extension $\widehat{D}_2 = (\overline{D}_2 + V_2, g_2)$ on $\mathscr{Y}_{0,O_L}$. All divisors on $\mathscr{Y}_0$ are Cartier divisors since it is regular. It follows that $\overline{D}_2$ and V_2 are Cartier divisors on $\mathscr{Y}_{0,O_L}$ since they are pull-backs of Cartier divisors. Hence the intersections $i(D_1, D_2)$ and $j(D_1, D_2)$ are well-defined. To verify the equality, we first decompose the pairing on any desingularization of $\mathscr{Y}_{0,O_L}$, and then use the projection formula.

Assume that D_1 and D_2 have disjoint supports. Then we have further decompositions to local heights:

$$i(D_1, D_2) = \sum_{v\in S_F} i_v(D_1, D_2)\log N_v, \qquad j(D_1, D_2) = \sum_{v\in S_F} j_v(D_1, D_2)\log N_v$$

with

$$i_v(D_1,D_2)=\frac{1}{\#S_{L_v}}\sum_{w\in S_{L_v}}i_w(D_1,D_2),\qquad j_v(D_1,D_2)=\frac{1}{\#S_{L_v}}\sum_{w\in S_{L_v}}j_w(D_1,D_2).$$

Here S_F denote the set of all places of F, and S_{L_v} denotes the set of places of L lying over v. And i_w and j_w are local intersection multiplicities of i and j over the model $\mathscr{Y}_{0,O_L}$.

Fix an embedding $\bar{F}\hookrightarrow\bar{F}_v$, or equivalently fix an extension $\bar{v}$ of the valuation v to $\bar{F}$. By varying the fields as in the global case, we have well-defined pairings $i_{\bar{v}}$ and $j_{\bar{v}}$ on $\mathrm{Div}(X_{\bar{F}_v})$ for proper intersections. It is the same as considering intersections on the model $\mathscr{Y}_{0,O_{\bar{F}_v}}$. The formulae above have the following equivalent forms:

$$i_v(D_1,D_2)=\int_{\mathrm{Gal}(\bar{F}/F)}i_{\bar{v}}(D_1^\sigma,D_2^\sigma)d\sigma,\qquad j_v(D_1,D_2)=\int_{\mathrm{Gal}(\bar{F}/F)}j_{\bar{v}}(D_1^\sigma,D_2^\sigma)d\sigma.$$

Here the integral on the Galois group takes the Haar measure with total volume one.

If D_1,D_2 have common irreducible components, then $i(D_1,D_2)$ can't be decomposed to a sum of local heights while the decomposition for $j(D_1,D_2)$ is still valid. This case does not happen in the computation of this paper. In any case, j_v is identically zero if v is archimedean or the model $\mathscr{Y}_0$ is smooth over all primes of L_0 dividing v.

4.5 Decomposition of the height series

Go back to the setting of Shimura curve X_U. Our goal in the geometric side is to compute

$$Z(g,(t_1,t_2))=\langle Z_*(g)(t_1-\xi_{t_1}),t_2-\xi_{t_2}\rangle_{\mathrm{NT}},\qquad t_1,t_2\in C_U.$$

Here we write $\xi_t=\deg(t)\xi$ for simplicity. In this subsection, we decompose the above pairing into a sum of local heights and some global pairings with ξ.

Arithmetic models

Fix an open compact subgroup U which is small enough. Recall that X_U has a canonical regular integral model $\mathscr{X}_U$ over O_F. At each finite place v of F, the base change $X_{U,v}$ over O_{F_v} will parameterizes p-divisible groups with level structures. Locally at a geometric point, $X_{U,v}$ is the universal deformation of the represented p-divisible group.

The Hodge bundle $\mathscr{L}_U$ can be extended to a metrized line bundle on $\mathscr{X}_U$. More precisely, at an archimedean place, the metric can defined by using Hodge structure or equivalently normalized such that its pull-back on $\Omega^1_{\mathscr{H}/\mathbb{C}}$ takes the form:

$$\|f(z)dz\|=4\pi\cdot\mathrm{Im}z\cdot|f(z)|.$$

At a finite place, we may take an extension $\mathscr{L}_{U,v}$ by using the fact that $\mathscr{L}_U$ is twice of the cotangent bundle of the divisible groups. The resulting bundle on $\mathscr{X}_U$ with metrics at archimedean places is denoted by $\widehat{\mathscr{L}}_U$. Also we have an arithmetic class $\widehat{\xi}_U$ induced from $\widehat{\mathscr{L}}_U$.

We will denote by $\langle \cdot, \cdot \rangle$ the $\widehat{\xi}_U$-admissible pairing explained in last subsection. To consider the decomposition $i + j$ and its corresponding local components, we need an integral model over a field where the CM points are rational. Let $H = H_U$ be the minimal field extension over E which contains the fields of definition of all $t \in C_U$. Then H is an abelian extension over E given by the reciprocity law. We will use the regular integral model $\mathscr{Y}_U$ of X_U over O_H introduced in the following to get the decomposition $i + j$.

Without loss of generality, assume that U_v is of the form $(1 + \varpi_v^r O_{\mathbb{B}_v})^\times$ for every finite place v, where $O_{\mathbb{B}_v}$ is a maximal order of $\mathbb{B}_v$. To describe $\mathscr{Y}_U$, it suffices to describe the corresponding local model $\mathscr{Y}_{U,w} = \mathscr{Y}_U \times O_{H_w}$ for any finite place w of H. Let v be the place of F induced from w. Then $\mathscr{Y}_{U,w}$ is defined by the following process:

- Let $U^0 = U^v U_v^0$ with $U_v^0 = O_{\mathbb{B}_v}^\times$ the maximal compact subgroup. Then X_{U^0} has a canonical regular model $\mathscr{X}_{U^0,v}$ over O_{F_v}. It is smooth if $\mathbb{B}_v$ is the matrix algebra. Let $\mathscr{X}'_U$ be the normalization of $\mathscr{X}_{U^0,v}$ in the function field of X_{U,H_w};
- Make a minimal desingularization of $\mathscr{X}'_U$ to get $\mathscr{Y}_{U,w}$.

With the model $\mathscr{Y}_U$, we can always write

$$\langle \beta, t \rangle = -i(\beta, t) - j(\beta, t), \quad \beta \in \mathrm{CM}_U, t \in C_U.$$

And we can further write i, j in terms of their corresponding local components if $\beta \neq t$.

Decomposition of the kernel Function

Go back to

$$Z(g, (t_1, t_2)) = \langle Z_*(g)(t_1 - \xi_{t_1}), t_2 - \xi_{t_2} \rangle_{\mathrm{NT}}, \qquad t_1, t_2 \in C_U.$$

We first write

$$Z(g, (t_1, t_2)) = \langle Z_*(g)t_1, t_2 \rangle - \langle Z_*(g)t_1, \xi_{t_2} \rangle - \langle Z_*(g)\xi_{t_1}, t_2 \rangle + \langle Z_*(g)\xi_{t_1}, \xi_{t_2} \rangle.$$

The last three terms will be taken care of next subsection. Here we look at the first term, which is the main term of the kernel function. By the model $\mathscr{Y}$ over O_H, we can decompose

$$\langle Z_*(g)t_1, t_2 \rangle = -i(Z_*(g)t_1, t_2) - j(Z_*(g)t_1, t_2).$$

The j-part is always a sum of local pairings over places and Galois orbits. So is the i-part if there is no self-intersections occur.

Let us first figure out the contribution of self-intersections. Recall that

$$Z_*(g)t_1 = \sum_{a \in F^\times} \sum_{x \in \mathbb{B}_f^\times / U} r(g)\phi(x)_a [t_1 x].$$

Apparently $[t_1x] = [t_2]$ as CM points on X_U if and only if $x \in t_1^{-1}t_2E^\times U$. It follows that the coefficient of $[t_2]$ in $Z_*(g)t_1$ is equal to

$$\sum_{a\in F^\times}\sum_{x\in t_1^{-1}t_2E^\times U/U} r(g)\phi(x)_a \quad = \quad \sum_{a\in F^\times}\sum_{y\in E^\times/(E^\times\cap U)} r(g)\phi(t_1^{-1}t_2y)_a.$$

As in the analytic side, under the global degeneracy assumption §3.2, $r(g)\phi(t_1^{-1}t_2y)_a = 0$ for all $y \in E(\mathbb{A})$. In particular, the part of self-intersection disappears and we can decompose the i-part into a sum of local pairings.

In the following we list decomposition of $i(Z_*(g)t_1, t_2)$ as a sum of local heights, under the degeneracy assumption above. All the notations and decompositions apply to $j(Z_*(g)t_1, t_2)$ even without the degeneracy assumption.

Note that Galois conjugates of points in C_U over E are described easily by multiplication by elements of $T(F)\backslash T(\mathbb{A}_f)$ via the reciprocity law. It is convenient to group local intersections according to places of E. We write:

$$\begin{aligned} i(Z_*(g)t_1, t_2) \quad &= \quad \frac{1}{2}\sum_{\nu\in S_E} i_\nu(Z_*(g)t_1, t_2)\log N_\nu, \\ i_\nu(Z_*(g)t_1, t_2) \quad &= \quad \fint_{T(F)\backslash T(\mathbb{A}_f)} i_{\bar\nu}(Z_*(g)tt_1, tt_2)dt. \end{aligned}$$

Here S_E denotes the set of all places of E. The integral on $T(F)\backslash T(\mathbb{A}_f)$ takes the Haar measure with total volume one, which has the same effect as the Galois group $\mathrm{Gal}(\bar E/E)$. The definition of $i_{\bar\nu}$ depends on fixed embeddings $H \hookrightarrow \bar E$ and $\bar E \hookrightarrow \bar E_\nu$, and can be viewed as intersections on $\mathscr{Y}_U \times_{O_H} O_{\bar E_\nu}$.

To compare with the analytic kernel, we also need to group the pairing in terms of places of F. We have:

$$\begin{aligned} i(Z_*(g)t_1, t_2) \quad &= \quad \sum_{v\in S_F} i_v(Z_*(g)t_1, t_2)\log N_v, \\ i_v(Z_*(g)t_1, t_2) \quad &= \quad \fint_{T(F)\backslash T(\mathbb{A}_f)} i_{\bar v}(Z_*(g)tt_1, tt_2)dt, \\ i_{\bar v}(Z_*(g)t_1, t_2) \quad &= \quad \frac{1}{\#S_{E_v}}\sum_{\nu\in S_{E_v}} i_{\bar\nu}(Z_*(g)t_1, t_2). \end{aligned}$$

Here S_{E_v} denotes the set of places of E lying over v. It has one or two elements.

The local pairing

$$i_{\bar v}(Z_*(g)t_1, t_2) \quad = \quad \sum_{a\in F^\times}\sum_{x\in \mathbb{B}_f^\times/U} r(g)\phi(x)_a \sum_v i_{\bar v}(t_1x, t_2)$$

is our main goal next section. We will divide it into a few cases and discuss them in different subsections. We will have explicit expression for $i_{\bar v}$ in the case that v is archimedean or the Shimura curve has good reduction at v.

4.6 Hecke action on arithmetic Hodge classes

Recall that

$$Z(g,(t_1,t_2)) = \langle Z_*(g)t_1, t_2\rangle - \langle Z_*(g)\xi_{t_1}, t_2\rangle + \langle Z_*(g)\xi_{t_1}, \xi_{t_2}\rangle - \langle Z_*(g)t_1, \xi_{t_2}\rangle.$$

We have considered a decomposition of the first term on the right-hand side in the last subsection.

In this subsection we consider the remaining three terms. We will show that their difference with the holomorphic projection $\mathscr{P}r'(\mathscr{J}'(0,g,\phi))$ can be approximated by a finite sum of Eisenstein series related to $J(s,g,\phi)$. The weight-two Eisenstein series $J(s,g,\phi)$ studied in §3.6 comes to the stage because it gives the degree of the generating function $Z(g,\phi)$ by Proposition 4.2.2.

The main results

Here we list the main results. The first two terms are very simple. They are exactly equal to non-constant parts of some Eisenstein series.

Lemma 4.6.1.

$$\begin{aligned}\langle Z_*(g,\phi)\xi_{t_1}, t_2\rangle &= -\frac{1}{2}\kappa_U J_*(0,g,r(t_1,t_2)\phi)\ \langle \xi, 1\rangle,\\ \langle Z_*(g,\phi)\xi_{t_1}, \xi_{t_2}\rangle &= -\frac{1}{2}\kappa_U J_*(0,g,r(t_1,t_2)\phi)\ \langle \xi_1, \xi_1\rangle.\end{aligned}$$

Here $\langle \xi, 1\rangle$ means the admissible pairing between ξ and the CM point $[1]$ in C_U, and ξ_1 means the sum of some components of ξ determined by $[1]$. They key is that both pairings on the right are independent of t_1, t_2.

The hard part is the last term $\langle Z_*(g)t_1, \xi_{t_2}\rangle$. It turns out that unramified local components of it match with the v-part $J'(0,g,r(t_1,t_2)\phi)(v)$ introduced in Proposition 3.7.2. Recall that in the proposition we have obtained the holomorphic projection formula

$$\mathscr{P}r'(\mathscr{J}'(0,g,\phi)) = c\ J_*(0,g,\phi) - (J_*'(0,g,\phi) + \widetilde{J}_*'(0,g,\phi)) + 2\sum_{v\nmid\infty} J'(0,g,\phi)(v).$$

Let S be a finite set of non-archimedean places of F containing all places ramified in $\mathbb{B}$, all places ramified over $\mathbb{Q}$, and all places v such that U_v is not maximal. The final result is as follows:

Proposition 4.6.2. *Assume that ϕ_v is standard for all $v \notin S$ and degenerate for all $v \in S$. Then there exists a Schwartz function $\phi_S' \in \mathscr{S}(\mathbb{B}_S \times F_S^\times)$ such that*

$$\langle Z_*(g,\phi)t_1, \xi_{t_2}\rangle - \sum_{v\nmid\infty} J'(0,g,\phi)(v) = J(0,g,r(t_1,t_2)(\phi^S\otimes\phi_S')), \quad \forall g\in 1_S\mathrm{GL}_2(\mathbb{A}^S).$$

In the following, we will first show Lemma 4.6.1, review some results of integral models of Hecke correspondences, and then prove Proposition 4.6.2.

Proof of Lemma 4.6.1

The correspondences $Z(x)$ are étale on the generic fibre, it keeps the canonical bundle up to a power under pull-back and push-forward. Then the hodge bundles are eigenvalues of all Hecke operators up to translation of components. More precisely, one has

$$Z(x)\xi_\beta = (\deg Z(x))\xi_{\beta x}, \quad \forall x \in \mathbb{B}_f^\times,\ \beta \in F_+^\times\backslash\mathbb{A}_f^\times/q(U).$$

It follows that for any $\alpha \in F_+^\times\backslash\mathbb{A}_f^\times/q(U)$,

$$Z_*(g)_\alpha\ \xi_\beta = \deg Z_*(g)_\alpha\ \xi_{\alpha\beta} = -\frac{1}{2}\kappa_U J_*(0,g,r(h_\alpha^{-1})\phi)\xi_{\alpha\beta}.$$

Here h_α is any element of $\mathbb{B}_f^\times \times \mathbb{B}_f^\times$ whose similitude represents α. The last identity follows from Proposition 4.2.2. Now we immediately have the following result.

As for the lemma, we first look at $\langle Z_*(g)\xi_{t_1}, t_2\rangle$. It is a sum of $\langle Z_*(g)_\alpha\xi_{t_1}, t_2\rangle$, which is nonzero only if $Z_*(g)_\alpha\xi_{t_1}$ and t_2 lie in the same geometrically connected component of X_U. It is true if and only if $\alpha \in q(t_1^{-1}t_2)F_+^\times q(U)$. It follows that

$$\langle Z_*(g)\xi_{t_1}, t_2\rangle = \langle Z_*(g)_{q(t_1^{-1}t_2)}\xi_{t_1}, t_2\rangle = -\frac{1}{2}\kappa_U J_*(0,g,r(t_1,t_2)\phi)\langle\xi_{t_2}, t_2\rangle.$$

Furthermore, $\langle\xi_{t_2}, t_2\rangle = \langle\xi, t_2\rangle = \langle\xi, 1\rangle$ is independent of $t_2 \in C_U$ since all elements of C_U are Galois conjugate.

Similarly, we have

$$\langle Z_*(g)\xi_{t_1}, \xi_{t_2}\rangle = -\frac{1}{2}\kappa_U J_*(0,g,r(t_1,t_2)\phi)\langle\xi_{t_2}, \xi_{t_2}\rangle.$$

By the same reason, $\langle\xi_{t_2}, \xi_{t_2}\rangle$ is independent of $t_2 \in C_U$.

Remark. All the equalities here are true for the corresponding constant terms, i.e., we can replace Z_*, J_* by Z, J everywhere.

Arithmetic Hecke operators

The rest of this subsection is devoted to treat $\langle Z_*(g)t_1, \xi_{t_2}\rangle$ for Proposition 4.6.2. Our treatment depends on extensions of Hecke operators to the integral model. Let S be a set of bad places described before the proposition. Then the models $\mathscr{X}_U$ and $\mathscr{Y}_U$ are smooth away from S. We will have "good extension" of $Z(x)$ away from S.

For any $x \in \mathbb{B}_f^\times$, let $\mathscr{Z}(x)$ be the Zariski closure of $Z(x)$ in $\mathscr{X}_U \times_{O_F} \mathscr{X}_U$. If $x \in (\mathbb{B}_f^S)^\times$, then many good properties of $\mathscr{Z}(x)$ are obtained in [Zh1]. For example, it has a canonical moduli interpretation, and satisfies:

(1) $\mathscr{Z}(x_1)$ commutes with $\mathscr{Z}(x_2)$ for any $x_1, x_2 \in (\mathbb{B}_f^S)^\times$;

(2) $\mathscr{Z}(x) = \prod_{v\notin S}\mathscr{Z}(x_v)$ for any $x \in (\mathbb{B}_f^S)^\times$;

(3) For any $x \in (\mathbb{B}_f^S)^\times$, both structure projections from $\mathscr{Z}(x)$ to $\mathscr{X}_U$ are finite everywhere, and étale above the set of places v with $x_v \in U_v$.

Here in (2), $\mathscr{Z}(x_v)$ means $\mathscr{Z}(x_v 1^v)$, and the product is only for non-archimedean places. We also use this convention for $D(x_v)$ introduced below.

Fix $x \in (\mathbb{B}_f^S)^\times$. Define an arithmetic class $D(x)$ on $\mathscr{X}_U$ by

$$D(x) := \mathscr{Z}(x)\hat{\xi} - \widehat{\mathscr{Z}(x)\xi} = \mathscr{Z}(x)\hat{\xi} - \deg Z(x)\ \widehat{\xi}.$$

Then $D(x)$ is a vertical divisor since it is zero on the generic fibre. We claim that $D(x)$ is a *constant divisor*, i.e., the pull-back of an arithmetic divisor D on $\mathrm{Spec}(O_F)$. Then it only depends on $\widehat{\deg}(D)$, and sometimes we identify it with this number.

Now we explain why $D(x)$ is a constant divisor. First, $D(x)$ is constant at archimedean places because the Petersson metric on the upper half plane is invariant under the action of $\mathrm{GL}_2(\mathbb{R})_+$. Now we look at non-archimedean places. Both structure morphisms of $\mathscr{Z}(x)$ are étale above S, so the pull-back and push-forward keeps the relative dualizing sheaf above S. Then the finite part of $D(x)$ is lying above primes not in S. Note that $\mathscr{X}_U$ is smooth outside S, its special fibres outside S are irreducible. Hence the finite part of $D(x)$ is a linear combination of these special fibres which are constant.

The constant divisor $D(x)$ satisfies a "product rule":

$$D(x) = \sum_{v \notin S} \deg Z(x^v)\ D(x_v), \quad x \in (\mathbb{B}_f^S)^\times.$$

In fact, for any $v \notin S$,

$$\mathscr{Z}(x)\hat{\xi} = \mathscr{Z}(x^v)(\deg Z(x_v)\hat{\xi} + D(x_v)) = \deg Z(x_v)\ \mathscr{Z}(x^v)\hat{\xi} + \deg Z(x^v)\ D(x_v).$$

Here $\mathscr{Z}(x^v)D(x_v) = \deg Z(x^v)\ D(x_v)$ because $D(x_v)$ is a constant divisor. By induction on the places, it is easy to get

$$\mathscr{Z}(x)\ \hat{\xi} = \deg Z(x)\ \hat{\xi} + \sum_{v \notin S} \deg Z(x^v)\ D(x_v).$$

It gives the product rule.

For any finite place v and positive integer n, denote by

$$T(p_v^n) = \{x \in \mathbb{B}_v^\times : \mathrm{ord}_v q(x) = n\}.$$

Then $T(p_v^n)$ gives the usual Hecke correspondence as a finite sum of some $Z(x)$. In particular, it makes sense to talk about the extension $\mathscr{T}(p_v^n)$ to X_U if $v \notin S$. Denote

$$\widetilde{D}(p_v^n) := \mathscr{T}(p_v^n)\hat{\xi} - \widehat{T(p_v^n)\xi} = \mathscr{T}(p_v^n)\hat{\xi} - (\deg T(p_v^n))\widehat{\xi}.$$

Then

$$\widetilde{D}(p_v^n) = \sum_{x_v \in U_v \backslash T(p_v^n)/U_v} D(x_v)$$

is a constant arithmetic divisor on $\mathscr{X}_U$. The following result is a rephrase of Proposition 4.3.2 in [Zh1].

Lemma 4.6.3.
$$\widetilde{D}(p_v^n) = \frac{1}{\kappa_U}\left(n\sum_{i=0}^{n} N_v^i - 2\sum_{i=0}^{n} iN_v^{n-i}\right)\log N_v.$$

The two parts of Proposition 4.3.2 in [Zh1] give exactly the finite component and the infinite component of $\kappa_U D_v$. Here κ_U appears due to the normalization $\mathscr{L} = \kappa_U \xi$. In the same way, we can have an explicit formula of $D(x)$ for any $x \in \mathbb{B}_f^\times$, but we don't need it here.

The result perfectly matches the result in Lemma 3.7.3 for the derivative of Whittaker functions, and they will lead to infinitely many cancellations in the comparison below. We don't simplify both results so that we can see that the infinite part (resp. v-adic part) of $\widetilde{D}$ corresponds to the special value (resp. special derivative) of $J_{a,v}$.

The following result is a consequence of the product rule and the projection formula. It will be used to decompose the pairing $\langle Z_*(g)t_1, \xi_{t_2}\rangle$.

Lemma 4.6.4. *For any $x \in \mathbb{B}_f^\times$ and $D \in \mathrm{Div}(X_{U,\bar{F}})$, we have*

$$\langle Z(x)D, \xi\rangle = \deg Z(x^S)\ \langle Z(x_S)D, \xi\rangle - \deg_{\mathrm{tot}}(D)\sum_{v\notin S}\deg Z(x^v)\ D(x_v).$$

Here $D(x_v)$ is viewed as a constant, and $\deg_{\mathrm{tot}}(D)$ is the sum of the degrees of D on all geometrically connected components.

Proof. We first reduce the problem to the case that D is defined over F. In deed, since ξ and $Z(x)$ are defined over F, both sides of the equality are invariant under Galois actions on D. So it suffices to show the result for the sum of all Galois conjugates of D.

Assume that D is defined over F. By §4.4,

$$\langle Z(x)D, \xi\rangle = \langle \overline{Z(x)D},\ \hat{\xi}\rangle.$$

Here $\overline{Z(x)D}$ is the Zariski closure in $\mathscr{X}_U$, and we denote the normalized intersection

$$\langle D_1, D_2\rangle := -D_1 \cdot D_2$$

for any arithmetic divisors D_1, D_2 on $\mathscr{X}_U$.

Use the model $\mathscr{Z}(x^S)$ away from S. We have

$$\overline{Z(x)D} = \overline{Z(x^S)Z(x_S)D} = \mathscr{Z}(x^S)\overline{Z(x_S)D}.$$

The second equality holds because $\mathscr{Z}(x^S)$ keeps Zariski closure by finiteness of its structure morphisms. By the projection formula, we get

$$\langle Z(x)D, \xi\rangle = \langle \mathscr{Z}(x^S)\overline{Z(x_S)D},\ \hat{\xi}\rangle = \langle \overline{Z(x_S)D},\ \mathscr{Z}(x^S)^t\ \hat{\xi}\rangle.$$

Here $\mathscr{Z}(x^S)^t$ denotes the transpose of $\mathscr{Z}(x^S)$ as correspondences.

We claim that $\mathscr{Z}(x^S)^t\ \hat{\xi} = \mathscr{Z}(x^S)\hat{\xi}$ as elements in $\widehat{\mathrm{Pic}}(\mathscr{X}_U)_{\mathbb{C}}$. It suffices to consider $\mathscr{Z}(x_v)$ for any $v \notin S$. We first have $\mathscr{Z}(x_v)^t = \mathscr{Z}(x_v^{-1})$ by $Z(x_v)^t = Z(x_v^{-1})$. Since U_v is maximal, we know that $U_v x_v^{-1} U_v = q(x_v)^{-1} U_v x_v U_v$. Then $Z(x_v)$ is the composition of $Z(x_v)^t$ with $Z(q(x_v))$. Here $Z(q(x_v))$ acts right multiplication by $q(x_v)$, which gives a Galois automorphism on $\mathscr{X}_U$ by the reciprocity law. It suffices to show $\mathscr{Z}(q(x_v))$ acts trivially on $\hat{\xi}$, or equivalently the constant $D(q(x_v)) = 0$. It is true since the automorphism has a finite order.

Therefore, we have

$$\langle Z(x)D, \xi\rangle = \langle\overline{Z(x_S)D}, \mathscr{Z}(x^S)\hat{\xi}\rangle.$$

By the product rule,

$$\mathscr{Z}(x^S)\hat{\xi} = \deg Z(x^S)\ \hat{\xi} + \sum_{v\notin S} \deg Z(x^{S,v})\ D(x_v).$$

We obtain

$$\begin{aligned}\langle Z(x)D, \xi\rangle =& \deg Z(x^S)\ \langle\overline{Z(x_S)D}, \hat{\xi}\rangle + \sum_{v\notin S} \deg Z(x^{S,v})\ \langle\overline{Z(x_S)D}, D(x_v)\rangle \\ =& \deg Z(x^S)\ \langle Z(x_S)D, \xi\rangle - \deg_{\mathrm{tot}}(D) \sum_{v\notin S} \deg Z(x^v)\ D(x_v).\end{aligned}$$

Here

$$\langle\overline{Z(x_S)D}, D(x_v)\rangle = -\deg_{\mathrm{tot}}(D) \deg Z(x_S)\ D(x_v),$$

because $D(x_v)$ is a constant divisor.

□

Comparison and Approximation

Now we are ready to prove Proposition 4.6.2. By Galois action of t via the reciprocity law, we have

$$\langle Z_*(g)t_1, \xi_{t_2}\rangle = \langle Z_*(g)[t_1 t_2^{-1}], \xi_1\rangle.$$

On the other hand, we have

$$J(0, g, r(t_1, t_2)\phi) = J(0, g, r(t_1 t_2^{-1}, 1)\phi).$$

So it suffices to consider $\langle Z_*(g)t,\ \xi_1\rangle$ for any $t \in C_U$.

After separating components, we get

$$\langle Z_*(g)t,\ \xi_1\rangle = \langle Z_*(g)_{q(1/t)}t,\ \xi_1\rangle = \langle Z_*(g)_{q(1/t)}t,\ \xi\rangle.$$

It is also equal to $\langle Z_*(g)_{q(1/t)}[1],\ \xi\rangle$ by the Galois action. Hence it only depends on the geometrically connected component of t in X_U.

The following proposition is a direct consequence of Lemma 4.6.4. Before the statement, we introduce the notation

$$P(a) = \{x \in P : q(x) = a\}$$

for any subset P of a quadratic space defined over a ring R and any $a \in R$. For example,

$$\mathbb{B}_v(a) = \{x \in \mathbb{B}_v : q(x) = a\}, \quad a \in F_v.$$

It is compatible with the definition of $\mathbb{B}_f(a)$ in §4.2.

Proposition 4.6.5.

$$\langle Z_*(g)t, \xi_1\rangle = \mathscr{A}(g,t) - \sum_{v \notin S} \mathscr{D}(g,t)(v).$$

Here

$$\mathscr{A}(g,t) = \sum_{u \in \mu_U' \backslash F^\times} \sum_{y \in \mathbb{B}_f^{\mathrm{ad}}/U^1} r(g,(t,1))\phi(y,u)\langle y_S, \xi\rangle,$$

$$\mathscr{D}(g,t)(v) = \sum_{u \in \mu_U' \backslash F^\times} \sum_{a \in F^\times} \sum_{y \in \mathbb{B}_f^v(a)/U^{1,v}} r(g,(t,1))\phi^v(y,u) D_a(g,t,u,\phi_v),$$

with

$$D_a(g,t,u,\phi_v) = \sum_{y \in K_v^t \backslash \mathbb{B}_v(a)} r(g,(t,1))\phi_v(y,u) D(t_v^{-1}y).$$

Proof. Recall that in §4.2 we have obtained

$$Z_*(g)_{q(1/t)} \quad = \sum_{u \in \mu_U' \backslash F^\times} \sum_{y \in K^t \backslash \mathbb{B}_f^{\mathrm{ad}}} r(g,(t,1))\phi(y,u) Z(t^{-1}y).$$

Then Lemma 4.6.4 yields the decomposition

$$\langle Z_*(g)t, \xi_1\rangle = \mathscr{A}'(g,t) - \sum_{v \notin S} \mathscr{D}'(g,t)(v),$$

where

$$\mathscr{A}'(g,t) = \sum_{u \in \mu_U' \backslash F^\times} \sum_{y \in K^t \backslash \mathbb{B}_f^{\mathrm{ad}}} r(g,(t,1))\phi(y,u) \deg Z((t^S)^{-1}y^S)\ \langle Z(t_S^{-1}y_S)[1], \xi\rangle,$$

$$\mathscr{D}'(g,t)(v) = \sum_{u \in \mu_U' \backslash F^\times} \sum_{y \in K^t \backslash \mathbb{B}_f^{\mathrm{ad}}} r(g,(t,1))\phi(y,u) \deg Z((t^v)^{-1}y^v) D(t_v^{-1}y_v).$$

We only need to check that $\mathscr{A}'(g,t) = \mathscr{A}(g,t)$ and $\mathscr{D}'(g,t)(v) = \mathscr{D}(g,t)(v)$.

First look at $\mathscr{A}'(g,t)$. Follow the way of obtaining the formula for $Z_*(g)_\alpha[z,\beta]$ in §4.2. Use the coset identity $Ut^{-1}yU/U = t^{-1}(K^t y/U^1)$, and the equality $\langle t_S^{-1}y_S, \xi\rangle = \langle y_S, \xi\rangle$ induced by Galois conjugation. As for $\mathscr{D}'(g,t)(v)$, write it as

$$\mathscr{D}'(g,t)(v) = \sum_{u \in \mu_U' \backslash F^\times} \sum_{a \in F^\times} \sum_{y \in K^t \backslash \mathbb{B}_f(a)} r(g,(t,1))\phi(y,u) \deg Z((t^v)^{-1}y^v) D(t_v^{-1}y_v).$$

Use the identity $Ut^{-1}yU/U = t^{-1}(K^t y/U^1)$ away from v.

□

The following is a detailed version of Proposition 4.6.2.

Proposition 4.6.6. *Assume as in Proposition 4.6.2 that ϕ_v is standard for $v \notin S$ and degenerate for all $v \in S$.*

(1) If $v \notin S$, then

$$\mathscr{D}(g,t)(v) = -J'(0,g,r(t,1)\phi)(v).$$

(2) There exists a Schwartz function $\phi'_S \in \mathscr{S}(\mathbb{B}_S \times F_S^\times)$ such that

$$\mathscr{A}(g,t) = J(0,g,r(t,1)(\phi^S \otimes \phi'_S)), \quad \forall g \in 1_S \mathrm{GL}_2(\mathbb{A}^S).$$

(3) If $v \notin S$, then there exists a Schwartz function $\phi''_v \in \mathscr{S}(\mathbb{B}_v \times F_v^\times)$ such that

$$J'(0,g,r(t,1)\phi)(v) = J(0,g,r(t,1)(\phi^v \otimes \phi''_v)), \quad \forall g \in 1_v \mathrm{GL}_2(\mathbb{A}^v).$$

Recall that

$$J'(0,g,\phi)(v) = \frac{1}{\zeta_v(2)} \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{a \in F^\times} J^{\circ\prime}_{1,v}(0, d^*(a)g, au) J_a^v(0,g,u).$$

To prepare for the proof, we first rearrange the summations so that they has the same type as $\mathscr{D}(g,t)(v)$. The process is very similar to Proposition 4.2.2 except for special attention at the place v. Let $y_{au^{-1}}$ be any fixed element in $\mathbb{B}^v$ with norm a. By the local Siegel–Weil formula,

$$J_a^v(0,g,u) = -\gamma_v |a^v| \int_{\mathbb{B}^{1,v}} r(g)\phi^v(by_{au^{-1}},u)db = -\gamma_v \mathrm{vol}(U^{1,v})\mathrm{vol}(\mathbb{B}^1_\infty)|a^v| \sum_{y \in \mathbb{B}^v_f(au^{-1})/U^{1,v}} r(g)\phi^v(y$$

Here $\gamma_v = \pm 1$ is the Weil index for $(\mathbb{B}_v, q)$ which is 1 if and only if $\mathbb{B}_v$ is split. It follows that

$$\begin{aligned} &J'(0,g,\phi)(v)\\ =&-\gamma_v \frac{\mathrm{vol}(U^1)\mathrm{vol}(\mathbb{B}^1_\infty)}{\zeta_v(2)\mathrm{vol}(U_v^1)} \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{a \in F^\times} |a|_v^{-1} J^{\circ\prime}_{1,v}(0,d^*(a)g,au) \sum_{y \in \mathbb{B}^v_f(au^{-1})/U^{1,v}} r(g)\phi^v(y,u)\\ =&-\frac{2\kappa_U^{-1}\gamma_v}{\zeta_v(2)\mathrm{vol}(U_v^1)} \sum_{u \in \mu'_U \backslash F^\times} \sum_{a \in F^\times} |au|_v^{-1} J^{\circ\prime}_{1,v}(0,d^*(au)g,au^2) \sum_{y \in \mathbb{B}^v_f(a)/U^{1,v}} r(g)\phi^v(y,u). \end{aligned}$$

The summation has the same type as $\mathscr{D}(g,t)(v)$.

Now we are ready to prove (1). Since v is unramified, $\gamma_v = 1$ and $\mathrm{vol}(U_v^1) = \zeta_v(2)^{-1}$. We have

$$J'(0,g,\phi)(v) = -2\kappa_U^{-1} \sum_{u \in \mu'_U \backslash F^\times} \sum_{a \in F^\times} |au|_v^{-1} J^{\circ\prime}_{1,v}(0,d^*(au)g,au^2) \sum_{y \in \mathbb{B}^v_f(a)/U^{1,v}} r(g)\phi^v(y,u).$$

It suffices to show

$$D_a(g,t,u,\phi_v)=2\kappa_U^{-1}|au|_v^{-1}J_{1,v}^{\circ\prime}(0,d^*(au)g,au^2,r(t)\phi_v). \tag{4.6.1}$$

The problem is purely local, so all notations mean their local components in this part. We first consider the case $g=1$ and $t=1$. Then

$$D_a(1,1,u,\phi_v)=\sum_{y\in K_v^1\backslash \mathbb{B}_v(a)}\phi_v(y,u)D(y).$$

It is nonzero only if $u\in O_{F_v}^\times$ and $v(a)\geq 0$. In that case,

$$D_a(1,1,u,\phi_v)=\sum_{y\in K_v^1\backslash O_{\mathbb{B}_v}(a)}D(y)=\sum_{y\in K_v\backslash T(p_v^{v(a)})}D(y)=\widetilde{D}(p_v^{v(a)}).$$

Then (4.6.1) holds in this case by the explicit results in Lemma 4.6.3 and Lemma 3.7.3.

Now we can shown (4.6.1) for general (a,g,t,u). We have already confirmed the case $(a,1,1,u)$. It is easy to extend it to $(a,g,1,u)$ for all $g\in \mathrm{GL}_2(O_{F_v})$. In fact, $\mathrm{GL}_2(O_{F_v})$ acts trivially on ϕ_v, and it is easy to see from the definition that both sides of (4.6.1) are the same as the case $g=1$. To extend to general g, use Iwasawa decomposition and consider the action of $P(F_v)$. It suffices to consider the behavior of both sides under the change $(a,g,t,u)\to(a,g'g,t't,u)$ for $g'\in P(F_v)$ and $t'\in E_v^\times$. The following lemma says that both sides transform in the same way. It proves (1).

Lemma 4.6.7. *The function $D_a(g,t,u,\phi_v)$ transfers in the following way:*

$$\begin{aligned}
D_a(g,t't,u,\phi_v) &= D_{a/q(t')}(g,t,uq(t'),\phi_v), && t'\in E_v^\times;\\
D_a(n(b)g,t,u,\phi_v) &= \psi_v(uab)\ D_a(g,t,u,\phi_v), && b\in F_v;\\
D_a(m(c)g,t,u,\phi_v) &= |c|^2\ D_a(g,c^{-1}t,c^2u,\phi_v), && c\in F_v^\times;\\
D_a(d(c)g,t,u,\phi_v) &= |c|^{-1}\ D_a(g,t,c^{-1}u,\phi_v), && c\in F_v^\times.
\end{aligned}$$

Furthermore, $|au|_v^{-1}J_{1,v}^{\circ\prime}(0,d^*(au)g,au^2,r(t)\phi_v)$ transfers in the same way as a function of (a,g,t,u).

Proof. We should compare it with Lemma 4.6.7. We only check the first equality. By definition,

$$D_a(g,t't,u,\phi_v)=\sum_{y\in K_v^{t't}\backslash \mathbb{B}_v(a)}r(g,(t't,1))\phi_v(y,u)D((t't)^{-1}y).$$

Replace y by $t'y$ in the above summation. Then the domain $K_v^{t't}\backslash\mathbb{B}_v(a)$ becomes $K_v^{t}\backslash\mathbb{B}_v(aq(t')^{-1}$ Thus the sum is equal to

$$\sum_{y\in K_v^{t}\backslash \mathbb{B}_v(aq(t')^{-1})}r(g,(t't,1))\phi_v(t'y,u)D(ty).$$

It is equal to $D_{a/q(t')}(g,t,uq(t'),\phi_v)$. As for the second function, it suffices to check the results for

$$|au|_v^{-1}J_{1,v}(s,d^*(au)g,au^2,r(t)\phi_v)=|au|_v^{-\frac{s}{2}-1}J_{au,v}(s,g,u,r(t)\phi_v).$$

It can be done similarly. □

Now we prove (2). The approximation method is very similar to Corollary 3.4.2. Recall that

$$\mathscr{A}(g,t)=\sum_{u\in\mu_U'\backslash F^\times}\sum_{y\in\mathbb{B}_f^{\mathrm{ad}}/U^1}r(g,(t,1))\phi(y,u)\langle y_S,\xi\rangle.$$

If we did not have the extra factor $\langle y_S,\xi\rangle$, then the above would be exactly $\deg Z_*(g)_{q(1/t)}$, which is equal to the Eisenstein series $-\dfrac{1}{2}\kappa_U J_*(0,g,r(t,1)\phi)$ as in Proposition 4.2.2.

In the case here, the extra factor complicates the components at S, but has no impact at other places. Similar to Corollary 3.4.2, we introduce a new function

$$\phi_S'(y,u):=\phi_S(y,u)\langle y,\xi\rangle,\quad (y,u)\in\mathbb{B}_S^\times\times F_S^\times.$$

It is apparently a Schwartz function of $(y,u)\in\mathbb{B}_S^\times\times F_S^\times$. Since ϕ_S is degenerate, $\phi_S'(y,u)$ vanishes when $q(y)$ is closed to zero. Therefore, ϕ_S' actually extends (by zero) to a Schwartz function on $\mathbb{B}_S\times F_S^\times$. By Galois conjugation of t_S, we have

$$r(t_S,1)\phi_S'(y,u)=r(t_S,1)\phi_S(y,u)\langle y,\xi\rangle.$$

Consider the new series

$$\sum_{u\in\mu_U'\backslash F^\times}\sum_{y\in\mathbb{B}_f^{\mathrm{ad}}/U^1}r(g,(t,1))\phi^S(y,u)\ r(g,(t,1))\phi_S'(y,u).$$

It is equal to $-\frac{1}{2}\kappa_U J(0,g,r(t,1)(\phi_S'\otimes\phi^S))$. And it is equal to the original series if $g_S=1$. Thus we have proved that if $g_S=1$,

$$\mathscr{A}(g,t)\quad=\quad-\frac{1}{2}\kappa_U J_*(0,g,r(t,1)(\phi_S'\otimes\phi^S)).$$

The constant term of the right-hand side is a multiple of

$$\sum_{u\in\mu_U^2\backslash F^\times}r(g,(t,1))(\phi_S'\otimes\phi^S)(0,u).$$

It is automatically zero as long as S is non-empty. In fact, it has the factor $r(t,1)\phi_v'(0,u)=0$ by the degeneracy of ϕ_v. It follows that

$$\mathscr{A}(g,t)\quad=\quad-\frac{1}{2}\kappa_U J(0,g,r(t,1)(\phi_S'\otimes\phi^S)),\quad \forall g\in 1_S\mathrm{GL}_2(\mathbb{A}^S).$$

It proves (2).

Now we prove (3) by a similar method. Recall

$$J'(0,g,\phi)(v) = c_v \sum_{u\in\mu_U'\backslash F^\times} \sum_{a\in F^\times} |au|_v^{-1} J_{1,v}^{\circ\prime}(0,d^*(au)g,au^2) \sum_{y\in\mathbb{B}_f^v(a)/U^{1,v}} r(g)\phi^v(y,u).$$

Here c_v is some nonzero constant. We claim that there is a Schwartz function $\phi_v''\in\mathscr{S}(\mathbb{B}_v\times F_v^\times)$ invariant under some open compact subgroup $U_v'^1$ of $\mathbb{B}_v^1$ such that

$$|au|_v^{-1}J_{1,v}^{\circ\prime}(0,d^*(au),au^2) = \sum_{y\in\mathbb{B}_v(a)/U_v'^1}\phi_v''(y,u), \quad \forall\ a,u\in F_v^\times. \tag{4.6.2}$$

Assume that it is true. Then for $g_v=1$, the series $J'(0,g,\phi)(v)$ is equal to

$$c_v \sum_{u\in\mu_U'\backslash F^\times} \sum_{a\in F^\times} \sum_{y_v\in\mathbb{B}_v(a)/U_v'^1} r(g_v)\phi_v''(y_v,u) \sum_{y^v\in\mathbb{B}_f^v(a)/U^{1,v}} r(g)\phi^v(y^v,u).$$

This series is a scalar multiple of $J(0,g,\phi^v\otimes\phi_v'')$. We can further extend the identity to the extra variable t since both sides of (4.6.2) transfers in the same way by the action of t.

It remains to show (4.6.2). Note that the problem is purely local. For convenience, denote

$$W(a,u) := |au|_v^{-1}J_{1,v}^{\circ\prime}(0,d^*(au),au^2).$$

It is also equal to

$$|au|_v^{-1}J_{au,v}^{\circ\prime}(0,1,u) - \frac{1}{2}|au|_v^{-1}\log|au|_v J_{au,v}^{\circ}(0,1,u).$$

It is zero if $v(u)$ is too big or too small. By Lemma 3.7.3 (2), $W(a,u)=0$ identically for all $a\in F_v^\times$ satisfying $v(a)>c$ or $v(a)<-c$. This is the key to construct ϕ_v''.

If $\mathbb{B}_v$ is a division algebra, define

$$\phi_v''(y,u) = \begin{cases} W(q(y),u), & (y,u)\in\mathbb{B}_v^\times\times F_v^\times;\\ 0, & y=0,\ u\in F_v^\times. \end{cases}$$

It is a schwartz function on $\mathbb{B}_v\times F_v^\times$ by the vanishing results of $W(a,u)$. By definition, it satisfies (4.6.2) by setting $U_v'^1=\mathbb{B}_v^1$.

If $\mathbb{B}_v$ is a matrix algebra, the definition above does not give a Schwartz function on $\mathbb{B}_v\times F_v^\times$ since $\mathbb{B}_v^1$ is non-compact any more. Fix one open and compact *subset* A of $\mathbb{B}_v^\times$ such that $A_a=\{y\in A: q(y)=a\}$ is nonempty for all $a\in F_v^\times$ with $W(a,u)\neq 0$. Define

$$\phi_v''(y,u) = \begin{cases} \mathrm{vol}(A_{q(y)})^{-1}W(q(y),u), & y\in A,\ u\in F_v^\times;\\ 0, & \text{otherwise.} \end{cases}$$

The right-hand side of (4.6.2) is essentially an integral on $B_v(a)$. It is easy to see that some constant multiple of ϕ_v'' satisfies the requirement.

5 Local heights of CM points

In last section, we have made the decomposition:

$$\langle Z_*(g)t_1, t_2\rangle = -\sum_v i_v(Z_*(g)t_1, t_2)\log N_v - \sum_v j_v(Z_*(g)t_1, t_2)\log N_v.$$

We further have

$$i_v(Z_*(g)t_1, t_2) = \fint_{T(F)\backslash T(\mathbb{A}_f)} i_{\bar{v}}(Z_*(g)tt_1, tt_2)dt.$$

The decomposition is valid under the global degeneracy assumption that ϕ is degenerate at two different finite places v_1, v_2 of F which are non-split in E and $g \in P(F_{v_1,v_2})\mathrm{GL}_2(\mathbb{A}^{v_1,v_2})$. We will make this assumption throughout this section.

The main goal of this section is to compute $i_{\bar{v}}(Z_*(g)t_1, t_2)$ and compare it with the local analytic kernel function $\mathscr{K}_\phi^{(v)}(g,(t_1,t_2))$ appeared in the decomposition of $I'(0,g)$ representing the series $L'(1/2,\pi,\chi)$. We will follow the work of Gross–Zagier [GZ] and its extension in [Zh2]. We divide the situation to the following four cases:

- archimedean case: v is archimedean;
- supersingular case: v non-split in E but split in $\mathbb{B}$;
- superspecial case: v is non-split in both E and $\mathbb{B}$;
- ordinary case: v is split in both E and $\mathbb{B}$.

They are studied in §5.1-5.4 respectively. For archimedean v and "good" supersingular v, we obtain an explicit formula which agrees with the analytic kernel. For "bad" supersingular v and all superspecial v, we can still write the local kernel as a pseudo theta series, and approximate it by a usual theta series under the condition that ϕ_v is degenerate. In the ordinary case, we show that the local kernel vanishes under the global degeneracy assumption.

In the last section §5.5, we will show that $j_{\bar{v}}(Z_*(g)t_1, t_2)$ can also be approximated by theta series or Eisenstein series.

5.1 Archimedean case

In this subsection we want to describe local heights of CM points at any archimedean place v. Denote $B = B(v)$ and fix an identification $B(\mathbb{A}_f) = \mathbb{B}_f$. We will use the uniformization

$$X_{U,v}(\mathbb{C}) = B_+^\times\backslash\mathscr{H}\times\mathbb{B}_f^\times/U.$$

We follow the treatment of Gross-Zagier [GZ]. See also [Zh2].

Multiplicity function

For any two points $z_1, z_2 \in \mathscr{H}$, the hyperbolic cosine of the hyperbolic distance between them is given by

$$d(z_1, z_2) = 1 + \frac{|z_1 - z_2|^2}{2\mathrm{Im}(z_1)\mathrm{Im}(z_2)}.$$

It is invariant under the action of $\mathrm{GL}_2(\mathbb{R})$. For any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, denote

$$m_s(z_1, z_2) = Q_s(d(z_1, z_2)),$$

where

$$Q_s(t) = \int_0^\infty \left(t + \sqrt{t^2 - 1}\cosh u\right)^{-1-s} du$$

is the Legendre function of the second kind. Note that

$$Q_0(1 + 2\lambda) = \frac{1}{2}\log(1 + \frac{1}{\lambda}), \quad \lambda > 0.$$

We see that $m_0(z_1, z_2)$ has the right logarithmic singularity.

For any two distinct points of

$$X_{U,v}(\mathbb{C}) = B_+^\times \backslash \mathscr{H} \times B^\times(\mathbb{A}_f)/U$$

represented by $(z_1, \beta_1), (z_2, \beta_2) \in \mathscr{H} \times B^\times(\mathbb{A}_f)$, we denote

$$g_s((z_1, \beta_1), (z_2, \beta_2)) = \sum_{\gamma \in \mu_U \backslash B_+^\times} m_s(z_1, \gamma z_2)\ 1_U(\beta_1^{-1}\gamma\beta_2).$$

It is easy to see that the sum is independent of the choice of the representatives $(z_1, \beta_1), (z_2, \beta_2)$, and hence defines a pairing on $X_{U,v}(\mathbb{C})$. Then the local height is given by

$$i_{\bar{v}}((z_1, \beta_1), (z_2, \beta_2)) = \widetilde{\lim}_{s \to 0}\ g_s((z_1, \beta_1), (z_2, \beta_2)).$$

Here $\widetilde{\lim}_{s \to 0}$ denotes the constant term at $s = 0$ of $g_s((z_1, \beta_1), (z_2, \beta_2))$, which converges for $\mathrm{Re}(s) > 0$ and has meromorphic continuation to $s = 0$ with a simple pole.

The definition above uses adelic language, but it is not hard to convert it to the classical language. We first observe that $g_s((z_1, \beta_1), (z_2, \beta_2)) \neq 0$ only if there is a $\gamma_0 \in B_+^\times$ such that $\beta_1 \in \gamma_0\beta_2 U$, which just means that $(z_1, \beta_1), (z_2, \beta_2)$ are in the same connected component. Assuming this, then $(z_2, \beta_2) = (z_2', \beta_1)$ where $z_2' = \gamma_0 z_2$. we have

$$g_s((z_1, \beta_1), (z_2, \beta_2)) = g_s((z_1, \beta_1), (z_2', \beta_1)) = \sum_{\gamma \in \mu_U \backslash B_+^\times} m_s(z_1, \gamma z_2')\ 1_U(\beta_1^{-1}\gamma\beta_1) = \sum_{\gamma \in \mu_U \backslash \Gamma} m_s(z_1, \gamma z$$

Here we denote $\Gamma = B_+^\times \cap \beta_1 U \beta_1^{-1}$. The connected component of these two points is exactly

$$B_+^\times \backslash \mathscr{H} \times B_+^\times \beta_1 U/U \approx \Gamma \backslash \mathscr{H}, \qquad (z, b\beta_1 U) \mapsto b^{-1}z.$$

The stabilizer of $\mathscr{H}$ in Γ is exactly $\Gamma \cap F^\times = \mu_U$. Now we see that the formula is the same as those in [GZ] and [Zh2].

Next we consider the special case of CM points. For any $\gamma \in B_{v,+}^\times - E_v^\times$, we have

$$1 + \frac{|z_0 - \gamma z_0|^2}{2\mathrm{Im}(z_0)\mathrm{Im}(\gamma z_0)} = 1 - 2\lambda(\gamma).$$

Here $\lambda(\gamma) = q(\gamma_2)/q(\gamma)$ is introduced at the end of the introduction.

Thus it is convenient to denote

$$m_s(\gamma) = Q_s(1 - 2\lambda(\gamma)), \qquad \gamma \in B_v^\times - E_v^\times.$$

For any two distinct CM points $\beta_1, \beta_2 \in \mathrm{CM}_U$, we obtain

$$g_s(\beta_1, \beta_2) = \sum_{\gamma \in \mu_U \backslash B_+^\times} m_s(\gamma)\ 1_U(\beta_1^{-1}\gamma\beta_2),$$

and

$$i_{\bar{v}}(\beta_1, \beta_2) = \widetilde{\lim}_{s\to 0} g_s(\beta_1, \beta_2).$$

Note that $m_s(\gamma)$ is not well-defined for $\gamma \in E^\times$. The above summation is understood to be

$$g_s(\beta_1, \beta_2) = \sum_{\gamma \in \mu_U \backslash B_+^\times,\ \beta_1^{-1}\gamma\beta_2 \in U} m_s(\gamma).$$

Then it still makes sense because $\beta_1 \neq \beta_2$ implies that $\beta_1^{-1}\gamma\beta_2 \notin U$ for all $\gamma \in E^\times$. Anyway, it is safer to write

$$g_s(\beta_1, \beta_2) = \sum_{\gamma \in \mu_U \backslash (B_+^\times - E^\times)} m_s(\gamma)\ 1_U(\beta_1^{-1}\gamma\beta_2).$$

Now the right-hand side is well-defined even for $\beta_1 = \beta_2$. In this case it can be interpreted as contribution of some local height by an arithmetic adjunction formula, but we don't need this fact here since there is no self-intersection in this paper by our degeneracy assumption of Schwartz functions.

Kernel function

We are going to compute the local height

$$i_{\bar{v}}(Z_*(g)t_1, t_2) \ = \ \sum_{a \in F^\times} \sum_{x \in \mathbb{B}_f^\times/U} r(g)\phi(x)_a i_{\bar{v}}(t_1 x, t_2).$$

It is well-defined under the global degeneracy assumption which kills the self-intersections. The goal is to show that it is equal to $\overline{\mathscr{K}}_\phi^{(v)}(g, (t_1, t_2))$ obtained in Proposition 3.7.1. We still assume the global degeneracy assumption.

Proposition 5.1.1. *For any* $t_1, t_2 \in C_U$,

$$i_{\bar v}(Z_*(g)t_1, t_2) := \sum_{a\in F^\times} \widetilde{\lim}_{s\to 0} \sum_{y\in\mu_U\backslash(B_+^\times - E^\times)} r(g,(t_1,t_2))\phi(y)_a m_s(y).$$

In particular, $i_{\bar v}(Z_*(g)t_1, t_2) = \overline{\mathscr{K}}_\phi^{(v)}(g,(t_1,t_2))$.

Proof. By the above formula,

$$\begin{aligned} i_{\bar v}(Z_*(g)t_1, t_2) &= \sum_{a\in F^\times}\sum_{x\in\mathbb{B}_f^\times/U} r(g)\phi(x)_a \,\widetilde{\lim}_{s\to 0} \sum_{\gamma\in\mu_U\backslash(B_+^\times - E^\times)} m_s(\gamma)\ 1_U(x^{-1}t_1^{-1}\gamma t_2) \\ &= \sum_{a\in F^\times} \widetilde{\lim}_{s\to 0} \sum_{\gamma\in\mu_U\backslash(B_+^\times - E^\times)} r(g)\phi(t_1^{-1}\gamma t_2)_a m_s(\gamma) \\ &= \sum_{a\in F^\times} \widetilde{\lim}_{s\to 0} \sum_{\gamma\in\mu_U\backslash(B_+^\times - E^\times)} r(g,(t_1,t_2))\phi(\gamma)_a m_s(\gamma). \end{aligned}$$

Here the second equality is obtained by replacing x by $t_1^{-1}\gamma t_2$. □

We want to compare the above result with the holomorphic projection

$$\overline{\mathscr{K}}_\phi^{(v)}(g,(t_1,t_2)) = \sum_{a\in F^\times} \widetilde{\lim}_{s\to 0} \sum_{y\in\mu_U\backslash(B_+^\times - E^\times)} r(g,(t_1,t_2))\phi(y)_a\ k_{v,s}(y)$$

computed in Proposition 3.7.1.

It amounts to compare

$$m_s(y) = Q_s(1-2\lambda(y))$$

with

$$k_{v,s}(y) = \frac{\Gamma(s+1)}{2(4\pi)^s}\int_1^\infty \frac{1}{t(1-\lambda_v(y)t)^{s+1}}dt.$$

By the result of Gross-Zagier,

$$\int_1^\infty \frac{1}{t(1-\lambda t)^{s+1}}dt = 2Q_s(1-2\lambda) + O(|\lambda|^{-s-2}), \quad \lambda\to-\infty,$$

and the error term vanishes at $s=0$. We conclude that

$$\overline{\mathscr{K}}_\phi^{(v)}(g,(t_1,t_2)) = i_{\bar v}(Z_*(g)t_1, t_2).$$

5.2 Supersingular case

Let v be a finite prime of F non-split in E but split in $\mathbb{B}$. We consider the local pairing $i_{\bar v}$, which depends on the fixed embeddings $H\subset \bar E\subset \bar E_v$ and the model $\mathscr{Y}_U$ over O_H. It is actually depends only on the local integral model $\mathscr{Y}_{U,w} = \mathscr{Y}_U\times_{O_H} O_{H_w}$ where w is the place of H induced by the embeddings. We will use the local multiplicity functions treated in Zhang [Zh2]. For more details, we refer to that paper.

Multiplicity function

Let $B = B(v)$ be the nearby quaternion algebra over F. Make an identification $B(\mathbb{A}^v) = \mathbb{B}^v$. Then the set of supersingular points on X_K over v is parameterized by

$$\mathbb{S}_U = B^\times \backslash (F_v^\times / \det(U_v)) \times (\mathbb{B}_f^{v\times}/U^v).$$

We have a natural isomorphism

$$\bar{v} : \mathrm{CM}_U = E^\times \backslash \mathbb{B}_f^\times / U \longrightarrow B^\times \backslash (B^\times \times_{E^\times} \mathbb{B}_v^\times / U_v) \times \mathbb{B}_f^{v\times}/U^v$$

sending β to $(1, \beta_v, \beta^v)$. The reduction map $\mathrm{CM}_U \to \mathbb{S}_U$ is given by taking norm on the first factor:

$$q : B^\times \times_{E^\times} \mathbb{B}_v^\times \longrightarrow F_v^\times, \quad (b, \beta) \longmapsto q(b)q(\beta).$$

The intersection pairing is given by a multiplicity function m on

$$\mathscr{H}_{U_v} := B_v^\times \times_{E_v^\times} \mathbb{B}_v^\times / U_v.$$

More precisely, the intersection of two points $(b_1, \beta_1), (b_2, \beta_2) \in \mathscr{H}_{U_v}$ is given by

$$g_v((b_1, \beta_1), (b_2, \beta_2)) = m(b_1^{-1} b_2, \beta_1^{-1} \beta_2).$$

The multiplicity function m is defined everywhere in $\mathscr{H}_{U_v}$ except at the image of $(1, 1)$. It satisfies the property

$$m(b, \beta) = m(b^{-1}, \beta^{-1}).$$

Lemma 5.2.1. *For any two distinct CM-points $\beta_1 \in \mathrm{CM}_U$ and $t_2 \in C_U$, their local height is given by*

$$i_{\bar{v}}(\beta_1, t_2) = \sum_{\gamma \in \mu_U \backslash B^\times} m(\gamma t_{2v}, \beta_{1v}^{-1}) 1_{U^v}((\beta_1^v)^{-1} \gamma t_2^v).$$

Proof. Like the archimedean case, we compute the height by pulling back to $\mathscr{H}_{U_v} \times \mathbb{B}_f^{v\times}$. The height is the sum over $\gamma \in \mu_U \backslash B^\times$ of the intersection of $(1, \beta_{1v}, \beta_1^v)$ with $\gamma(1, t_{2v}, t_2^v) = (\gamma, t_{2v}, \gamma t_2^v) = (\gamma t_{2v}, 1, \gamma t_2^v)$ on $\mathscr{H}_{U_v} \times \mathbb{B}_f^{v\times}$. □

Analogous to the archimedean case, the summation is well-defined for all $\beta_1 \neq t_2$. Indeed, assume that there is a $\gamma \in E^\times$ such that $(\beta_1^v)^{-1} \gamma t_2^v \in U^v$ and $m(\gamma t_{2v}, \beta_{1v}^{-1})$ is not well-defined. Then we must have $\gamma t_{2v} \in E_v^\times$ and $\beta_{1v}^{-1} \gamma t_{2v} \in U^v$. It forces $\gamma \in E^\times$ and $\gamma t_2 \in \beta_1 U$, which implies that $\beta_1 = t_2 \in \mathrm{CM}_U$.

The kernel function

Now we compute

$$i_{\bar{v}}(Z_*(g) t_1, t_2) = \sum_{a \in F^\times} \sum_{x \in \mathbb{B}_f^\times / U} r(g)\phi(x)_a i_{\bar{v}}(t_1 x, t_2).$$

As in the archimedean case, we assume that ϕ is degenerate at two different finite places v_1, v_2 of F which are non-split in E, and only consider $g \in P(F_{v_1,v_2})\mathrm{GL}_2(\mathbb{A}^{v_1,v_2})$.

By the above formula above, we have

$$i_{\bar v}(Z_*(g)t_1,t_2) \quad = \quad \sum_{a\in F^\times}\sum_{x\in \mathbb{B}_f^\times/U} r(g)\phi(x)_a \sum_{\gamma\in\mu_U\backslash B^\times} m(\gamma t_2, x^{-1}t_1^{-1})1_{U^v}(x^{-1}t_1^{-1}\gamma t_2).$$

Replace x^v by $t_1^{-1}\gamma t_2$ and then get

$$\begin{aligned}
&\sum_{a\in F^\times}\sum_{\gamma\in\mu_U\backslash B^\times} r(g)\phi^v(t_1^{-1}\gamma t_2)_a \sum_{x_v\in\mathbb{B}_v^\times/U_v} r(g)\phi_v(x_v)_a m(t_1^{-1}\gamma t_2, x^{-1})\\
=\ &\sum_{u\in\mu_U^2\backslash F^\times}\sum_{\gamma\in B^\times} r(g,(t_1,t_2))\phi^v(\gamma,u)\sum_{x_v\in\mathbb{B}_v^\times/U_v} r(g)\phi_v(x_v, uq(\gamma)/q(x_v))m(t_1^{-1}\gamma t_2, x^{-1}).
\end{aligned}$$

For convenience, we introduce:

Notation.

$$\begin{aligned}
m_{\phi_v}(y,u) &= \int_{\mathbb{B}_v^\times} m(y,x^{-1})\phi_v(x, uq(y)/q(x))dx\\
&= \int_{\mathbb{B}_v^\times} m(y^{-1},x)\phi_v(x, uq(y)/q(x))dx, \quad (y,u)\in(B_v-E_v)\times F_v^\times.
\end{aligned}$$

Notice that $m_{\phi_v}(y,u)$ is well-defined for $y\notin E_v$ since $m(y,x^{-1})$ has no singularity for such y. By this notation, we obtain:

Proposition 5.2.2.

$$i_{\bar v}(Z_*(g)t_1,t_2) = \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in B-E} r(g,(t_1,t_2))\phi^v(y,u)\ m_{r(g,(t_1,t_2))\phi_v}(y,u).$$

Here we can change the summation to $y\in B-E$ since the contribution of $y\in E$ is zero by the degeneracy assumption on our Schwartz functions. We should compare the following result with Lemma 3.3.2 for $k_{\phi_v}(g,y,u)$.

Lemma 5.2.3. *The function $m_{\phi_v}(y,u)$ behaves like Weil representation under the action of $P(F_v)\times(E_v^\times\times E_v^\times)$ on (y,u). Namely,*

$$m_{r(g,(t_1,t_2))\phi_v}(y,u) = r(g,(t_1,t_2))m_{\phi_v}(y,u), \quad (g,(t_1,t_2))\in P(F_v)\times(E_v^\times\times E_v^\times).$$

More precisely,

$$\begin{aligned}
m_{r(m(a))\phi_v}(y,u) &= |a|^2 m_{\phi_v}(ay,u), \quad a\in F_v^\times\\
m_{r(n(b))\phi_v}(y,u) &= \psi(buq(y))m_{\phi_v}(y,u), \quad b\in F_v\\
m_{r(d(c))\phi_v}(y,u) &= |c|^{-1}m_{\phi_v}(y,c^{-1}u), \quad c\in F_v^\times\\
m_{r(t_1,t_2)\phi_v}(g,y,u) &= m_{\phi_v}(t_1^{-1}yt_2, q(t_1t_2^{-1})u), \quad (t_1,t_2)\in E_v^\times\times E_v^\times
\end{aligned}$$

Proof. They follow from basic properties of the multiplicity function $m(x, y)$. We only verify the first identity:

$$\begin{aligned} m_{r(m(a))\phi_v}(y,u) &= \int_{\mathrm{GL}_2(F_v)} m(y^{-1},x) r(g_v)\phi_v(ax, uq(y)/q(x))|a|^2 dx \\ &= |a|^2 \int_{\mathrm{GL}_2(F_v)} m(y^{-1},a^{-1}x) r(g_v)\phi_v(x, uq(y)/q(a^{-1}x)) dx \\ &= |a|^2 \int_{\mathrm{GL}_2(F_v)} m((ay)^{-1},x) r(g_v)\phi_v(x, uq(ay)/q(x)) dx \\ &= |a|^2 m_{\phi_v}(ay,u). \end{aligned}$$

□

Unramified Case

Fixing an isomorphism $\mathbb{B}_v = M_2(F_v)$. In this subsection we compute $m_{\phi_v}(y, u)$ in the following unramified case:

1. ϕ_v is the characteristic function of $M_2(O_{F_v}) \times O_{F_v}^\times$;
2. U_v is the maximal compact subgroup $\mathrm{GL}_2(O_{F_v})$.

By [Zh2], there is a decomposition

$$\mathrm{GL}_2(F_v) = \coprod_{c=0}^{\infty} E_v^\times h_c \mathrm{GL}_2(O_{F_v}), \qquad h_c = \begin{pmatrix} 1 & 0 \\ 0 & \varpi^c \end{pmatrix} \tag{5.2.1}$$

We may assume that $\beta = h_c$. The following result is Lemma 5.5.2 in [Zh2]. There is a small mistake in the original statement. Here is the corrected one.

Lemma 5.2.4. *The multiplicity function $m(b, \beta) \neq 0$ only if $q(b)q(\beta) \in O_{F_v}^\times$. In this case, assume that $\beta \in E_v^\times h_c \mathrm{GL}_2(O_{F_v})$. Then*

$$m(b,\beta) = \begin{cases} \frac{1}{2}(\mathrm{ord}_v \lambda(b) + 1) & \text{if } c = 0; \\ N_v^{1-c}(N_v+1)^{-1} & \text{if } c > 0,\ E_v/F_v \text{ is unramified;} \\ \frac{1}{2} N_v^{-c} & \text{if } c > 0,\ E_v/F_v \text{ is ramified.} \end{cases}$$

Proposition 5.2.5. *The function $m_{\phi_v}(y, u) \neq 0$ only if $(y, u) \in O_{B_v} \times O_{F_v}^\times$. In this case,*

$$m_{\phi_v}(y,u) = \frac{1}{2}(\mathrm{ord}_v q(y_2) + 1).$$

Here $y = y_1 + y_2$ is the orthogonal decomposition introduced at the end of the introduction.

Proof. We will use Lemma 5.2.4. Recall that

$$m_{\phi_v}(y,u)=\sum_{x\in \mathrm{GL}_2(F_v)/U_v} m(y^{-1},x)\phi_v(x,uq(y)/q(x)).$$

Note that $m(y^{-1},x)\neq 0$ only if $\mathrm{ord}_v(q(x)/q(y))=0$. Under this condition, $\phi_v(x,uq(y)/q(x))\neq 0$ if and only if $u\in O_{F_v}^\times$ and $x\in M_2(O_{F_v})$. It follows that $m_{\phi_v}(y,u)\neq 0$ only if $u\in O_{F_v}^\times$ and $n=\mathrm{ord}(q(y))\geq 0$. Assuming these two conditions, we have

$$m_{\phi_v}(y,u)=\sum_{x\in M_2(O_{F_v})_n/U_v} m(y^{-1},x),$$

where $M_2(O_{F_v})_n$ denotes the set of integral matrices whose determinants have valuation n. Now we use decomposition (2.3), we obtain

$$m_{\phi_v}(y,u)=\sum_{c=0}^{\infty} m(y^{-1},h_c)\mathrm{vol}(E_v^\times h_c\mathrm{GL}_2(O_{F_v})\cap M_2(O_{F_v})_n).$$

We first consider the case that E_v/F_v is unramified. The set in the right hand side is non-empty only if $n-c$ is even and non-negative. In this case it is given by

$$\varpi^{(n-c)/2}O_{E_v}^\times h_c U_v.$$

The volume of this set is 1 if $c=0$ and $N_v^{c-1}(N_v+1)$ if $c>0$ by the computation of [Zh2, p. 101]. It follows that, for $c>0$ with $2\mid(n-c)$,

$$m(y^{-1},h_c)\mathrm{vol}(E_v^\times h_c\mathrm{GL}_2(O_{F_v})\cap M_2(O_{F_v})_n)=1.$$

If n is even,

$$m_{\phi_v}(y,u)=\frac{1}{2}(\mathrm{ord}_v\lambda(y)+1)+\frac{n}{2}=\frac{1}{2}(\mathrm{ord}_v q(y_2)+1).$$

If n is odd, $m_{\phi_v}(y,u)=\frac{1}{2}(n+1)$. It is easy to see that $\mathrm{ord}_v q(y_1)$ is even and $\mathrm{ord}_v q(y_2)$ is odd. Then $n=\mathrm{ord}_v q(y_2)$, since $n=\mathrm{ord}_v q(y)=\min\{\mathrm{ord}_v q(y_1),\mathrm{ord}_v q(y_2)\}$ is odd. We still get $m_{\phi_v}(y,u)=\frac{1}{2}(\mathrm{ord}_v q(y_2)+1)$ in this case.

Now assume that E_v/F_v is ramified. Then the condition that $2\mid(n-c)$ is unnecessary, and $\mathrm{vol}(E_v^\times h_c\mathrm{GL}_2(O_{F_v})\cap M_2(O_{F_v})_n)=N_v^c$. Thus

$$m_{\phi_v}(y,u)=\frac{1}{2}(\mathrm{ord}_v\lambda(y)+1)+n\cdot\frac{1}{2}=\frac{1}{2}(\mathrm{ord}_v q(y_2)+1).$$

□

We immediately see that in the unramified case, m_{ϕ_v} matches the analytic kernel k_{ϕ_v} computed in Proposition 3.4.1.

Proposition 5.2.6. *Let v be unramified as above with further conditions:*

- *E_v/F_v is ramified;*
- *the local different d_v is trivial.*

Then

$$k_{r(t_1,t_2)\phi_v}(g,y,u) = m_{r(g,(t_1,t_2))\phi_v}(y,u)\log N_v,$$

and thus

$$\mathscr{K}_\phi^{(v)}(g,(t_1,t_2)) = i_{\bar{v}}(Z_*(g)t_1,t_2)\log N_v.$$

Proof. The case $(g,t_1,t_2)=(1,1,1)$ follows from the above result and Corollary 3.4.1. It is also true for $g \in \mathrm{GL}_2(O_{F_v})$ since it is easy to see that such g has the same kernel functions as 1 for standard ϕ_v. For the general case, apply the action of $P(F_v)$ and $E_v^\times \times E_v^\times$. The equality follows from Proposition 3.3.2 and Lemma 5.2.3. □

Ramified case

Now we consider general U_v. By Proposition 5.2.5 for the unramified case, we know that m_{ϕ_v} may have logarithmic singularity around the boundary $E_v \times F_v^\times$. The singularity is caused by self-intersections in the computation of local multiplicity. However, we will see that there is no singularity if $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$ is degenerate.

Proposition 5.2.7. *Assume that $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$ and it is invariant under the right action of U_v. Then $m_{\phi_v}(y,u)$ can be extended to a Schwartz function for $(y,u) \in B_v \times F_v^\times$.*

Proof. By the choice of ϕ_v, there is a constant $c>0$ such that $\phi_v(x,u)\neq 0$ only if $-c < v(q(x)) < c$ and $-c < v(u) < c$. Recall that

$$m_{\phi_v}(y,u) = \int_{\mathbb{B}_v^\times} m(y^{-1},x)\phi_v(x,uq(y)/q(x))dx.$$

In order that $m(y^{-1},x)\neq 0$, we have to make $q(y)/q(x) \in q(U_v)$ and thus $v(q(y)) = v(q(x))$. It follows that $\phi_v(x,uq(y)/q(x)) \neq 0$ only if $-c < v(u) < c$. The same is true for $m_{\phi_v}(y,u)$ by looking at the integral above. Then it is easy to see that $m_{\phi_v}(y,u)$ is Schwartz for $u \in F_v^\times$.

On the other hand, $m_{\phi_v}(y,u)\neq 0$ only if $-c < v(q(y)) < c$, since $\phi_v(x,uq(y)/q(x))\neq 0$ only if $-c < v(q(x)) < c$. Extend m_{ϕ_v} to $B_v \times F^\times$ by taking zero outside $B_v^\times \times F^\times$. We only need to show that it is locally constant in $B_v^\times \times F^\times$.

We have $\phi_v(E_vU_v, F_v^\times)=0$ by the degeneracy assumption and the invariance of ϕ_v under U_v. Thus

$$m_{\phi_v}(y,u) = \int_{\mathbb{B}_v^\times} m(y^{-1},x)(1-1_{E_v^\times U_v}(x))\phi_v(x,uq(y)/q(x))dx.$$

It is locally constant in $B_v^\times \times F_v^\times$, since $m(y^{-1},x)(1-1_{E_v^\times U_v}(x))$ is locally constant as a function on $B_v^\times \times \mathbb{B}_v^\times$. This completes the proof. □

As in the analytic case, we want to approximate the above pseudo-theta series for $i_{\bar{v}}(Z_*(g)t_1, t_2)$ by the usual theta series

$$\theta(g, (t_1, t_2), m_{\phi_v} \otimes \phi^v) = \sum_{u \in \mu_K^2 \backslash F^\times} \sum_{y \in V} r(g, (t_1, t_2)) m_{\phi_v}(y, u)\; r(g, (t_1, t_2)) \phi^v(y, u).$$

The following result is parallel to Corollary 3.4.2.

Corollary 5.2.8. *Assume the the global degeneracy assumption and that ϕ is degenerate at v. Then*

$$i_{\bar{v}}(Z_*(g, \phi)t_1, t_2) = \theta(g, (t_1, t_2), m_{\phi_v} \otimes \phi^v), \quad \forall g \in 1_{v, v_1, v_2} \mathrm{GL}_2(\mathbb{A}^{v, v_1, v_2}).$$

5.3 Superspecial case

Let v be a finite prime of F non-split in both $\mathbb{B}$ and E, and we consider the local height $i_{\bar{v}}(Z_*(g)t_1, t_2)$. The Shimura curve always has a bad reduction at v due to the ramification of the quaternion algebra. We only control the singularities as in Proposition 5.2.7. It is enough for approximation since there are only finitely many places non-split in $\mathbb{B}$.

As in the supersingular case, we assume that ϕ is degenerate at two different finite places v_1, v_2 of F which are non-split in E, and only consider $g \in P(F_{v_1, v_2}) \mathrm{GL}_2(\mathbb{A}^{v_1, v_2})$. Most of the definitions and computations are similar to the supersingular case, and we will mainly them. Meanwhile, we will pay special attention to the parts that are different to the supersingular case.

Kernel function

Denote by $B = B(v)$ the nearby quaternion algebra. We fix identifications $B_v \simeq M_2(F_v)$ and $B(\mathbb{A}_f^v) \simeq \mathbb{B}_f^v$. The intersection pairing is given by a multiplicity function m on

$$\mathscr{H}_{U_v} := B_v^\times \times_{E_v^\times} \mathbb{B}_v^\times / U_v.$$

More precisely, the intersection of two points $(b_1, \beta_1), (b_2, \beta_2) \in \mathscr{H}_{U_v}$ is given by

$$g_v((b_1, \beta_1), (b_2, \beta_2)) = m(b_1^{-1} b_2, \beta_1^{-1} \beta_2).$$

The multiplicity function m is defined everywhere on $\mathscr{H}_{U_v}$ except at the image of $(1, 1)$. It satisfies the property

$$m(b, \beta) = m(b^{-1}, \beta^{-1}).$$

For any two distinct CM-points $\beta_1 \in \mathrm{CM}_U$ and $t_2 \in C_U$, their local height is given by

$$i_{\bar{v}}(\beta_1, t_2) = \sum_{\gamma \in \mu_U \backslash B^\times} m(\gamma t_{2v}, \beta_{1v}^{-1}) 1_{U^v}((\beta_1^v)^{-1} \gamma t_2^v).$$

Analogous to Proposition 5.2.2, we have the following result.

Proposition 5.3.1. *For $g \in P(F_{v_1,v_2})\mathrm{GL}_2(\mathbb{A}^{v_1,v_2})$,*

$$i_{\bar{v}}(Z_*(g)t_1, t_2) = \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{y \in B-E} r(g,(t_1,t_2))\phi^v(y,u)\ m_{r(g,(t_1,t_2))\phi_v}(y,u).$$

Here we use the same notation:

$$m_{\phi_v}(y,u) = \int_{\mathbb{B}_v^\times} m(y^{-1},x)\phi_v(x, uq(y)/q(x))dx, \quad (y,u) \in (B_v - E_v) \times F_v^\times.$$

Lemma 5.2.3 is still true. It says that the action of $P(F_v) \times (E_v^\times \times E_v^\times)$ on $m_{r(g,(t_1,t_2))\phi_v}(y,u)$ behaves like Weil representation.

The following is a basic result used to control the singularity of the series. Its proof is given in the next two subsections.

Lemma 5.3.2. *(1) If v is unramified in E, then $m(b,\beta) \neq 0$ only if*

$$\mathrm{ord}_v(q(b)q(\beta)) = 0, \quad b \in F_v^\times \mathrm{GL}_2(O_{F_v}).$$

(1) If v is ramified in E, then $m(b,\beta) \neq 0$ only if

$$\mathrm{ord}_v(q(b)q(\beta)) = 0, \quad b \in F_v^\times \mathrm{GL}_2(O_{F_v}) \bigcup \begin{pmatrix} & 1 \\ \varpi_v & \end{pmatrix} F_v^\times \mathrm{GL}_2(O_{F_v}).$$

The main result below is parallel to Proposition 5.2.7.

Proposition 5.3.3. *Assume $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$ is invariant under the right action of U_v. Then $m_{\phi_v}(y,u)$ can be extended to a Schwartz function for $(y,u) \in B_v \times F_v^\times$.*

Proof. The proof is very similar to Proposition 5.2.7. By the argument of Proposition 5.2.7, there is a constant $C > 0$ such that $m_{\phi_v}(y,u) \neq 0$ only if $-C < v(q(y)) < C$ and $-C < v(u) < C$. Extend m_{ϕ_v} to $B_v \times F^\times$ by taking zero outside $B_v^\times \times F^\times$. The same method shows that it is locally constant on $B_v^\times \times F^\times$. It is compactly supported in y by Lemma 5.3.2 since $v(q(y))$ is bounded. □

As in the analytic case and the supersingular case, denote

$$\theta(g,(t_1,t_2), m_{\phi_v} \otimes \phi^v) = \sum_{u \in \mu_K^2 \backslash F^\times} \sum_{y \in V} r(g,(t_1,t_2))m_{\phi_v}(y,u)\ r(g,(t_1,t_2))\phi^v(y,u).$$

Then it approximates the original series as in Corollary 3.4.2 and Corollary 5.2.8.

Corollary 5.3.4. *Assume the the global degeneracy assumption and that ϕ is degenerate at v. Then*

$$i_{\bar{v}}(Z_*(g,\phi)t_1, t_2) = \theta(g,(t_1,t_2), m_{\phi_v} \otimes \phi^v), \quad \forall g \in 1_{v,v_1,v_2}\mathrm{GL}_2(\mathbb{A}^{v,v_1,v_2}).$$

Support of the multiplicity function: unramified quadratic extension

Here we verify Lemma 5.3.2 assuming that v is unramified in E. The case that v is ramified in E is slightly different and considered next subsection. The idea is very simple: two points in $\mathscr{H}_{U_v} = B_v^\times \times_{E^\times} \mathbb{B}_v^\times / U_v$ has a nonzero intersection only if they specializes to the same irreducible component of the special fibre in the related formal neighborhood.

We first look at the case of full level, for which the integral model is very clear by the Cherednik-Drinfeld uniformization. We can easily have an explicit expression for the multiplicity function, but we don't need it in this paper.

Assume that $U_v = O_{\mathbb{B}_v}^\times$ is maximal and U^v is sufficiently small. By the reciprocity law, all points in CM_U are defined over $\widehat{E}_v^{\mathrm{ur}}$, the completion of the maximal unramified extension of E_v. We have $\widehat{E}_v^{\mathrm{ur}} = \widehat{F}_v^{\mathrm{ur}}$ since E_v is unramified over F_v. In particular, the field H_w is unramified over F_v, and the model $\mathscr{Y}_{U,w} = \mathscr{X}_U \times_{O_F} O_{H_w}$ since the later is still regular. It suffices to compute the intersections over $\mathscr{X}_U \times_{O_F} O_{\widehat{F}_v^{\mathrm{ur}}}$.

The rigid analytic space X_U^{an} has the uniformization

$$X_U^{\mathrm{an}} \widehat{\otimes} \widehat{F}_v^{\mathrm{ur}} = B^\times \backslash (\Omega \widehat{\otimes} \widehat{F}_v^{\mathrm{ur}}) \times \mathbb{Z} \times \mathbb{B}_f^{v\times} / U^v.$$

Here Ω is the rigid analytic upper half plane over F_v. More importantly, it has the integral version

$$\widehat{\mathscr{X}_U} \widehat{\otimes} O_{\widehat{F}_v^{\mathrm{ur}}} = B^\times \backslash (\widehat{\Omega} \widehat{\otimes} O_{\widehat{F}_v^{\mathrm{ur}}}) \times \mathbb{Z} \times \mathbb{B}_f^{v\times} / U^v.$$

Here $\widehat{\mathscr{X}_U}$ denotes the formal completion of the integral model $\mathscr{X}_U$ along the special fibre over v, and $\widehat{\Omega}$ is the formal model of Ω over O_{F_v} obtained by successive blowing-ups of rational points on the special fibres of the scheme $\mathbb{P}^1_{O_{F_v}}$.

The formal model $\widehat{\Omega}$ is regular and semistable. Its special fibre is a union of $\mathbb{P}^1$'s indexed by scalar equivalence class of O_{F_v}-lattices of F_v^2, and acted transitively by $\mathrm{GL}_2(F_v)$. Hence these irreducible components are parametrized by

$$\mathrm{GL}_2(F_v)/F_v^\times \mathrm{GL}_2(O_{F_v}).$$

It follows that the set $\mathscr{V}_U$ of irreducible components of $\mathscr{X}_U \times O_{\widehat{F}_v^{\mathrm{ur}}}$ can be indexed as

$$\mathscr{V}_U = B^\times \backslash (\mathrm{GL}_2(F_v)/F_v^\times \mathrm{GL}_2(O_{F_v})) \times \mathbb{Z} \times \mathbb{B}_f^{v\times} / U^v.$$

Consider the set of CM-points

$$\mathrm{CM}_U = E^\times \backslash B^\times(\mathbb{A}_f)/U = B^\times \backslash (B^\times \times_{E^\times} \mathbb{B}_v^\times / U_v) \times \mathbb{B}_f^{v\times} / U^v.$$

The embedding $\mathrm{CM}_U \to X_U^{\mathrm{an}}$ is given by

$$\mathscr{H}_{U_v} \longrightarrow (\Omega \widehat{\otimes} \widehat{F}_v^{\mathrm{ur}}) \times \mathbb{Z}, \quad (b, \beta) \longmapsto (bz_0, \mathrm{ord}_v(q(b)q(\beta))).$$

Here $\mathscr{H}_{U_v} = B_v^\times \times_{E^\times} \mathbb{B}_v^\times / U_v$ is the space where the multiplicity function is defined, and z_0 is a point in $\Omega(E_v)$ fixed by $E_v^\times$. Since all CM-points are defined over $\widehat{F}_v^{\mathrm{ur}}$, their reductions are smooth points. The reduction map $\mathrm{CM}_U \to \mathscr{V}_U$ is given by the $B_v^\times$-equivariant map

$$\mathscr{H}_{U_v} \longrightarrow (\mathrm{GL}_2(F_v)/F_v^\times \mathrm{GL}_2(O_{F_v})) \times \mathbb{Z}, \quad (b, \beta) \longmapsto (bb_0, \mathrm{ord}_v(q(b)q(\beta))).$$

Here b_0 is represents the irreducible component of the reduction of z_0.

Consider the multiplicity function $m(b,\beta)$. It is equal to the intersection of Zariski closures of the points (b,β) and $(1,1)$ in $(\widehat{\Omega}\widehat{\otimes}O_{\widehat{F}_v^{\mathrm{ur}}})\times\mathbb{Z}$. If their intersection is nonzero, they have to lie in the same irreducible component on the special fibre. It is true if and only if $b\in F_v^\times \mathrm{GL}_2(O_{F_v})$ and $\mathrm{ord}_v(q(b)q(\beta))=0$. It gives the lemma.

Next, we assume that $U_v=1+p_v^rO_{\mathbb{B}_v}$ is general. Denote $U^0=O_{\mathbb{B}_v}^\times\times U^v$. By the construction in §4.5, there is a morphism $\mathscr{Y}_{U,w}\to\mathscr{X}_{U^0,v}$. It induces a map $\mathscr{V}_U\to\mathscr{V}_{U^0}$ on the sets of irreducible components on the special fibres. Composing with the reduction map $\mathrm{CM}_U\to\mathscr{V}_U$, we obtain a map $\mathrm{CM}_U\to\mathscr{V}_{U^0}$ which is also the composition of $\mathrm{CM}_U\to\mathrm{CM}_{U^0}$ and $\mathrm{CM}_{U^0}\to\mathscr{V}_{U^0}$. Hence it is induced by the $B_v^\times$-equivariant map

$$\mathscr{H}_{U_v}\longrightarrow(\mathrm{GL}_2(F_v)/F_v^\times\mathrm{GL}_2(O_{F_v}))\times\mathbb{Z},\quad (b,\beta)\longmapsto(bb_0,\mathrm{ord}_v(q(b)q(\beta))).$$

It has the same form as above. Then $m(b,\beta)$ is nonzero only if (b,β) and $(1,1)$ has the same image in the map, which still implies $\mathrm{ord}_v(q(b)q(\beta))=0$ and $b\in F_v^\times\mathrm{GL}_2(O_{F_v})$.

Support of the multiplicity function: ramified quadratic extension

Assume that v is ramified in E. We consider Lemma 5.3.2. Similar to the unramified case, the general case essentially follows from the case of full level.

We first assume that $U_v=O_{\mathbb{B}_v}^\times$ is maximal. Then all points in CM_U are still defined over $\widehat{E}_v^{\mathrm{ur}}$, and $H_w\subset\widehat{E}_v^{\mathrm{ur}}$. But $\widehat{E}_v^{\mathrm{ur}}$ is a quadratic extension of $\widehat{F}_v^{\mathrm{ur}}$ this time. The model $\mathscr{Y}_{U,w}$ is obtained from $\mathscr{X}\times_{O_F}O_{H_w}$ by blowing-up all the ordinary double points on the special fibre.

We consider uniformizations over $\widehat{E}_v^{\mathrm{ur}}$. The uniformization on the generic fibre does not change:

$$X_U^{\mathrm{an}}\widehat{\otimes}\widehat{E}_v^{\mathrm{ur}}=B^\times\backslash(\Omega'\widehat{\otimes}\widehat{E}_v^{\mathrm{ur}})\times\mathbb{Z}\times\mathbb{B}_f^{v\times}/U^v.$$

Here $\Omega'=\Omega\widehat{\otimes}E_v$. Let $\widehat{\Omega}'$ be the blowing-up of all double points on the special fibre of $\widehat{\Omega}\widehat{\otimes}O_{E_v}$. It is regular and semistable. Then the formal completion of $\widehat{\mathscr{Y}_{U,w}}$ along its special fibre is uniformized by

$$\widehat{\mathscr{Y}_{U,w}}\widehat{\otimes}O_{\widehat{E}_v^{\mathrm{ur}}}=B^\times\backslash(\widehat{\Omega}'\widehat{\otimes}O_{\widehat{E}_v^{\mathrm{ur}}})\times\mathbb{Z}\times\mathbb{B}_f^{v\times}/U^v.$$

The special fibre of $\widehat{\Omega}'$ consists of the strict transforms of the irreducible components on the special fibre of $\widehat{\Omega}$ and exceptional components coming from the blowing-up. The reduction map sends CM_U to the set $\mathscr{V}_U'$ of exceptional components. The exceptional components are indexed by double points in $\widehat{\Omega}$, and each double point corresponds to a pair of adjacent lattices in F_v^2. The action of $\mathrm{GL}_2(F_v)$ on the double points is transitive. Then $\mathscr{V}_U'\cong\mathrm{GL}_2(F_v)/S_v$ where S_v is the stabilizer of any double point.

Similar to the unramified case, the reduction map $\mathrm{CM}_U\to\mathscr{V}_U'$ is given by the $\mathrm{GL}_2(F_v)$-equivariant map

$$\mathscr{H}_{U_v}\longrightarrow(\mathrm{GL}_2(F_v)/S_v)\times\mathbb{Z},\quad (b,\beta)\longmapsto(bb_0,\mathrm{ord}_v(q(b)q(\beta))).$$

Use the same argument that two points intersects on the special fibre if and only if they reduce to the same irreducible component. We see that $m(b,\beta) \neq 0$ only if $b \in S_v$ and $\mathrm{ord}_v(q(b)q(\beta)) = 0$. It suffices to bound S_v.

Take S_v to be the stabilizer of the double point corresponding to the edge between the lattices $O_{F_v} \oplus O_{F_v}$ and $O_{F_v} \oplus \varpi_v O_{F_v}$. The action of $h_v = \begin{pmatrix} 0 & 1 \\ \varpi_v & 0 \end{pmatrix}$ switch these two lattices. Then it is easy to see that

$$S_v \subset F_v^\times \mathrm{GL}_2(O_{F_v}) \cup h_v F_v^\times \mathrm{GL}_2(O_{F_v}).$$

The result is verified in this case.

We remark that the group S_v is generated by the center $F_v^\times$, the element h_v, and the subgroup

$$\Gamma_0(p_v) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_{F_v}) : c \in p_v \right\}.$$

Now we consider the general case $U_v = 1 + p_v^r O_{\mathbb{B}_v}$. It is similar to the unramified case. Still compare it with $U^0 = O_{\mathbb{B}_v}^\times \times U^v$. The reduction map $\mathrm{CM}_U \to \mathscr{V}'_{U^0}$ is given by the $B_v^\times$-equivariant map

$$\mathscr{H}_{U_v} \longrightarrow (\mathrm{GL}_2(F_v)/S_v) \times \mathbb{Z}, \quad (b,\beta) \longmapsto (bb_0, \mathrm{ord}_v(q(b)q(\beta))).$$

The multiplicity function $m(b,\beta)$ is nonzero only if $\mathrm{ord}_v(q(b)q(\beta)) = 0$ and $b \in F_v^\times \mathrm{GL}_2(O_{F_v})$.

5.4 Ordinary case

In this subsection we consider the case that v is a finite prime of F split in E. The local height is expected to vanish because there is no corresponding v-part in the analytic kernel in this case.

Proposition 5.4.1. *Under the global degeneracy assumption,* $i_{\bar v}(Z_*(g)t_1, t_2) = 0$.

Let ν_1 and ν_2 be the two primes of E lying over v. They corresponds to two places w_1 and w_2 of H via our fixed embedding $\bar E \hookrightarrow \bar F_v$. For $\ell = 1, 2$, the intersection multiplicity $i_{\bar\nu_\ell}$ is computed on the model $\mathscr{Y}_U \times_{O_H} O_{\bar E_{\nu_\ell}}$ where the fibre product is taken according to the fixed inclusions $H \subset \bar E \subset \bar E_{\nu_\ell}$. It is actually a base change of the local integral model $\mathscr{Y}_{U,w} = \mathscr{Y}_U \times_{O_H} O_{H_w}$ for w a place of H induced by ν_ℓ.

By the embedding $E_v \to \mathbb{B}_v$ we see that $\mathbb{B}_v$ is split. Fix an identification $\mathbb{B}_v \cong \mathrm{M}_2(F_v)$ under which $E_v = \begin{pmatrix} F_v & \\ & F_v \end{pmatrix}$. Assume that ν_1 corresponds to the ideal $\begin{pmatrix} F_v & \\ & 0 \end{pmatrix}$ and ν_2 corresponds to $\begin{pmatrix} 0 & \\ & F_v \end{pmatrix}$ of E_v. It suffices to show that $i_{\bar\nu_1}(Z_*(g)t_1, t_2) = 0$.

We still make use of results of [Zh2]. The reduction map of CM-points to ordinary points at ν_1 is given by

$$E^\times \backslash \mathbb{B}_f^\times / U \longrightarrow E^\times \backslash (N(F_v) \backslash \mathrm{GL}_2(F_v)) \times \mathbb{B}_f^{v\times} / U.$$

The intersection multiplicity is a function $g_v : (N(F_v)U_v/U_v)^2 \to \mathbb{R}$. An explicit expression of g_v for general U_v are proved in [Zh2, Lemma 6.3.2]. But we don't need it here. The local height pairing of two distinct CM points $\beta_1, \beta_2 \in E^\times \backslash \mathbb{B}_f^\times / U$ is given by

$$i_{\bar{\nu}_1}(\beta_1, \beta_2) = \sum_{\gamma \in \mu_U \backslash E^\times} g_{\nu_1}(\beta_1, \gamma\beta_2) 1_{U^v}(\beta_1^{-1}\gamma\beta_2).$$

Unlike other cases, the above summation has nothing to do with the nearby quaternion algebra. It is only a "small" sum for $\gamma \in E$. This is the key for the the vanishing of the local height series.

Now we can look at

$$i_{\bar{\nu}_1}(Z_*(g)t_1, t_2) = \sum_{a \in F^\times} \sum_{x \in \mathbb{B}_f^\times / U} r(g)\phi(x)_a \sum_{\gamma \in \mu_U \backslash (E^\times - t_1 x U_v t_2^{-1})} g_{\nu_1}(t_1 x, \gamma t_2) 1_{U^v}(x^{-1} t_1^{-1} \gamma t_2).$$

A nonzero term has to satisfy $x^v \in t_1^{-1}\gamma t_2 U^v$. Then $r(g)\phi(x)_a$ has a factor

$$r(g_{v_1})\phi_{v_1}(x)_a = r(g_{v_1})\phi_{v_1}(t_1^{-1}\gamma t_2)_a.$$

But we know it is zero for $g \in P(F_{v_1})$ since $t_1^{-1}\gamma t_2 \in E_{v_1}$ and ϕ_{v_1} is degenerate.

5.5 The j-part

Now we consider $j_{\bar{v}}(Z_*(g)t_1, t_2)$ for a non-archimedean place v of F. It is nonzero only if X_U has a bad reduction at v. We will show that it can be approximated by Eisenstein series or theta series by the global degeneracy assumption that ϕ is degenerate at v_1, v_2 and the local degeneracy assumption that ϕ is degenerate at v.

The pairing $j_{\bar{v}}(Z_*(g)t_1, t_2)$ is the average of $j_{\bar{\nu}}(Z_*(g)t_1, t_2)$ for each non-archimedean place ν of E lying over v. Similar to the computation of the i-part, the intersection is computed on the model $\mathscr{Y}_{U,w}$ introduced in §4.5. Here w is the place of H induced by $\bar{\nu}$. By the definition in §4.4,

$$j_{\bar{\nu}}(Z_*(g)t_1, t_2) = \frac{1}{[H:F]} \overline{Z_*(g)t_1} \cdot V_{t_2}$$

Here $\overline{Z_*(g)t_1}$ is the Zariski closure in $\mathscr{Y}_{U,w}$ and V_{t_2} is a linear combination of irreducible components in the special fibres of $\mathscr{Y}_{U,w}$ which gives the $\hat{\xi}$-admissible arithmetic extension of t_2.

It suffices to treat $(Z_*(g)t, C)$ for any $t \in C_U$ and any irreducible component C in special fibre of $\mathscr{Y}_{U,w}$. Here we use the notation $(D, C) = \frac{1}{[H:F]}\overline{D} \cdot C$ for convenience. It is essentially a question about the reduction of CM-points in CM_U to irreducible components on the special fibre on the model $\mathscr{Y}_{U,w} \times O_{\bar{E}_\nu}$. Many cases have been described explicitly in [Zh2].

By the construction in §4.5, there is a morphism $\mathscr{Y}_{U,w} \to \mathscr{X}_{U^0,v}$, where $U^0 = O_{\mathbb{B}_v}^\times \times U^v$. By this map, we classify the irreducible component C on the special fibre into the following three categories:

- C is *ordinary* if the image of C in $\mathscr{X}_{U^0,v}$ is not a point.
- C is *supersingular* if v is split $\mathbb{B}$ and C maps to a point in $\mathscr{X}_{U^0,v}$. Notice this point must be supersingular.
- C is *superspecial* if v is non-split in $\mathbb{B}$.

Let $\mathscr{V}_U^{\mathrm{ord}}$, $\mathscr{V}_U^{\mathrm{sing}}$, and $\mathscr{V}_U^{\mathrm{spe}}$ denote the set of these components.

Ordinary Components

We first consider $(Z_*(g)t, C)$ in the case that C is an ordinary component. It is nonzero only if points in CM_U reduce to ordinary components, which happens exactly when E is split at v. Let ν_1, ν_2 be the two places of E above v, and we use the convention of the splitting $E_v = F_v \oplus F_v$ at the beginning of §5.4.

It suffices to consider $j_{\bar{\nu}_1}$. We will see that it can be approximated by the Eisenstein series J in some way. The treatment is very similar to Proposition 4.2.2 by separating geometrically connected components.

By Lemma 5.4.2 in [Zh2], the ordinary components are parameterized by

$$\mathscr{V}_U^{\mathrm{ord}} = F_+^\times \backslash \mathbb{A}_f^\times / q(U) \times P(F_v) \backslash \mathrm{GL}_2(F_v)/U_v.$$

Note that the first double coset is exactly the set of geometrically connected components. The reduction $\mathrm{CM}_U \longrightarrow \mathscr{V}_U^{\mathrm{ord}}$ is given by the natural map:

$$\begin{array}{rcl} E^\times \backslash \mathbb{B}_f^\times / U & \longrightarrow & F_+^\times \backslash \mathbb{A}_f^\times / q(U) \times P(F_v) \backslash \mathrm{GL}_2(F_v)/U_v, \\ g & \longmapsto & (\det g, g_v). \end{array}$$

For any $\beta \in \mathrm{CM}_U$, the intersection $(\beta, C) \neq 0$ only if β and C are in the same geometrically connected component. Once this is true, it is given by a locally constant function l_C for $\beta_v \in \mathbb{B}_v^\times$. Moreover, the function l_C factors through $P(F_v)\backslash \mathrm{GL}_2(F_v)/U_v$. In summary, we have

$$(\beta, C) = l_C(\beta_v) 1_{F_+^\times q(\beta_C) q(U)}(q(\beta)).$$

Here $\beta_C \in \mathbb{B}_f^\times$ is any element such that $q(\beta_C)$ gives the geometrically connected component containing C.

Therefore, we have

$$(Z_*(g)t, C) = (Z_*(g)_\alpha t, C)$$

where $\alpha = q(t)^{-1} q(\beta_C)$. By the result in §4.2,

$$Z_*(g)_\alpha t = \sum_{u \in \mu_U' \backslash F^\times} \sum_{y \in \mathbb{B}_f^{\mathrm{ad}}/U^1} r(g, (\beta_C^{-1} t, 1)) \phi(y, u)\ [\beta_C y].$$

Thus

$$(Z_*(g)t, C) \;=\; \sum_{u\in\mu_U'\backslash F^\times}\sum_{a\in F_+^\times}\sum_{y\in\mathbb{B}_f(a)/U^1} r(g,(\beta_C^{-1}t,1))\phi(y,u)l_C(\beta_C y).$$

Proposition 4.2.2 can be viewed as the case that $l_C\equiv 1$, in which case we have shown that the above the Eisenstein series is equal to the Eisenstein series $-\frac{1}{2}\kappa_U J_*(0,g,r(\beta_C^{-1}t,1)\phi)$.

In the case here, the extra factor l_C complicates the v-adic component of the Schwartz function, but has no impact at other places. The function

$$l_{\phi_v}(y,u)=l_{C,\phi_v}(y,u):=\phi(y,u)l_C(y)$$

is still a Schwartz function of $(y,u)\in\mathbb{B}_v\times F_v^\times$ since l_C is smooth. Consider the new series

$$\sum_{u\in\mu_U'\backslash F^\times}\sum_{a\in F_+^\times}\sum_{y\in\mathbb{B}_f(a)/U^1} r(g,(\beta_C^{-1}t,1))\phi^v(y,u)r(g,(\beta_C^{-1}t,1))l_{\phi_v}(y,u).$$

It is equal to the original $(Z_*(g)t,C)$ if $g_v=1$, and we can check that it is actually true for all $g_v\in P(F_v)$. Here we need the fact that l_C is invariant under the left action of $T(F_v)\subset P(F_v)$. On the other hand, it is equal to a scalar multiple of the Einsenstein series $J_*(0,g,r(t\beta_C^{-1},1)(\phi^v\otimes l_{\phi_v}))$ since Proposition 4.2.2 applies to it. Therefore, we have shown that

$$(Z_*(g)t,C) \;=\; -\frac{1}{2}\kappa_U J_*(0,g,r(\beta_C^{-1}t,1)(\phi^v\otimes l_{\phi_v})),\quad \forall g\in P(F_v)\mathrm{GL}_2(\mathbb{A}^v).$$

Now we look at the constant term

$$J_0(0,g,r(\beta_C^{-1}t,1)(\phi^v\otimes l_{\phi_v}))=\sum_{u\in\mu_U^2\backslash F^\times} r(g,(\beta_C^{-1}t,1))(\phi^v\otimes l_{\phi_v})(0,u).$$

It automatically zero if $g_v\in P(F_v)$ since in the case $r(g,(\beta_C^{-1}t,1))l_{\phi_v}(0,u)=0$ by the degeneracy assumption of ϕ_v. Hence, we end up with

$$(Z_*(g,\phi)t,C) \;=\; -\frac{1}{2}\kappa_U J(0,g,r(\beta_C^{-1}t,1)(\phi^v\otimes l_{\phi_v})),\quad \forall g\in P(F_{v,v_1,v_2})\mathrm{GL}_2(\mathbb{A}^{v,v_1,v_2}).$$

It is approximated by an Eisenstein series.

Supersingular components

Now we consider $(Z_*(g)t,C)$ in the case that C is a supersingular component. It is nonzero only if points in CM_U reduce to supersingular components, which happens exactly when both $\mathbb{B}$ and E are non-split at v. The treatment is similar to §5.2. Denote $B=B(v)$ and fix an isomorphism $B(\mathbb{A}_f^v)\simeq\mathbb{B}_f^v$ as usual.

The key is to characterize the reduction map

$$\mathrm{CM}_U \longrightarrow \mathscr{V}_U^{\mathrm{sing}} \longrightarrow \mathbb{S}_{U^0}.$$

Here $\mathbb{S}_{U^0}$ is the set of supersingular points in $\mathscr{X}_{U^0,v}$. Recall that

$$\mathrm{CM}_U = E^\times \backslash \mathbb{B}_f^\times / U = B^\times \backslash (B^\times \times_{E^\times} \mathbb{B}_v^\times) \times (\mathbb{B}_f^v)^\times / U \hookrightarrow B^\times \backslash \mathscr{H}_v \times (\mathbb{B}_f^v)^\times / U$$

where $\mathscr{H}_v = B_v^\times \times_{E_v^\times} \mathbb{B}_v^\times$. We also have the parametrization

$$\mathbb{S}_{U^0} = B^\times \backslash \mathbb{B}_f^\times / U^0 = B^\times \backslash F_v^\times \times (\mathbb{B}_f^v)^\times / U^0.$$

Then the reduction map $\mathrm{CM}_U \to \mathbb{S}_{U^0}$ is given by product of determinants:

$$\mathscr{H}_v \longrightarrow F_v^\times, \qquad (b, \beta) \longmapsto q(b)q(\beta).$$

Fix one supersingular point z on the special fibre of $\mathscr{X}_{U^0,v}$. The formal completion Ω^0 of $\mathscr{X}_{U^0,v}$ along z is isomorphic to the universal neighborhood of the formal O_{F_v}-module of height two. Then the formal completion $\widehat{\mathscr{X}_{U^0,v}}$ of $\mathscr{X}_{U^0,v}$ along its supersingular locus $\mathbb{S}_{U^0}$ is given by

$$\widehat{\mathscr{X}_{U^0,v}} = B_0^\times \backslash \Omega^0 \times (\mathbb{B}_f^v)^\times / U^v = B^\times \backslash \Omega_0 \times (\mathbb{B}_f^v)^\times / U^v$$

where $B_{0,v}^\times = \{b \in B_v : \mathrm{ord}_v q(b) = 0\}, B_0^\times = B \cap B_{0,v}^\times$, and $\Omega_0 = B_v^\times \times_{B_{0,v}^\times} \Omega^0$.

Let $\widetilde{\Omega}$ be the minimal desingularization of $\Omega_0 \widehat{\otimes} O_{H_w}$. It admits an action by $B_v^\times \times U_v$. Then the formal completion of $\mathscr{Y}_{U,w}$ along the union of fibers in $\mathscr{V}_U^{\mathrm{sing}}$ can be described as

$$\widehat{\mathscr{Y}_U} = B^\times \backslash \widetilde{\Omega} \times (\mathbb{B}_f^v)^\times / U.$$

Let $\widetilde{\mathscr{V}}$ be the set of irreducible components on the special fibre of $\widetilde{\Omega}$. It also admits an action by $B_v^\times \times U_v$. Then we have a description

$$\mathscr{V}_U^{\mathrm{sing}} = B^\times \backslash \widetilde{\mathscr{V}} \times (\mathbb{B}_f^v)^\times / U.$$

Our conclusion is that the map

$$\mathrm{CM}_U \longrightarrow \mathscr{V}_U^{\mathrm{sing}} \longrightarrow \mathbb{S}_{U^0}$$

is given by $(B_v^\times \times U_v)$-equivariant maps

$$\mathscr{H}_v \longrightarrow \widetilde{\mathscr{V}} \longrightarrow F_v^\times.$$

We are now applying the above result to compute the intersection pairing $(Z_*(g)t, C)$. It is very similar to our treatment of the i-part. Let $(C_0, \beta_C) \in \widetilde{\mathscr{V}} \times (\mathbb{B}_f^v)^\times$ be a couple representing $C \in \mathscr{V}_U^{\mathrm{sing}}$. The intersection with C_0 defines a locally constant function l_{C_0} on

$\mathscr{H}_{U_v} = B_v^\times \times_{E_v^\times} \mathbb{B}_v^\times/U_v$. Unlike the multiplicity function m, l_{C_0} has no singularity on $\mathscr{H}_{U_v}$. For any CM-point $\beta \in \mathrm{CM}_U$, the intersection pairing is given by

$$(\beta, C) = \sum_{\gamma\in\mu_U\backslash B^\times} l_{C_0}(\gamma, \beta_v) 1_{U^v}(\beta_C^{-1}\gamma\beta^v).$$

Hence, we obtain

$$(Z_*(g)t, C) = \sum_{a\in F^\times}\sum_{x\in\mathbb{B}_f^\times/U} r(g)\phi(x)_a \sum_{\gamma\in\mu_U\backslash B^\times} l_{C_0}(\gamma, t_v x_v) 1_{U^v}(\beta_C^{-1}\gamma t^v x^v).$$

Now we convert the above to a pseudo-theta series. The process is the same as the i-part. We sketch it here. Change the order of the summations. Note that $1_{U^v}(\beta_C^{-1}\gamma t^v x^v) \neq 0$ if and only if $x^v \in t^{-1}\gamma^{-1}\beta_C U^v$. Put it into the sum. We have

$$(Z_*(g)t, C) = \sum_{a\in F^\times}\sum_{\gamma\in\mu_U\backslash B^\times} r(g)\phi^v(t^{-1}\gamma^{-1}\beta_C)_a \sum_{x\in\mathbb{B}_v^\times/U_v} r(g)\phi_v(x)_a l_{C_0}(\gamma, t_v x_v).$$

Denote

$$l_{\phi_v}(y,u) = l_{C_0,\phi_v}(y,u) := \int_{\mathbb{B}_v^\times} \phi_v\left(x, \frac{uq(y)}{q(x)}\right) l_{C_0}(y^{-1}, x)dx.$$

Then

$$\begin{aligned}(Z_*(g)t, C) &= \sum_{a\in F^\times}\sum_{y\in\mu_U\backslash B^\times} r(g,(t,\beta_C))\phi^v(y)_a l_{r(g,(t,1))\phi_v}(y)_a \\ &= \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in B^\times} r(g,(t,\beta_C))\phi^v(y,u) l_{r(g,(t,1))\phi_v}(y,u).\end{aligned}$$

It is a pseudo-theta series.

We claim that if $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$, then l_{ϕ_v} extends to a Schwartz function for $(y,u) \in B_v \times F_v^\times$. The proof of Proposition 5.2.7 applies here. We only explain that $v(q(y)^{-1}q(x))$ is a constant on the support of $l_{C_0}(y^{-1}, x)$, which is needed for l_{ϕ_v} to be compactly supported. In fact, $l_{C_0}(y^{-1}, x) \neq 0$ only if the point $(y^{-1}, x) \in \mathscr{H}_v$ and $C_0 \in \mathscr{V}_H$ have the same image in $F_v^\times$. It determines the coset $q(y)^{-1}q(x)q(U_v)$ in $F_v^\times$ uniquely.

Similar to all the pseudo-theta series we treated before, our conclusion is

$$(Z_*(g,\phi)t, C) = \theta(g, (t,\beta_C), r(\beta_c^{-1}, 1) l_{\phi_v} \otimes \phi^v), \quad \forall g \in P(F_{v,v_1,v_2})\mathrm{GL}_2(\mathbb{A}^{v,v_1,v_2}).$$

In our particular case, C is in the geometrically connected components of t_2, and thus we can take $\beta_C = t_2$.

Superspecial components

Now we consider $(Z_*(g)t, C)$ in the case that C is a superspecial component. It happens when v is non-split in $\mathbb{B}$. Resume the notations in the treatment of the i-part. It is similar to the supersingular case.

The curve $\mathscr{X}_{U^0,v}$ has the explicit uniformization as formal schemes:

$$\widehat{\mathscr{X}_{U^0,v}}\widehat{\otimes}O_{\widehat{F}_v^{\mathrm{ur}}} = B^\times\backslash(\widehat{\Omega}\widehat{\otimes}O_{\widehat{F}_v^{\mathrm{ur}}}) \times \mathbb{Z} \times (\mathbb{B}_f^v)^\times/U^v.$$

For general levels, the uniformization is easily done in the level of rigid spaces:

$$X_U^{\mathrm{an}}\widehat{\otimes}\widehat{F}_v^{\mathrm{ur}} = B^\times\backslash\Sigma_r \times \mathbb{B}_f^{v\times}/U^v.$$

Here Σ_r is some etale rigid-analytic cover of $(\Omega\widehat{\otimes}\widehat{F}_v^{\mathrm{ur}}) \times \mathbb{Z}$ depending on r. Take the normalization of the formal scheme $(\widehat{\Omega}\widehat{\otimes}\widehat{O}_{F_v^{\mathrm{ur}}}) \times \mathbb{Z}$ in the rigid space $\Sigma_r\widehat{\otimes}_{\widehat{F}_v^{\mathrm{ur}}}\widehat{H}_w^{\mathrm{ur}}$, and make a minimal resolution of singularities. We obtain a regular formal scheme $\widehat{\Sigma}_r$ over $O_{H_w^{\mathrm{ur}}}$. The construction is compatible with the algebraic construction of $\mathscr{Y}_{U,w}$, i.e.,

$$\widehat{\mathscr{Y}_{U,w}}\widehat{\otimes}O_{\widehat{H}_w^{\mathrm{ur}}} = B^\times\backslash\widehat{\Sigma}_r \times \mathbb{B}_f^{v\times}/U^v.$$

Here $\widehat{\mathscr{Y}_{U,w}}$ is the formal completion of the $\mathscr{Y}_{U,w}$ along its special fibre. The uniformizations here are not explicit at all, but we only need some group-theoretical properties.

Let $\widetilde{\mathscr{V}}$ be the set of irreducible components of $\widehat{\Sigma}_r$. Then the reduction $\mathrm{CM}_U \to \mathscr{V}^{\mathrm{spe}}$ is given by a $B_v^\times$-equivariant map $\mathscr{H}_{U_v} \to \widetilde{\mathscr{V}}$. Assume the global degeneracy assumption and assume that ϕ is degenerate at v. The same calculation as in supersingular case will show that $(Z_*(g)t, C)$ can be approximated be a theta series on the quadratic space B.

6 Proof of the main result

In this section we prove the main result Theorem 1.3.1. We first prove the main identity Theorem 6.1.1 between the analytic and the geometric kernel functions. We argue that the computations and approximations we have done are enough to make the conclusion. It is easy to see that the approximation we obtain for the difference of the kernel functions becomes an equality of automorphic forms by the modularity. Then we make an comparison between the theta correspondences defined by Hecke operators and the normalized theta correspondences defined in §2.2. This comparison is done by the Lefschetz trace formula and the lcoal Siegel–Weil formula.

6.1 Identity of kernel functions

Let $\phi \in \mathscr{S}(\mathbb{B}\times\mathbb{A}^\times)$ be a Schwartz function with ϕ_∞ is standard. Recall that we have two kernel functions:

$$I(s,g,\chi,\phi) = \int_{[T]} \widetilde{I}(s,g,r(t,1)\phi)\chi(t)dt,$$
$$Z(g,\chi,\phi) = \int_{[T]} \widetilde{Z}(g,(t,1),\phi)\chi(t)dt.$$

Both of them have the same central character as π. We are ready to show the following result.

Theorem 6.1.1. *For any $\varphi \in \pi$,*

$$(I'(0,g,\chi,\phi),\ \varphi(g))_{\mathrm{Pet}} = 2\,(Z(g,\chi,\phi),\ \varphi(g))_{\mathrm{Pet}}\,.$$

Now we prove Theorem 6.1.1 by gathering all the computation we have made together. We need to prove that

$$(I'(0,g,\chi) - 2Z(g,\chi),\ \varphi(g))_{\mathrm{pet}} = 0.$$

Equivalently, we show that

$$(\mathscr{P}r I'(0,g,\chi) - 2Z(g,\chi),\ \varphi(g))_{\mathrm{pet}} = 0.$$

For convenience of readers, we list the main results in last three sections.

The holomorphic projections $\mathscr{P}r$ and $\mathscr{P}r'$ are introduced in §3.6. By Proposition 3.6.2, we see that the derivative $\mathscr{J}'(0,g,\chi)$ of the Eisenstein series $\mathscr{J}(s,g,\chi)$ cancels the growth of $I'(0,g,\chi)$. Then by Proposition 3.6.1,

$$\mathscr{P}r I'(0,g,\chi) = \mathscr{P}r' I'(0,g,\chi) - \mathscr{P}r' \mathscr{J}'(0,g,\chi).$$

Note that both terms on the right-hand side can fail to be automorphic since $\mathscr{P}r'$ is just an algorithm. We can further write the right-hand side as an integral of

$$\mathscr{P}r' I'(0,g,r(t,1)\phi) - \mathscr{P}r' \mathscr{J}'(0,g,r(t,1)\phi)$$

against the character χ for the variable t on $[T]$.

On the other hand, we have

$$Z(g,(t,1)) = \langle Z_*(g)t, 1\rangle - \langle Z_*(g)\xi_t, 1\rangle + \langle Z_*(g)\xi_t, \xi_1\rangle - \langle Z_*(g)t, \xi_1\rangle.$$

The pairings on the right-hand side are the $\hat{\xi}$-admissible pairings. To get $Z(g,\chi)$, we need an integral of t on $[T]$ against the character χ.

Putting these two parts together, we have

$$\mathscr{P}rI'(0,g,\chi) - 2Z(g,\chi) = \int_{[T]} \Delta(g,(t,1))dt$$

where

$$\begin{aligned}\Delta(g,(t,1)) \quad = \quad & \mathscr{P}r'I'(0,g,r(t,1)\phi) - 2\langle Z_*(g)t, 1\rangle \\ & -\mathscr{P}r'\mathscr{J}'(0,g,r(t,1)\phi) + 2\langle Z_*(g)t, \xi_1\rangle \\ & +2\langle Z_*(g)\xi_t, 1\rangle - 2\langle Z_*(g)\xi_t, \xi_1\rangle.\end{aligned}$$

Both terms on the third line above are equal to scalar multiples of the Eisenstein series $J_*(0,g,r(t,1)\phi)$ by Lemma 4.6.1. To control the first line and the second line, we need some degeneracy assumption.

Let S be a finite set of non-archimedean places of F containing all places ramified in $\mathbb{B}$, all places ramified in E, all places ramified over $\mathbb{Q}$, and all places v such that U_v is not maximal. Assume that ϕ_v is standard for all $v \notin S$ and degenerate for all $v \in S$. To match with the global degeneracy assumption, we assume that S contains at least two places non-split in E.

We first gather the results for the second line in the expression of $\Delta(g,(t,1))$. The holomorphic projection $\mathscr{P}r'\mathscr{J}'(0,g,\phi)$ is computed in Proposition 3.7.2. By Proposition 4.6.2, we see that the second line is equal to

$$J'_*(0,g,r(t,1)\phi) + \widetilde{J}'_*(0,g,r(t,1)\phi) + J(0,g,r(t,1)(\phi^S \otimes \phi'_S))$$

for all $g \in 1_S\mathrm{GL}_2(\mathbb{A}^S)$. Here ϕ'_S is some Schwartz function in $\mathscr{S}(\mathbb{B}_S \times F_S^\times)$.

It remains to consider $\mathscr{P}r'I'(0,g,r(t,1)\phi) - 2\langle Z_*(g)t, 1\rangle$, the first line in the expression of $\Delta(g,(t,1))$. By Proposition 3.7.1, for all $g \in 1_S\mathrm{GL}_2(\mathbb{A}^S)$,

$$\mathscr{P}r'I'(0,g,r(t,1)\phi) = -\sum_{v|\infty} \overline{I'}(0,g,r(t,1)\phi)(v) - \sum_{v\nmid\infty \text{ nonsplit}} I'(0,g,r(t,1)\phi)(v).$$

Here

$$\begin{aligned}\overline{I'}(0,g,r(t,1)\phi)(v) &= 2\fint_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} \overline{\mathscr{K}}_\phi^{(v)}(g,(tt',t'))dt', \qquad v|\infty, \\ I'(0,g,r(t,1)\phi)(v) &= 2\fint_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} \mathscr{K}_\phi^{(v)}(g,(tt',t'))dt', \qquad v\nmid\infty.\end{aligned}$$

Recall that $\mathscr{K}_\phi^{(v)}$ is certain pseudo theta series on the nearby quaternion algebra $B(v)$, and $\overline{\mathscr{K}}_\phi^{(v)}$ has a similar expression.

On the other hand, in §4.5 we have the decomposition

$$\langle Z_*(g)t, 1\rangle = -\sum_v i_v(Z_*(g)t, 1)\log N_v - \sum_{v\in S} j_v(Z_*(g)t, 1)\log N_v,$$

where

$$\begin{aligned} i_v(Z_*(g)t, 1) &= \fint_{T(F)\backslash T(\mathbb{A}_f)} i_{\bar v}(Z_*(g)tt', t')dt', \\ j_v(Z_*(g)t, 1) &= \fint_{T(F)\backslash T(\mathbb{A}_f)} j_{\bar v}(Z_*(g)tt', t')dt'. \end{aligned}$$

The decomposition is valid for all $g \in 1_S\mathrm{GL}_2(\mathbb{A}^S)$. It is not true for all $g \in \mathrm{GL}_2(\mathbb{A})$ because of the appearance of the self-intersection.

Since the archimedean parts are standard, we see that the average integrals

$$\fint_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} = \fint_{T(F)\backslash T(\mathbb{A}_f)} = \fint_{C_U}.$$

These integrals are essentially finite sums on C_U. It makes the difference

$$\mathscr{P}r' I'(0, g, r(t,1)\phi) - 2\langle Z_*(g)t, 1\rangle$$

very convenient to handle. Below is a checklist of the results. All the equalities are valid for all $g \in 1_S\mathrm{GL}_2(\mathbb{A}^S)$ and all $t_1, t_2 \in T(\mathbb{A}_f)$.

(1) For $v|\infty$, Proposition 5.1.1 shows that

$$\overline{\mathscr{K}}_\phi^{(v)}(g, (t_1, t_2)) = i_{\bar v}(Z_*(g)t_1, t_2).$$

(2) For $v \notin S$ finite and non-split in E, Proposition 5.2.6 shows that

$$\mathscr{K}_\phi^{(v)}(g, (t_1, t_2)) = i_{\bar v}(Z_*(g)t_1, t_2)\log N_v.$$

(3) For $v \in S$ finite and non-split in E, by Corollary 3.4.2,

$$\mathscr{K}_\phi^{(v)}(g, (t_1, t_2)) = \theta(g, (t_1, t_2), k_{\phi_v} \otimes \phi^v).$$

By Corollary 5.2.8 and Corollary 5.3.4,

$$i_{\bar v}(Z_*(g)t_1, t_2) = \theta(g, (t_1, t_2), m_{\phi_v} \otimes \phi^v).$$

(4) For v finite and split in E, Proposition 5.4.1 shows that

$$i_{\bar{v}}(Z_*(g)t_1, t_2) = 0.$$

(5) For $v \in S$, $j_{\bar{v}}$ is treated in §5.5. It can be approximated as follows:

- If v is split in E, then

$$j_{\bar{v}}(Z_*(g)t_1, t_2) = J(0, g, r(t_1, t_2)(l_{\phi_v} \otimes \phi^v))$$

  for some Schwartz function $l_{\phi_v} \in \mathscr{S}(\mathbb{B}_v \times F_v^\times)$.
- If v is non-split in E, then

$$j_{\bar{v}}(Z_*(g)t_1, t_2) = \theta(g, (t_1, t_2), l_{\phi_v} \otimes \phi^v)$$

  for some Schwartz function $l_{\phi_v} \in \mathscr{S}(B(v)_v \times F_v^\times)$.

We remark that the results in (1), (2) and (4) are proved by explicit computation. Other results are proved by approximation method where the degeneracy assumption of ϕ at v is needed to control singularities.

In summary, $\Delta(g, (t, 1))$ is equal to a finite linear combinations of theta series and Eisenstein series for $g \in 1_S \mathrm{GL}_2(\mathbb{A}^S)$. It implies the corresponding result for $\mathscr{P}r I'(0, g, \chi) - 2Z(g, \chi)$. More precisely, for $g \in 1_S \mathrm{GL}_2(\mathbb{A}^S)$,

$$\begin{aligned} & \mathscr{P}r I'(0, g, \chi, \phi) - 2Z(g, \chi, \phi) \\ = \;& J'(0, g, \chi, \phi) + \widetilde{J}'(0, g, \chi, \phi) + J(0, g, \chi, \phi^S \otimes \widetilde{\phi}'_S) + \sum_{v \in S \text{ nonsplit}} \theta(g, \chi, \widetilde{\phi}''_v \otimes \phi^v). \end{aligned}$$

Here $\widetilde{\phi}'_S \in \mathscr{S}(\mathbb{B}_S \times F_S^\times)$ and $\widetilde{\phi}''_v \in \mathscr{S}(B(v)_v \times F_v^\times)$ are some Schwartz functions. We can replace J_* by J since the constant term is killed by the degeneracy assumption as before.

Now we can conclude that the above equality is true for all $g \in \mathrm{GL}_2(\mathbb{A})$. In fact, the equality is true for all $g \in \mathrm{GL}_2(F)\mathrm{GL}_2(\mathbb{A}^S)$ since both sides are automorphic. Then it is true for all $g \in \mathrm{GL}_2(\mathbb{A})$ because $\mathrm{GL}_2(F)\mathrm{GL}_2(\mathbb{A}^S)$ is dense in $\mathrm{GL}_2(\mathbb{A})$.

This result easily implies Theorem 6.1.1 for all ϕ satisfying the degeneracy assumption. In fact, the Eisenstein series and their derivatives are automatically perpendicular to φ. It remains to check that $\theta(g, \chi, \widetilde{\phi}''_v \otimes \phi^v)$ is perpendicular to φ. Note that the theta series is defined on the nearby quaternion algebra $B(v)$. Then the result follows from the criterion of the existence of local χ-linear vectors by the result of Tunnell [Tu] and Saito [Sa], Proposition 1.1.1.

In the end, we extend Theorem 6.1.1 to general Schwartz functions. It is easy to see that the result of the theorem depends only on the theta lifting $\theta(\phi \otimes \varphi)$ in $\pi' \otimes \widetilde{\pi}'$. It suffices to show that any element of $\pi' \otimes \widetilde{\pi}'$ is equal to a linear combination of $\theta(\phi \otimes \varphi)$ for some $\varphi \in \pi$ and $\phi \in \mathscr{S}(\mathbb{B} \times \mathbb{A}^\times)$ satisfying the degeneracy assumption associated to some S.

In fact, let α be any nonzero pure tensor in $\pi' \otimes \widetilde{\pi}'$. Then by theta lifting, it gives a surjective $\mathrm{GL}_2(\mathbb{A})$-equivariant map $l : \mathscr{S}(\mathbb{B} \times \mathbb{A}^\times) \to \pi$. By Proposition 3.2.1, there is a ϕ satisfying the degeneracy assumption for some S such that the image $\varphi = l(\phi)$ is nonzero. Then α must be a multiple of $\theta(\phi \otimes \varphi)$. We conclude that the theorem is true for all φ, ϕ.

6.2 Hecke and theta correspondences

For any $\phi \in \mathscr{S}(\mathbb{B} \times \mathbb{A}^\times)$ we can define a generating series $Z(\phi)$ of Hecke operators, i.e. an element in

$$\mathrm{Ch}^1(X \times X) \otimes \mathscr{A}^{(2)}(\mathrm{GL}_2(\mathbb{A})).$$

Here $\mathrm{Ch}^1(X \times X)$ denote the direct limit of $\mathrm{Ch}^1(X_U \times X_U)$ under pull-backs and $\mathscr{A}^{(2)}(\mathrm{GL}_2(\mathbb{A}))$ denote the space of automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ which is discrete of weight 2 at all archimedean places. For φ a form in an irreducible space π of $\mathscr{A}^{(2)}(\mathrm{GL}_2(\mathbb{A}))$ invariant under $F_\infty^\times$, we will have a Hecke operator

$$\widetilde{Z}(\phi \otimes \varphi) := \int_{[\mathrm{GL}_2]} \widetilde{Z}(r(g)\phi)\varphi(g) dg.$$

Here the notation $\int_{[G]}$ is defined in Introduction. We want to understand the relation between this series and theta series $\theta(\phi \otimes \varphi)$.

By definition, the cycle $\widetilde{Z}(\phi \otimes \varphi)$ is the direct limit of certain cycles $Z(\phi \otimes \varphi)_U$ on $X_U \times X_U$ where U is an open subgroup of $\mathbb{B}^\times/D$ such that ϕ is invariant over $U \times U$. Each $Z(\phi \otimes \varphi)_U$ defines an endomorphism $Z(\phi)_U$ of $\mathrm{Jac}(X_U)$. Under the projections between X_U's, this morphism define a morphism between inverse and direct limits:

$$\widetilde{Z}(\phi \otimes \varphi) : \quad \mathrm{Alb}(X) := \lim_{\overleftarrow{U}} \mathrm{Jac}(X_U) \longrightarrow \mathrm{Jac}(X) := \lim_{\overrightarrow{U}} \mathrm{Jac}(X_U).$$

On the other hand, using the measure on $\mathbb{B}^\times/D$ introduced in the introduction, $\deg(\mathscr{L}_U)^{-1}Z$ defines an endomorphism:

$$R(\phi \otimes \varphi) : \quad \mathrm{Jac}(X) \otimes \mathbb{C} \longrightarrow \mathrm{Jac}(X) \otimes \mathbb{C}.$$

More precisely, the cycle $R(\phi \otimes \varphi)$ is represented by an element in $\mathscr{S}(\mathbb{B}_f^\times)$ which acts on $\mathrm{Jac}(X)$ by usual right translation.

The image $\mathbb{T}^{(2)}$ of $\mathscr{S}(\mathbb{B}^\times/D)$ in $\mathrm{End}(\mathrm{Jac}(X)) \otimes \mathbb{C}$ is exactly $\widetilde{\mathscr{S}}(\mathbb{B}^\times/D)$ introduced in the introduction, i.e, isomorphic to the direct sum of $\sigma \otimes \widetilde{\sigma}$ where σ are Jacquet–Langlands correspondence of finite parts weight 2 forms on $\mathrm{GL}_2(\mathbb{A})$. To see that we may fix an archimedean place and consider the induced action on cotangent space $\Gamma(X_\tau, \Omega^1)$. It is easy to see that this isomorphic to direct sum of σ.

If we take standard archimedean components of ϕ and φ, we just define a homomorphism

$$R : \quad \mathscr{S}(\mathbb{B} \otimes \mathbb{A}^\times) \otimes \pi \longrightarrow \oplus\, \sigma \otimes \widetilde{\sigma}.$$

As this map is equivariant under the action by $\mathbb{B}^\times \times \mathbb{B}^\times \times \mathrm{GL}_2(\mathbb{A})$, this morphism must have image in $\mathrm{JL}(\pi) \otimes \mathrm{JL}(\widetilde{\pi})$:

$$R_\pi : \quad \mathscr{S}(\mathbb{B}_f \times \mathbb{A}^\times) \otimes \pi \longrightarrow \mathrm{JL}(\pi) \times \mathrm{JL}(\widetilde{\pi}).$$

Also this morphism must be a multiple of the theta lifiting normalized in §2.2:

$$R_\pi = c(\pi)\theta_\pi$$

for some $c(\pi) \in \mathbb{C}$. Notice that θ_π was originally defined over $\widetilde{\mathscr{S}}(\mathbb{V} \times \mathbb{A}^\times) \otimes \pi$ but fact through the quotient $\mathscr{S}(\mathbb{V} \times \mathbb{A}^\times) \otimes \pi$ since π has trivial central characters at infinities. The goal of the next few sections is to prove the following:

Proposition 6.2.1.

$$c(\pi) = \frac{L(1, \pi, ad)}{2\zeta(2)}.$$

Lefschetz trace formula

To compute $c(\pi)$, it suffices to compose with composition map $\mathrm{tr} : \mathrm{JL}(\pi_f) \otimes \mathrm{JL}(\widetilde{\pi}_f) \longrightarrow \mathbb{C}$. Write $\tau(g, \phi)$ as the generating series

$$\tau(g, \phi) := \mathrm{tr} \cdot \widetilde{Z}(r(g)\phi)$$

where trace is on direct sum of weight 2 representations $\oplus\sigma_f$. Then $\tau(\phi)$ is a cusp form of weight 2. Then we have

$$\mathrm{tr}R(\phi \otimes \varphi) = (\tau(\phi), \varphi).$$

To get a formula $\tau(\phi)$, we use Lefshetz fixed point theorem and an identification

$$\oplus\sigma_f = \Gamma(X_\tau, \Omega^1)$$

for a real place τ. Recall that $\tau(\phi)$ is a linear combination of Hecke operators $Z(x, u)_U$ which acts on $\Gamma(X_U, \Omega^1)$ which has trace $t(x, u)_U \in F$.

Since

$$H^1(X_\tau, \mathbb{C}) = \Gamma(X_\tau, \Omega^1) \oplus \Gamma(X_\tau, \bar{\Omega}^1),$$

$\tau \circ t(x, u)$ plus its complex conjugate is the trace of $Z(x, u)_U$ on $H^1(X_\tau, \mathbb{Q})$. As τ is real, it follows that

$$t(x, u)_U = \frac{1}{2}\mathrm{tr}(Z(x, u)_U, H^1(X_\tau, \mathbb{Q})).$$

Using Leftschets fixed point theorem,

$$\begin{aligned} \deg \Delta_U^* Z(x, u)_U &= \sum_i (-1)^i \mathrm{tr}(Z(x, u)_U, H^i(X_\tau, \mathbb{Q})) \\ &= 2 \deg Z(x, u)_U - \mathrm{tr}(Z(x, u)_U, H^1(X_\tau, \mathbb{Q})) \end{aligned}$$

where Δ_U is the diagonal embedding $X_U \longrightarrow X_U \times X_U$ and $\deg Z(x, u)_U$ is the (equal) degree of two projections of $Z(x, u)_U$ to X_U. Combining this we obtain:

$$t(x, u)_U = \deg Z(x, u)_U - \frac{1}{2} \deg \Delta_U^* Z(x, u)_U.$$

Normalize this degree by the factor $(\deg \mathscr{L}_U)^{-1}$, we obtain

$$\tau(\phi) = \deg \widetilde{Z}(\phi) - \frac{1}{2} \deg \Delta^* \widetilde{Z}(\phi).$$

The series $\deg \widetilde{Z}(\phi)$ is essentially an Eisenstein series, thus we have

$$\mathrm{tr} R(\phi \otimes \varphi) = -\frac{1}{2}(\deg \Delta^* \widetilde{Z}(\phi), \varphi).$$

To compute $\Delta^* Z(\phi)$ we will compute its pull-back $Z(\phi)_1$ to the Shimura variety $M'_{K'}$ defined by group $\mathrm{GSpin}(\mathbb{V})$ as calculated in §4.2 and the pull-back formula in our composito paper, section 3. Let $\mu_U = F^\times \cap U$ and $\mu'_U = F_+^\times \cap q(U)$. Assume that U is sufficiently small so that $\mu'_U = \mu_U^2$. Then $Z(\phi)_1$ has the following expression:

$$\widetilde{Z}(\phi)_1 = [\mathbb{A}_f^\times : F^\times K_Z] \mathrm{vol}(K_Z) \sum_{u \in \mu_U^2 \backslash F^\times} Z(\phi_u)$$

where $Z(\phi_u)$ is the generating series on $M'_{K'}$ for the function $\phi_u(x) := \phi(x, u)$ on $\mathbb{V}$. Write $\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1$ be the orthogonal decomposition corresponding to the constant matrices $\mathbb{V}_1 = \mathbb{A}$ and trace free matrices $\mathbb{V}_0$. The diagonal $X_U \longrightarrow X_U \times M'_{K'}$ corresponding to the subgroup $\mathbb{B}_f^\times = \mathrm{GSpin}(\mathbb{V}_0)$. Thus by pull-back formula we have

$$Z(\phi_u)|_{X_U} = Z(\theta_1(\phi_u))$$

where $Z(\theta(\phi_u))$ is the generating series of CM-points on X_U for the function

$$\theta_1(\phi_u)(x_0) = \sum_{x_1 \in V_1} \phi(x_0 + x_1, u).$$

In summary we have just shown the following:

$$\Delta^* \widetilde{Z}(\phi) = [\mathbb{A}_f^\times : F^\times K_Z] \mathrm{vol}(K_Z) \sum_{u \in \mu_U^2 \backslash F^\times} Z(\theta_1(\phi_u))$$

and

$$\tau(\phi) = [\mathbb{A}_f^\times : F^\times K_Z] \mathrm{vol}(K_Z) \sum_{u \in \mu_U^2 \backslash F^\times} \mathrm{vol}(U) \deg_{X_U} Z(\theta_1(\phi_u)).$$

6.3 Degree and Siegel–Weil formulae

Let $\phi \in \mathscr{S}(\mathbb{V}_0)$ which is invariant under $\mathrm{GSpin}(\mathbb{V}_0)$. We form the generating series of CM-cycles as $Z(\phi)_U = \sum_{x \in U \backslash \mathbb{V}_0} \phi(x) Z(x)_U$ where U is any open subset over which ϕ is invariant. Since ϕ is invariant under the center of $\mathbb{B}^\times$, we will always assume that U contains $\mathbb{B}_\infty^\times \cdot Z(\mathbb{A})$. In particular, it contains D. Then $Z(\phi)_U$ form an element $Z(\phi)$ in the direct limit of $\mathrm{Ch}^1(X_U)_\mathbb{C}$. We want to compute the degree $\deg Z(\phi) = (\deg \mathscr{L}_U)^{-1} \deg Z(\phi)_U$. Recall that the measure on $\mathbb{B}^\times / D$ has been fixed in the introduction. The computation is very similar to Proposition 4.2.2.

Fix one embedding $\tau : F \longrightarrow \mathbb{R}$ and write the Shimura curve using a *coherent quaternion algebra* B over F which has ramification set $\Sigma \setminus \{\tau\}$. In this way, the Shimura curve has an presentation at τ as

$$X_{U,\tau}^{\mathrm{an}} = B^\times \backslash \mathscr{H}^\pm \times \mathbb{B}_f^\times / U$$

here B is an quaternion algebra over F with ramification set $\Sigma\setminus\{\tau\}$, and an isomorphism $B\otimes\mathbb{A}_f\simeq\mathbb{B}_f$. Let V denote quadratic space B with norm q. For every $x\in\mathbb{V}_0$ with norm in $F_+^\times$, there is a $(y,g)\in V_0\times\mathbb{B}_f^\times$ such that $x=g^{-1}y$, we have the following expression of the generating series

$$Z(\phi)_U=\phi(0)Z(0)_U+\sum_{y\in B^\times\backslash(V_0-\{0\})}\sum_{g\in\mathbb{B}_{y,f}^\times\backslash\mathbb{B}_f^\times/U}\phi(g^{-1}y)Z(y,g)_U$$

here $\mathbb{B}_{y,f}$ is the commutator of $y\in\mathbb{B}_f$ which is actually equal to $\mathbb{A}_f[y]$ and $Z(y,g)$ is the cycle represented by points $\mathscr{H}_y^\pm\times\mathbb{B}_{y,f}^\times gU$, where $\mathscr{H}_y^\pm$ are the two points perpenticular to y.

As a distribution, the pull-back of $Z(y,g)_U$ on $\mathscr{H}^\pm\times\mathbb{B}_f^\times$ has form

$$\sum_{\gamma\in B^\times/B_y^\times}1_{\mathscr{H}_{\gamma y}^\pm}(\tau)1_{\mathbb{B}_{\gamma y}^\times\gamma gU}(h),\qquad\tau\in\mathscr{H}^\pm,h\in\mathbb{B}_f^\times.$$

The condition that $h\in\mathbb{B}_{\gamma y}^\times\gamma gU$ is equivalent to $g\in\mathbb{B}_y^\times\gamma^{-1}hU$. Thus the non-singular part of $Z(\phi)_U$ is represented by

$$\sum_{y\in B^\times\backslash(V_0-\{0\})}\sum_{\gamma\in B^\times/B_y^\times}\phi(h^{-1}\gamma y)1_{\mathscr{H}_{\gamma y}^\pm}(\tau)=\sum_{y\in V_0-\{0\}}\phi(h^{-1}y)1_{\mathscr{H}_y^\pm}(\tau).$$

The normalized degree of this divisor is given by the integration of this function on

$$B^\times\mathbb{A}_f^\times\backslash\mathscr{H}^\pm\times\mathbb{B}_f^\times$$

against the product of the discrete measure on $\mathscr{H}^\pm$ and the chosen measure on $\mathbb{B}_f^\times$. Since $\mathscr{H}_y^\pm$ contains exactly two points for each y, the degree of non-singular part $Z_*(\phi)$ of $Z(\phi)$ is given by

$$2\int_{B^\times\mathbb{A}_f^\times\backslash\mathbb{B}_f^\times}\sum_{y\in V_0-\{0\}}\phi(h^{-1}y)dh=2\sum_{y\in B^\times\backslash(V_0-\{0\})}\mathrm{vol}(B_y^\times\mathbb{A}_f^\times\backslash\mathbb{B}_y^\times)\int_{\mathbb{B}_y^\times\backslash\mathbb{B}_f^\times}\phi(h^{-1}y)dh.$$

Here the decomposition depends on the choice of the Haar measure on $\mathbb{B}_{y,f}^\times$. We will take a measure on $\mathbb{B}_{y,f}^\times$ so that its maximal compact subgroup takes volume 1. Then the volume above is the relative class number $h(t)$ of the field $E_t=F(\sqrt{-t})$. Notice that the last expression does not depend on the choice of B. Indeed, it can be written as a Fourier expansion:

$$\sum_{t\in F_+^\times}2h(t)\int_{\mathbb{V}(t)}\phi(x)d_{\mathbb{B}}x$$

where $\mathbb{V}(t)$ is the set of elements in $\mathbb{V}$ with norm t which can be identified with $\mathbb{B}_y^\times\backslash\mathbb{B}^\times$ after chosen any point $x_t\in\Omega_t$. Here we take measures at archimedean places so that both $\mathbb{B}_v^\times/F_v^\times$ and $E_{t,v}^\times/F_v^\times$ have measure 1.

Recall the Eisesntein series for the Metaplectic group $\widetilde{\mathrm{SL}}_2(\mathbb{A})$:

$$E(s,g,\phi)=\sum_{\gamma\in P^1(F)\backslash \mathrm{SL}_2(F)}\delta(\gamma g)^s r(\gamma g)\phi(0).$$

We want to show that

$$\deg Z(\phi)=\kappa\cdot E(0,1,\phi)$$

for some constant κ depending on $\mathbb{B}$ only. We abbreviate $E(0,1,\phi)=E(\phi)$.

For any $a\in F_+^\times$, consider the a-th Fourier coefficient

$$E_a(\phi)=\int_{F\backslash\mathbb{A}}E\left(0,n(b),\phi\right)\psi(-ab)db=\int_{\mathbb{A}}r(wn(b))\phi(0)\psi(-ab)db.$$

By the local Siegel–Weil formula in Theorem 2.1.2,

$$E_a(\phi)=-\int_{\mathbb{V}(a)}\phi(x)d_\psi x.$$

Here the negative sign comes from the Weil index which equals -1 because of the incoherence.

In the following we want to compare the measure $d_{\mathbb{B}}x$ and $d_\psi x$ on $\mathbb{V}(a)$. Since both measures are invariant under conjugation by $\mathbb{B}^\times$, their ratio $c(a,\mathbb{B})$ is a constant which can be decomposed as a product of local ratios:

$$d_{\mathbb{B}_v}x_v=c(a,\mathbb{B}_v,\psi_v)d_{\psi_v}x_v.$$

Siegel–Weil formula

To understand these local constants, we use Siegel–Weil formula for a totally definite quaternion algebra B over F. We define the theta series for V_0, the trace 0 part of B. For any function $\phi\in\mathscr{S}(V_0(\mathbb{A}))$ we can define theta series $\theta(g,h,\phi)$ and Eisenstein series as usual. The Siegel–Weil formula gives

$$\int_{B^\times\mathbb{A}^\times\backslash B_{\mathbb{A}}^\times}\theta(g,h)dh=\mathrm{vol}(B^\times\mathbb{A}^\times\backslash B_{\mathbb{A}}^\times)E(0,g,\phi).$$

If we compute the Fourier coefficient by the same method as above we obtain

$$\int\theta_a(1,h)dh=h(a)\int_{\mathbb{V}(a)}\phi(x)d_{B_{\mathbb{A}}}x$$

and

$$E_a(\phi)=\int_{\mathbb{V}(a)}\phi(x)d_\psi x.$$

It follows that

$$h(a)c(a,B_{\mathbb{A}})=\mathrm{vol}(B^\times\mathbb{A}^\times\backslash B_{\mathbb{A}}^\times).$$

This in particular shows that $c(a,\mathbb{B})/c(a,\mathbb{B}')$ is independent of a and ψ for any two coherent quaternion algebras $\mathbb{B}$ and $\mathbb{B}'$. For each v, let $\delta(a,\psi_v)$ denote the ratio of

$$c(a, D_v, \psi_v)/c(a, M_2(F_v), \psi_v)$$

here D_v is the quaternion division algebra over F_v. Then $c(a,\mathbb{B})/c(a,\mathbb{B}')$ is a product of even numbers of $\delta(a,\psi_v)^{\pm}$. Thus by varies $\mathbb{B}$ and $\mathbb{B}'$, we see that $\delta_v(a,\psi_v)$ is independent of a and ψ_v. This implies that $h(a)c(a,\mathbb{B})$ is independent of $\mathbb{B}$ even for incoherent $\mathbb{B}$. Thus we have shown that

$$\deg Z(\phi) = \kappa \cdot E(0,1,\phi)$$

for some κ depending only on $\mathbb{B}$.

Proposition 6.3.1.

$$\kappa = -1.$$

We compare the constant term of the above identity. The constant term of $\deg Z(\phi)$ is $\phi(0)\deg Z(0)$ while the constant term of $E(0,e,\phi)$ is $\phi(0)$. In this way, we obtain the following:

$$\kappa = \deg Z(0) = \frac{1}{\deg \mathscr{L}_U} \deg Z(0)_U.$$

Recall that $Z(0)_U$ is the negative Hodge bundle on X_U thus we have $\kappa = -1$.

6.4 Completion of proof

Now we are ready to compute $\mathrm{tr}R(\phi\otimes\varphi)$ which is given by

$$\mathrm{tr}R(\phi\otimes\varphi) = -\frac{1}{2}(\deg \Delta^*\widetilde{Z}(\phi), \varphi).$$

with normalized measure on $\mathbb{B}^\times/D$, we have

$$\begin{aligned}\deg \Delta^*\widetilde{Z}(\phi) &= -\,[\mathbb{A}_f^\times : F^\times K_Z]\mathrm{vol}(K_Z) \sum_{u\in\mu_K^2\backslash F^\times} E(0,e,\theta_1(\phi_u))\\ &= -\,I_1(0,g,\phi)\end{aligned}$$

where

$$I_1(s,g,\phi) = [\mathbb{A}_f^\times : F^\times K_Z]\mathrm{vol}(K_Z) \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s \sum_{(x,u)\in\mu_K\backslash V_1\times F^\times} r(\gamma)\phi(x,u)$$

is the minus the mixed Eisenstein-theta series for the coherent subspace V_1.

Let $\widetilde{\phi}\in\widetilde{\mathscr{S}}(\mathbb{V}\times\mathbb{A}^\times)$ be such that

$$\int_{Z(F_\infty)} r(z)\widetilde{\phi} = \phi.$$

Define

$$I_1(s,g,\widetilde{\phi}) = \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s \sum_{(x,u)\in V_1\times F^\times} r(\gamma)\widetilde{\phi}(x,u)$$

then we have and equal to

$$I_1(s,g,\phi) = [Z(\mathbb{A}_f):Z(F)K_Z]\int_{Z(F)Z(F_\infty)Z_K/Z(F)} I(s,zg,\widetilde{\phi})dz.$$

The computation of the trace is straightforward:

$$\begin{aligned}\mathrm{tr}R(\phi\otimes\varphi) =& \frac{1}{2}\int_{\mathbb{A}^\times \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})}\int_{F^\times\backslash\mathbb{A}^\times} \varphi(zg)I(0.zg,\widetilde{\phi})dz\\ =&\frac{1}{2}\int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})}\varphi(g)I(0,g,\widetilde{\phi})dg.\end{aligned}$$

The integral inside has been computed before and it equals to

$$\int_{N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} W_{-1}(g)\phi(1,1)dg = \frac{L(1,\pi,ad)}{\zeta(2)}\mathscr{F}\theta(\phi\otimes\widetilde{\varphi}).$$

In summary, we have shown:

$$R(\phi\otimes\varphi) = \frac{L(1,\pi,ad)}{2\zeta(2)}\theta(\phi\otimes\widetilde{\varphi}).$$

Now we can deduce the main theorem from Theorem 6.1.1. Recall definitions

$$I(s,g,\chi,\phi) = \int_{[T]} I(s,g,r(t,1)\phi)\chi(t)dt,$$

$$\widetilde{Z}(g,(t_1,t_2),\phi) = \langle \widetilde{Z}(g,\phi)[t_1],[t_2]\rangle_{NT}$$

and

$$Z(g,\chi,\phi) = \int_{[T]}\widetilde{Z}(g,(t,1),\phi)\chi(t)dt = \frac{1}{2L(1,\eta)}\langle\widetilde{Z}(g,\phi)Y_\chi,Y_\chi\rangle_{\mathrm{NT}}.$$

. Here the integration on $[T]$ means integration on $T(F)\backslash T(\mathbb{A})/Z(F_\infty)Z_K$ and a multiplication by $[\mathbb{A}_f^\times : F^\times Z_K]$. Then Theorem 6.1.1 gives

$$(I'(0,g,\chi,\phi),\varphi)_{Pet} = 2(Z(g,\chi,\phi),\varphi)_{Pet}.$$

Let $\widetilde{\phi}$ be a lift of ϕ in $\widetilde{\mathscr{S}}(\mathbb{V}\times\mathbb{A}^\times)$. Then we have

$$I(s,g,\chi,\phi) = I(s,g,\chi,\widetilde{\phi}) := \int_{T(\mathbb{A})/T(F)} I(s,g,r(t,1)\phi)\chi(t)dt.$$

By proposition 2.3.2, the left hand side of the main identity can be written as

$$(I'(0,g,\chi,\phi),\varphi)_{Pet} = P'(0,\chi,\widetilde{\phi},\varphi) = \frac{L'(1/2,\pi,\chi)}{2L(1,\eta)} \prod_v \alpha_v(\theta(\widetilde{\phi}_v \otimes \varphi_v)).$$

By our computation in the last section, the right hand side of the main identity is

$$\frac{1}{L(1,\eta)}\langle \widetilde{Z}(\phi\otimes\varphi)Y_\chi, Y_\chi\rangle_{NT} = \frac{L(1,\pi,ad)}{2L(1,\eta)\zeta(2)}\langle T(\theta(\widetilde{\phi}\otimes\varphi))Y_\chi, Y_\chi\rangle_{NT}.$$

Thus we have the main theorem in our paper taking a linear combination of ϕ_i and φ_i so that

$$f\otimes\widetilde{f} = \sum_i \theta(\phi_i\otimes\varphi_i).$$

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A Conjecture of André and Oort

Andrei Yafaev and Bas Edixhoven

Galois orbits and equidistribution of special subvarieties : towards the André-Oort conjecture. *

Emmanuel Ullmo Andrei Yafaev

Abstract

In this paper we develop a strategy and some technical tools for proving the André-Oort conjecture. We give lower bounds for the degrees of Galois orbits of geometric components of special subvarieties of Shimura varieties, assuming the Generalised Riemann Hypothesis. We proceed to show that sequences of special subvarieties whose Galois orbits have bounded degrees are equidistributed in a suitable sense.

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1 Introduction.

The main motivation for this paper is the André-Oort conjecture stated below.

Conjecture 1.1 (André-Oort) *Let S be a Shimura variety and let Σ be a set of special points in S. Every irreducible component of the Zariski closure of Σ is a special (or Hodge type) subvariety of S.*

There are two main approaches to this conjecture which proved fruitful in some cases. One, due to Edixhoven and Yafaev (see [7] and [18]), relies on the Galois properties of special points and geometric properties of images of subvarieties of Shimura varieties by Hecke correspondences. The other, due to Clozel and Ullmo (see [2]), aims at proving that certain sequences of special subvarieties are equidistributed in a certain sense. This approach uses some deep theorems from ergodic theory. The purpose of this paper is to explain how to combine these two approaches in order to obtain a definite strategy for attacking the André-Oort conjecture and to provide certain ingredients. This strategy and the results of this paper are subsequently used in [9] by Klingler and Yafaev to prove the André-Oort conjecture assuming the Generalised Riemann Hypothesis (GRH).

To explain the alternative, we need to introduce some terminology. Let S be a connected component of a Shimura variety. There is a Shimura datum (G, X) and a compact open subgroup K of $G(\mathbb{A}_f)$ such that S is a connected component of

$$\mathrm{Sh}_K(G, X) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K.$$

Without loss of generality, we may and do assume that S is the image of $X^+ \times \{1\}$ in $\mathrm{Sh}_K(G, X)$ (where X^+ is a fixed connected component of X). A special subvariety Z of S is associated to a sub-Shimura datum (H, X_H) of (G, X). More precisely Z is an irreducible component of the image of $\mathrm{Sh}_{K\cap H(\mathbb{A}_f)}(H, X_H)$ in $\mathrm{Sh}_K(G, X)$ contained in S. Furthermore, as will be explained later, we will always be able to assume that H is the generic Mumford-Tate group on X_H.

Let E be some number field over which S admits a canonical model and such that E contains the reflex field $E(H, X_H)$. Let Z be a special subvariety of S. Then Z is a connected component of the image of $\mathrm{Sh}_{H(\mathbb{A}_f)\cap K}(H, X_H)$ in $\mathrm{Sh}_K(G, X)$ where (H, X_H) is a sub-Shimura datum of (G, X) such that H

is the generic Mumford-Tate group of X_H. We also assume that E contains the reflex field $E(H, X_H)$.

By degree of the Galois orbit of Z, denoted, $\deg(\mathrm{Gal}(\overline{E}/E)\cdot Z)$, we mean the degree of $\mathrm{Gal}(\overline{E}/E)\cdot Z$ with respect to the Baily-Borel compactification of $\mathrm{Sh}_K(G, X)$. If Z is a special point, then $\deg(\mathrm{Gal}(\overline{E}/E)\cdot Z)$ is simply the number of $\mathrm{Gal}(\overline{E}/E)$ conjugates of Z.

The "philosophy" of this paper is the following alternative. Let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of special subvarieties of S. After possibly replacing (Z_n) by a subsequence and assuming the GRH for CM-fields, at least one of the following cases occurs.

1. The sequence $\deg(\mathrm{Gal}(\overline{E}/E)\cdot Z)$ tends to infinity as $n \to \infty$ (and therefore Galois-theoretic and geometric techniques can be used).

2. The sequence of probability measures (μ_n) canonically associated with (Z_n) weakly converges to some μ_Z, the probability measure canonically associated with a special subvariety Z of S. Moreover, for every n large enough, Z_n is contained in Z. In other words, the sequence (Z_n) is equidistributed.

Which of the two cases occurs depends on the geometric nature of the subvarieties Z_n. Let us explain this in more detail.

A special subvariety Z defined by a Shimura datum (H, X_H) as before is called strongly special (see [2]) if the image of the group H in the adjoint group G^{ad} is semisimple. Note that the condition (b) in the definition of "strongly special" ([2], 4.1) is in fact implied by the first (see [15] Rem. 3.9, or the proof of the theorem 3.8 of this paper). Clozel and Ullmo proved in [2] that if the subvarieties Z_n are strongly special than the second case of the alternative occurs. This result is unconditional.

On the other extreme, if H is a torus, then Z is a special point. If (Z_n) is a sequence of special points, then the first case of the alternative occurs (and the second in general does not ! A sequence of special points is usually not equidistributed). This uses the GRH but we believe that one might be able to get rid of this assumption. We also prove the equidistribution result unconditionally in the case where the subvarieties Z_n satisfy an additional assumption. In the paper [18], lower bounds for Galois orbits of special points are given and used to prove the André-Oort conjecture for curves. However, these bounds are not strong enough to prove that they are unbounded for a general infinite sequence of special points.

The first thing we do in this paper is we give lower bounds for the degree of Galois orbits of *non-strongly* special subvarieties (Theorem 2.13). A special case of our theorem (when H is a torus) is an improvement upon [18]. In the case where H is a torus, we can show that given an infinite set Σ of special points, the Galois orbit of the point x is unbounded as x ranges through Σ. Lower bounds obtained in [18] do not allow to prove such statement.

We now explain our lower bounds in detail. Let N be an integer. Let H be the generic Mumford-Tate group on X_H and let T be its connected centre. Suppose that T is a non-trivial torus. Let L_T be the splitting field of T. Let K_T^m be the maximal compact open subgroup of $T(\mathbb{A}_f)$. Note that K_T^m is a product of maximal compact open subgroups $K_{T,p}^m$ of $T(\mathbb{Q}_p)$ for all primes p. Let K_T be the compact open subgroup $T(\mathbb{A}_f) \cap K$ of $T(\mathbb{A}_f)$. We may assume that K is a product of compact open subgroups K_p of $G(\mathbb{Q}_p)$ in which case K_T is also a product of compact open subgroups $K_{T,p}$ of $T(\mathbb{Q}_p)$. We define $i(T)$ to be the number of primes p such that $K_{T,p} \neq K_{T,p}^m$. We show (thm. 2.13) that there is an absolute constant B such that for every component Z of the image of $\mathrm{Sh}_{H(\mathbb{A}_f)\cap K}(H, X_H)$ in S

$$\deg(\mathrm{Gal}(\overline{E}/E) \cdot Z) \gg B^{i(T)} |K_T^m / K_T| \log(|\mathrm{disc}(L_T)|)^N.$$

The next task we carry out is the analysis of conditions, under which a given sequence of special subvarieties Z_n is such that $\deg(\mathrm{Gal}(\overline{E}/E) \cdot Z_n)$ is bounded. We translate this condition into explicit conditions on the Shimura data defining the Z_n. We introduce the notion of a T-special subvariety. Suppose that G is semisimple of adjoint type and fix a subtorus T of G such that $T(\mathbb{R})$ is compact. A T-sub-Shimura datum (H, X_H) of (G, X) is a sub-Shimura datum such that H^{der} is non-trivial and $T = Z(H)^0$ is the connected centre of H. A T-special subvariety is a special subvariety defined by a T-sub-Shimura datum. Fix an integer N. We show (thm. 3.9) that there is a finite set $\{T_1, \ldots, T_r\}$ of subtori of G such that any special subvariety Z with $\deg(\mathrm{Gal}(\overline{E}/E) \cdot Z) \leq N$ is T_i-special for some $i = 1, \ldots, r$. This result crucially relies on the results of Gille and Moret-Bailly [8] provided in the appendix.

Finally, using the ergodic methods of [2], we prove that if the degree of $Gal(\overline{E}/E) \cdot Z_n$ is bounded (when n varies), then the second case of the alternative occurs. We actually show (thm. 3.8) that, for a fixed T, a sequence of T-special subvarieties is equidistributed in the sense explained above.

The alternative explained above is used in the forthcoming paper by Klingler and the second author [9] to prove the following theorem which is the most general result on the André-Oort conjecture obtained so far.

Theorem 1.2 *Let (G, X) be a Shimura datum and K a compact open subgroup of $G(\mathbb{A}_f)$. Let Σ be a set of special points in $\mathrm{Sh}_K(G, X)$. We make one of the two following assumptions:*

1. *Assume the Generalised Riemann Hypothesis (GRH) for CM fields.*
2. *Assume that there exists a faithful representation $G \hookrightarrow \mathrm{GL}_n$ such that with respect to this representation, the Mumford-Tate groups $MT(s)$ lie in one $\mathrm{GL}_n(\mathbb{Q})$-conjugacy class as s ranges through Σ.*

Then every irreducible component of the Zariski closure of Σ in $\mathrm{Sh}_K(G, X)$ is a special subvariety.

Klingler and Yafaev started working together on this conjecture in 2003 trying to generalise the Edixhoven-Yafaev strategy to the general case of the André-Oort conjecture. In the process two main difficulties occurred. One is the question of irreducibility of transforms of subvarieties under Hecke correspondences. This problem is dealt with in the forthcoming paper by Klingler and Yafaev, this allows to deal with cases where the first case of the alternative explained above occurs.

The other difficulty was dealing with sets of special subvarieties which are defined over number fields of bounded degree. We deal with this difficulty in the present paper. In fact, we show that this is precisely when the second case of the alternative occurs. This strategy : combination of Galois theoretic and ergodic techniques was discovered by the authors of this paper while the second author was visiting the University of Paris-Sud in January-February 2005. We tested our strategy on the case of subvarieties of a product of modular curves (see [16]).

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2 Degrees of Galois orbits of special subvarieties.

In this section we give lower bounds for the degrees of Galois orbits of non-strongly special subvarieties.

2.1 Preliminaries on special subvarieties and reciprocity morphisms.

We start by recaling some facts about special subvarieties, reciprocity morphisms and Galois action on the geometric components of Shimura varieties. If Z is a topological space, we denote by $\pi_0(Z)$ the set of connected components of Z.

Let (G, X) be a Shimura datum. We assume that that G is semisimple of adjoint type. We fix a faithful representation of G which allows us to view G as a closed subgroup of some GL_n. Let K be a compact open subgroup of $G(\mathbb{A}_f)$ which is contained in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$. We also assume that K is a product of compact open subgroups K_p of $G(\mathbb{Q}_p)$.

Let (H, X_H) be a sub-Shimura datum of (G, X). We suppose that H is not semisimple. Let T be the connected centre of H, so that T is a non-trivial torus and H is an almost direct product TH^{der}.

Let K_H be the compact open subgroup $H(\mathbb{A}_f) \cap K$ of $H(\mathbb{A}_f)$. We first describe the Galois action on the set of components of $\mathrm{Sh}_{K_H}(H, X_H)$. We refer to the sections 2.4-2.6 of [4] for details and proofs. Let $\pi_0(H, K_H)$ be the set of geometric components of $Sh_{K_H}(H, X_H)$. Recall that $\pi_0(H, X_H)$ is $H(\mathbb{Q})^+ \backslash H(\mathbb{A}_f)/K_H$ where $H(\mathbb{Q})^+$ is the stabilizer of a connected component of X_H in $H(\mathbb{Q})$. Let $E = E_H$ be the reflex field of (H, X_H) and $T_E := \mathrm{Res}_{E_H/\mathbb{Q}}\mathbb{G}_{mE_H}$.

The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E_H)$ on $\pi_0(H, K_H)$ is given by the reciprocity morphism

$$r_{(H,X_H)}\colon \mathrm{Gal}(\overline{\mathbb{Q}}/E_H) \longrightarrow \pi_0(\pi(H))$$

where for any reductive group N over $\mathbb{Q}$

$$\pi(N) = N(\mathbb{A}_f)/N(\mathbb{Q})\rho(\widetilde{N}(\mathbb{A}_f)).$$

Here $\rho\colon \widetilde{N} \longrightarrow N^{\mathrm{der}}$ denotes the universal covering of N^{der}. The morphism $r_{(H,X_H)}$ factors through $\mathrm{Gal}(\overline{\mathbb{Q}}/E_H)^{\mathrm{ab}}$ which is identified via global class field theory to $\pi_0(T_E(\mathbb{R})) \times \pi_0(\pi(T_E))$. Let C be the torus H/H^{der}. To (H, X_H) one associates two Shimura data $(C, \{x\})$ and $(H^{\mathrm{ad}}, X_{H^{\mathrm{ad}}})$. The field E_H is the composite of $E(C, \{x\})$ and $E(H^{\mathrm{ad}}, X_{H^{\mathrm{ad}}})$. There are morphisms of Shimura data

$$\theta^{\mathrm{ab}}\colon (H, X_H) \longrightarrow (C, \{x\}) \text{ and } \theta^{\mathrm{ad}}\colon (H, X_H) \longrightarrow (H^{\mathrm{ad}}, X_{H^{\mathrm{ad}}}).$$

Note that $(C, \{x\})$ is a special Shimura datum. Let $r_{(C,\{x\})}$ be the reciprocity morphism associated with $(C, \{x\})$. The morphism θ^{ab} induces a morphism $\pi_0(\pi(H)) \to \pi_0(\pi(C))$. This morphism preceded by $r_{(H,X_H)}$ is $r_{(C,\{x\})}$. We let F be the Galois closure of E_H. Note that the degree of F over $\mathbb{Q}$ is bounded uniformly on (H, X_H). We will keep the notations and assumptions introduced above throughout this section.

It is convenient and sometimes essential to make the assumption that H is the generic Mumford-Tate group on X_H. Below we prove a lemma which will allow us to make this assumption. Let H' be the generic Mumford-Tate group on X_H. By definition, H' is a subgroup of H. Furthermore, $H'^{\mathrm{der}} = H^{\mathrm{der}}$. Let x be an element of X_H and let $X_{H'}$ be the $H'(\mathbb{R})$-orbit of x. Then $X_{H'} = X_H$ and (H', X_H) is a sub-Shimura datum of (H, X_H). let E_H be the reflex field of (H, X_H). Note that E_H is also the reflex field of (H', X_H). Indeed, E_H is the field of definition of the $H(\mathbb{C})$-conjugacy class of $h_{\mathbb{C}}(z, 1)$ for h in X_H which is the same as the $H'(\mathbb{C})$-conjugacy class of $h_{\mathbb{C}}(z, 1)$.

Lemma 2.1 *Let* $\Gamma := G(\mathbb{Q})^+ \cap K$ *and* S *be the component* $\Gamma\backslash X^+$ *of* $\mathrm{Sh}_K(G, X)$. *Note that* S *is the image of* $X^+ \times \{1\}$ *in* $\mathrm{Sh}_K(G, X)$. *Let* V *be a special subvariety of* S. *There exists a sub-Shimura datum* (H_V, X_V) *of* (G, X) *such that* H_V *is the generic Mumford-Tate group on* X_V *and* V *is the image of a connected component of* $\mathrm{Sh}_{K\cap H_V(\mathbb{A}_f)}(H_V, X_V)$ *in* $\mathrm{Sh}_K(G, X)$.

Proof. There exists a sub-Shimura datum $(H, X_H) \subset (G, X)$, such that V is the image of $X_H^+ \times \{h\}$ in $\mathrm{Sh}_K(G, X)$ for some $h \in H(\mathbb{A}_f)$. As this image is contained in S, there exists $g \in G(\mathbb{Q})^+$ and $k \in K$ such that $h = gk$. Let $H_g = g^{-1}Hg$ and X_{H_g} be the conjugacy class of $g^{-1}x_0$ for some $x_0 \in X_H^+$. Let $X_{H_g}^+$ be the connected component of X_{H_g} containing $g.x_0$. Then (H_g, X_{H_g}) is a sub-Shimura datum of (G, X) and V is also the image of $X_{H_g}^+ \times \{1\}$ in $\mathrm{Sh}_K(G, X)$. Let H_V be the generic Mumford-Tate group on X_{H_g} and $X_V = X_{H_g}$. By the previous discussion (H_V, X_V) is a sub-Shimura datum such that H_V is the Mumford-Tate group on X_V and V is the image of $X_V^+ \times \{1\}$ in $\mathrm{Sh}_K(G, X)$. □

In view of this lemma we will only consider in the rest of this section sub-Shimura data $(H, X_H) \subset (G, X)$ such that H is the generic Mumford-Tate group on X_H.

Lemma 2.2 *Let (H, X_H) and K_H be as above, with H being the generic Mumford-Tate group on X_H. Let $f\colon Sh_{K_H}(H, X_H) \longrightarrow Sh_K(G, X)$ be the morphism induced by the inclusion (H, X_H) into (G, X).*

The morphism

$$f\colon Sh_{K_H}(H, X_H) \longrightarrow f(Sh_{K_H}(H, X_H))$$

is generically finite of degree uniformly bounded when (H, X_H) varies. Furthermore, if K is neat, then f is generically injective. In particular, the number of geometric components of $Sh_{K_H}(H, X_H)$ is, up to a uniform (on (H, X_H)) constant, is equal to the number of geometric components of its image in $Sh_K(G, X)$.

Proof. First note that it suffices to prove that the morphism f is generically injective when K is neat. Indeed, any compact open subgroup K of $G(\mathbb{A}_f)$ contains a neat compact open subgroup K'. Using the generic injectivity of $Sh_{K'_H}(H, X_H) \longrightarrow Sh_{K'}(G, X)$, one easily sees that the degree of f is bounded by the index of K' in K.

Suppose that K is neat. Let $\overline{(x_1, h_1)}$ and $\overline{(x_2, h_2)}$ be two points of $Sh_{K_H}(H, X_H)$ having the same image by f. We suppose that $MT(x_1) = MT(x_2) = H$.

There exist an element q of $G(\mathbb{Q})$ and an element k of K such that $x_2 = qx_1$ and $h_2 = qh_1k$.

The fact that $MT(x_1) = MT(x_2) = H$ implies that q belongs to the normalizer $N_G(H)(\mathbb{Q})$ of H in G. Let us check that the group $N_G(H)$ is reductive. There is an element x of X that factors through $N_G(H)$. Then $x(\mathbb{S})$ normalizes the unipotent radical R_u of $N_G(H)$ hence $Lie(R_u)$ is a rational polarisable Hodge structure and the Killing form is non degenerate on $Lie(R_u)$. It follows that R_u is reductive and therefore is trivial. The group $G' := N_G(H)/H$ has the property that $G'(\mathbb{R})$ is compact. Indeed, the centralizer $Z_G(H)(\mathbb{R})$ is compact because it stabilizes a point of a hermitian symmetric domain and as $N_G(H)$ is reductive, the images of $Z_G(H)^0(\mathbb{R})$ and $N_G(H)^0(\mathbb{R})$ in $G'(\mathbb{R})$ coincide.

The equality $h_2 = qh_1k$ shows that q belongs to $H(\mathbb{A}_f)K$. It follows that the image $\overline{q}$ of q in $G'(\mathbb{Q})$ is contained in a compact subgroup of $G'(\mathbb{A}_f)$. As $G'(\mathbb{R})$ is compact, this group is finite. As K is neat, this group is trivial. It follows that q belongs to $H(\mathbb{Q})$ and k to $K_H = H(\mathbb{A}_f) \cap K$. We conclude that the points $\overline{(x_1, h_1)}$ and $\overline{(x_2, h_2)}$ of $Sh_{K_H}(H, X_H)$ are equal. This finishes the proof. □

Recall that T is the connected centre of H and C is H/H^{der}. Note that there is an isogeny $T \longrightarrow C$ with kernel $T \cap H^{\mathrm{der}}$, given by the restriction of the quotient map $H \longrightarrow H/H^{der}$ to T. We will make use of the following lemma.

Lemma 2.3 *The order of the group $T \cap H^{\mathrm{der}}$ is uniformly bounded as (H, X_H) ranges through the sub-Shimura data of (G, X) with H^{der} connected.*

Proof. As $T \cap H^{\mathrm{der}}$ is contained in the centre of H^{der}, we just need a uniform bound on orders of the centres of connected semi-simple subgroups of G. Let L be a connected semi-simple subgroup of G and let D_L be the Dynkin diagram of $L_{\mathbb{C}}$. As the rank of $L_{\mathbb{C}}$ is bounded by the rank of $G_{\mathbb{C}}$, there are only finitely many possibilities for D_L. For each of these possibilities, the order of the centre of $L_{\mathbb{C}}$ is bounded by the index of the lattice of roots in the lattice of weights. □

We now prove some uniformity results regarding the characters occurring in the representation $T \subset \mathrm{GL}_n$ and the reciprocity morphism $r_{(C,x)}$.

As the degree of F is uniformly bounded, we may assume that the Galois group of F over $\mathbb{Q}$ is isomorphic to a fixed abstract group M. Let T_F be the torus $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F}$. We write $H = TH^{\mathrm{der}}$ and we let $\mu\colon \mathbb{G}_{m,\mathbb{C}} \longrightarrow H_{\mathbb{C}}$ be the character $h_{\mathbb{C}}(z, 1)$ where h is an element of X_H such that $MT(h) = H$.

The composition of μ with $H \longrightarrow C$ gives a cocharacter $\mathbb{G}_{m,\mathbb{C}} \longrightarrow C_{\mathbb{C}}$ which we denote μ_C. The cocharacter μ_C is defined over F. Each σ in M defines a character χ_σ and a cocharacter μ_σ of the torus T_F. The character (resp. cocharacter) group of T_F is generated by the χ_σ (resp. μ_σ). We enumerate the elements of M, thus getting a "canonical" basis for the character (respectively cocharacter) group of the torus T_F. There is a natural pairing

$$<,>: X^*(T_F) \times X_*(T_F) \longrightarrow \mathbb{Z}$$

defined by $<\chi_\sigma, \mu_\tau> = \delta_{\sigma,\tau}$ for all σ, τ in M. The reciprocity morphism $r_{(C,\{x\})}: T_F \longrightarrow C$ induces the morphism $r_{C*}: X_*(T_F) \to X_*(C)$ which sends the cocharacter μ_σ to $\sigma(\mu_C)$. The reciprocity $r_{(C,x)}$ induces an injection $X^*(C) \subset X^*(T_F)$. We identify $X^*(C)$ with its image in $X^*(T_F)$.

The fact that the isogeny $\alpha: T \longrightarrow C$ has uniformly bounded degree, say n, implies that there is a surjective morphism $r: T_F \longrightarrow T$ such that

$$\alpha \circ r = r^n_{(C,x)}$$

The morphism r identifies $X^*(T)$ with a submodule of $X^*(T_F)$. We will consider the coordinates of the characters in $X^*(T)$ with respect to the canonical basis of $X^*(T_F)$ described previously.

Lemma 2.4 *The coordinates of characters χ of T intervening in the representation $T \subset \mathrm{GL}_n$, with respect to the basis described above, are bounded uniformly on (H, X_H).*

The size of the torsion of $X^*(T_F)/X^*(T)$ is bounded uniformly on (H, X_H).

Proof. The second statement is a direct consequence of the first.

Let $T_{H^{\mathrm{der}}}$ be a maximal torus of $H^{\mathrm{der}}_{\mathbb{C}}$ such that μ factors through $T_{\mathbb{C}} T_{H^{\mathrm{der}}}$. Let $\widetilde{T_{\mathbb{C}}}$ be the almost direct product $T_{\mathbb{C}} T_{H^{\mathrm{der}}}$, the torus $\widetilde{T_{\mathbb{C}}}$ is a maximal torus of $H_{\mathbb{C}}$.

Let R be the root system associated to (T, H^{der}). There are only a finite, uniformly bounded number of possibilities for R. The representation of H, induces a representation of H^{der}. The dimensions of the irreducible factors of this representation are uniformly bounded hence there is only a finite (uniformly bounded) number of characters of $T_{H^{\mathrm{der}}}$ that intervene in the representation.

As $T \cap H^{\mathrm{der}}$ is finite, we have a direct sum decomposition

$$X^*(\widetilde{T_{\mathbb{C}}})_{\mathbb{Q}} = X^*(T_{\mathbb{C}})_{\mathbb{Q}} \oplus X^*(T_{H^{\mathrm{der}}})_{\mathbb{Q}}$$

Let χ be a character of $\widetilde{T}$ that intervenes in the representation $\widetilde{T} \subset \mathrm{GL}_{n\mathbb{C}}$. The direct sum decomposition above gives the decomposition $\chi = \chi_T + \chi_{H^{\mathrm{der}}}$.

Let $r_H \colon L_{\mathbb{C}} \longrightarrow \widetilde{T_{\mathbb{C}}}$ be the morphism induced by μ. The values taken by the $< \chi, r_{H*}\mu_\sigma >$ are the p such that $V_{\mathbb{C}}^{p,q}$ is non-zero. Hence they are finite in number and uniformly bounded. On the other hand, we have

$$< \chi, r_{H*}\mu_\sigma > = < \chi_T, r_{H*}\mu_\sigma > + < \chi_{H^{\mathrm{der}}}, r_{H*}\mu_\sigma >$$

where χ_T and $\chi_{H^{\mathrm{der}}}$ are the restrictions of χ to T and $T_{H^{\mathrm{der}}}$ respectively. In the decomposition

$$r_{H*}\mu_\sigma = (r_{H*}\mu_\sigma)_T + (r_{H*}\mu_\sigma)_{H^{\mathrm{der}}}$$

there is only a finite number of possibilities for $(r_{H*}\mu_\sigma)_{H^{\mathrm{der}}}$. This is a consequence of the theory of symmetric spaces. To see this, we decompose the root system R into irreducible factors R_i. The components of the $r_{H*}\mu_\sigma$ on R_i are either trivial or correspond to minuscule weights of the dual root system $R_i^\vee$.

It follows that $< \chi_{H^{\mathrm{der}}}, r_{H*}\mu_\sigma >$ takes only finitely many values and so does $< \chi_T, r_{H*}\mu_\sigma >$. We finish by noticing that these $< \chi_T, r_{H*}\mu_\sigma >$ are precisely the coordinates of the characters intervening in $T \subset \mathrm{GL}_n$ with respect to our chosen basis. □

Finally we prove the following result.

Proposition 2.5 *There is an integer A such that for any (H, X_H) and F as above and for any $x \in T(\mathbb{A}_f)$, the image of x^A in $\pi_0(\pi(H))$ is contained in $r_{(H,X_H)}(Gal(\overline{F}/F))$.*

Proof. Consider as before the morphism of algebraic tori

$$r \colon T_F \longrightarrow T.$$

Using the previous lemma and the proof of the theorem 2.6 of [18] we see that there is a uniform integer h such that for all prime p the index of $r(\mathbb{Q}_p \otimes F^*)$ in $T(\mathbb{Q}_p)$ and that of $r(\mathbb{Z}_p \otimes O_F^*)$ in the maximal open compact subgroup $K_{T,p}^m$ of $T(\mathbb{Q}_p)$ are bounded by h.

Let $x \in T(\mathbb{A}_f)$, by the previous discussion x^h is in $r((\mathbb{A}_f \otimes F)^*)$. By construction of r, the image of x^h in $\pi_0(\pi(C))$ is $r_{(C,x)}(\sigma)$ for some $\sigma \in Gal(\overline{F}/F)$. Consider $r_{(H,X_H)}(\sigma) \in \pi_0(\pi(H))$. The image of x^h in $\pi_0(\pi(H))$ and $r_{(H,X_H)}(\sigma)$ have the same image in $\pi_0(\pi(C))$. By lemma 7.2.3 of [9], the kernel of the map $\pi(H) \longrightarrow \pi(C)$ is killed by a uniform power, the conclusion follows. □

2.2 Lower bounds for the degree of Galois orbits of special subvarieties.

We now deal with the problem of bounding (below) the degree of Galois orbits of geometric components of special subvarieties of $Sh_K(G, X)$. We assume that $K \subset G(\mathbb{A}_f)$ is neat and of the form $K = \prod_p K_p$ for some compact open subgroups K_p of $G(\mathbb{Q}_p)$. Note that in the general case we can always find a subgroup K' of K of finite index with these properties. We fix a faithful representation of G which allows us to view G as a closed subgroup of some GL_n. We may and do assume that K is contained in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$

Let M be a projective variety over $\mathbb{C}$, Y be a subvariety of M and $\mathcal{L}$ be an ample line bundle on M. Then $\deg_{\mathcal{L}}(Y)$ is the degree of Y associated to $\mathcal{L}$.

Let (G, X) be a Shimura datum, $K \subset G(\mathbb{A}_f)$ be a neat open compact subgroup. The Baily-Borel compactification of $\mathrm{Sh}_K(G, X)$ is denoted $\overline{Sh_K(G, X)}$. Let $\mathcal{L}_K = \mathcal{L}_K(G, X)$ be the ample line bundle on $\overline{Sh_K(G, X)}$ extending the line bundle of holomorphic differential forms of maximal degree on $\mathrm{Sh}_K(G, X)$. We say that $\mathcal{L}_K$ is the Baily-Borel line bundle on $\overline{Sh_K(G, X)}$. Let Y be a subvariety of $\overline{Sh_K(G, X)}$, we write $\deg(Y) = \deg_{\mathcal{L}_K}(Y)$ the degree of Y computed with the Baily-Borel line bundle. Let Z be a subvariety of $\mathrm{Sh}_K(G, X)$ and $\overline{Z}$ be its Zariski closure in $\overline{Sh_K(G, X)}$ we'll write $\deg(Z)$ for $\deg(\overline{Z})$.

Definition 2.6 *Let (H, X_H) be a sub-Shimura datum of (G, X) such that H is the generic Mumford-Tate group on X_H. Let $K_H = K \cap H(\mathbb{A}_f)$ and F be as above a field over which $\mathrm{Sh}_{K_H}(H, X_H)$ is defined and has a canonical model. Let V be a geometric irreducible component of the image of $Sh_{K_H}(H, X_H)$ in $\mathrm{Sh}_K(G, X)$.*

We define the degree of the Galois orbit of V, denoted $\deg(Gal(\overline{F}/F) \cdot V)$ to be the degree of the subvariety $\mathrm{Gal}(\overline{F}/F) \cdot V$ of $\mathrm{Sh}_K(G, X)$ with respect to the line bundle $\mathcal{L}_K$

Note that when H is a torus (and hence V is a special point), $\deg(Gal(\overline{F}/F) \cdot V)$ is simply the number of conjugates of V under $Gal(\overline{F}/F)$.

Let V be a geometric component of $Sh_{K_H}(H, X_H)$. We'll use the same notation for V and it's image in $Sh_K(G, X)$. This will be harmless for our purpose in view of lemma 2.2. Let K_T^m be the maximal compact open subgroup of $T(\mathbb{A}_f)$. We consider the compact open subgroup $K_H^m := K_T^m K_H$ of

$H(\mathbb{A}_f)$. Note that as both K_H and K_T^m are product of compact open subgroups of $H(\mathbb{Q}_p)$, the group K_H^m is a product of compact open subgroups of $H(\mathbb{Q}_p)$. We replace the group K_H^m by a neat open compact subgroup as in Lemma 4.1.2 of [9] applied to K_H^m and $l = 3$. The index of K_H^m/K_H than changes by a uniform quantity (bounded by $|\mathrm{GL}_n(\mathbb{F}_3)|$).

Lemma 2.7 *The morphism*

$$\pi \colon \mathrm{Sh}_{K_H}(H, X_H) \longrightarrow \mathrm{Sh}_{K_H^m}(H, X_H)$$

is finite of degree $|K_H^m/K_H|$.

Proof. Let $\overline{(x, g)}$ be a point of $\mathrm{Sh}_{K_H^m}(H, X_H)$. The preimage of $\overline{(x, g)}$ is $\overline{(x, gK_H^m)}$ in $Sh_{K_H}(H, X_H)$. Suppose

$$\overline{(x, g)} = \overline{(x, gk)}$$

with $k \in K_H^m$. There exist q in $H(\mathbb{Q})$ and $k' \in K_H$ such that $qx = x$ and $g = qgkk'$. The first condition implies that q is in a compact subgroup of $H(\mathbb{R})$ and the second condition implies that q is in the neat compact open subgroup $gK_H^m g^{-1}$ of $H(\mathbb{A}_f)$. These two conditions imply that q is trivial. Therefore $k = (k')^{-1} \in K_H$. □

The next lemma splits the degree of $\mathrm{Gal}(\overline{F}/F)$ into two pieces that we will estimate separately.

Lemma 2.8 *The degree of the Galois orbit* $\mathrm{Gal}(\overline{F}/F) \cdot V$ *is at least the degree of* $\mathrm{Gal}(\overline{F}/F) \cdot V \cap \pi^{-1}\pi(V)$ *times the number of* $\mathrm{Gal}(\overline{F}/F)$ *conjugates of* $\pi(V)$.

Proof. We first note that by cor 4.2.10 of [9] it suffices to prove the lemma for the *internal* degree i.e. degree calculated with respect to $\mathcal{L}_{K_H}$. In the rest of the proof by degree we mean internal degree. We need to check that the degree of $\mathrm{Gal}(\overline{F}/F) \cdot V \cap \pi^{-1}(\sigma(\pi(V))$ with $\sigma \in \mathrm{Gal}(\overline{F}/F)$ is independent of σ.

Fix a σ in $\mathrm{Gal}(\overline{F}/F)$. Note that the group K_T^m/K_T acts by automorphisms on $Sh_{K_H}(H, X_H)$. Moreover for all $\alpha \in K_T^m/K_T$ we have $\alpha^*\mathcal{L}_{K_H} = \mathcal{L}_{K_H}$. By the projection formula, if V_i is a component of $\pi^{-1}(\sigma\pi(V))$ then $\deg_{\mathcal{L}_{K_H}}(V_i) = \deg_{\mathcal{L}_{K_H}}(\sigma V)$.

It follows that

$$\deg_{\mathcal{L}_{K_H}}(\pi^{-1}(\sigma\pi(V))) = \deg_{\mathcal{L}_{K_H}}(\sigma V)\cdot|\pi^{-1}(\sigma\pi(V))|.$$

Similarly

$$\deg_{\mathcal{L}_{K_H}}(Gal(\overline{F}/F)\cdot V\cap\pi^{-1}\sigma\pi(V)) = \deg_{\mathcal{L}_{K_H}}(\sigma V)\cdot|Gal(\overline{F}/F)\cdot V\cap\pi^{-1}\sigma\pi(V)|$$

The proof is finished by noticing that

$$\deg_{\mathcal{L}_{K_H}}(\sigma V) = \deg_{\mathcal{L}_{K_H}}(V)$$

and

$$|Gal(\overline{F}/F)\cdot V\cap\pi^{-1}(\sigma\pi(V))| = |Gal(\overline{F}/F)\cdot V\cap\pi^{-1}\pi(V)|$$

□

We first deal with the second piece. Let K_C^m be the maximal open compact subgroup of $C(\mathbb{A}_f)$. The number of components of the Galois orbit of $\pi(V)$ is at least the size of the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ in $\pi(H)/K_H^m$ by $r_{(H,X_H)}$ which is at least the size of the image of $r_{(C,x)}((F\otimes\mathbb{A}_f)^*)$ in $\pi(C)/K_C^m = C(\mathbb{Q})\backslash C(\mathbb{A}_f)/K_C^m$.

By lemma 2.4, $X^*(T)$ has a basis $(\chi_1,\dots,\chi_d)$ such that the coordinates of the χ_i in the canonical basis $(\chi_\sigma)_{\sigma:F\to\mathbb{C}}$ of $X^*(T_F)$ are uniformly bounded. By lemma 2.3, $X^*(C)$ has a basis $(\chi'_1,\dots,\chi'_d)$ such that the coordinates of the χ'_i in the canonical basis of $X^*(T_F)$ are uniformly bounded. As $(C,\{x\})$ is a Shimura datum of CM type there exists an integer λ such that for all $i\in\{1,\dots,d\}$ $\chi'_i\overline{\chi'_i} = \lambda\sum_{\sigma:F\to\mathbb{C}}\chi_\sigma$. By the previous discussion the integer λ is uniformly bounded. We are now in the situation of the theorem 2.13 of [18]. This theorem implies the following.

Proposition 2.9 *Assume the GRH for CM fields. Let N be a positive integer. Let L_C be the splitting field of C. The size of the image of $r_{(C,\{x\})}((\mathbb{A}_f\otimes L_C)^*)$ in $C(\mathbb{Q})\backslash C(\mathbb{A}_f)/K_C^m$ is at least a constant depending on N only times* $(\log|\mathrm{disc}(L_C)|)^N$.

We have proved the following.

Proposition 2.10 *Assume the GRH for CM fields. Let N be a positive integer. Let L_C be the splitting field of C. The number of components of $Gal(\overline{F}/F)\cdot\pi(V)$ is at least a constant depending on N only times* $(\log|\mathrm{disc}(L_C)|)^N$

Now we deal with the second 'piece' : estimating the Galois degree in the fibre over $\pi(V)$. We prove the following key proposition.

Proposition 2.11 *Let K_T be the compact open subgroup $T(\mathbb{A}_f) \cap K$. The group K_T is a product of compact open subgroups $K_{T,p}$ of $T(\mathbb{Q}_p)$. We let $i(T)$ be the number of primes p such that $K_{T,p} \neq K^m_{T,p}$.*

There exists a uniform real constant $B > 0$ such that

$$|Gal(\overline{F}/F) \cdot V \cap \pi^{-1}\pi(V)| \geq B^{i(T)} |\pi^{-1}\pi(V)|$$

where $|Gal(\overline{F}/F) \cdot V \cap \pi^{-1}\pi(V)|$ is the number of Galois conjugates of V contained in the fibre $\pi^{-1}\pi(V)$ and $|\pi^{-1}\pi(V)|$ is the number of components of the fibre.

Proof. The fibre $\pi^{-1}\pi(V)$ has a transitive action of K^m_T/K_T. By the proposition 2.5, the number of Galois conjugates of V contained in one fibre is at least the size of the orbit of V under the action of Θ_A, where Θ_A is the the image of the morphism $x \mapsto x^A$ (with A as in 2.5) on K^m_T/K_T.

We have

$$|\pi^{-1}\pi(V)| = |(K^m_T/K_T) \cdot V| \leq |(K^m_T/K_T)/\Theta_A||\Theta_A \cdot V|$$

and

$$|Gal(\overline{F}/F) \cdot V \cap \pi^{-1}\pi(V)| \geq |\Theta_A \cdot V|$$

To finish the proof we hence need to show that the kernel of the map $x \mapsto x^A$ on K^m_T/K_T is bounded by $D^{i(T)}$ where D is uniform. It will then suffice to set $B = 1/D$.

Since K^m_T/K_T is the product of the $K^m_{T,p}/K_{T,p}$, it is enough to prove that the order of the kernel of the A-th power morphism on $K^m_{T,p}/K_{T,p}$ for each p is bounded uniformly on T and p.

Let p be a prime. Let E be the splitting field of T. Using a basis of the character group of T, one can embed T into a product of finite and uniformly bounded number of tori $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E}$ where E is a number field of uniformly bounded degree over $\mathbb{Q}$. It follows that K^m_T and K_T are subgroups of the product of the $(\mathbb{Z}_p \otimes O_E)^*$. These groups are free $\mathbb{Z}_p$-modules of uniformly bounded rank r, therefore the group $K^m_{T,p}/K_{T,p}$ is a finite abelian group, product of at most r cyclic factors. It follows that the size of the kernel of A-th power map on $K^m_{T,p}/K_{T,p}$ is bounded by $D := A^r$. □

Suppose that H is a torus, i.e. V is a point. Then $|\pi^{-1}\pi(V)| = |K_T^m/K_T|$ and propositions 2.10 and 2.11 put together give

$$\deg(\mathrm{Gal}(\overline{F}/F).V) \gg B^{i(T)}|K_T^m/K_T|\log(|\mathrm{disc}(L_C)|)^N.$$

We now turn to the case where H^{der} is nontrivial. We prove the following:

Proposition 2.12

$$\deg(\pi^{-1}\pi(V)) \geq |K_T^m/K_T|$$

Proof. Let Z be the subvariety $\pi^{-1}\pi(V)$ of $Sh_{K_H}(H, X_H)$. Let $\mathcal{L}_{K_H}$ and $\mathcal{L}_{K_H^m}$ be the Baily-Borel line bundles on $Sh_{K_H}(H, X_H)$ and $Sh_{K_H^m}(H, X_H)$ respectively. The morphism of Shimura varieties $\pi\colon Sh_{K_H}(H, X_H) \longrightarrow Sh_{K_H^m}(H, X_H$ extends to a proper morphism

$$\overline{\pi}\colon \overline{Sh_{K_H}(H, X_H)} \longrightarrow \overline{Sh_{K_H^m}(H, X_H)}$$

which is generically finite of degree $|K_T^m/K_T|$ by lemma 2.7. Furthermore $\pi^*\mathcal{L}_{K_H^m} \cong \mathcal{L}_{K_H}$. The projection formula gives

$$\deg_{\mathcal{L}_{K_H}}(Z) = \deg_{\pi^*\mathcal{L}_{K_H^m}}(Z) = \deg_{\mathcal{L}_{K_H^m}}(\pi_\star Z) = [K_T : K_T^m]\deg_{\mathcal{L}_{K_H^m}}(\pi(Z)) \geq [K_T : I$$

On another hand, according to [9], cor 4.2.10 we have

$$\deg_{\mathcal{L}_{K_H}}(Z) \leq \deg_{\mathcal{L}_K}(Z) = \deg(Z).$$

We deduce that

$$\deg(Z) \geq [K_T : K_T^m].$$

□

In the proof of the lemma 2.8, we have seen that

$$\deg_{\mathcal{L}_{K_H}}(Gal(\overline{F}/F)\cdot V \cap \pi^{-1}\pi(V)) = \deg_{\mathcal{L}_{K_H}}(V)\cdot |Gal(\overline{F}/F)\cdot V \cap \pi^{-1}\pi(V)|$$

The propositions 2.11 and 2.12 combined together now give

$$\deg_{\mathcal{L}_{K_H}}(Gal(\overline{F}/F)\cdot V \cap \pi^{-1}\pi(V)) \geq B^{i(T)}|K_T^m/K_T|.$$

Putting all previous ingredients together we get:

Theorem 2.13 *Assume the GRH for CM fields. There exists a real number B such that the following holds. Let (H, X_H) be a sub-Shimura datum of (G, X) such that H is the generic Mumford-Tate group on X_H. Let E be a field over which $\mathrm{Sh}_K(G, X)$ admits a canonical model (for example $E = E(G, X)$ the reflex field of (G, X)). Let K_H be $H(\mathbb{A}_f) \cap K$. Let T be the connected centre of H. We suppose that T is non-trivial.*

Then for every geometric component V of the image of $Sh_{K_H}(H, X_H)$ in $Sh_K(G, X)$, and for any positive integer N,

$$\deg(\mathrm{Gal}(\overline{E}/E).V) \geq c_N B^{i(T)} \cdot |K_T^m/K_T| \cdot (\log(|\mathrm{disc}(L_C)|))^N. \qquad (1)$$

for a real constant c_N depending only on N.

Remark 2.14 The proof of theorem 2.13 actually shows the more precise results:

$$\deg_{\mathcal{L}_{K_H}}(Gal(\overline{E}/E) \cdot V) \geq c_N B^{i(T)} \cdot |K_T^m/K_T| \cdot (\log(|\mathrm{disc}(L_C)|))^N.$$

This will not be important for the purpose of this paper but will be useful in the forthcoming paper by Klingler and Yafaev [9].

The proof of the theorem shows also that the degree of the Galois orbit of V is $\geq c_N \max(1, B^{i(T)} \cdot |K_T^m/K_T|) \cdot (\log(|\mathrm{disc}(L_C)|))^N$ for a real constant c_N depending only on N.

In the case where we consider subvarieties V such that the associated tori T lie in one $\mathrm{GL}_n(\mathbb{Q})$-conjugacy class with respect to some faithful representation $G \hookrightarrow \mathrm{GL}_n$, we do not need to assume the GRH. Indeed, in this case the field L_C is fixed and hence the term involving it is constant. We only used the GRH to obtain this term.

3 Special subvarieties whose degree of Galois orbits are bounded.

3.1 Equidistribution of T-special subvarieties.

Let (G, X) be a Shimura datum with G semisimple of adjoint type and let K be an open compact subgroup of $G(\mathbb{A}_f)$. Let $\Gamma = G(\mathbb{Q})^+ \cap K$ and $S = \Gamma \backslash X^+$ a fixed component of $\mathrm{Sh}_K(G, X)$. Note that S is the image of $X^+ \times \{1\}$ in $\mathrm{Sh}_K(G, X)$.

If $(H, X_H) \subset (G, X)$ is a sub-Shimura datum, we denote by $\mathrm{MT}(X_H)$ the generic Mumford-Tate group on X_H. If $H' = \mathrm{MT}(X_H)$, then $H' \subset H$, $H'^{\mathrm{der}} = H^{\mathrm{der}}$ and $Z(H')^0 \subset Z(H)^0$. Moreover X_H is the $H'(\mathbb{R})$-conjugacy class of $x \in X_H$ and $x(\mathbb{S}) \subset H'(\mathbb{R})$. Therefore (H', X_H) is a sub-Shimura-datum of (H, X_H). We sometimes use the notation $X_{H'}$ instead of X_H.

Definition 3.1 Let $T_{\mathbb{Q}}$ be a torus such that $T(\mathbb{R})$ is compact. A T-sub-Shimura datum (H, X_H) of (G, X) is a sub-Shimura datum such that H^{der} is non trivial and T is the connected center of the generic Mumford-Tate group $H' = TH^{der}$ of X_H. Note that in this definition T may be trivial. In this case the generic Mumford-Tate group H' of X_H is semi-simple.

Definition 3.2 A T-special subvariety of S is a geometric component Z of the image of $\mathrm{Sh}_{K \cap H(\mathbb{A}_f)}(H, X_H)$ contained in S for a T-sub-Shimura datum $(H, X_H) \subset (G, X)$. In this case, we say that Z is associated to (H, X_H). If Z is associated to (H, X_H), we say that Z is standard if there exists a connected component X_H^+ of X_H contained in X^+ such that Z is the image of $X_H^+ \times \{1\}$ in S. If Z is standard, then we have:

$$Z \simeq \Gamma \backslash \Gamma X_H^+ \simeq (\Gamma \cap H(\mathbb{R})^+) \backslash X_H^+.$$

We denote by Σ_T the set of T-special subvarieties of S.

Lemma 3.3 *A standard T-special subvariety Z is associated to a sub-Shimura datum (H, X_H) such that $H = \mathrm{MT}(X_H) = T.H^{\mathrm{der}}$.*

If Z is associated to (H_1, X_{H_1}) and Z is standard, then Z is the image of $X_{H_1}^+ \times \{1\}$ in S for some connected component $X_{H_1}^+$ of X_{H_1} contained in X^+. Write $H = \mathrm{MT}(X_{H_1})$, then $X_H = X_{H_1}$ and Z is also associated to (H, X_H) and is standard.

Lemma 3.4 *Recall that Σ_T is the set of T-special subvarieties. Let $\alpha \in \Gamma$ and $T_\alpha = \alpha T \alpha^{-1}$. Then $\Sigma_{T_\alpha} = \Sigma_T$.*

Proof. Let (H, X_H) be a T-sub-Shimura datum of (G, X). Fix $x \in X_H$. Let $H_\alpha = \alpha H \alpha^{-1}$ and X_{H_α} be the $H_\alpha(\mathbb{R})$-conjugacy class of $\alpha.x$. Then (H_α, X_{H_α}) is a T_α-sub-Shimura datum and the images of $\mathrm{Sh}_{K \cap H(\mathbb{A}_f)}(H, X_H)$ and $\mathrm{Sh}_{K \cap H_\alpha(\mathbb{A}_f)}(H_\alpha, X_{H_\alpha})$ in $\mathrm{Sh}_K(G, X)$ coincide. $\square$

Lemma 3.5 *There exists a finite subset* $\{r_1, \dots, r_k\}$ *of* $G(\mathbb{A}_f)$ *such that any* T*-special subvariety of* S *is a component of the image by the Hecke operator* T_{r_i} *of a standard* T*-special subvariety.*

Proof. We have a finite double coset decomposition

$$Z_G(T)(\mathbb{A}_f) = \cup_{i=1}^k Z_G(T)(\mathbb{Q})^+ \; r_i \; Z_G(T)(\mathbb{A}_f) \cap K.$$

Let Z be a T special subvariety associated to a T-sub-Shimura datum (H, X_H). Fix a connected component X_H^+ of X_H contained in X^+. Then Z is the image in S of $X_H^+ \times \{h\}$ for some $h \in H(\mathbb{A}_f)$. Note that $X_H = H(\mathbb{Q})X_H^+$, this is a consequence of the fact that $H(\mathbb{Q})$ is dense in $H(\mathbb{R})$.

By definition of a T-sub-Shimura datum, $T \subset Z(H)$ (where $Z(H)$ is the centre of H) and therefore $H \subset Z_G(T)$.

We can find $z \in Z_G(T)(\mathbb{Q})^+$, $k \in Z_G(T)(\mathbb{A}_f) \cap K$ and $i \in \{1, \dots, r\}$ such that $h = zr_ik$. Therefore Z is in the image of $z^{-1}.X_H^+ \times \{r_i\}$ in S.

Write $X_H = H(\mathbb{R}).x$ for some $x \in X_H$, $H_z = z^{-1}Hz$ and $X_{H_z} = H_z(\mathbb{R}).(z^{-1}.x)$. Then (H_z, X_{H_z}) is a sub-Shimura datum. The generic Mumford-Tate group of X_{H_z} is

$$MT(X_{H_z}) = z^{-1}MT(X_H)z = z^{-1}(TH^{der})z = T.z^{-1}H^{der}z.$$

Therefore (H_z, X_{H_z}) is a T-sub-Shimura datum. Note that $z^{-1}X_H^+$ is a connected component of X_{H_z}. Note also that because $z \in Z_G(T)(\mathbb{Q})^+$, $z^{-1}X_H^+$ is contained in X^+.

Let Z_0 be the standard T-special subvariety associated to (H_z, X_{H_z}). Then Z is a component of $T_{r_i}.Z_0$. □

The algebraic group $Z_G(T)$ is reductive and connected as a centralizer of a torus. Let

$$Z_G(T) = \tilde{T}L_1 \dots L_r$$

be the decomposition of $Z_G(T)$ as an almost direct product of $\mathbb{Q}$-simple factors.

Let $L_{\mathbb{Q}} \simeq \widetilde{T}L_1 \dots L_s$ be the almost direct product of $\widetilde{T}$ and of the L_i's such that $L_i(\mathbb{R})$ is not compact. We have

$$H \subset Z_G(T) = Z_G(\widetilde{T})$$

and as the almost $\mathbb{Q}$-simple factors H_i of H are such that $H_i(\mathbb{R})$ aren't compact their projections on the L_i with $L_i(\mathbb{R})$ compact are trivial. We deduce from this that $H \subset L$. Let X_L be the $L(\mathbb{R})$-conjugacy class of some $x \in X_H$.

Lemma 3.6 *The pair (L, X_L) is a T-sub-Shimura datum such that*

$$(H, X_H) \subset (L, X_L).$$

Proof. The proof of ([2] proposition 3.2) shows that (L, X_L) is a Shimura datum. As H is contained in L, $(H, X_H) \subset (L, X_L)$. We write $H' = MT(X_H)$ and $L' = MT(X_L)$. We have an inclusion of sub-Shimura datum

$$(H', X_H) \subset (L', X_L).$$

By definition $T = Z(H')^0 \subset L'$ and T commutes with L', therefore $T \subset Z(L')^0$. Fix $x \in X_H$, then X_L is the L^{der}-conjugacy-class of x. By definition of the generic Mumford-Tate group of X_H we know that

$$x(\mathbb{S})(\mathbb{R}) \subset (T.H^{der})(\mathbb{R}) \subset (T.L^{der})(\mathbb{R}).$$

We then see that for any $y \in X_L$ we have

$$y(\mathbb{S})(\mathbb{R}) \subset (T.L^{der})(\mathbb{R}).$$

Therefore $L' = MT(X_L) \subset T.L^{der}$ and $Z(L')^0 \subset T$. Finally $T = Z(L')^0$ and (L, X_L) is a T-sub-Shimura datum. □

The following lemma will be useful later.

Lemma 3.7 *Let (M, X_M) be a sub-Shimura datum of (G, X). Then there exist at most finitely many Y such that (M, Y) is a sub-Shimura datum of (G, X). Moreover as the M vary among connected reductive groups the number of Y is uniformly bounded.*

Proof. Let $X_{1,M}$ and $X_{2,M}$ such that $(M, X_{1,M})$ and $(M, X_{2,M})$ are sub-Shimura data of (G, X). Fix $x_i \in X_{i,M}$ and $\alpha \in G(\mathbb{R})$ such that

$$x_2 = \alpha.x_1 = \alpha x_1 \alpha^{-1}.$$

Let $K_i = Z_G(x_i(\sqrt{-1}))(\mathbb{R})$ the associated maximal compacts of $G(\mathbb{R})$. We have the Cartan decompositions:

$$G(\mathbb{R}) = P_1 K_1 = P_2 K_2 \quad \text{and} \quad M(\mathbb{R}) = P_1 \cap M\ K_1 \cap M = M(\mathbb{R}) = P_2 \cap M\ K_2 \cap M.$$

We then have $K_2 = \alpha K_1 \alpha^{-1}$ and $P_2 = \alpha P_1 \alpha^{-1}$. As the Cartan decompositions are conjugate in $M(\mathbb{R})$, there exists $h \in M(\mathbb{R})$ such that

$$K_2 \cap M = h(K_1 \cap M)h^{-1} \quad \text{and} \quad P_2 \cap M = h(P_1 \cap M)h^{-1}.$$

Let $\gamma = h^{-1}\alpha = p.k$ with $p \in P_1$ and $k \in K_1$. Then

$$(\star) \quad K_1 \cap M = pK_1p^{-1} \cap M \quad \text{and} \quad P_1 \cap M = pP_1p^{-1} \cap M.$$

By ([15] lemma 3.11) we have the following:

1. Let $\{p, q, r\}$ be elements of P_1 such that $pqp^{-1} = r$ then $p^2q = qp^2$.
2. Let $p \in P_1$ and $\{k_1, k_2\} \in K_1$ such that $pk_1p^{-1} = k_2$ then $p^2k_1 = k_1p^2$.

Then $(\star)$ and (1) implies that $p^2 \in Z_G(P_1)(\mathbb{R}) \cap M$ and $(\star)$ and (2) implies that $p^2 \in Z_G(K_1)(\mathbb{R}) \cap M$. We then find that

$$p^2 \in Z_G(M)(\mathbb{R}) \subset Z_G(x_1(\sqrt{-1}))(\mathbb{R}) = K_1$$

so $p^2 \in P_1 \cap K_1$ is trivial and $p = 1$.

We now know that $\alpha = h\gamma$ with $h \in H(\mathbb{R})$ and $\gamma \in K_1$. Fix a set of representative $\{\gamma_1, \ldots, \gamma_r\}$ in K_1 of K_1/K_1^+. As K_1^+ fixes x_1 we obtain that for some $i \in \{1, \ldots, r\}$, $\gamma_i.x_1 \in X_{2,M}$. This finishes the proof of the lemma and of the proposition. □

Theorem 3.8 *Fix a torus $T_{\mathbb{Q}}$ with $T(\mathbb{R})$ compact. Let (Z_n) be a sequence of T-special subvarieties of S. Let $(\mu_n) = (\mu_{Z_n})$ be the associated sequence of probability measures. There exists a T-special subvariety Z of S and a subsequence (Z_{n_k}) such that (μ_{n_k}) converges weakly to μ_Z. Moreover Z contains Z_{n_k} for all k large enough.*

Proof. Using the lemmas (3.3) and (3.5), we may assume that Z_n is a standard T-special subvariety of S associated to a T-special sub-Shimura datum (H_n, X_n) with $H_n = \mathrm{MT}(X_n) = TH_n^{\mathrm{der}}$.

Let (H_n, X_n) be the sequence of T-sub-Shimura datum associated to (Z_n). Using the lemmas 3.7 and 3.6 we may assume that for all $n \in \mathbb{N}$, (H_n, X_n) is a sub-Shimura datum of the T-Shimura datum (L, X_L).

Therefore we may assume that (Z_n) is contained in a fixed component S_L of $\mathrm{Sh}_{L(\mathbb{A}_f)\cap K}(L, X_L)$. Then (Z_n) is a sequence of strongly special subvarieties of S_L in the sense of [2] 4.1. Let $(L^{ad}, X_{L^{ad}})$ be the adjoint Shimura datum and K_L^{ad} a compact open subgroup containing the image of $L(\mathbb{A}_f) \cap K$ in $L^{ad}(\mathbb{A}_f)$. We recall that Z_n is a strongly special subvariety of S_L if and only if its image Z_n^{ad} in $\mathrm{Sh}_{K_L^{ad}}(L^{ad}, X_{L^{ad}})$ is strongly special. As T is the connected center of H_n and T is contained in the center of L we see that Z_n^{ad} is defined by a sub-Shimura datum (H_n', X_n') of $(L^{ad}, X_{L^{ad}})$ with H_n' semi-simple and that Z_n^{ad} is strongly special.

Note that the condition (b) in the definition of "strongly special" ([2] 4.1) is in fact implied by the first: let (F, X_F) be a sub-Shimura datum of an adjoint Shimura datum (G, X) with F semi-simple. Let $\alpha : \mathbb{S} \to F_{\mathbb{R}}$ be a element of X_F and $K_\alpha = Z_G(\alpha(\sqrt{-1}))$ be the associated maximal compact subgroup of $G(\mathbb{R})$. Then $Z_G(F)(\mathbb{R}) \subset Z_G(\alpha(\sqrt{-1}))$ is compact. Therefore $Z_G(F)$ is $\mathbb{Q}$-anisotropic (even $\mathbb{R}$-anisotropic) and (F, X_F) satisfies the condition (b") of ([2] 4.1) which is equivalent to the condition (b).

The theorem 4.6 of [2] proves that after possibly having replaced (Z_n) by a subsequence; there exists a special subvariety $Z \subset S_L$ such that (μ_{Z_n}) converges weakly to μ_Z and $Z_n \subset Z$ for all $n \gg 0$. We can find a sub-Shimura datum (H, X_H) associated to Z such that for any n large enough the following inclusions of Shimura datum hold:

$$(H_n, X_n) \subset (H, X_H) \subset (L, X_L)$$

We once more write $L' = MT(X_L)$ and $H' = MT(X_H)$. Then

$$(H_n, X_n) \subset (H', X_H) \subset (L', X_L)$$

It is now easy to deduce that $Z(H') = Z(H_n) = Z(L')$ for every n large enough and consequently Z is a T-special subvariety. □

3.2 Special subvarieties whose Galois orbits have bounded degrees.

Let $S = \Gamma\backslash X^+$ be a component of $\mathrm{Sh}_K(G, X)$. We fix as in the previous sections a faithful representation $G \subset \mathrm{GL}(V_{\mathbb{Q}})$ on a n dimensional $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$. We fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}}$ such that $K \subset \mathrm{GL}_n(\widehat{\mathbb{Z}})$. For any algebraic subgroup H of G, we let $H_{\mathbb{Z}}$ (resp. $H_{\mathbb{Z}_p}$) be the Zariski-closure of H in $\mathrm{GL}_{n,\mathbb{Z}} = \mathrm{GL}(V_{\mathbb{Z}})$ (resp. $\mathrm{GL}_{n,\mathbb{Z}_p}$).

Fix a number field F such that S admits a canonical model over F. The aim of this section is to prove the following theorem which merely provides a justification for the seemingly unnatural definition of T-special subvarieties.

Theorem 3.9 *Assume the GRH for CM fields. Let N be an integer. There exists a finite set $\{T_1, \dots, T_r\}$ of $\mathbb{Q}$-tori of G with the following property. Let Z be a special subvariety of S such that $\mathrm{Gal}(\overline{F}/F) \cdot Z$ has degree at most N. Then Z is a T_i-special subvariety for some $i \in \{1, \dots, r\}$.*

Let $\Sigma_F = \Sigma_{F,N}$ be the set of special subvarieties Z of S such that $\deg(\mathrm{Gal}(\overline{F}/F).Z)$ is bounded by N. Let $Z \in \Sigma_F$. By lemma 2.1 we may assume that Z is associated to a sub-Shimura datum (H, X_H) such that H is the generic Mumford-Tate group on X_H.

Let $C \simeq H/H^{\mathrm{der}}$ and let L_C be the splitting field of C. By the theorem 2.13 and the remark following its statement, the discriminant $|\mathrm{disc}(L_C)|$ is bounded when Z varies in Σ_F. To prove the theorem 3.9, it suffices to consider the set of $Z \in \Sigma_F$ such that the corresponding L_C is fixed.

Lemma 3.10 *Let $\mathbb{T}_F$ be the set of $\mathbb{Q}$-tori T of G such that there exists $Z \in \Sigma_F$ associated with a sub-Shimura datum (H, X_H) such that $T = Z(\mathrm{MT}(X_H))$. Then $\mathbb{T}_F$ is contained in a finite union of $\mathrm{GL}_n(\mathbb{Q})$-conjugacy classes.*

Proof. The assumption of this lemma implies that the discriminant of L_C is bounded and therefore we can assume that the torus $L := \mathrm{Res}_{L_C/\mathbb{Q}}\mathbb{G}_m$ is fixed. As before, we identify $X^*(T)$ with a submodule of $X^*(L)$ via a "lifting" of the reciprocity $r_{(T,\{x\})}$. By the lemma 2.4, there is only a finite number of possibilities for the set of characters occurring in the representations $T \subset \mathrm{GL}_n$. Each of these possibilities corresponds to an isomorphism class of such a representation and hence to a $\mathrm{GL}_n(\mathbb{Q})$-orbit of a torus T. It follows that the set $\mathbb{T}_F$ as in the statements lies in a finite number of such orbits. □

We need in fact the following more precise result:

Proposition 3.11 *The set $\mathbb{T}_F$ is contained in a finite union of $\mathrm{GL}_n(\mathbb{Z})$-conjugacy classes.*

To prove this proposition, we will analyze the variation of $B^{i(T)} \cdot |K_T^m/K_T|$ as T ranges through the set tori that lie in one $\mathrm{GL}_n(\mathbb{Q})$-conjugacy class.

Lemma 3.12 *Let $T_0 \in \mathbb{T}_F$ and Σ_0 the $\mathrm{GL}_n(\mathbb{Q})$-orbit of T_0. For all $T \in \Sigma_0$ we have the lower bound*

$$B^{i(T)} \cdot |K_T^m/K_T| \gg \prod_{\{p : K_{T,p}^m \neq K_{T,p}\}} cp$$

where c is a uniform constant.

There exists a uniform constant C_0 such that for all prime number $p > C_0$ and all $T \in \Sigma_0 \cap \mathbb{T}_F$, the Zariski-closure $T_{\mathbb{Z}_p}$ of T in $\mathrm{GL}(V_{\mathbb{Z}_p})$ is a torus and there exists $\alpha_p \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that $T_{\mathbb{Z}_p} = \alpha_p T_{0\mathbb{Z}_p} \alpha_p^{-1}$.

Proof. Let p a prime such that p unramified in L_C, K_p is $G(\mathbb{Z}_p)$ for the $\mathbb{Z}_p$-structure given by our fixed representation of G and such that $T_{0\mathbb{Z}_p}$ is a torus. These conditions are verified for almost all p.

Let $g \in \mathrm{GL}_n(\mathbb{Q})$ such that $T = gT_0g^{-1} \in \Sigma_0$ is such that $K_{T,p}^m \neq K_{T,p}$. These conditions are equivalent to the fact that $T_{\mathbb{Z}_p}$ is not a torus. The conjugation morphism $x \mapsto gxg^{-1}$ establishes a bijection between $K_{T,p}^m/K_{T,p}$ and $K_{T_0,p}^m/T_0(\mathbb{Q}_p) \cap g\mathrm{GL}_n(\mathbb{Z}_p)g^{-1}$ where $K_{T_0,p}^m$ is the maximal compact open subgroup of $T_0(\mathbb{Q}_p)$. This last index is the size of the orbit $T_0(\mathbb{Z}_p) \cdot g\mathbb{Z}_p^n$. The fact that $T_{\mathbb{Z}_p}$ is not a torus implies that $T_{0,\mathbb{Z}_p}$ does not fix the lattice $g\mathbb{Z}_p^n$ in the sense of [7], section 3.3. Now the proposition 4.3.9 of [7] implies that this index is at least a uniform constant times p. We conclude by noticing that $|K_T^m/K_T|$ is the product of the $i(T)$ local indices.

Using theorem 2.13, we see that there exists an integer C_0 such that for all $T \in \Sigma_0 \cap \mathbb{T}_F$ and all prime $p > C_0$, $K_{T,p} = K_{T,p}^m$. Let $T \in \Sigma_0 \cap \mathbb{T}_F$, then $T_{\mathbb{Z}_p}$ is a torus. Let $g \in \mathrm{GL}_n(\mathbb{Q})$ such that $T = gT^0g^{-1}$. The previous discussion shows that $T_{0,\mathbb{Z}_p}$ fixes the lattice $g\mathbb{Z}_p^n$. By ([7] lemma 3.3.1), there exists $c \in Z_{\mathrm{GL}_n}(T)(\mathbb{Q}_p)$ and $\alpha_p \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that $g_p = c\alpha_p$. Therefore $T_{\mathbb{Z}_p} = \alpha_p T_{0\mathbb{Z}_p} \alpha_p^{-1}$ for some $\alpha_p \in \mathrm{GL}_n(\mathbb{Z}_p)$.

□

The proposition 3.11 will follow from the following proposition whose proof was communicated to us by Laurent Clozel.

Proposition 3.13 *(Clozel) Let G be a reductive group over $\mathbb{Q}_p$, $T \subset G$ a torus and let $H = Z_G(T)$. Let K be a fixed compact open subgroup of $G(\mathbb{Q}_p)$ and let $K_T = K_T^m$ be the maximal compact subgroup of $T(\mathbb{Q}_p)$. The function*

$$I(g) = |K_T/T(\mathbb{Q}_p) \cap g^{-1}Kg| \to \infty$$

as $g \to \infty$ in $G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$. Let W be a set of $g \in G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$ such that $I(g)$ is bounded. The image of W in $G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$ is finite.

Proof. As $T(\mathbb{Q}_p) \cap g^{-1}Kg$ is a compact open subgroup of $T(\mathbb{Q}_p)$, $T(\mathbb{Q}_p) \cap g^{-1}Kg$ is contained in K_T. For $g \in G(\mathbb{Q}_p)$ and $h \in H(\mathbb{Q}_p)$ we find that

$$T(\mathbb{Q}_p) \cap h^{-1}g^{-1}Kgh = h^{-1}(hT(\mathbb{Q}_p)h^{-1} \cap g^{-1}Kg)h = h^{-1}(T(\mathbb{Q}_p) \cap g^{-1}Kg)h = T \cap g^{-1}Kg$$

as h commutes to T. So $I(g)$ is well defined on $G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$.

Let $\mathbf{1}_K$ the characteristic function of K on $G(\mathbb{Q}_p)$. Let μ_T the normalized measure on K_T. Then $I(g) \to \infty$ if and only if

$$\int_{K_T} \mathbf{1}_K(gtg^{-1})\ d\mu_T \longrightarrow 0.$$

We just have to prove that for t outside a subset of K_T of μ_T-measure 0:

$$\mathbf{1}_K(gtg^{-1}) \to 0.$$

Let $T^{reg} \subset T(\mathbb{Q}_p)$ be the set

$$T^{reg} = \{t \in T(\mathbb{Q}_p) \mid Z_G(t) = Z_G(T) = H\}.$$

For $t \in T^{reg}$ we have an homeomorphism

$$\pi_t :\ G(\mathbb{Q}_p)/H(\mathbb{Q}_p) \to O(t)$$
$$g \mapsto gtg^{-1}$$

where $O(t)$ denotes the orbit of t under $G(\mathbb{Q}_p)$. As t is semi-simple this orbit is closed and the map π_t is proper. In this way we get that for $g \to \infty$ $\mathbf{1}_K(gtg^{-1}) = 0$. So the following lemma finishes the proof of the proposition. □

Lemma 3.14 *The set of $t \in K_T$ such that $t \notin T^{reg}$ is of μ_T-measure 0.*

This last lemma is a consequence of [12], 2.1.11.

We can now finish the proof of the proposition 3.11. Let $T_0 \in \mathbb{T}_F$ and Σ_0 the $\mathrm{GL}_n(\mathbb{Q})$-conjugacy class of T_0. Let $T_{0,\mathbb{Z}}$ be the Zariski closure of T_0 in $\mathrm{GL}_{n,\mathbb{Z}}$. By lemma 3.10, we just need to prove that $\Sigma_0 \cap \mathbb{T}_F$ is contained in a finite union of $\mathrm{GL}_n(\mathbb{Z})$-conjugacy classes. By lemma 3.12, there exists $C_0 > 0$ such that for all $T \in \Sigma_0 \cap \mathbb{T}_F$ and all prime number $p > C_0$ there exists $\alpha_p \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that $T_{\mathbb{Z}_p} = \alpha_p T_{0\mathbb{Z}_p} \alpha_p^{-1}$.

Let $g \in \mathrm{GL}_n(\mathbb{Q})$ be such that $T := gT_0g^{-1} \in \mathbb{T}_F \cap \Sigma_0$. By theorem 2.13

$$|K^m_{T,p}/K_{T,p}| = |K^m_{T_0,p}/T_0(\mathbb{Q}_p) \cap g^{-1}K_p g|$$

is bounded when T varies in $\Sigma_0 \cap \mathbb{T}_F$. Using the proposition 3.13, we see that for all prime number $p \leq C_0$ there exists a finite subset W_p of

$$\mathrm{GL}_{n,\mathbb{Z}_p} \backslash \mathrm{GL}_n(\mathbb{Q}_p)/Z_{\mathrm{GL}_n}(T_0)(\mathbb{Q}_p)$$

such that the image of g in $\mathrm{GL}_{n,\mathbb{Z}_p}\backslash\mathrm{GL}_n(\mathbb{Q}_p)/Z_{\mathrm{GL}_n}(T_0)(\mathbb{Q}_p)$ is contained in W_p.

We therefore just need to prove that the set of tori $T = gT_0g^{-1} \in \Sigma_0 \cap \mathbb{T}_F$ such that the image g_p in W_p is fixed for all $p \leq C_0$ is contained in a finite union of $\mathrm{GL}_n(\mathbb{Z})$-conjugacy class.

If this set is non empty, there exists $T_1 \in \Sigma_0 \cap \mathbb{T}_F$ such that for all prime p and all T in this set there exists $\alpha_p \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that $T_{\mathbb{Z}_p} = \alpha_p T_{1\mathbb{Z}_p} \alpha_p^{-1}$. By the results of the appendix by Gille and Moret-Bailly ([8] cor. 6.4) the set of tori under consideration is contained in a finite union of $\mathrm{GL}_n(\mathbb{Z})$-conjugacy classes.

Proposition 3.15 *The set $\mathbb{T}_F$ is a finite union of Γ-conjugacy classes.*

This proposition finishes the proof of the theorem 3.9: Fix $T_1, \ldots, T_s$ a system of representatives of the Γ-conjugacy classes in $\mathbb{T}_F$. In view of the lemma 3.4, any $Z \in \Sigma_F$ is a T_i special subvariety.

Before starting the proof the proposition, we need to define the "type" of a torus. Let $\mathcal{S}$ be a finite set of places of $\mathbb{Q}$ and let A be the ring of $\mathcal{S}$-integers. Let $\overline{A}$ be the integral closure of A inside $\overline{\mathbb{Q}}$. Suppose that G_A is a smooth reductive model of $G_{\mathbb{Q}}$ and that T_A is a torus inside G_A. Then $Z_{G_A}(T_A)$ is a connected reductive subgroup of G_A of maximal reductive rank containing a maximal torus T_A^{max}. By [5] (Exp. XXII prop 2.2) $T_{\overline{A}}^{max}$ is a split maximal torus of $G_{\overline{A}}$. One can describe $Z_{G_{\overline{A}}}(T_{\overline{A}})$ using roots of $(G_{\overline{A}}, T_{\overline{A}}^{max})$ which are trivial on $\tilde{T_{\overline{A}}} = Z(Z_{G_{\overline{A}}}(T_{\overline{A}}))$. Hence, there exists at most finitely many $G(\overline{A})$-conjugacy classes of groups of this form. If T_A is a A-torus in G_A the type of T_A is the $G(\overline{A})$-conjugacy class of $Z_{G_{\overline{A}}}(T_{\overline{A}})$ (compare with ([5], exp. XXII sec. 2)) .

We only need to prove the proposition for a subset $\mathbb{T}'_F$ of $\mathbb{T}_F$ such that the tori in $\mathbb{T}'_F$ belong to a fixed $\mathrm{GL}_n(\mathbb{Z})$-conjugacy class of a torus $T_0 \in \mathbb{T}'_F$. Let A be the ring $\mathbb{Z}[\frac{1}{s}]$ where s is the product of primes p belonging to the finite set $\mathcal{S}$ of primes such that either $T_{0\mathbb{Z}_p}$ is not a torus or the Zariski-closure of G in $GL(n)_{\mathbb{Z}_p}$ is not reductive and smooth.

The Zariski closures G_A of G and $T_{0,A}$ of T_0 in $\mathrm{GL}_{n,A}$ are smooth. As we work in a fixed $\mathrm{GL}_n(\mathbb{Z})$-conjugacy class all the tori in $\mathbb{T}'_F$ have a smooth Zariski closure in GL_{nA}. We therefore may assume that all the tori in $\mathbb{T}'_F$ have the same type. Let $\tilde{T_0} = Z(Z_G(T_0))$, then $Z_G(T_0) = Z_G(\tilde{T_0})$ also has a smooth Zariski-closure in GL_{nA}.

If $T \in \mathbb{T}'_F$, we write $\tilde{T} = Z(Z_G(T))$. Then $\tilde{T}_A$ and $\tilde{T}_{0,A}$ are some A-subtori of G_A locally conjugate in the fppf topology. The corollary 1.11 of the appendix of this paper by Gille and Moret-Bailly [8] tells us that there is at most finitely many $G(A)$-conjugacy-classes of such subtori. We may therefore assume that for any $T \in \mathbb{T}'_F$ the associated A-torus $\tilde{T}_A$ is conjugate to $\tilde{T}_{0,A}$ by an element of $G(A)$.

Let $\alpha \in G(A)$ such that $\tilde{T}_A = \alpha \tilde{T}_{0,A} \alpha^{-1}$. Then

$$Z_{G_A}(\tilde{T}_A) = Z_{G_A}(T_A) = \alpha Z_{G_A}(\tilde{T}_{0,A})\alpha^{-1}.$$

Over $\mathbb{Q}$ we get $Z_G(T) = \alpha Z_G(T_0)\alpha^{-1}$. Let L and L_0 be the reductive subgroups of $Z_G(T)$ and $Z_G(T_0)$ obtained by removing the $\mathbb{R}$-compact $\mathbb{Q}$-factors of $Z_G(T)$ and $Z_G(T_0)$ as described before the lemma 3.6. Let (L, X_L) and (L_0, X_{L_0}) be the associated Shimura datum (see 3.6). Using lemma 3.7 we may assume that for any $T \in \mathbb{T}'_F$, α induces an isomorphism of Shimura datum between (L_0, X_{L_0}) and (L, X_L). Therefore the generic Mumford-Tate group $MT(X_L)$ of X_L equals $\alpha MT(X_{L_0})\alpha^{-1}$. As a consequence we have

$$T = Z(MT(X_L)) = \alpha T_0 \alpha^{-1}.$$

The proposition 3.13 of Clozel shows that for all prime $p \in \mathcal{S}$ the image α_p of α in $G(\mathbb{Q}_p)/Z_G(T_0)(\mathbb{Q}_p)$ is contained in a finite union of $G(\mathbb{Z}_p)$-orbits. We may therefore assume that for all $p \in \mathcal{S}$ any torus T in $\mathbb{T}'_F$ is conjugate to T_0 by an element of $G(\mathbb{Z}_p)$. As T and T_0 are also conjugate by an element of $G(\mathbb{Z}_p)$ for all $p \notin S$ the corollary 1.11 of the appendix of Gille and Moret-Bailly [8] tells us that T is contained in a finite union of $G(\mathbb{Z})$-orbits. As Γ is of finite index in $G(\mathbb{Z})$, T is contained in a finite union of Γ-orbits.

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THE ANDRÉ-OORT CONJECTURE.

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Abstract. In this paper we prove, assuming the Generalized Riemann Hypothesis, the André-Oort conjecture on the Zariski closure of sets of special points in a Shimura variety. In the case of sets of special points satisfying an additional assumption, we prove the conjecture without assuming the GRH.

Contents

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1. Introduction

1.1. **The André-Oort conjecture.** The purpose of this paper is to prove, under certain assumptions, the André-Oort conjecture on special subvarieties of Shimura varieties.

Before stating the André-Oort conjecture we provide some motivation from algebraic geometry. Let Z be a smooth complex algebraic variety and let $\mathcal{F} \longrightarrow Z$ be a variation of polarizable $\mathbb{Q}$-Hodge structures on Z (for example $\mathcal{F} = R^i f_* \mathbb{Q}$ for a smooth proper morphism $f : Y \longrightarrow Z$). To every $z \in Z$ one associates a reductive algebraic $\mathbb{Q}$-group $\mathbf{MT}(z)$, called the Mumford-Tate group of the Hodge structure $\mathcal{F}_z$. This group is the stabilizer of the Hodge classes in the rational Hodge structures tensorially generated by $\mathcal{F}_z$ and its dual. A point $z \in Z$ is said to be Hodge generic if $\mathbf{MT}(z)$ is maximal. If Z is irreducible, two Hodge generic points of Z have the same Mumford-Tate group, called the generic Mumford-Tate group $\mathbf{MT}_Z$. The complement of the Hodge generic locus is a countable union of closed irreducible algebraic subvarieties of Z, each not contained in the union of the others. This is proved in [7]. Furthermore, it is shown in [34] that when Z is defined over $\overline{\mathbb{Q}}$ (and under certain simple assumptions) these components are also defined over $\overline{\mathbb{Q}}$. The irreducible components of the intersections of these subvarieties are called *special subvarieties* (or subvarieties of Hodge type) of Z relative to $\mathcal{F}$. Special subvarieties of dimension zero are called *special points*.

Example : Let Z be the modular curve $Y(N)$ (with $N \geq 4$) and let $\mathcal{F}$ be the variation of polarizable $\mathbb{Q}$-Hodge structures $R^1 f_* \mathbb{Q}$ of weight one on Z associated to the universal elliptic curve $f : E \longrightarrow Z$. Special points on Z parametrize elliptic curves with complex multiplication. The generic Mumford-Tate group on Z is $\mathbf{GL}_{2,\mathbb{Q}}$. The Mumford-Tate group of a special point corresponding to an elliptic curve with complex multiplication by a quadratic imaginary field K is the torus $\mathrm{Res}_{K/\mathbb{Q}}\mathbf{G}_{\mathbf{m},K}$ obtained by restriction of scalars from K to $\mathbb{Q}$ of the multiplicative group $\mathbf{G}_{\mathbf{m},K}$ over K.

The general Noether-Lefschetz problem consists in describing the geometry of these special subvarieties, in particular the distribution of special points. Griffiths transversality condition prevents, in general, the existence of moduli spaces for variations of polarizable $\mathbb{Q}$-Hodge structures. Shimura varieties naturally appear as solutions to such moduli problems with additional data (c.f. [11], [12], [20]). Recall that a $\mathbb{Q}$-Hodge structure on a $\mathbb{Q}$-vector space V is a structure of $\mathbf{S}$-module on $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$, where $\mathbf{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G}_{\mathbf{m},\mathbb{C}}$. In other words it is a morphism of real algebraic groups

$$h \colon \mathbf{S} \longrightarrow \mathbf{GL}(V_{\mathbb{R}}) \ .$$

The Mumford-Tate group $\mathbf{MT}(h)$ is the smallest algebraic $\mathbb{Q}$-subgroup $\mathbf{H}$ of $\mathbf{GL}(V)$ such that h factors through $\mathbf{H}_{\mathbb{R}}$. A *Shimura datum* is a pair $(\mathbf{G}, X)$, with $\mathbf{G}$ a linear connected reductive group over $\mathbb{Q}$ and X a $\mathbf{G}(\mathbb{R})$-conjugacy class in the set of morphisms of real algebraic groups $\mathrm{Hom}(\mathbf{S}, \mathbf{G}_{\mathbb{R}})$, satisfying the "Deligne's conditions" [12, 1.1.13]. These conditions imply, in particular, that the connected components of X are Hermitian symmetric domains and that $\mathbb{Q}$-representations of $\mathbf{G}$ induce polarizable variations of $\mathbb{Q}$-Hodge structures on X. A morphism of Shimura data from $(\mathbf{G}_1, X_1)$ to $(\mathbf{G}_2, X_2)$ is a $\mathbb{Q}$-morphism $f : \mathbf{G}_1 \longrightarrow \mathbf{G}_2$ that maps X_1 to X_2.

Given a compact open subgroup K of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ (where $\mathbf{A}_{\mathrm{f}}$ denotes the ring of finite adèles of $\mathbb{Q}$) the set $\mathbf{G}(\mathbb{Q})\backslash(X \times \mathbf{G}(\mathbf{A}_{\mathrm{f}})/K)$ is naturally the set of $\mathbb{C}$-points of a quasi-projective variety over $\mathbb{C}$, denoted $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. The projective limit $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}} = \varprojlim_K \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is a $\mathbb{C}$-scheme on which $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ acts continuously by multiplication on the right (c.f. section 4.1.1). The multiplication by $g \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ on $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ induces an algebraic correspondence T_g on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, called a *Hecke correspondence*. One easily shows that a subvariety $V \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is *special* (with respect to some variation of Hodge structure associated to a $\mathbb{Q}$-representation of $\mathbf{G}$) if and only if there is a Shimura datum $(\mathbf{H}, X_{\mathbf{H}})$, a morphism of Shimura data $f : (\mathbf{H}, X_{\mathbf{H}}) \longrightarrow (\mathbf{G}, X)$ and an element $g \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ such that V is an irreducible component of the image of the morphism :

$$\mathrm{Sh}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}} \xrightarrow{\mathrm{Sh}(f)} \mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}} \xrightarrow{\cdot g} \mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \ .$$

It can also be shown that the Shimura datum $(\mathbf{H}, X_{\mathbf{H}})$ can be chosen in such a way that $\mathbf{H}$ is the generic Mumford-Tate group on $X_{\mathbf{H}}$. A *special* point is a special subvariety of dimension zero. One sees that a point $\overline{(x,g)} \in \mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$ (where $x \in X$ and $g \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$) is *special* if and only the group $\mathbf{MT}(x)$ is commutative (in which case $\mathbf{MT}(x)$ is a torus).

Given a special subvariety V of $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$, the set of special points of $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$ contained in V is dense in V for the strong (and in particular for the Zariski) topology. Indeed, one shows that V contains a special point, say s. Let $\mathbf{H}$ be a reductive group defining V and let $\mathbf{H}(\mathbb{R})^+$ denotes the connected component of the identity in the real Lie group $\mathbf{H}(\mathbb{R})$. The fact that $\mathbf{H}(\mathbb{Q}) \cap \mathbf{H}(\mathbb{R})^+$ is dense in $\mathbf{H}(\mathbb{R})^+$ implies that the "$\mathbf{H}(\mathbb{Q}) \cap \mathbf{H}(\mathbb{R})^+$-orbit" of s, which is contained in V, is dense in V. This "orbit" (sometimes referred to as the Hecke orbit of s) consists of special points. The André-Oort conjecture is the converse statement.

Definition 1.1.1. *Given a set Σ of subvarieties of $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$ we denote by $\boldsymbol{\Sigma}$ the subset $\cup_{V\in\Sigma} V$ of $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$.*

Conjecture 1.1.2 (André-Oort)**.** *Let $(\mathbf{G},X)$ be a Shimura datum, K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and let Σ a set of special points in $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$. Then every irreducible component of the Zariski closure of $\boldsymbol{\Sigma}$ in $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$ is a special subvariety.*

One may notice an analogy between this conjecture and the so-called Manin-Mumford conjecture (first proved by Raynaud) which asserts that irreducible components of the Zariski closure of a set of *torsion* points in an Abelian variety are translates of Abelian subvarieties by torsion points. There is a large (and constantly growing) number of proofs of the Manin-Mumford conjecture.

1.2. **The results.** Our main result is the following :

Theorem 1.2.1. *Let $(\mathbf{G},X)$ be a Shimura datum, K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and let Σ be a set of special points in $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$. We make* **one** *of the following assumptions :*

1. *Assume the Generalized Riemann Hypothesis (GRH) for CM fields.*
2. *Assume that there exists a faithful representation $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ such that with respect to this representation, the Mumford-Tate groups $\mathbf{MT}_s$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class as s ranges through Σ.*

Then every irreducible component of the Zariski closure of $\boldsymbol{\Sigma}$ in $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$ is a special subvariety.

In fact we prove the following

Theorem 1.2.2. *Let* $(\mathbf{G}, X)$ *be a Shimura datum,* K *a compact open subgroup of* $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ *and let* Σ *be a set of special subvarieties in* $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$*. We make* **one** *of the following assumptions :*

(1) Assume the Generalized Riemann Hypothesis (GRH) for CM fields.

(2) Assume that there exists a faithful representation $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ *such that with respect to this representation, the generic Mumford-Tate groups* $\mathbf{MT}_V$ *of* V *lie in one* $\mathbf{GL}_n(\mathbb{Q})$*-conjugacy class as* V *ranges through* Σ*.*

Then every irreducible component of the Zariski closure of $\boldsymbol{\Sigma}$ *in* $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ *is a special subvariety.*

The case of theorem 1.2.2 where Σ is a set of special points is theorem 1.2.1.

1.3. **The history of the André-Oort conjecture.** For history and results obtained before 2002, we refer to the introduction of [16]. We just mention that conjecture 1.1.2 was stated by André in 1989 in the case of an irreducible curve of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing a Zariski dense set of special points, and in 1995 by Oort for irreducible subvarieties of moduli spaces of polarized Abelian varieties containing a Zariski-dense set of special points.

Let us mention some results we will use in the course of our proof.

In [9] (further generalized in [31] and [33]), the conclusion of the theorem 1.2.2 is proved for sets Σ of *strongly special* subvarieties in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ without assuming (1) or (2) (c.f. section 2). The statement is proved using ergodic theoretic techniques.

Using Galois-theoretic techniques and geometric properties of Hecke correspondences, Edixhoven and the second author (see [17]) proved the conjecture for curves in Shimura varieties containing infinite sets of special points satisfying our assumption (2). Subsequently, the second author (in [37]) proved the André-Oort conjecture for curves in Shimura varieties assuming the GRH. The main new ingredient in [37] is a theorem on lower bounds for Galois orbits of special points. In the work [15], Edixhoven proves, assuming the GRH, the André-Oort conjecture for products of modular curves. In [36], the second author proves the André-Oort conjecture for sets of special points satisfying an additional condition.

The authors started working together on this conjecture in 2003 trying to generalize the Edixhoven-Yafaev strategy to the general case of the André-Oort conjecture. In the process two main difficulties occur. One is the question of irreducibility of transforms of subvarieties under Hecke correspondences. This problem is dealt with in sections 6 and 7. The other difficulty consists in dealing with higher dimensional special subvarieties. Our strategy is to proceed by induction on the generic dimension of elements of Σ. The main ingredient for controlling the induction was the discovery by Ullmo's and the second author in [33] of a possible combination of Galois theoretic and ergodic techniques. It took form while the second author was visiting the University of Paris-Sud in January-February 2005.

1.4. **Conventions.** In this paper a complex algebraic variety is a reduced scheme over $\mathbb{C}$, not necessarily irreducible. A subvariety is always assumed to be a closed subvariety.

2. Equidistribution and Galois orbits.

In this section we recall a crucial ingredient in the proof of the theorem 1.2.2 : an alternative discovered in [33] concerning the use of ergodic theoretic methods or Galois geometric methods for attacking conjecture 1.1.2. We start with three definitions :

Definition 2.0.1. *Let $(\mathbf{G}, X)$ be a Shimura datum, K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let V be a special subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, defined by the Shimura subdatum $(\mathbf{H}_V, X_{\mathbf{H}_V})$ of $(\mathbf{G}, X)$. By lemma 2.1 of [33], one can assume that $\mathbf{H}_V$ is the generic Mumford-Tate group on V. The special subvariety V of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ can be canonically written as the image $f(\Gamma_V \backslash X^+_{\mathbf{H}_V})$, where $X^+_{\mathbf{H}_V}$ denotes a connected component of $X_{\mathbf{H}_V}$, Γ_V is an arithmetic subgroup of the stabilizer $\mathbf{H}_V(\mathbb{R})_+$ of $X^+_{\mathbf{H}_V}$ in $\mathbf{H}_V(\mathbb{R})$, $\Gamma_V \backslash X^+_{\mathbf{H}_V}$ is an Hermitian locally symmetric space and $f : \Gamma_V \backslash X^+_{\mathbf{H}_V} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is a finite morphism of complex algebraic varieties. We define μ_V to be the probability measure on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C})$ supported on V, push-forward by f of the standard probability measure on the Hermitian locally symmetric space $\Gamma_V \backslash X^+_{\mathbf{H}_V}$ induced by the Haar measure on $\mathbf{H}_V(\mathbb{R})_+$.*

Definition 2.0.2. *Let $(\mathbf{G}, X)$ be a Shimura datum and $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ a compact open subgroup. Given a complex subvariety $Z \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ we will denote by $\deg_{L_K} Z$ the degree of the compactification $\overline{Z} \subset \overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ with respect to the natural line bundle L_K on the Baily-Borel compactification $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ (c.f. section 4.2).*

Definition 2.0.3. *Let $(\mathbf{G}, X)$ be a Shimura datum, K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let $\lambda : \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ be the adjoint morphism and $\mathbf{T}$ be an $\mathbb{R}$-anisotropic $\mathbb{Q}$-subtorus of $\mathbf{G}^{\mathrm{ad}}$.*

A special subvariety V of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is called $\mathbf{T}$-special if the torus $\mathbf{T}$ is the connected center of the group $\lambda(\mathbf{H}_V)$, where $\mathbf{H}_V$ denotes the generic Mumford-Tate group of V.

In the case where $\mathbf{T}$ is the trivial torus, one says that V is strongly special.

Remark 2.0.4. The definition of *strongly special* given in [9] requires moreover that $\lambda(\mathbf{H}_V)$ is not contained in a proper parabolic subgroup of $\mathbf{G}^{\mathrm{ad}}$ but as explained in [31, rem. 3.9] this last condition is automatically satisfied.

With these three definitions, the alternative discovered in [33] can roughly be explained as follows. Assume the GRH. Let F be a number field over which $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ admits a canonical model (c.f. section 4.1.2) and let $(V_n)_{n\in\mathbb{N}}$ be a sequence of special subvarieties of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$.

- If there exists a finite collection $\{\mathbf{T}_1, \cdots, \mathbf{T}_r\}$ of $\mathbb{R}$-anisotropic $\mathbb{Q}$-subtori of $\mathbf{G}^{\mathrm{ad}}$ such that each V_n, $n \in \mathbb{N}$, is $\mathbf{T}_i$-special for some $i \in \{1, \cdots, r\}$, then the sequence (V_n) is equidistributed in the following sense. After possibly passing to a subsequence the sequence of probability measures μ_{V_n} weakly converges to the probability measure μ_V of some special subvariety V and for n large, V_n is contained in V.

  This implies that irreducible components of the Zariski-closure of $\bigcup_{n\in\mathbb{N}} V_n$ in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ are special.
- otherwise the function $\deg_{L_K}(\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot V_n)$ is an unbounded function of n and we can try Galois-theoretic methods for studying the Zariski-closure of $\bigcup_{n\in\mathbb{N}} V_n$ in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$.

We now explain this alternative in more details.

2.1. **Equidistribution results.** Let $\mathbf{G}$ be a reductive $\mathbb{R}$-group. Ratner's classification of probability measures on homogeneous spaces of the form $\Gamma\backslash\lambda(\mathbf{G})(\mathbb{R})^+$ (where Γ denotes a lattice in $\lambda(\mathbf{G})(\mathbb{R})^+$), ergodic under some unipotent flows [27], and Dani-Margulis recurrence lemma [10] enable Clozel and Ullmo [9] to prove the following equidistribution result in the strongly special case, generalized by Ullmo and Yafaev [33, theorem 3.8] to the $\mathbf{T}$-special case :

Theorem 2.1.1 (Clozel-Ullmo, Ullmo-Yafaev)**.** *Let $(\mathbf{G}, X)$ be a Shimura datum, K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let $\mathbf{T}$ be an $\mathbb{R}$-anisotropic $\mathbb{Q}$-subtorus of $\mathbf{G}^{\mathrm{ad}}$. Let $(V_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbf{T}$-special subvarieties of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Let μ_{V_n} be the canonical probability measure on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ supported by V_n. There exists a $\mathbf{T}$-special subvariety V and a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ weakly converging to μ_V. Furthermore V contains V_{n_k} for all k sufficiently large. In particular, the irreducible components of the Zariski closure of a set of $\mathbf{T}$-special subvarieties of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ are special.*

Remarks 2.1.2. (1) Note that a special point, whose Mumford-Tate group is a non-central torus, is not strongly special. Moreover, given an $\mathbb{R}$-anisotropic $\mathbb{Q}$-subtorus $\mathbf{T}$ of $\mathbf{G}^{\mathrm{ad}}$, the Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ contains only a finite number of $\mathbf{T}$-special points (c.f. [33, lemma 5.5]). Thus theorem 2.1.1 says nothing *directly* on the André-Oort conjecture.

(2) In fact the conclusion of the theorem 2.1.1 is simply not true for special points : they are dense for the Archimedian topology in $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$, so just consider a sequence of special points converging to a non-special point in $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$ (or diverging to a cusp if $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$ is non-compact). In this case the corresponding sequence of Dirac delta measures will converge to the Dirac delta measure of the non-special point (respectively escape to infinity).

(3) There is a so-called equidistribution conjecture which implies André-Oort and much more. A sequence (x_n) of points of $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$ is called *strict* if any for any proper special subvariety V of $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$, the set

$$\{n : x_n \in V\}$$

is finite. Let E be a field of definition of canonical model of $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$. To any special point x, one associates a probability measure Δ_x on $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$ as follows :

$$\Delta_x = \frac{1}{|\mathrm{Gal}(\overline{E}/E)(x)|} \sum_{\sigma\in \mathrm{Gal}(\overline{E}/E)} \delta_{\sigma(x)}$$

where $\delta_{\sigma(x)}$ is the Dirac measure at the point $\sigma(x)$ and $|\mathrm{Gal}(\overline{E}/E)(x)|$ denotes the cardinality of the Galois orbit $\mathrm{Gal}(\overline{E}/E)(x)$. The equidistribution conjecture predicts that if (x_n) is a strict sequence of special points, then the sequence of measures Δ_{x_n} weakly converges to the canonical probability measure attached to $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}(\mathbb{C})$. This statement implies the André-Oort conjecture. The equidistribution conjecture is known for modular curves and is completely open in general. For more on this, we refer to the survey [32].

(4) In [9] and [33] the theorem 2.1.1 is proven in the case where $\mathbf{G}$ is a semi-simple group of adjoint type. The general case is an easy corollary, c.f. appendix A.

2.2. **Galois orbits of non-strongly special subvarieties.** In this paragraph, we recall the lower bound obtained in [33] for the degree of the Galois orbit of a non-strongly special subvariety in a Shimura variety $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$.

Definition 2.2.1. *Let $(\mathbf{G},X)$ be a Shimura datum. Let $K=\prod_{p \text{ premier}} K_p$ be a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let V be a special subvariety of a Shimura variety $\mathrm{Sh}_K(\mathbf{G},X)_{\mathbb{C}}$. We denote by :*

- $\mathbf{H}_V \subset \mathbf{G}$ *the generic Mumford-Tate group of V and $(\mathbf{H}_V, X_V)$ the Shimura subdatum of $(\mathbf{G},X)$ defining V.*
- $E_{\mathbf{H}_V}$ *the reflex field of $(\mathbf{H}_V, X_{\mathbf{H}_V})$*
- $\mathbf{T}_V$ *the torus connected center of $\mathbf{H}_V$. The torus $\mathbf{T}_V$ is non-trivial if and only if V is non-strongly special.*

- $K^{\mathrm{m}}_{\mathbf{T}_V}$ *the maximal compact open subgroup of* $\mathbf{T}_V(\mathbf{A}_{\mathrm{f}})$.
- $K_{\mathbf{T}_V}$ *the compact open subgroup* $\mathbf{T}_V(\mathbf{A}_{\mathrm{f}}) \cap K \subset K^{\mathrm{m}}_{\mathbf{T}_V}$.
- $i(\mathbf{T}_V)$ *the number of primes* p *such that* $K^{m}_{\mathbf{T}_{V,p}} \neq K_{\mathbf{T}_{V,p}}$.
- $\mathbf{C}_V$ *the torus* $\mathbf{H}_V/\mathbf{H}_V^{\mathrm{der}}$ *isogenous to* $\mathbf{T}_V$.
- $d_{\mathbf{T}_V}$ *the absolute value of the discriminant of the splitting field* L_V *of* $\mathbf{C}_V$*, and* n_V *the absolute degree of* L_V.
- $\beta_V := \log |d_{\mathbf{T}_V}|$. *In particular* $\beta_V = 0$ *if* V *is strongly special.*

One of the main ingredients of our proof is the following lower bounds for the degree of Galois orbits of non-strongly special subvarieties obtained in [33, theorem 2.13] :

Theorem 2.2.2 (Ullmo-Yafaev)**.** *Let* $(\mathbf{G}, X)$ *be a Shimura datum. Let* $K = \prod_{p \text{ premier}} K_p$ *be a compact open subgroup of* $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$.

Assume the GRH for CM fields. There exists a real number $B > 0$ *and, for each positive integer* N*, a real number* $C(N) > 0$ *such that the following holds.*

Let $(\mathbf{H}, X_{\mathbf{H}})$ *be a Shimura subdatum of* $(\mathbf{G}, X)$ *and* V *be a connected component of* $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, X_{\mathbf{H}})_{\mathbb{C}}$*, where* $K_{\mathbf{H}} \subset \mathbf{H}(\mathbf{A}_{\mathrm{f}})$ *denotes the compact open subgroup* $K \cap \mathbf{H}(\mathbf{A}_{\mathrm{f}})$. *Then the following inequality holds :*

$$\deg_{L_{K_{\mathbf{H}}}}(\mathrm{Gal}(\overline{\mathbb{Q}}/E_{\mathbf{H}_V}) \cdot V) > C(N) \cdot B^{i(\mathbf{T}_V)} \cdot |K^{m}_{\mathbf{T}_V}/K_{\mathbf{T}_V}| \cdot \beta_V^N \quad . \tag{2.1}$$

Furthermore, if one considers only the subvarieties V *such that the associated tori* $\mathbf{T}_V$ *lie in one* $\mathbf{GL}_n(\mathbb{Q})$*-conjugacy class, then the assumption of the GRH can be dropped.*

2.3. **The alternative.** Throughout the paper we will be using the following notations.

Definition 2.3.1. *Let* $(\mathbf{G}, X)$ *be a Shimura datum. Let* $K = \prod_{p \text{ premier}} K_p$ *be a compact open subgroup of* $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. *Let* V *be a special subvariety of* $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ *with generic Mumford-Tate group* $\mathbf{H}_V$.

If V *is strongly special we let* $\alpha_V = 0$.

If V *is a non-strongly special subvariety, we set with the notations of definition 2.2.1 and theorem 2.2.2*

$$\alpha_V := B^{i(\mathbf{T}_V)} \cdot |K^{m}_{\mathbf{T}_V}/K_{\mathbf{T}_V}| \quad .$$

The alternative roughly explained in the introduction to section 2 can now be formulated in the following theorem (easy adaptation of [33, theor. 3.9]) :

Theorem 2.3.2. *Let* $(\mathbf{G}, X)$ *be a Shimura datum. Let* $K = \prod_{p \text{ premier}} K_p$ *be a compact open subgroup of* $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$.

Assume the GRH for CM fields.

Let Σ be a set of special subvarieties V of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ such that $\alpha_V \beta_V$ is bounded as V ranges through Σ. There exists a finite set $\{\mathbf{T}_1, \cdots, \mathbf{T}_r\}$ of $\mathbb{Q}$-subtori of $\mathbf{G}$ such that any V in Σ is $\mathbf{T}_i$-special for some $i \in \{1, \cdots, r\}$.

Furthermore, if one considers only the subvarieties V such that the associated tori $\mathbf{T}_V$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class, then the assumption of the GRH can be dropped.

Proof. Of course we can assume that all the subvarieties V in Σ are non-strongly special. Moreover, if $\alpha_V \beta_V$ is bounded, then clearly α_V is bounded.

As noticed in [33, prop. 3.11] there exist $A > 0$ and $c > 0$ such that for any $V \in \Sigma$ we have

$$\alpha_V > A \cdot \prod_{\{p \,:\, K^m_{\mathbf{T}_V} \neq K_{\mathbf{T}_V}\}} c \cdot p \ .$$

In particular α_V bounded implies that $i(\mathbf{T}_V) = |\{p : K^m_{\mathbf{T}_V} \neq K_{\mathbf{T}_V}\}|$ is also bounded. As $\alpha_V \geq B^{i(\mathbf{T}_V)}$, $i(\mathbf{T}_V)$ is bounded and $\alpha_V \beta_V$ is bounded, we obtain that $\beta_V = \log(d_{\mathbf{T}_V})$ is also bounded. Thus if $L_{\mathbf{C}_V}$ denotes the splitting field of $\mathbf{C}_V$ (or $\mathbf{T}_V$, their splitting field is the same), its discriminant is bounded.

To prove the theorem, it is thus enough to replace Σ with the set of $V \in \Sigma$ with fixed field $L_{\mathbf{C}_V}$. The proof is then the same as [33, theorem 3.9] starting with [33, lemma 3.10]. □

3. Reduction and strategy.

From now on we will use the following convenient terminology :

Definition 3.0.3. *Let $(\mathbf{G}, X)$ be a Shimura datum and K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let Σ be a set of special subvarieties of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. A subset Λ of Σ is called a modification of Σ if $\mathbf{\Lambda}$ and $\mathbf{\Sigma}$ have the same Zariski-closure in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Given a subtorus $\mathbf{T}$ of $\mathbf{G}$ we say that Σ is $\mathbf{T}$-special if any element in Σ is a $\mathbf{T}$-special subvariety.*

3.1. First reduction.

We first have the following obvious reduction of the proof of theorem 1.2.2 :

Theorem 3.1.1. *Let $(\mathbf{G}, X)$ be a Shimura datum and K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let Z be a subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Suppose that Z contains a Zariski dense set $\mathbf{\Sigma}$, which is a union of special subvarieties V, $V \in \Sigma$, all of the same dimension $n(\Sigma)$.*

We make **one** *of the following assumptions :*

1. *Assume the Generalized Riemann Hypothesis (GRH) for CM fields.*
2. *Assume that there is a faithful representation $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ such that with respect to this representation, the connected centers $\mathbf{T}_V$ of the generic Mumford-Tate groups $\mathbf{H}_V$ of V lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class as V ranges through Σ.*

Then

(a) *The variety Z contains a Zariski dense set Σ' of special subvarieties of constant dimension $n(\Sigma') > n(\Sigma)$.*

(b) *Furthermore, if Σ satisfies the condition (2), one can choose Σ' also satisfying (2).*

Proposition 3.1.2. *Theorem 3.1.1 implies the main theorem 1.2.2.*

Proof. Let Σ as in the main theorem1.2.2. Without loss of generality one can assume that the Zariski closure Z of $\mathbf{\Sigma}$ is irreducible. Moreover by Noetherianity one can assume that all the $V \in \Sigma$ have the same dimension $n(\Sigma)$.

Notice that the assumption (2) of the theorem 1.2.2 implies the assumption (2) of the theorem 3.1.1. We then apply theorem 3.1.1,(a) to Σ : the subvariety Z contains a Zariski-dense set $\mathbf{\Sigma}'$ of special subvarieties V', $V' \in \Sigma'$, of constant dimension $n(\Sigma') > n(\Sigma)$.

By theorem 3.1.1,(b) one can replace Σ by Σ'. Applying this process recursively and as $n(\Sigma') \leq \dim(Z)$, we conclude that Z is special. □

3.2. **Second reduction.** Part (b) of theorem 3.1.1 is easy, we deal with it in section 5. Part (a) of theorem 3.1.1 can itself be reduced to the following main theorem (we refer to section 4 for a reminder on reflex fields and the definition of the connected component $S_K(\mathbf{G}, X)_{\mathbb{C}}$, and to definition 6.0.4 for the (usual) definition of an F-irreducible F-variety) :

Theorem 3.2.1. *Let $(\mathbf{G}, X)$ be a Shimura datum and K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Let F be a number field containing the reflex field $E(\mathbf{G}, X)$.*

Let Z be a Hodge-generic F-irreducible F-subvariety of the connected component $S_K(\mathbf{G}, X)_{\mathbb{C}}$ of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Suppose that Z contains a Zariski dense set $\mathbf{\Sigma}$, which is a union of special subvarieties V, $V \in \Sigma$, all of the same dimension $n(\Sigma)$ and such that for any modification Σ' of Σ the set $\{\alpha_V \beta_V,\ V \in \Sigma'\}$ is unbounded.

We make **one** *of the following assumptions :*

(1) Assume the Generalized Riemann Hypothesis (GRH) for CM fields.

(2) Assume that there is a faithful representation $\mathbf{G} \hookrightarrow \mathbf{GL}_n$ such that with respect to this representation, the connected centers $\mathbf{T}_V$ of the generic Mumford-Tate groups $\mathbf{H}_V$ of V lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class as V ranges through Σ.

After possibly replacing Σ by a modification, for every V in Σ there exists a special subvariety V' such that $V \subsetneq V' \subset Z$.

Proposition 3.2.2. *Theorem 3.2.1 implies theorem 3.1.1* (a).

Proof. Let Z be as in theorem 3.1.1.

We can assume that the variety Z is Hodge-generic. To fulfill this condition, replace $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ by the smallest special subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing Z (c.f. [17, prop.2.1]). This comes down to replacing $\mathbf{G}$ with the generic Mumford-Tate group on Z. This does not change α_V and β_V.

We can also assume that Z is contained in $S_K(\mathbf{G}, X)_{\mathbb{C}}$: proving theorem 3.1.1 for Z is equivalent to proving theorem 3.1.1 for each irreducible component of Z, thus we can assume Z is irreducible. As proving theorem 3.1.1 for Z is also equivalent to proving theorem 3.1.1 for any irreducible component of its image under some Hecke correspondence, we can ensure Z is contained in $S_K(\mathbf{G}, X)_{\mathbb{C}}$.

As Z contains a Zariski-dense set of special points, and any special point is $\overline{\mathbb{Q}}$-valued, the variety Z is defined over some number field $F \subset \mathbb{C}$ containing the reflex field $E(\mathbf{G}, X)$: $Z = Z_F \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \mathbb{C}$.

Let $Z_F = (Z_F)_1 \cup \cdots \cup (Z_F)_l$ be the decomposition of Z_F in F-irreducible components, thus $Z = \cup_{i=1}^l Z_i$, with $Z_i := (Z_F)_i \times_{\mathrm{Spec}\, F} \mathrm{Spec}\mathbb{C}$, $1 \leq i \leq l$. This decomposition induces a decomposition $\Sigma = \cup_{i=1}^l \Sigma_i$, where Σ_i is the set of special subvarieties V, $V \in \Sigma$, such that $V \subset Z_i$. Each Z_i, $1 \leq i \leq l$, contains the Zariski dense set $\mathbf{\Sigma_i} = \cup_{V \in \Sigma_i} V$ and proving theorem 3.1.1 for Z is equivalent to proving theorem 3.1.1 for each Z_i, $1 \leq i \leq l$.

Fix $i \in \{1, \cdots, l\}$. If for some modification Σ_i' of Σ_i the set $\{\alpha_V \beta_V,\ V \in \Sigma_i'\}$ is bounded, by theorem 2.3.2 and by Noetherianity there exists a $\mathbb{Q}$-subtorus $\mathbf{T}$ of $\mathbf{G}^{\mathrm{ad}}$ and a $\mathbf{T}$-special modification of Σ_i. Applying theorem 2.1.1 one obtains that every geometrically irreducible component of Z_i is special, which proves theorem 3.1.1 for Z_i.

Thus we can assume that Z_i, $1 \leq i \leq l$, satisfies the hypothesis of theorem 3.2.1 and we have reduced the proof of theorem 3.1.1 to the case where Z satisfies the assumptions of theorem 3.2.1.

Let Σ' be the set of the special subvarieties V' obtained from theorem 3.2.1 applied to Z. Thus Z contains the Zariski-dense set $\mathbf{\Sigma'} = \cup_{V' \in \Sigma'} V'$. After possibly replacing Σ' by a modification, we can assume by Noetherianity of Z that the subvarieties in Σ' have the same dimension $n(\Sigma')$. Of course $n(\Sigma') > n(\Sigma)$. This proves the theorem 3.1.1 assuming theorem 3.2.1. □

3.3. **Reminder about the proof of theorem 1.2.1 in the case where Z is a curve.** The strategy for proving theorem 3.2.1 is fairly complicated. We first recall the strategy developed in [17] in the case where Z is a curve. In the next section we explain why this strategy *cannot* be directly generalized to higher dimensional cases.

Without loss of generality one can assume that the group $\mathbf{G}$ is adjoint, Z is Hodge generic (i.e. its generic Mumford-Tate group is equal to $\mathbf{G}$), and Z is contained in the connected component $S_K(\mathbf{G}, X)_{\mathbb{C}}$ of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. The proof of the theorem 1.2.1 in the case where Z is a curve relies on three ingredients.

3.3.1. The first one is a geometric *criterion* for a Hodge generic subvariety Z to be special in terms of Hecke correspondences. Given a Hecke correspondence T_m, $m \in \mathbf{G}(\mathbf{A}_f)$ (c.f. section 4.1.1) we denote by T_m^0 the correspondence it induces on $S_K(\mathbf{G}, X)_{\mathbb{C}}$. This correspondence decomposes as $T_m^0 = \sum_i T_{q_i}$, where the q_i's are elements of $\mathbf{G}(\mathbb{Q})_+ \cap KmK$ defined by

$$\mathbf{G}(\mathbb{Q})_+ \cap KmK = \coprod \Gamma_K q_i^{-1} \Gamma_K \ .$$

Theorem 3.3.1. [17, theorem 7.1] *Let* $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ *be a Shimura variety, with* $\mathbf{G}$ *semi-simple of adjoint type. Let* $Z \subset S_K(\mathbf{G}, X)_{\mathbb{C}}$ *be an Hodge-generic closed subvariety of the connected component* $S_K(\mathbf{G}, X)_{\mathbb{C}}$ *of* $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. *Suppose there exist a prime* l *and an element* $m \in \mathbf{G}(\mathbb{Q}_l)$ *such that the neutral component* $T_m^0 = \sum_{i=1}^n T_{q_i}$ *of the Hecke correspondence* T_m *associated with* m *has the following properties :*

(1) $Z \subset T_m^0 Z$.
(2) For any $i \in \{1, \cdots n\}$, *the variety* $T_{q_i} Z$ *is irreducible.*
(3) For any $i \in \{1, \cdots n\}$ *the* $T_{q_i} + T_{q_i^{-1}}$*-orbit is dense in* $S_K(\mathbf{G}, X)$.

Then $Z = S_K(\mathbf{G}, X)$, *in particular* Z *is special.*

From (1) and (2) one deduces the existence of one index i such that $Z = T_{q_i} Z$. It follows that Z contains an $T_{q_i} + T_{q_i^{-1}}$-orbit. The equality $Z = S_K$ follows from (3).

In the case where Z is a curve one proves the existence of a prime l and of an element $m \in \mathbf{G}(\mathbb{Q}_l)$ satisfying these properties as follows. The property (3) is satisfied for essentially any m. The property (2), which is crucial for this strategy, is obtained by showing that for any prime l outside a finite set of primes $\mathcal{P}_Z$ and any $q \in \mathbf{G}(\mathbb{Q})^+ \cap (\mathbf{G}(\mathbb{Q}_l) \times \prod_{p \neq l} K_l)$, the variety $T_q Z$ is irreducible. This is an easy corollary of a result due independently to Weisfeiler and Nori (c.f. theorem 4.3.3) applied to the Zariski closure of the image of the monodromy representation. This result implies that for all l except those in a finite set $\mathcal{P}_Z$, the closure in $\mathbf{G}(\mathbb{Q}_l)$ of the image of the monodromy representation for the $\mathbb{Z}$-variation of Hodge structure on the smooth locus Z^{sm} of Z coincides with the closure of $K \cap \mathbf{G}(\mathbb{Q})^+$ in $\mathbf{G}(\mathbb{Q}_l)$ of the open compact subgroup $K \subset \mathbf{G}(\mathbf{A}_f)$. To prove the property (1) one uses Galois orbits of special points contained in Z and the fact that Hecke correspondences commute with the Galois action. First one notices that Z is defined over a number field F, finite extension of the reflex field $E(\mathbf{G}, X)$ (c.f. section 4.1.2). If $s \in Z$ is a special point, r_s the associated reciprocity morphism and $m \in \mathbf{G}(\mathbb{Q}_l)$ belongs to $r_s((\mathbb{Q}_l \otimes F)^*) \subset \mathbf{MT}(s)(\mathbb{Q}_l)$ then the Galois orbit $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot s$ is contained in the intersection $Z \cap T_m Z$. If this intersection is proper its cardinality $Z \cap T_m Z$ is essentially the degree $[K_l : K_l \cap m K_l m^{-1}]$ of the correspondence T_m. To find l and m such that $Z \subset T_m Z$

it is then enough to exhibit $m \in r_s((\mathbb{Q}_l \otimes F)^*)$ such that the cardinality $|\mathrm{Gal}(\overline{\mathbb{Q}}/F).s|$ is larger than $[K_l : K_l \cap mK_l m^{-1}]$. This is dealt with by the next two ingredients :

3.3.2. The second ingredient claims the existence of "unbounded" Hecke correspondences of controlled degree defined by elements in $r_s((\mathbb{Q}_l \otimes F)^*)$:

Theorem 3.3.2. [17, corollary 7.4.4] *There exists an integer k such that for all $s \in \Sigma$ and for any prime l such that $\mathbf{MT}(s)_{\mathbb{F}_l}$ is a split torus, there exists $m \in r_s((\mathbb{Q}_l \otimes F)^*) \subset \mathbf{MT}(s)(\mathbb{Q}_l)$ such that*

(1) for any factor $\mathbf{G}_i$ of $\mathbf{G}$ the image of m in $\mathbf{G}_i(\mathbb{Q}_l)$ is not in a compact subgroup.
(2) $[K_l : K_l \cap mK_l m^{-1}] \ll l^k$

3.3.3. The third ingredient is a lower bound for $|\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot s|$ due to Edixhoven, and improved in theorem 2.2.2.

3.3.4. Finally using this lower bound for $|\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot s|$ and the effective Chebotarev theorem consequence of GRH one proves the existence for any special point $s \in \Sigma$ with a sufficiently big Galois orbit of a prime l outside $\mathcal{P}_Z$, splitting $\mathbf{MT}(s)$, such that $\mathbf{MT}(s)_{\mathbb{F}_l}$ is a torus and such that $|\mathrm{Gal}(\overline{\mathbb{Q}}/F).s| \gg l^k$. Effective Chebotarev is not needed under the assumption that the $\mathbf{MT}(s)$, $s \in \Sigma$, are isomorphic. The reason being that in this case, the splitting field of the $\mathbf{MT}(s)$ is constant and the classical Chebotarev

We then choose an m satisfying the conditions of the theorem 3.3.2. As $|\mathrm{Gal}(\overline{\mathbb{Q}}/F).s| \gg [K_l : K_l \cap mK_l m^{-1}]$ one obtains $Z \subset T_m Z$ and by the criterion 3.3.1 the subvariety Z is special.

3.4. **Strategy for proving the theorem 3.2.1 : the general case.** Our strategy for dealing with the general case of the theorem 3.2.1 is as follows :

Let $(\mathbf{G}, X)$ be a Shimura datum and K a compact open subgroup of $\mathbf{G}(\mathbf{A}_\mathrm{f})$. Let Z be a subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Suppose that Z contains a Zariski dense set $\mathbf{\Sigma}$, which is a union of special subvarieties V, $V \in \Sigma$, all of the same dimension $n(\Sigma)$ and such that the set $\{\alpha_V \beta_V,\ V \in \Sigma\}$ is unbounded.

Notice that the idea of the proof of [17] generalizes to the case where $\dim Z = n(\Sigma) + 1$ (c.f. section 8.4.1). In the general case, for a V in Σ with $\alpha_V \beta_V$ sufficiently large we want to exhibit V' special subvariety in Z containing V properly.

Our first step (section 6) is geometric : we give a criterion (theorem 6.1) similar to criterion 3.3.1 saying that an inclusion $Z \subset T_m Z$, for a prime l and an element $m \in \mathbf{H}_V(\mathbb{Q}_l)$ satisfying certain conditions, implies that V is properly contained in a special subvariety V' of Z.

The criterion we need has to be much more subtle than the one in [17]. In the characterization of [17], in order to obtain the irreducibility of $T_m Z$ the prime l must be outside some finite set $\mathcal{P}_Z$ of primes. It seems impossible to make the set of bad primes $\mathcal{P}_Z$ explicit in terms of numerical invariants of Z, except in a few cases where the Chow ring of the Baily-Borel compactification of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is easy to describe (like the case considered by Edixhoven, where $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is a product $\prod_{i=1}^n X_i$ of modular curves, and where he shows that for a k-dimensional subvariety Z dominant on all factors X_i, $1 \leq i \leq n$, the bad primes $p \in \mathcal{P}_Z$ are smaller than the supremum of the degree of the projections of Z on the k-factors $X_{1_1} \times \cdots \times X_{i_k}$ of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$). In particular that characterization is not suitable for our induction.

Our criterion 6.1 for an irreducible subvariety Z containing a non-strongly special subvariety V and satisfying $Z \subset T_m Z$ for some $m \in \mathbf{T}_V(\mathbb{Q}_l)$ to contain a special subvariety V' containing V properly does no longer require the irreducibility of $T_m Z$. In particular it is valid for *any* prime l, outside $\mathcal{P}_Z$ or not. Instead we notice that the inclusion $Z \subset T_m Z$ implies that Z contains the image Z' in $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ of the $\langle K_l', (k_1 m k_2)^n \rangle$-orbit of (one irreducible component of) the preimage of V in the pro-l-covering of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Here k_1 and k_2 are some elements of K_l, n some positive integer and K_l' the l-adic closure of the image of the monodromy of Z. If the group $\langle K_l', (k_1 m k_2)^n \rangle$ is not compact, then the irreducible component of Z' containing V contains a special subvariety V' of Z containing V properly.

The main problem with this criterion is that the group $\langle K_l', k_1 m k_2 \rangle$ can be compact, containing K_l' with very small index. This is the case in Edixhoven's counter-example [14, Remark 7.2]. In this case $\mathbf{G} = \mathrm{PGL}_2 \times \mathrm{PGL}_2$, $K_l' := \Gamma_0(l) \times \Gamma_0(l)$ and $k_1 m k_2$ is $w_l \times w_l$, the product of two Atkin-Lehner involutions. The index $[\langle K_l', k_1 m k_2 \rangle : K_l']$ is four.

Our second step (section 7) consists in getting rid of this problem and is purely group-theoretic. We notice that if K_l is *not a maximal* compact open subgroup but is contained in an *Iwahori* subgroup of $\mathbf{G}(\mathbb{Q}_l)$, then for "many" m in $\mathbf{T}_V(\mathbb{Q}_l)$ the element $k_1 m k_2$ is not contained in a compact subgroup for any k_1 and k_2 in K_l. This is our theorem 7.1 about the existence of adequate Hecke correspondences. The proof relies on simple properties of the Bruhat-Tits decomposition of $\mathbf{G}(\mathbb{Q}_l)$.

Our third step (section 8) is Galois-theoretic and geometric. We use theorem 2.2.2, theorem 6.1, theorem 7.1 to show (under one of the assumptions of theorem 3.1.1) that the existence of a prime number l satisfying certain conditions forces a subvariety Z of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing a non-strongly special subvariety V to contain a special subvariety V' containing V properly. The proof is a nice geometric induction on $r = \dim Z - \dim V$.

Our last step (section 9) is number-theoretic : we complete the proof of the theorem 3.2.1 and hence of theorem 1.2.2 by exhibiting, using effective Chebotarev under GRH (or usual Chebotarev under the second assumption of theorem 1.2.2), a prime l satisfying our desiderata. For this step it is crucial that both the index of an Iwahori subgroup in a maximal compact subgroup of $\mathbf{G}(\mathbb{Q}_l)$ and the degree of the correspondence T_m are bounded by a uniform power of l.

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4. Preliminaries.

4.1. **Notations.** In this section we define some notations and recall some standard facts about Shimura varieties that we will use in this paper. We refer to [11], [12], [20] for details.

As far as groups are concerned, reductive algebraic groups are assumed to be connected. The exponent ^{0} denotes the algebraic neutral component and the exponent $^+$ the topological neutral component. Thus if $\mathbf{G}$ is a $\mathbb{Q}$-algebraic group $\mathbf{G}(\mathbb{R})^+$ denotes the topological neutral component of the real Lie group of $\mathbb{R}$-points $\mathbf{G}(\mathbb{R})$. We also denote by $\mathbf{G}(\mathbb{Q})^+$ the intersection $\mathbf{G}(\mathbb{R})^+ \cap \mathbf{G}(\mathbb{Q})$. When $\mathbf{G}$ is reductive we denote by $\mathbf{G}^{\mathrm{ad}}$ the adjoint group of $\mathbf{G}$ (the quotient of $\mathbf{G}$ by its center) and by $\mathbf{G}(\mathbb{R})_+$ the preimage in $\mathbf{G}(\mathbb{R})$ of $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$. The notation $\mathbf{G}(\mathbb{Q})_+$ denotes the intersection $\mathbf{G}(\mathbb{R})_+ \cap \mathbf{G}(\mathbb{Q})$. In particular when $\mathbf{G}$ is adjoint then $\mathbf{G}(\mathbb{Q})^+ = \mathbf{G}(\mathbb{Q})_+$.

4.1.1. *Shimura varieties.* Let $(\mathbf{G}, X)$ be a Shimura datum. We fix X^+ a connected component of X. Given K a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ one obtains the homeomorphic decomposition

$$\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} = \mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbf{A}_{\mathrm{f}})/K \simeq \coprod_{g \in \mathcal{C}} \Gamma_g \backslash X^+ \quad , \tag{4.1}$$

where $\mathcal{C}$ denotes a set of representatives for the (finite) double coset space $\mathbf{G}(\mathbb{Q})_+ \backslash \mathbf{G}(\mathbf{A}_{\mathrm{f}})/K$, and Γ_g denotes the arithmetic subgroup $gKg^{-1} \cap \mathbf{G}(\mathbb{Q})_+$ of $\mathbf{G}(\mathbb{Q})_+$. We denote by Γ_K the group Γ_e corresponding to the identity element $e \in \mathcal{C}$ and by $S_K(\mathbf{G}, X)_{\mathbb{C}} = \Gamma_K \backslash X^+$ the corresponding connected component of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$.

The Shimura variety $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ is the $\mathbb{C}$-scheme projective limit of the $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, K compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. The group $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ acts continuously on the right on $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$. The set of $\mathbb{C}$-points is

$$\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}(\mathbb{C}) = \frac{\mathbf{G}(\mathbb{Q})}{\mathbf{Z}(\mathbb{Q})} \backslash (X \times \mathbf{G}(\mathbf{A}_{\mathrm{f}})/\overline{\mathbf{Z}(\mathbb{Q})}) \quad ,$$

where $\mathbf{Z}$ denotes the center of $\mathbf{G}$ and $\overline{\mathbf{Z}(\mathbb{Q})}$ is the closure of $\mathbf{Z}(\mathbb{Q})$ in $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ [12, prop.2.1.10]. The action of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ on the right is given by : $\overline{(x,h)} \stackrel{\cdot g}{\longrightarrow} \overline{(x, h \cdot g)}$. For $m \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$, we denote by T_m the Hecke correspondence

$$\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \longleftarrow \mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}} \stackrel{\cdot m}{\longrightarrow} \mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \quad .$$

4.1.2. *Reciprocity morphisms and canonical models.* Given $(\mathbf{G}, X)$ a Shimura datum, where X is the $\mathbf{G}(\mathbb{R})$-conjugacy class of $h : \mathbf{S} \longrightarrow \mathbf{G}_{\mathbb{R}}$, we denote by $\mu_h : \mathbf{G}_{m,\mathbb{C}} \longrightarrow \mathbf{G}_{\mathbb{C}}$ the $\mathbb{C}$-morphism of $\mathbb{Q}$-groups obtained by composing the embedding of tori

$$\begin{array}{ccc} \mathbf{G}_{m,\mathbb{C}} & \longrightarrow & \mathbf{S}_{\mathbb{C}} \\ z & \longrightarrow & (z, 1) \end{array}$$

with $h_{\mathbb{C}}$. Let $E(\mathbf{G}, X)$ be the field of definition of the $\mathbf{G}(\mathbb{C})$-conjugacy class of μ_h (the reflex field of $(\mathbf{G}, X)$). In the case where $\mathbf{G}$ is a torus $\mathbf{T}$ and $X = \{h\}$ we denote by

$$r_{(\mathbf{T}, \{h\})} : \mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{a}b} \longrightarrow \mathbf{T}(\mathbf{A}_{\mathrm{f}})/\overline{\mathbf{T}(\mathbb{Q})}$$

the reciprocity morphism defined in [12, 2.2.3] for any field $E \subset \mathbb{C}$ containing $E(\mathbf{T}, \{h\})$. Let $x = \overline{(h, g)}$ be a special point in $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ image of the pair $(h : \mathbf{S} \longrightarrow \mathbf{T} \subset \mathbf{G}, g) \in X \times \mathbf{G}(\mathbf{A}_{\mathrm{f}})$. The field $E(h) = E(\mathbf{T}, \{h\})$ depends only on h and is an extension of $E(\mathbf{G}, X)$ [12, 2.2.1]. The Shimura variety $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ admits a unique model $\mathrm{Sh}(\mathbf{G}, X)$ over $E(\mathbf{G}, X)$ such that the $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$-action on the right is defined over $E(\mathbf{G}, X)$, the special points are algebraic and if $x = \overline{(h, g)}$ is a special point of $\mathrm{Sh}(\mathbf{G}, X)(\mathbb{C})$ then an element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E(h)) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/E(\mathbf{G}, X))$ acts on x by $\sigma(x) = \overline{(h, \tilde{r}(\sigma) g)}$, where $\tilde{r}(\sigma) \in \mathbf{T}(\mathbf{A}_{\mathrm{f}})$ is any lift of $r_{(\mathbf{T}, \{h\})}(x) \in \mathbf{T}(\mathbf{A}_{\mathrm{f}})/\overline{\mathbf{T}(\mathbb{Q})}$, c.f. [12, 2.2.5]. This defines a canonical

$E(\mathbf{G}, X)$-model $\mathrm{Sh}_K(\mathbf{G}, X)$ for $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, K compact open subgroup in $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. For $m \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ the Hecke correspondence T_m is defined over $E(\mathbf{G}, X)$. We will denote by $\pi_K : \mathrm{Sh}(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)$ the natural projection.

4.1.3. *Galois action on the set of connected components of a Shimura variety.* Let $\pi_0(\mathbf{G})$ be the set of geometrically irreducible components of $\mathrm{Sh}(\mathbf{G}, X)$. This set is a principal homogeneous space under the Abelian group $\pi(\mathbf{G}) = \mathbf{G}(\mathbf{A}_{\mathrm{f}})/\mathbf{G}(\mathbb{Q})\rho\widetilde{\mathbf{G}}(\mathbf{A}_{\mathrm{f}})$ where $\rho \colon \widetilde{\mathbf{G}} \longrightarrow \mathbf{G}^{\mathrm{der}}$ is the universal cover (c.f. [12, 2.1.14]). The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E(\mathbf{G}, X))$ on $\pi_0(\mathbf{G})$ factors through $\mathrm{Gal}(\overline{\mathbb{Q}}/E(\mathbf{G}, X))^{\mathrm{ab}}$ and is given by the reciprocity morphism (c.f. [12, 2.6])

$$r_{(\mathbf{G}, X_{\mathbf{G}})} : \mathrm{Gal}(\overline{\mathbb{Q}}/E(\mathbf{G}, X))^{\mathrm{ab}} \longrightarrow \pi(\mathbf{G}) \ .$$

Let $\mathbf{C} := \mathbf{G}/\mathbf{G}^{\mathrm{der}}$ be the quotient of $\mathbf{G}$ by its derived subgroup. If the group $\mathbf{G}$ is not semi-simple (the neutral component $\mathbf{T}$ of the center of $\mathbf{G}$ is a non-trivial torus), then $\mathbf{C}$ is a torus and the projection $\mathbf{G} \longrightarrow \mathbf{C}$ induces an isogeny $\mathbf{T} \longrightarrow \mathbf{C}$. Let x be any element of X, and let $\overline{x}$ be the morphism $\overline{x} \colon \mathbf{S} \longrightarrow \mathbf{C}_{\mathbb{R}}$ obtained by composing x with the projection $\mathbf{G}_{\mathbb{R}} \longrightarrow \mathbf{C}_{\mathbb{R}}$. The pair $(\mathbf{C}, \{\overline{x}\})$ is a special Shimura datum. The reflex field $E(\mathbf{C}, \{\overline{x}\})$ contains $E(\mathbf{G}, X)$ and the reciprocity morphism

$$r_{(\mathbf{C}, \overline{x})} \colon \mathrm{Gal}(\overline{\mathbb{Q}}/E(\mathbf{C}, \overline{x}))^{\mathrm{ab}} \longrightarrow \pi(\mathbf{C}) \ .$$

is the morphism $r_{(\mathbf{G}, X)}$ composed with the natural morphism $\pi(\mathbf{G}) \longrightarrow \pi(\mathbf{C})$.

4.1.4. *The Shimura variety at a prime l.* Let l be a prime. Suppose $K^l \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}}^l)$ is a compact open subgroup, where $\mathbf{A}_{\mathrm{f}}^l$ denotes the ring of finite adèles outside l.

Definition 4.1.1. *We denote by* $\mathrm{Sh}_{K^l}(\mathbf{G}, X)$ *the* $E(\mathbf{G}, X)$*-scheme* $\varprojlim \mathrm{Sh}_{K^l \cdot U_l}(\mathbf{G}, X)$ *for* U_l *compact open subgroup of* $\mathbf{G}(\mathbb{Q}_l)$.

The scheme $\mathrm{Sh}_{K^l}(\mathbf{G}, X)$ identifies with the quotient $\mathrm{Sh}(\mathbf{G}, X)/K^l$. It admits a continuous $\mathbf{G}(\mathbb{Q}_l)$-action on the right. Given a compact open subgroup $U_l \subset \mathbf{G}(\mathbb{Q}_l)$ we denote by $\pi_{U_l} : \mathrm{Sh}_{K^l}(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_{K^l U_l}(\mathbf{G}, X)$ the canonical projection.

4.1.5. *Neatness.* Let $\mathbf{G}$ be a linear algebraic group over $\mathbb{Q}$. We recall the definition of *neatness* for subgroups of $\mathbf{G}(\mathbb{Q})$ and its generalization to subgroups of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$. We refer to [3] and [24, 0.6] for more details.

Given an element $g \in \mathbf{G}(\mathbb{Q})$ let $\mathrm{Eig}(g)$ be the subgroup of $\overline{\mathbb{Q}}^*$ generated by the eigenvalues of g. We say that $g \in \mathbf{G}(\mathbb{Q})$ is *neat* if the subgroup $\mathrm{Eig}(g)$ is torsion-free. A subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is neat if any element of Γ is neat. In particular such a group is torsion-free.

Given an element $g_p \in \mathbf{G}(\mathbb{Q}_p)$ let $\mathrm{Eig}_p(g_p)$ be the subgroup of $\overline{\mathbb{Q}_p}^*$ generated by all eigenvalues of g_p. Let $\overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}_p}$ be some embedding and consider the torsion part $(\overline{\mathbb{Q}}^* \cap$

$\mathrm{Eig}_p(g_p))_{\mathrm{tors}}$. Since every subgroup of $\overline{\mathbb{Q}}^*$ consisting of roots of unity is normalized by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, this group does not depend on the choice of the embedding $\overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}_p}^*$. We say that g_p is *neat* if

$$(\overline{\mathbb{Q}}^* \cap \mathrm{Eig}_p(g))_{\mathrm{tors}} = \{1\} \ .$$

We say that $g = (g_p)_p \in \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ is neat if

$$\bigcap_p (\overline{\mathbb{Q}}^* \cap \mathrm{Eig}_p(g_p))_{\mathrm{tors}} = \{1\} \ .$$

A subgroup $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ is neat if any element of K is neat. Of course if the projection K_p of K in $\mathbf{G}(\mathbb{Q}_p)$ is neat then K is neat. Notice that if K is a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ then all of the Γ_g in the decomposition (4.1) are.

Neatness is preserved by conjugacy and intersection with an arbitrary subgroup. Moreover if $\rho : \mathbf{G} \longrightarrow \mathbf{H}$ is a $\mathbb{Q}$-morphism of linear algebraic $\mathbb{Q}$-groups and $g \in \mathbf{G}(\mathbb{Q})$ (resp. $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$) is neat then its image $\rho(g)$ is also neat.

We recall the following well-known lemma :

Lemma 4.1.2. *Let $K = \prod_p K_p$ be a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and let $l \geq 3$ be a prime number. There exists an open subgroup K'_l of K_l such that the subgroup $K' := K'_l \times \prod_{p \neq l} K_p$ of K is neat.*

Proof. As noticed above if K'_l is neat then $K' := K'_l \times \prod_{p \neq l} K_p$ is neat. As a subgroup of a neat group is neat, it is enough to show that a special maximal compact open subgroup $K_l \subset \mathbf{G}(\mathbb{Q}_l)$ contains a neat subgroup K'_l with finite index. By [24, p.5] one can take, $K'_l = K_l^{(1)}$ the first congruence kernel. □

4.2. **Baily-Borel compactification and degrees of subvarieties.** In this section we recall the results we will need on projective geometry of Shimura varieties. We will also prove a proposition (proposition 4.2.10) which compares the degrees of a subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)$ with respect to two different line bundles.

4.2.1. *Baily-Borel compactification.* Let $(\mathbf{G}, X)$ be a Shimura datum. Given $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ a neat compact open subgroup, let $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ the corresponding complex Shimura variety.

Definition 4.2.1. *We denote by $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ the Baily-Borel compactification of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, c.f.* [2].

The Baily-Borel compactification $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ is a normal projective variety with boundary $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}} - \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ of complex codimension at least 2 if $\mathbf{G}$ does not have $\mathbb{Q}$-simple factors of dimension 3. The following proposition summarizes basic properties of $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ that we will use.

Proposition 4.2.2. *(1) The line bundle of holomorphic forms of maximal degree on X descends to $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ and extends uniquely to an ample line bundle L_K on $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ such that, at the generic points of the boundary components of codimension one, it is given by n-th powers of forms with logarithmic poles. Let K_1 and K_2 be neat compact open subgroups of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and g in $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ such that $K_2 \subset gK_1g^{-1}$. Then the morphism from $\mathrm{Sh}_{K_2}(\mathbf{G}, X)_{\mathbb{C}}$ to $\mathrm{Sh}_{K_1}(\mathbf{G}, X)_{\mathbb{C}}$ induced by g extends to a morphism $f : \overline{\mathrm{Sh}_{K_2}(\mathbf{G}, X)_{\mathbb{C}}} \longrightarrow \overline{\mathrm{Sh}_{K_1}(\mathbf{G}, X)_{\mathbb{C}}}$, and the line bundle $f^*L_{K_1}$ is canonically isomorphic to L_{K_2} .*

(2) The canonical model $\mathrm{Sh}_K(\mathbf{G}, X)$ of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ over the reflex field $E(\mathbf{G}, X)$ admits a unique extension to a model $\overline{\mathrm{Sh}_K(\mathbf{G}, X)}$ of $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ over $E(\mathbf{G}, X)$. The line bundle L_K is naturally defined over $E(\mathbf{G}, X)$.

(3) Let $\varphi : (\mathbf{H}, Y) \longrightarrow (\mathbf{G}, X)$ be a morphism of Shimura data and $K_{\mathbf{H}} \subset \mathbf{H}(\mathbf{A}_{\mathrm{f}})$, $K_{\mathbf{G}} \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ neat compact open subgroups with $\varphi(K_{\mathbf{H}}) \subset K_{\mathbf{G}}$. Then the canonical map $\phi : \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y) \longrightarrow \mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G}, X)$ induced by φ extends to a morphism still denoted by $\phi : \overline{\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)} \longrightarrow \overline{\mathrm{Sh}_{K_{\mathbf{G}}}(\mathbf{G}, X)}$.

Proof. The first statement is [2, lemma 10.8] and [24, prop.8.1, sections 8.2, 8.3]. The second one is [24, theor.12.3.a]. The third statement is [28, theorem p.231] (over $\mathbb{C}$) and [24, theor. 12.3.b] (over $E(\mathbf{G}, X)$). □

Definition 4.2.3. *Given a complex subvariety $Z \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ we will denote by $\deg_{L_K} Z$ the degree of the compactification $\overline{Z} \subset \overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$ with respect to the line bundle L_K. We will write $\deg Z$ when it is clear to which line bundle we are referring to.*

Remark 4.2.4. More generally given a connected semi-simple algebraic $\mathbb{Q}$-group $\mathbf{G}$ of Hermitian type (and of non-compact type) with associated Hermitian domain X and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ a neat arithmetic lattice, the Baily-Borel compactification $\overline{\Gamma\backslash X}$ of the quasi-projective complex variety $\Gamma\backslash X$ and the bundle L_Γ on $\overline{\Gamma\backslash X}$ are well-defined.

4.2.2. *Comparison of degrees for sub-Shimura data.*

Proposition 4.2.5. *Let $\phi : \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_{K'}(\mathbf{G}', X')_{\mathbb{C}}$ be a morphism of Shimura varieties associated to a Shimura subdatum $\varphi : (\mathbf{G}, X) \longrightarrow (\mathbf{G}', X')$, a neat compact open subgroup K of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and a neat compact open subgroup K' of $\mathbf{G}'(\mathbf{A}_{\mathrm{f}})$ containing $\varphi(K)$. Let $\Lambda_{K,K'}$ denotes the line bundle $\phi^*L_{K'} \otimes L_K^{-1}$ on $\overline{\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}}$. Then for any subvariety Z of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ one has the inequality $\deg_{\Lambda_{K,K'}} Z \geq 0$.*

This proposition is a corollary of the following

Proposition 4.2.6. *Let $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$ be a $\mathbb{Q}$-morphism of connected semi-simple algebraic $\mathbb{Q}$-groups of Hermitian type (and of non-compact type) inducing a holomorphic* *totally geodesic embedding of the associated Hermitian domains* $\phi : X^+ \longrightarrow X'^+$. *Let* $\Gamma \subset \mathbf{G}(\mathbb{Q})$ *be a neat arithmetic lattice and* $\Gamma' \subset \mathbf{G}'(\mathbb{Q})$ *a neat arithmetic lattice containing* $\varphi(\Gamma)$. *Let* $\Lambda_{\Gamma,\Gamma'}$ *denote the line bundle* $\phi^* L_{\Gamma'} \otimes L_\Gamma^{-1}$ *on* $\overline{\Gamma\backslash X}$. *Then for any subvariety* Z *of* $\Gamma\backslash X$ *one has the inequality* $\deg_{\Lambda_{\Gamma,\Gamma'}} Z \geq 0$.

Proposition 4.2.6 implies the proposition 4.2.5. To prove proposition 4.2.5 one can assume without loss of generality that the subvariety Z is irreducible, that Z is contained in the connected component $S_K = \Gamma_K\backslash X^+$ and that $\phi : \mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C} \longrightarrow \mathrm{Sh}_{K'}(\mathbf{G}', X')_\mathbb{C}$ maps S_K to $S_{K'} = \Gamma_{K'}\backslash X'^+$. The morphism of reductive $\mathbb{Q}$-groups $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$ induces a $\mathbb{Q}$-morphism $\overline{\varphi} : \mathbf{G}^{\mathrm{der}} \longrightarrow \mathbf{G}'^{\mathrm{ad}}$ of semi-simple $\mathbb{Q}$-groups. Let Γ denote the neat lattice $\mathbf{G}^{\mathrm{der}}(\mathbb{Q}) \cap K \subset \mathbf{G}^{\mathrm{der}}(\mathbb{Q})$ and Γ' the neat lattice of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q})$ image of $\Gamma_{K'}$. Notice that $\Gamma'\backslash X'^+ = \Gamma_{K'}\backslash X'^+$. Consider the diagram

(4.2)
$$\begin{array}{ccc} \Gamma\backslash X^+ & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\phi\circ\pi} & \\ \Gamma_K\backslash X^+ & \xrightarrow[\phi]{} & \Gamma'\backslash X'^+ \end{array}$$

with π the natural finite étale map. The proposition 4.2 (1) easily extends to this setting :

$$\pi^*(L_{\Gamma_K}) = L_\Gamma \quad .$$

By the projection formula the inequality $\deg_{\Lambda_{K,K'}} Z \geq 0$ is implied by the inequality $\deg_{\Lambda_{\Gamma,\Gamma'}} \pi^{-1}(Z) \geq 0$. This inequality follows from proposition 4.2.6.

□

Proof of the proposition 4.2.6. First notice that by the projection formula and by proposition 4.2 (1), we can assume that the group $\mathbf{G}$ is simply connected and the group $\mathbf{G}'$ is adjoint.

Let $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$ be the decomposition of $\mathbf{G}$ into $\mathbb{Q}$-simple factors. Let $\varphi_i : \mathbf{G}_i \longrightarrow \mathbf{G}'$, $1 \leq i \leq r$ denote the components of $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$. If $\Gamma_1 \subset \Gamma$ is a finite index subgroup and $p : \Gamma_1\backslash X^+ \longrightarrow \Gamma\backslash X^+$ is the corresponding finite étale morphism, by proposition 4.2 the line bundle $\Lambda_{\Gamma_1,\Gamma'}$ corresponding to $\phi \circ p$ is isomorphic to $p^*\Lambda_{\Gamma,\Gamma'}$. The fact that $\deg_{\Lambda_{\Gamma,\Gamma'}} Z \geq 0$ is implied by $\deg_{\Lambda_{\Gamma_1,\Gamma'}} p^{-1}Z \geq 0$. Thus we can assume that $\Gamma = \Gamma_1 \times \cdots \times \Gamma_r$, with Γ_i a neat arithmetic subgroup of $\mathbf{G}_i(\mathbb{Q})$. The variety $\Gamma\backslash X^+$ decomposes into a product

$$\Gamma\backslash X^+ = \Gamma_1\backslash X_1^+ \times \cdots \times \Gamma_r\backslash X_r^+$$

and the line bundle $\Lambda_{\Gamma,\Gamma'}$ on $\overline{\Gamma\backslash X^+}$ decomposes as

$$\Lambda_{\Gamma,\Gamma'} = \Lambda_{\Gamma_1,\Gamma'} \boxtimes \cdots \boxtimes \Lambda_{\Gamma_r,\Gamma'} \ ,$$

with $\Lambda_{\Gamma_i,\Gamma'} = \phi_i^* L_{\Gamma'} \otimes L_{\Gamma_i}^{-1}$ the corresponding line bundle on $\overline{\Gamma_i\backslash X_i^+}$. Let $p_i : \overline{\Gamma\backslash X^+} \longrightarrow \overline{\Gamma_i\backslash X_i^+}$ be the natural projection. As

$$\deg_{\Lambda_{\Gamma,\Gamma'}} Z = \sum_{i=1}^{r} \deg_{p_i^*\Lambda_{\Gamma_i,\Gamma'}} Z \ ,$$

the proposition follows from the following one applied to each of the $\mathbf{G}_i$, $1 \leq i \leq r$. □

Proposition 4.2.7. *Assume that* $\mathbf{G}$ *is* $\mathbb{Q}$*-simple.*

(1) If $\mathbf{G}$ *is* $\mathbb{Q}$*-anisotropic then the line bundle* $\Lambda_{\Gamma,\Gamma'}$ *on the smooth complex projective variety* $\Gamma\backslash X^+$ *admits a metric of non negative curvature.*

(2) If $\mathbf{G}$ *is* $\mathbb{Q}$*-isotropic then either the line bundle* $\Lambda_{\Gamma,\Gamma'}$ *on* $\overline{\mathrm{Sh}_K(\mathbf{G},X)}$ *is trivial or it is ample.*

Proof. Let $\mathbf{G}' = \mathbf{G}'_1 \times \cdots \times \mathbf{G}'_{r'}$ be the decomposition of $\mathbf{G}'$ into $\mathbb{Q}$-simple factor and $\varphi_j : \mathbf{G} \longrightarrow \mathbf{G}'_j$, $1 \leq j \leq r'$, the components of $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$. By naturality of L_Γ and $L_{\Gamma'}$ (c.f. proposition 4.2) one can assume that $\Gamma' = \Gamma'_1 \times \cdots \Gamma'_{r'}$. Accordingly one has

$$\Gamma'\backslash X'^+ = \Gamma'_1\backslash {X'_1}^+ \times \cdots \times \Gamma'_{r'}\backslash {X'_{r'}}^+ \ .$$

As $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$ is injective and $\mathbf{G}$ is $\mathbb{Q}$-simple we can without loss of generality assume that $\varphi_1 : \mathbf{G} \longrightarrow \mathbf{G}'_1$ is injective. As

$$\Lambda = (\phi_1^* L_{\Gamma'_1} \otimes L_\Gamma^{-1}) \otimes \phi_2^* L_{\Gamma'_2} \otimes \cdots \phi_{r'}^* L_{\Gamma'_r} \ ,$$

and the $L_{\Gamma'_j}$, $j \geq 2$, are ample on $\overline{\Gamma'_j\backslash {X'_j}^+}$ it is enough to prove the statement replacing $\Lambda_{\Gamma,\Gamma'}$ by $\phi_1^* L_{\Gamma'_1} \otimes L_\Gamma^{-1}$. Thus we can assume $\mathbf{G}'$ is $\mathbb{Q}$-simple.

By the adjunction formula the line bundle $\Lambda_{\Gamma,\Gamma'|\Gamma\backslash X^+}$ restriction of $\Lambda_{\Gamma,\Gamma'}$ coincides with $\Lambda^{\max} N^*$, where N denotes the automorphic bundle on $\Gamma\backslash X^+$ associated to the normal bundle of X in X' and N^* denotes its dual. A classical computation shows that the automorphic line bundle $\Lambda_{\Gamma,\Gamma'|\Gamma\backslash X^+}$ admits a Hermitian metric of non-negative curvature. This is enough to conclude the proof of the proposition in the case $\mathbf{G}$ is $\mathbb{Q}$-anisotropic.

Suppose now $\mathbf{G}$ is $\mathbb{Q}$-isotropic. For simplicity we denote $\Lambda_{\Gamma,\Gamma'}$ by Λ from now on. We have to prove that the boundary components of $\overline{\Gamma\backslash X^+}$ do not essentially modify the positivity of $\Lambda_{|\Gamma\backslash X^+}$. We use the notation and the results of Dynkin [13], Ihara [19] and Satake [29]. Let $X = X_1 \times \cdots \times X_r$ (resp. $X' = X'_1 \times \cdots \times X'_{r'}$) be the decomposition of X (resp. X') into irreducible factors. Each X_i (resp. X'_j) is the Hermitian symmetric domain associated to an $\mathbb{R}$-isotropic $\mathbb{R}$-simple factor $\mathbf{G}_i$ (resp. $\mathbf{G}'_j$) of $\mathbf{G}_\mathbb{R}$ (resp. $\mathbf{G}'_\mathbb{R}$). The group $\mathbf{G}_{\mathbb{R}}$ (resp. $\mathbf{G}'_{\mathbb{R}}$) decomposes as $\mathbf{G}_0 \times \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$ (resp. $\mathbf{G}'_0 \times \mathbf{G}'_1 \times \cdots \times \mathbf{G}'_{r'}$) with $\mathbf{G}_0$ (resp. $\mathbf{G}'_0$) an $\mathbb{R}$-anisotropic group. Let $\mathbf{m}$ (resp. $\mathbf{m}'$) be the r-tuple (resp. r'-tuple) of non-negative integers defining the automorphic line bundle L_K (resp. $L_{K'}$) (c.f. [29, lemma 2]) and M_ϕ be the $r' \times r$-matrix with integral coefficients associated to $\varphi : \mathbf{G} \hookrightarrow \mathbf{G}'$ (c.f. [29, section 2.1]). The automorphic line bundle $\Lambda_{|\Gamma\backslash X^+}$ on $\Gamma\backslash X^+$ is associated to the r-tuple of integers $\boldsymbol{\lambda} = \mathbf{m}' M_\varphi - \mathbf{m}$ (where $\mathbf{m}$ and $\mathbf{m}'$ are seen as row vectors). It admits a locally homogeneous Hermitian metric of non-negative curvature if and only if $\lambda_i \geq 0$, $1 \leq i \leq r$ (in which case we say that $\boldsymbol{\lambda}$ is non-negative).

Lemma 4.2.8. *The row vector $\boldsymbol{\lambda}$ is non-negative.*

Proof. As $\mathbf{G}$ and $\mathbf{G}'$ are defined over $\mathbb{Q}$, both $\mathbf{m}$ and $\mathbf{m}'$ are of rational type by [29, p.301]. So $m_i = m$ for all i, $m'_j = m'$ for all j. The equality $\boldsymbol{\lambda} = \mathbf{m}' M_\varphi - \mathbf{m}$ can be written in coordinates

$$\forall i \in \{1, \cdots, r\}, \quad \lambda_i = \sum_{1 \leq j \leq r'} m_{j,i}\, m' - m \quad , \tag{4.3}$$

with $M_\varphi = (m_{j,i})$. Fix i in $\{1, \cdots r\}$ and let prove that $\lambda_i \geq 0$. As the $m_{i,j}$'s and m' are non-negative, it is enough to exhibit one j, $1 \leq j \leq r'$, with $m_{j,i}\, m' - m \geq 0$. Choose j such that the component $\varphi_{i,j} : X_i \longrightarrow X'_j$ of the map $\varphi : X_1 \times \cdots \times X_r \longrightarrow X'_1 \times \cdots \times X'_{r'}$ induced by $\varphi : \mathbf{G} \longrightarrow \mathbf{G}'$ is an embedding. Recall that with the notation of [29, p.290] one has

$$m_i = < H_{1,i}, H_{1,i} >_i \quad ,$$

where $\mathfrak{h}_i$ denotes the chosen Cartan subalgebra of $\mathfrak{g}_i(\mathbb{R})$ and $<,>_i$ denotes the canonical scalar product on $\sqrt{-1}\mathfrak{h}_i$. This gives the equality :

$$m_{j,i}\, m'_j - m_i = < \phi_j(H_{1,i}), \phi_j(H_{1,i}) >_j - < H_{1,i}, H_{1,i} >_i \quad . \tag{4.4}$$

As $\mathbf{G}_i$ is $\mathbb{R}$-simple, any two invariant non-degenerate forms on $\sqrt{-1}\mathfrak{h}_i$ are proportional : there exists a positive real constant $c_{i,j}$ (called by Dynkin [13, p.130] the index of $\varphi_{i,j}$: $\mathbf{G}_i \longrightarrow \mathbf{G}_j$) such that

$$\forall X, Y \in \sqrt{-1}\mathfrak{h}_i, \quad < \phi_j(X), \phi_j(Y) >_j = c_{i,j} < X, Y >_i \quad .$$

Equation (4.4) thus gives :

$$m_{j,i}\, m'_j - m_i = (c_{i,j} - 1) < H_{1,i}, H_{1,i} >_i \quad . \tag{4.5}$$

By [13, theorem 2.2. p.131] the constant $c_{i,j}$ is a positive integer. Thus $m_{j,i}\, m'_j - m_i$ is non-negative and this finishes the proof that $\boldsymbol{\lambda}$ is non-negative. $\square$

By [29, cor.2 p.298] the sum $M = \sum_{1\leq j\leq r'} m_{j,i}$ is independent of i ($1 \leq i \leq r$). This implies that $\boldsymbol{\lambda}$ is of rational type : one of the λ_i is non-zero if and only if all are. In this case $\boldsymbol{\lambda}$ is positive of rational type and Λ is ample on $\overline{\Gamma\backslash X^+}$ by [29, theor.1].

If $\boldsymbol{\lambda} = 0$, the line bundle $\Lambda_{|\Gamma\backslash X^+}$ is trivial. As $\mathbf{G}$ is $\mathbb{Q}$-simple, if $\mathbf{G}$ is not locally isomorphic to $\mathbf{SL}_2$ the line bundle Λ on $\overline{\Gamma\backslash X^+}$ is trivial.

The last case is treated in the following lemma :

Lemma 4.2.9. *If $\boldsymbol{\lambda} = 0$ and $\mathbf{G}$ is locally isomorphic to $\mathbf{SL}_2$, then $\phi : \mathbf{G} \longrightarrow \mathbf{G}'$ is a local isomorphism and the line bundle Λ on $\overline{\Gamma\backslash X^+}$ is trivial.*

Proof. It follows from the equation (4.3) that there exists a unique integer j such that the morphism $\varphi_j : \mathbf{G}_{\mathbb{R}} \longrightarrow \mathbf{G}_j$ is non trivial. In particular $\mathbf{G}'$ is $\mathbb{R}$-simple. Moreover the equation (4.5) implies that index c of $\phi : \mathbf{G} \longrightarrow \mathbf{G}'$ is equal to 1. Thus by [13, theorem 6.2 p.152] the Lie algebra $\mathfrak{g}$ is a regular subalgebra of $\mathfrak{g}'$. If $\mathbf{G}'_{\mathbb{R}}$ is classical, the equality [13, (2.36) p.136] shows that necessarily $\phi : \mathbf{G} \longrightarrow \mathbf{G}'$ is a local isomorphism. In particular the line bundle Λ on $\overline{\Gamma\backslash X^+}$ is trivial. If the group $\mathbf{G}'_{\mathbb{R}}$ is an exceptional simple Lie group of Hermitian type (thus E_6 or E_7), Dynkin shows in [13, Tables 16, 17 p.178-179] that there is a unique realization of $\mathfrak{g}$ as a regular subalgebra of $\mathfrak{g}'$ of index 1. However this realization is not of Hermitian type : the coefficient $\alpha'_1(\varphi(H_1))$ is zero. Thus this case is impossible. □

This finishes the proof of proposition 4.2.7.

□

Corollary 4.2.10. *Let $\phi : \mathrm{Sh}_K(\mathbf{G}, X) \longrightarrow \mathrm{Sh}_{K'}(\mathbf{G}', X')$ be a morphism of Shimura varieties associated to a Shimura subdatum $\varphi : (\mathbf{G}, X) \longrightarrow (\mathbf{G}', X')$, K' a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and $K = K' \cap \mathbf{G}(\mathbf{A}_{\mathrm{f}})$. Then for any subvariety Z of $\mathrm{Sh}_K(\mathbf{G}, X)$ whose irreducible components are Hodge generic one has $\deg_{L_K} Z \leq \deg_{L_{K'}} \phi(Z)$.*

Proof. As the irreducible components of Z are Hodge generic in $\mathrm{Sh}_K(\mathbf{G}, X)$ we know by lemma 4.2 in [33] (and its proof) that $\phi_{|Z} : Z \longrightarrow Z'$ is generically injective. In particular by the projection formula one has

$$\deg_{L_{K_{\mathbf{G}'}}} Z' = \deg_{\phi^* L_{K'}} Z \ .$$

So the inequality $\deg_{L_K} Z \leq \deg_{L_{K'}} Z'$ is equivalent to the inequality $\deg_\Lambda Z \geq 0$ proven in the proposition 4.2.6. □

4.3. ***p*-adic closure of Zariski-dense groups.** We recall the following well-known result :

Proposition 4.3.1. *Let H be a finitely generated subgroup of $\mathbf{GL}_n(\mathbb{Z})$ and let $\mathbf{H}$ be the Zariski-closure of H in $\mathbf{GL}_{n,\mathbb{Z}}$. Suppose that $\mathbf{H}$ is semi-simple. Then for any prime number p the closure of H in $\mathbf{H}(\mathbb{Z}_p)$ is open.*

Proof. The case H finite is obvious. Suppose that H is infinite. Since $\mathbf{H}(\mathbb{Z}_p)$ is compact and H is infinite, the closure H_p of H in $\mathbf{H}(\mathbf{Z}_p)$ is not discrete. Then it is a p-adic analytic group and it has a Lie algebra L which is a Lie subalgebra of the Lie algebra Lie $\mathbf{H}$ of $\mathbf{H}$ and projects non-trivially on any factor of Lie $\mathbf{H}$. By construction L is invariant under the adjoint action of H, thus also under the adjoint action of the Zariski-closure $\mathbf{H}$ of H. Therefore $L =$ Lie $\mathbf{H}$, which implies that H_p is open in $\mathbf{H}(\mathbb{Z}_p)$. □

Remark 4.3.2. The easy proposition 4.3.1 can be strengthened to the following remarkable theorem, due independently to Weisfeiler and Nori, which was used in [17] but which we will not need :

Theorem 4.3.3 ([35], [23])**.** *Let H be a finitely generated subgroup of $\mathbf{GL}_n(\mathbb{Z})$ and let $\mathbf{H}$ be the Zariski-closure of H in $\mathbf{GL}_{n,\mathbb{Z}}$. Suppose that $\mathbf{H}(\mathbb{C})$ has finite fundamental group. Then the closure of H in $\mathbf{GL}_n(\mathbf{A}_{\mathrm{f}})$ is open in the closure of $\mathbf{H}(\mathbb{Z})$ in $\mathbf{GL}_n(\mathbf{A}_{\mathrm{f}})$.*

5. Inclusion of sub-Shimura datum.

In this section we prove the following proposition which implies part (b) of the theorem 3.1.1.

Proposition 5.1. *Suppose that the set Σ in the theorem 3.2.1 is such that with respect to a faithful representation $\rho\colon \mathbf{G} \longrightarrow \mathbf{GL}_n$ the centers $\mathbf{T}_V$ of the generic Mumford-Tate groups $\mathbf{H}_V$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-orbit as V ranges through Σ. Then the set Σ' obtained in the proposition 3.2.2 admits a modification Σ'' such that the centers $\mathbf{T}_{V'}$ of the generic Mumford-Tate groups $\mathbf{H}_{V'}$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-orbit as V' ranges through Σ''.*

We first prove the following general fact about Shimura data which will also be used at another point in this paper.

Lemma 5.2. *Let $(\mathbf{G}, X)$ be a Shimura datum such that $\mathbf{G}$ is the generic Mumford-Tate group on X and $(\mathbf{H}, X_{\mathbf{H}})$ be a sub-Shimura datum of $(\mathbf{G}, X)$. Let $\mathbf{T}$ (resp. $\mathbf{Z}$) be the connected center of $\mathbf{G}$ (resp. $\mathbf{H}$). Then*

$$\mathbf{T} \subset \mathbf{Z} \ .$$

Proof. The proof uses in a crucial way the fact that $\mathbf{G}$ is the generic Mumford-Tate group on X. We write

$$\mathbf{G} = \mathbf{T}\mathbf{G}^{\mathrm{der}} \ .$$

As $\mathbf{T} \cap \mathbf{H}$ is contained in the center $\mathbf{Z}$ of $\mathbf{H}$, we can write

$$\mathbf{H} = (\mathbf{T} \cap \mathbf{H})\mathbf{H}'$$

for some subgroup $\mathbf{H}'$ of $\mathbf{H}$. Clearly $\mathbf{H}' \subset \mathbf{G}^{\mathrm{der}}$.

Fix α an element of X that factors through $\mathbf{H}_{\mathbb{R}} = (\mathbf{T} \cap \mathbf{H})_{\mathbb{R}} \mathbf{H}'_{\mathbb{R}}$. As X is the $\mathbf{G}(\mathbb{R})$ conjugacy class of α any element $x \in X$ is of the form $g\alpha g^{-1} = \alpha^g$ for some g of $\mathbf{G}(\mathbb{R})$. Thus x factors through

$$((\mathbf{T} \cap \mathbf{H})_{\mathbb{R}})^g \, (\mathbf{G}^{\mathrm{der}}_{\mathbb{R}})^g = (\mathbf{T} \cap \mathbf{H})_{\mathbb{R}} \, \mathbf{G}^{\mathrm{der}}_{\mathbb{R}} \quad .$$

It follows that the Mumford-Tate group of x is contained in $(\mathbf{T} \cap \mathbf{H})\mathbf{G}^{\mathrm{der}}$. For x Hodge generic, we obtain

$$(\mathbf{T} \cap \mathbf{H})\mathbf{G}^{\mathrm{der}} = \mathbf{G} \quad ,$$

hence $\mathbf{T} \cap \mathbf{H} = \mathbf{T}$, therefore $\mathbf{T} \subset \mathbf{Z}$. □

To prove the proposition, first note that an inclusion of special subvarieties $V \subset V'$ corresponds to an inclusion of Shimura data $(\mathbf{H}, X_{\mathbf{H}}) \subset (\mathbf{H}', X_{\mathbf{H}'})$. The lemma above implies that the centers $\mathbf{T}'$ of the groups $\mathbf{H}'$ are contained in a $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class of a fixed torus $\mathbf{T}$. It follows that the tori $\mathbf{T}'$ are split by the same field L. As there are only finitely many subfields of L, a modification of Σ' satisfies the condition that the splitting field of the tori $\mathbf{T}'$ is constant, say L. As in the discussion before the lemma 2.4 of [33], we identify $X^*(\mathbf{T}')$ with a submodule of $X^*(\mathrm{Res}_{L/\mathbb{Q}} \mathbf{G}_{\mathbf{m}L})$ which has a canonical basis. By the lemma 2.4 of [33], the coordinates of the characters (with respect to this basis) occurring in the representation $\mathbf{T}' \subset \mathbf{GL}_n$ are uniformly bounded. It follows that the tori $\mathbf{T}'$ lie in finitely many $\mathbf{GL}_n(\mathbb{Q})$-conjugacy classes. The result follows.

6. The geometric criterion.

We prove in this section that for certain elements $m \in \mathbf{G}(\mathbb{Q}_l)$, the inclusion $Z \subset T_m Z$ implies that Z contains a special subvariety V' containing V properly.

Definition 6.0.4. *Let $(\mathbf{G}, X)$ be a Shimura datum, $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ a compact open subgroup. Let $F \subset \mathbb{C}$ be a number field containing the reflex field $E(\mathbf{G}, X)$. We use the following common abuse of notation : a subvariety $Z \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is called an F-irreducible F-subvariety if $Z = Z_F \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \mathbb{C}$, where $Z_F \subset \mathrm{Sh}_K(\mathbf{G}, X)_F$ is an irreducible closed subscheme.*

Our main theorem in this section is the following :

Theorem 6.1. *Let* $(\mathbf{G}, X)$ *be a Shimura datum,* $K = \prod_{p \text{ prime}} K_p \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ *an open compact subgroup of* $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$*. We assume there exists a prime* p_0 *such that the compact open subgroup* $K_{p_0} \subset \mathbf{G}(\mathbb{Q}_{p_0})$ *is neat. Let* F *be a number field containing the field of definition of* $S_K(\mathbf{G}, X)_{\mathbb{C}}$*.*

Let V *be a non-strongly special subvariety of* $S_K(\mathbf{G}, X)_{\mathbb{C}}$ *contained in a Hodge-generic* F*-irreducible* F*-subvariety* Z *of* $S_K(\mathbf{G}, X)_{\mathbb{C}}$*.*

Let l *be a prime number splitting* $\mathbf{T}_V$ *and* m *an element of* $\mathbf{T}_V(\mathbb{Q}_l)$*. We assume that the compact open subgroup* K *is of the form* $K = K^l \cdot K_l$*, where* K^l *is a compact open subgroup of* $\mathbf{G}(\mathbf{A}_{\mathrm{f}}^l)$ *and* K_l *is a compact open subgroup of* $\mathbf{G}(\mathbb{Q}_l)$*.*

Suppose that Z *satisfies the conditions*

(1) $Z \subset T_m Z$.
(2) *for every* k_1 *and* k_2 *in* K_l *the image of* $k_1 m k_2$ *in* $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$ *generates an unbounded subgroup of* $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$

Then Z *contains a special subvariety* V' *containing* V *properly.*

Proof.

Lemma 6.2. *Suppose theorem 6.1 is true for any* $(\mathbf{G}, X)$ *Shimura datum with* $\mathbf{G}$ *of adjoint type. Then theorem 6.1 is true.*

Proof. Let $\mathbf{G}$, X, K, V, Z, F, l and m as in the statement of theorem 6.1. Let $\lambda\colon \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ be the natural morphism. Let $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ be the adjoint Shimura datum attached to $(\mathbf{G}, X)$ and let $K^{\mathrm{ad}} = \prod_{p \text{ prime}} K_p^{\mathrm{ad}}$ be the compact open subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbf{A}_{\mathrm{f}})$ defined as follows :

(1) $K_{p_0}^{\mathrm{ad}} \subset \mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_{p_0})$ is the compact open subgroup image of K_{p_0} by λ.
(2) $K_l^{\mathrm{ad}} \subset \mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$ is the compact open subgroup image of K_l by λ.
(3) If $p \neq p_0, l$, K_p^{ad} is a maximal compact open subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_p)$ containing the image of K_p by λ.

The group K^{ad} is neat because K_{p_0}, and therefore $K_{p_0}^{\mathrm{ad}}$, is. As the reflex field $E(\mathbf{G}, X)$ contains the reflex field $E(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$, there is a finite morphism of Shimura varieties

$$f : \mathrm{Sh}_K(\mathbf{G}, X)_F \longrightarrow \mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_F$$

Let V^{ad} be the image $f_{\mathbb{C}}(V)$. As V is non-strongly special, V^{ad} is a non-strongly special subvariety of $S_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$. Thus $\mathbf{T}_{V^{\mathrm{ad}}} = \lambda(\mathbf{T}_V)$ is a non-trivial torus.

We define the F-irreducible subvariety Z_F^{ad} of $\mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_F$ as the image of Z_F in $\mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_F$ by this morphism. Of course $Z^{\mathrm{ad}} := Z_F \times_F \mathbb{C}$ is contained in $S_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$. Let m^{ad} be the image of m in $\mathbf{T}_V^{\mathrm{ad}}(\mathbb{Q}_l)$. The inclusion $Z \subset T_m Z$ implies that $Z^{\mathrm{ad}} \subset T_{m^{\mathrm{ad}}} Z^{\mathrm{ad}}$.

As $\mathbf{G}^{\mathrm{ad}}$ is of adjoint type, we can apply the theorem 6.1 to $\mathbf{G}^{\mathrm{ad}}$, X^{ad}, K^{ad}, V^{ad}, Z^{ad}, F, l and m^{ad}. So Z^{ad} contains a special subvariety V'^{ad} containing V^{ad} properly. As irreducible components of the preimage by a finite morphism of a special subvariety are special, Z contains a special subvariety V' containing V properly. □

From now on, we will assume the group $\mathbf{G}$ *to be of adjoint type.* Moreover for simplicity of notation we replace in this proof the field $E(\mathbf{G}, X)$ by the field F. Thus $\mathrm{Sh}(\mathbf{G}, X)$ denotes the canonical model of $\mathrm{Sh}(\mathbf{G}, X)_{\mathbb{C}}$ over F, $S_{K^l}(\mathbf{G}, X)$ is the connected component, of $\mathrm{Sh}_{K^l}(\mathbf{G}, X) = \mathrm{Sh}_{K^l}(\mathbf{G}, X)_F$ (image of $X^+ \times \{1\}$), etc. Moreover we will drop the label $(\mathbf{G}, X)$ when it is obvious what Shimura datum we are referring to

Lemma 6.3. *Let* $Z = Z_1 \cup \cdots \cup Z_n$ *be the decomposition of* Z *into geometrically irreducible components. Each irreducible component* Z_i*,* $1 \leq i \leq n$*, is Hodge-generic.*

Proof. As Z is Hodge-generic, at least one irreducible component, say Z_1, is Hodge generic. As Z_F is irreducible, any irreducible component Z_j, $1 \leq j \leq n$, is of the form Z_1^σ for some element $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. As the conjugate under any Galois element of a special subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is still special, one gets the result. □

We fix a $\mathbb{Z}$-structure on $\mathbf{G}$ and its subgroups by choosing a finitely generated free $\mathbb{Z}$-module W, a faithful representation $\xi \colon \mathbf{G} \hookrightarrow \mathbf{GL}(W_{\mathbb{Q}})$ and considering Zariski-closures in the $\mathbb{Z}$-group-scheme $\mathbf{GL}(W)$. We choose the representation ξ in such a way that K is contained in $\mathbf{GL}(\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} W)$ (i.e. K stabilizes $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} W$). This induces canonically a $\mathbb{Z}$-variation of Hodge structure on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$: c.f. [17, section 3.2].

Let z be a Hodge-generic point of the smooth locus Z_1^{sm} of Z_1. Let $\pi_1(Z_1^{\mathrm{sm}}, z)$ be the topological fundamental group of Z_1^{sm} at the point z. The representation $\xi \colon \mathbf{G} \longrightarrow \mathbf{GL}(W_{\mathbb{Q}})$ induces a polarizable variation of $\mathbb{Z}$-Hodge structure $\mathcal{F}$ on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, in particular on its irreducible component $S_K(\mathbf{G}, X)_{\mathbb{C}}$. We choose a point $\widetilde{z}$ of X lying above z. This choice canonically identifies the fiber at z of the locally constant sheaf underlying $\mathcal{F}$ with the $\mathbb{Z}$-module W. The action of $\pi_1(Z_1^{\mathrm{sm}}, z)$ on this fiber is described by the monodromy representation

$$\rho \colon \pi_1(Z_1^{\mathrm{sm}}, z) \longrightarrow \Gamma = \pi_1(S_K(\mathbf{G}, X)_{\mathbb{C}}, z) = \mathbf{G}(\mathbb{Q}) \cap K \xrightarrow{\xi} \mathbf{GL}(W) \ .$$

As Γ is Zariski-dense in $\mathbf{G}_{\mathbb{C}}$ the algebraic monodromy group is $\mathbf{G}_{\mathbb{C}}$. As Z is Hodge-generic the group $\rho(\pi_1(Z_1^{\mathrm{sm}}, z))$ is Zariski-dense in $\mathbf{G}_{\mathbb{C}}$ by [1, theor. 1.4].

Let l be a prime as in the statement. The proposition 4.3.1 implies that the l-adic closure of $\rho(\pi_1(Z_1^{\mathrm{sm}}, z))$ in $\mathbf{G}(\mathbb{Q}_l)$ is a compact open subgroup $K'_l \subset K_l$.

Write $K = K^l K_l$ with $K^l = \prod_{p \neq l} K_p$. Let $\pi_{K_l} \colon \mathrm{Sh}_{K^l} \longrightarrow \mathrm{Sh}_K$ be the Galois étale cover of group K_l as defined in section 4.1.1. Let $\widetilde{Z_1}$ be an irreducible component of the preimage of $Z_1^{\rm sm}$ in Sh_{K^l} and let $\widetilde{V}$ be an irreducible component of the preimage of V in $\widetilde{Z_1}$. As $\pi_{K_l} : \mathrm{Sh}_{K^l} \longrightarrow \mathrm{Sh}_K$ is étale the smooth locus $\widetilde{Z_1}^{\rm sm}$ of $\widetilde{Z_1}$ naturally identifies with an irreducible component $\widetilde{Z_1^{\rm sm}}$ of $\pi_{K_l}^{-1}(Z_1^{\rm sm})$.

The idea of the proof is to show that the inclusion $Z \subset T_m Z$ implies that $\widetilde{Z_1}$ is stabilized by a "big" group and then consider the orbit of $\widetilde{V}$ under the action of this group.

Lemma 6.4. *The variety $\widetilde{Z_1}$ is stabilized by the group K'_l. The set of irreducible components of $\pi_{K_l}^{-1}(Z_1)$ naturally identifies with the finite set K_l/K'_l.*

Proof. We replace Z_1 by $Z_1^{\rm sm}$. Let $\widetilde{z}$ be a geometric point of $\widetilde{Z_1^{\rm sm}}$ lying over z. Let $\varpi(Z_1, z)$ denote the algebraic fundamental group of Z_1^{sm} at z. The set of irreducible components of $\pi_{K_l}^{-1}(Z_1)$ naturally identifies with the quotient $K_l/\rho_{\rm alg}(\varpi(Z_1^{\rm sm}, z))$, where $\rho_{\rm alg}\colon \varpi(Z_1^{\rm sm}, z) \longrightarrow K_l \subset \mathbf{G}(\mathbb{Q}_l)$ denotes the (continuous) monodromy representation of the K_l-pro-étale cover $\pi_{K_l} : \pi_{K_l}^{-1}(Z_1^{\rm sm}) \longrightarrow Z_1^{\rm sm}$. The group $\varpi(Z_1, z)$ naturally identifies with the profinite completion of $\pi_1(Z_1^{\rm sm}, z)$. One has the commutative diagram

$$\begin{array}{ccc} \pi_1(Z_1^{\rm sm}, z) & \xrightarrow{\ \rho\ } & \mathbf{G}(\mathbb{Q}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ \varpi_1(Z_1^{\rm sm}, z) & \xrightarrow[\rho_{\rm alg}]{} & \mathbf{G}(\mathbb{Q}_l) \end{array} \tag{6.1}$$

where $i : \pi_1(Z_1^{\rm sm}, z) \longrightarrow \varpi_1(Z_1^{\rm sm}, z)$ and $j : \mathbf{G}(\mathbb{Q}) \longrightarrow \mathbf{G}(\mathbb{Q}_l)$ denote the natural homomorphisms. As $i(\pi_1(Z_1^{\rm sm}, z))$ is dense in $\varpi_1(Z_1^{\rm sm}, z)$ and $\rho_{\rm alg}$ is continuous one deduces that $\rho_{\rm alg}(\varpi_1(Z_1^{\rm sm}, z)) = K'_l$. Thus the set of irreducible components of $\pi_{K_l}^{-1}(Z_1^{\rm sm})$ identifies with K_l/K'_l and $\widetilde{Z_1^{\rm sm}}$ is K'_l-stable. $\square$

Lemma 6.5. *There exist elements k_1, k_2 of K_l and an integer $n \geq 1$ such that*

$$\widetilde{Z_1} = \widetilde{Z_1} \cdot (k_1 m k_2)^n$$

Proof. The inclusion $Z \subset T_m Z$ implies that for every geometrically irreducible component Z_i, $1 \leq i \leq n$, of Z, there is a geometric irreducible component $\widetilde{Z}_i$ of $\pi_{K_l}^{-1}(Z_i)$ which is also a geometric irreducible component of the preimage of $T_m Z$ by $\pi_{K_l} : \mathrm{Sh}_{K^l} \longrightarrow \mathrm{Sh}_K$. As the geometric irreducible components of $\pi_{K_l}^{-1}(T_m Z)$ are of the form $\widetilde{Z}_i \cdot (k_1 m k_2)$, $k_1, k_2 \in K_l$, there exists an index i, $1 \leq i \leq n$, and two elements k_1, k_2 in K_l such that

$$\widetilde{Z_1} = \widetilde{Z}_i \cdot k_1 m k_2 \ .$$

As Z is F-irreducible there exists σ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that $Z_i = \sigma(Z_1)$. As the morphism $\pi_{K_l} : \mathrm{Sh}_{K_l} \longrightarrow \mathrm{Sh}_K$ is defined over F, the subvariety $\sigma(\widetilde{Z_1})$ of Sh_{K^l} satisfies $\pi_{K_l}(\sigma(\widetilde{Z_1})) =$

Z_i. Thus the subvarieties $\sigma(\widetilde{Z_1})$ and $\widetilde{Z}_i$ of Sh_{K^l} are both irreducible components of $\pi_{K_l}^{-1}(Z_i)$. Thus there exists an element k of K_l such that

$$\widetilde{Z}_i = \sigma(\widetilde{Z_1}) \cdot k \quad .$$

By replacing k_1 with kk_1, we obtain k_1, k_2 in K_l such that

$$\widetilde{Z_1} = \sigma(\widetilde{Z_1}) \cdot (k_1 m k_2) \quad . \tag{6.2}$$

As the $\mathbf{G}(\mathbf{A}_\mathrm{f})$-action is defined over F, the previous equation implies :

$$\forall\, i \in \mathbb{N}, \quad \widetilde{Z_1} = \sigma^i(\widetilde{Z_1}) \cdot (k_1 m k_2)^i \quad . \tag{6.3}$$

As the set of connected components of Z is finite, there exists a positive integer m such that $\sigma^m(Z_1) = Z_1$. Thus the Abelian group $(\sigma^m)^{\mathbf{Z}}$ acts on the set of irreducible components of $\pi_{K_l}^{-1}(Z_1)$. By the previous lemma this set is finite. So there exists a positive integer n (multiple of m) such that $\sigma^n(\widetilde{Z_1}) = \widetilde{Z_1}$. The equality (6.3) applied to $i = n$ gives the proposition. □

From the lemmas 6.4 and 6.5 one obtains the

Corollary 6.6. *Let U_l be the group $\langle K_l', (k_1 m k_2)^n \rangle$. The variety $\widetilde{Z_1}$ is stabilized by U_l.*

To conclude the proof of theorem 6.1 we first study the group U_l in more detail. Let $\mathbf{G} = \prod_{i=1}^s \mathbf{G}_i$ be the decomposition of $\mathbf{G}$ into $\mathbb{Q}$-simple factors. Let $p_i : \mathbf{G} \longrightarrow \mathbf{G}_i$ denote the natural projections. By the hypothesis on m the group U_l is unbounded in $\mathbf{G}(\mathbb{Q}_l)$. After possibly renumbering the factors, we can assume that $p_1(U_l)$ is unbounded in $\mathbf{G}_1(\mathbb{Q}_l)$. In particular the torus $p_1(\mathbf{T})$ is non-trivial. Indeed if it was trivial, then the group $p_1(U_l)$ would be contained in $p_1(K_l)$ which is compact and therefore bounded. Choose a simple $\mathbb{Q}_l$-factor $\mathbf{H}_1$ of $\mathbf{G}_{1,\mathbb{Q}_l}$ such that the image of U_l under the projection $h_1 : \mathbf{G}_{\mathbb{Q}_l} \longrightarrow \mathbf{H}_1$ is unbounded in $\mathbf{H}_1(\mathbb{Q}_l)$. Let $\tau : \widetilde{\mathbf{G}_{\mathbb{Q}_l}} \longrightarrow \mathbf{G}_{\mathbb{Q}_l}$ (resp. $\tau_1 : \widetilde{\mathbf{H}_1} \longrightarrow \mathbf{H}_1$) be the universal cover of $\mathbf{G}_{\mathbb{Q}_l}$ (resp. $\mathbf{H}_1$).

Sublemma 6.7. *The group $U_l \cap \mathbf{H}_1(\mathbb{Q}_l)$ contains the group $\tau_1(\widetilde{\mathbf{H}_1}(\mathbb{Q}_l))$ with finite index.*

Proof. Let $\widetilde{h_1} : \widetilde{\mathbf{G}_{\mathbb{Q}_l}} \longrightarrow \widetilde{\mathbf{H}_1}$ be the canonical projection. Let $\widetilde{U_l} = \tau^{-1}(U_l) \subset \widetilde{\mathbf{G}_{\mathbb{Q}_l}}(\mathbb{Q}_l)$. As U_l is an open non-compact subgroup of $\mathbf{G}_{\mathbb{Q}_l}(\mathbb{Q}_l)$, the group $\widetilde{U_l}$ is open non-compact in $\widetilde{\mathbf{G}_{\mathbb{Q}_l}}(\mathbb{Q}_l)$. As $h_1(U_l)$ is non-compact in $\mathbf{H}_1(\mathbb{Q}_l)$ the projection $\widetilde{h_1}(\widetilde{U_l})$ is open non-compact in the group $\widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$. As the group $\widetilde{\mathbf{H}_1}$ is simple and simply connected, we obtain by the theorem (T) of [26] the equality $\widetilde{h_1}(\widetilde{U_l}) = \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$. This implies that the group $\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$ is normal in $\widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$: given $h \in \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$, let $g \in \widetilde{U_l}$ satisfying $\widetilde{h_1}(g) = h$. Then

$$(\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l))^h = (\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l))^g = (\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)) \quad .$$

As the group $\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$ is an open normal subgroup of $\widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$ and the group $\widetilde{\mathbf{H}_1}$ is simply-connected, we obtain the equality $\widetilde{U_l} \cap \widetilde{\mathbf{H}_1}(\mathbb{Q}_l) = \widetilde{\mathbf{H}_1}(\mathbb{Q}_l)$. As τ_1 is an isogeny of algebraic groups, we get that $U_l \cap \mathbf{H}_1(\mathbb{Q}_l)$ contains $\tau_1(\widetilde{\mathbf{H}_1}(\mathbb{Q}_l))$ with finite index. □

Definition 6.8. *We replace U_l by its subgroup $\tau_1(\widetilde{\mathbf{H}_1}(\mathbb{Q}_l))$. We denote by V' the Zariski-closure $\overline{\pi_{K_l}(\widetilde{V} \cdot U_l)}^{\mathrm{Zar}}$.*

Lemma 6.9. *The subvariety V' of Z is special.*

Proof. Without loss of generality we can assume that $K = K_1 \times \cdots \times K_s$, where K_i, $1 \leq i \leq s$, is a compact open subgroup of $\mathbf{G}_i(\mathbf{A}_{\mathrm{f}})$.

Let $(\mathbf{G}_{>1}, X_{>1})$ be the product of Shimura data $(\prod_{i=2}^{s} \mathbf{G}_i, \prod_{i=2}^{s} X_i)$, and $K_{>1}$ be the compact open subgroup $\prod_{i=2}^{s} K_i$ of $\mathbf{G}_{>1}(\mathbf{A}_{\mathrm{f}})$. The connected component $S_K(\mathbf{G}, X)_{\mathbb{C}}$ of the Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ decomposes as a product

$$S_K(\mathbf{G}, X)_{\mathbb{C}} = S_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}} \times S_{K_{>1}}(\mathbf{G}_{>1}, X_{>1})_{\mathbb{C}}$$

with $S_{K_{>1}}(\mathbf{G}_{>1}, X_{>1})_{\mathbb{C}} = \prod_{i=2}^{s} S_{K_i}(\mathbf{G}_i, X_i)_{\mathbb{C}}$. By [17, th.6.1] one obtains

$$V' = S_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}} \times V_{>1} \ ,$$

where $V_{>1}$ denotes the special subvariety of $S_{K_{>1}}(\mathbf{G}_{>1}, X_{>1})_{\mathbb{C}}$ projection of V. In particular V' is special. □

Lemma 6.10. *The subvariety V' of Z contains V properly.*

Proof. As the Mumford-Tate group $\mathbf{H}$ of V centralizes the torus $\mathbf{T}$, the projection $\mathbf{H}_1$ of $\mathbf{H}$ on $\mathbf{G}_1$ centralizes the non-trivial torus $\mathbf{T}_1$ projection of $\mathbf{T}$ on $\mathbf{G}_1$. In particular $\mathbf{H}_1$ is a proper algebraic subgroup of $\mathbf{G}_1$. But as

$$V' = S_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}} \times V_{>1} \ ,$$

the group $\mathbf{G}_1$ is a direct factor of the Mumford-Tate group of V'. □

□

7. Existence of suitable Hecke correspondences.

In this section we prove under some conditions on the compact open subgroup K_l the existence of Hecke correspondences of small degree candidates for applying theorem 6.1 assuming the Galois orbits of V is sufficiently big.

Definition 7.0.5. *Let $\mathbf{G}$ be a reductive $\mathbb{Q}$-group and $\mathbf{T} \subset \mathbf{G}$ a split torus. Let l be a prime number. A compact open subgroup U_l of $\mathbf{G}(\mathbb{Q}_l)$ is said to be* in good position *with respect to $\mathbf{T}$ if $U_l \cap \mathbf{T}(\mathbb{Q}_l)$ is the maximal compact open subgroup of $\mathbf{T}(\mathbb{Q}_l)$.*

Our main result in this section is the following :

Theorem 7.1. *Let* $(\mathbf{G}, X)$ *be a Shimura datum,* $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ *a neat open compact subgroup of* $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ *and* F *a number field containing the field of definition of* $S_K(\mathbf{G}, X)_{\mathbb{C}}$*. There is a positive integer* k *such that the following holds.*

Let V *be a non-strongly special subvariety contained in a Hodge-generic* F*-irreducible* F*-subvariety* Z *of* $S_K(\mathbf{G}, X)_{\mathbb{C}}$*.*

Let l *be a prime number splitting* $\mathbf{T}_V$ *and* m *an element of* $\mathbf{T}_V(\mathbb{Q}_l)$*. We assume that the compact open subgroup* K *is of the form* $K = K^l \cdot K_l$*, where* K^l *is a compact open subgroup of* $\mathbf{G}(\mathbf{A}_{\mathrm{f}}^l)$ *and* K_l *is a compact open subgroup of* $\mathbf{G}(\mathbb{Q}_l)$ *contained in an Iwahori subgroup* I_l *of* $\mathbf{G}(\mathbb{Q}_l)$ *(c.f. next paragraph) in good position with respect to* $\mathbf{T}_V$*.*

Then there exists an element $m \in \mathbf{T}_V(\mathbb{Q}_l)$ *satisfying the following conditions :*

(1) $\mathrm{Gal}(\overline{F}/F) \cdot V \subset Z \cap T_m Z$.
(2) *For every* $k_1, k_2 \in K_l$ *the image of* $k_1 m k_2$ *in* $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$ *generates an unbounded subgroup of* $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$*.*
(3) $[K_l : K_l \cap m K_l m^{-1}] < l^k$.

Remark 7.0.6. As notices in the introduction, the restriction $K_l \subset I_l$ is a necessary condition. One easily constructs counter-example to the conclusion of theorem 7.1 if $K_l \subset \mathbf{G}(\mathbb{Q}_l)$ is special maximal compact open.

7.1. **Some properties of Iwahori subgroups.** We refer to [5], [6] and [18] for basic facts about buildings, Iwahori subgroups and Iwahori-Hecke algebras.

We first recall the definition on an Iwahori subgroup. Let l be a prime number. Let $\mathbf{G}$ be a reductive algebraic isotropic $\mathbb{Q}_l$-group and $\mathbf{A} \subset \mathbf{G}$ a maximal split torus of $\mathbf{G}$. We denote by $\mathbf{M} \subset \mathbf{G}$ the centralizer of $\mathbf{A}$ in $\mathbf{G}$. We choose $\mathbf{P} = \mathbf{M} \cdot \mathbf{N}$ a minimal parabolic subgroup of $\mathbf{G}$, where $\mathbf{N}$ denotes the unipotent radical of $\mathbf{P}$. Let $\mathcal{X}$ be the (extended) Bruhat-Tits building of $\mathbf{G}$, $\mathcal{A} \subset \mathcal{X}$ the apartment of $\mathcal{X}$ associated to $\mathbf{A}$. Let $K_l^{\mathrm{m}} \subset \mathbf{G}(\mathbb{Q}_l)$ be a special maximal subgroup of $\mathbf{G}(\mathbb{Q}_l)$ such that $K_{l,\mathbf{A}}^{\mathrm{m}} = K_l^{\mathrm{m}} \cap \mathbf{A}(\mathbb{Q}_l)$ is the maximal compact open subgroup of $\mathbf{A}(\mathbb{Q}_l)$. We denote by $x_0 \in \mathcal{A}$ the unique K_l-fixed vertex in $\mathcal{X}$, by $\mathcal{C} \subset \mathcal{A}$ the unique Weyl chamber with apex at x_0 whose stabilizer at infinity is $\mathbf{P}(\mathbb{Q}_l)$, by C the unique chamber (or alcove) of $\mathcal{C}$ having x_0 for one of its vertices and by $I_l \subset K_l$ the Iwahori subgroup fixing C pointwise.

Remark 7.1.1. Strictly speaking (i.e. with the notations of Bruhat-Tits [5]) the group I_l as defined above is an Iwahori subgroup only in the case where the group $\mathbf{G}$ is simply-connected. Our terminology is a well-established abuse of notations.

Definition 7.1.2. *We denote by* $\mathrm{ord}_{\mathbf{M}} : \mathbf{M}(\mathbb{Q}_l) \longrightarrow X_*(\mathbf{M})$ *the homomorphism characterized by*

$$< \mathrm{ord}_{\mathbf{M}}(m), \alpha >= \mathrm{ord}_{\mathbb{Q}_l}(\alpha(m)) \quad ,$$

where $\mathrm{ord}_{\mathbb{Q}_l}$ *denotes the normalized (additive) valuation in* $\mathbb{Q}_l$ *and* $X_*(\mathbf{M})$ *denotes the group of cocharacters of* $\mathbf{M}$*. We denote by* $\Lambda \subset X_*(\mathbf{M})$ *the free* $\mathbf{Z}$*-module* $\mathrm{ord}_{\mathbf{M}}(\mathbf{M}(\mathbb{Q}_l))$*.*

The group $\mathbf{M}(\mathbb{Q}_l)$ (in particular the group $\mathbf{A}(\mathbb{Q}_l)$) acts on $\mathcal{A}$ via Λ-translations.

Definition 7.1.3. *Let* $\Lambda^+ \subset \Lambda$ *be the positive cone associated to the Weyl chamber* $\mathcal{C}$*.*

Elements of Λ^+ acting on $\mathcal{A}$ map $\mathcal{C}$ to $\mathcal{C}$.

Proposition 7.1.4. *Let* m *be an element of* $\mathbf{A}(\mathbb{Q}_l)$ *with non-trivial image* $\mathrm{ord}_M(m) \in \Lambda^+$*. Then for any elements* $i_1, i_2 \in I_l$*, the element* $i_1 m i_2 \in \mathbf{G}(\mathbb{Q}_l)$ *is not contained in a compact subgroup of* $\mathbf{G}(\mathbb{Q}_l)$*.*

Proof. Let W_0 be the finite Weyl group of $\mathbf{G}$, let W be the modified affine Weyl group associated to $\mathcal{A}$ and Ω the finite subgroup of W taking the chamber C to itself. Let $\Delta = \{\alpha_1, \cdots, \alpha_m\}$ be the set of affine roots on $\mathcal{A}$ which are positive on C and whose null set H_α is a wall of C. For $\alpha \in \Delta$ we denote by S_α the reflexion of $\mathcal{A}$ along the wall H_α. The group W is generated by Ω and the S_α's, $\alpha \in \Delta$. It identifies with the semi-direct product $W_0 \ltimes \Lambda$ (c.f. [6, p.140]).

Recall the Bruhat-Tits decomposition :

$$\mathbf{G}(\mathbb{Q}_l) = I_l \cdot W \cdot I_l \quad . \tag{7.1}$$

Let $r : \mathbf{G}(\mathbb{Q}_l) \longrightarrow W$ be the map sending $g \in \mathbf{G}(\mathbb{Q}_l)$ to the unique $r(g) \in W$ such that $r(g) \in I_l g I_l$. Geometrically speaking the map r essentially coincides with the retraction $\rho_{\mathcal{A},C}$ of the Bruhat-Tits building $\mathcal{X}$ with center the chamber C onto the apartment $\mathcal{A}$ ([5, I, theor.2.3.4]).

Let $\mathcal{H}(\mathbf{G}, I_l)$ be the Hecke algebra (for the convolution product) of bi-I_l-invariant compactly supported continuous complex functions on $\mathbf{G}(\mathbb{Q}_l)$. By the equation (7.1) this is an associative algebra with a vector space basis $T_w = 1_{I_l w I_l}$, $w \in W$, where $1_{I_l w I_l}$ denotes the characteristic function of the double coset $I_l w I_l$. A presentation of the algebra $\mathcal{H}(\mathbf{G}, I_l)$ with generators T_ω, $\omega \in \Omega$, and T_α, $\alpha \in \Delta$, is given in [6, theorem 3.6 p.142] (or [4, p.242-243]). Given $w \in W$ let $l(w) \in \mathbb{N}$ be the number of hyperplanes H_α separating the two chambers C and wC. One obtains in particular (c.f. [6, theorem 3.6 (b)] or [3, section 3.2, 1) and 6)]) :

$$\forall w, w' \in W, \quad T_w \cdot T_{w'} = T_{ww'} \ \text{ if l(ww')= l(w) +l(w')} \quad . \tag{7.2}$$

Let $\delta \in X^*(\mathbf{M})$ be the determinant of the adjoint action of $\mathbf{M}$ on the Lie algebra of $\mathbf{N}$. For $\lambda \in \Lambda^+ \subset W$ one easily shows the equality :

$$l(\lambda) = <\delta, \lambda> \quad . \tag{7.3}$$

In particular any two elements λ, μ in $\Lambda^+ \subset W$ satisfy $l(\lambda \cdot \mu) = l(\lambda) + l(\mu)$. Thus the equation (7.2) implies the relation :

$$T_\lambda T_\mu = T_{\lambda+\mu} \quad . \tag{7.4}$$

Remark 7.1.5. Equality (7.4) is stated in [18, (1.15)]) for the Iwahori-Hecke algebra of a split adjoint group, but generalizes easily.

Let m, i_1, i_2 as in the statement of the proposition and denote by g the element $i_1 m i_2 \in \mathbf{G}(\mathbb{Q}_l)$. By equation (7.4) one has the equality :

$$r(g^n) = n \cdot r(g) = n \cdot \mathrm{ord}_{\mathbf{M}}(m) \quad .$$

This implies that the chamber $\rho_{\mathcal{A},C}(g^n C) = n \cdot \mathrm{ord}_{\mathbf{M}}(m) \cdot C$ leaves any compact of $\mathcal{A}$ as n tends to infinity. As a corollary the chamber $g^n C$ of $\mathcal{X}$ also leaves any compact of $\mathcal{X}$ when n tends to infinity. This proves that the group $g^{\mathbb{Z}}$ is not contained in a compact subgroup of $\mathbf{G}(\mathbb{Q}_l)$. □

7.2. **Some uniformity results.** In this section we prove some uniformity results concerning Shimura data and reciprocity morphisms. The fist is this simple observation :

Lemma 7.2.1. *Let $(\mathbf{G}, X)$ be a Shimura datum. There is constant R such that for any sub-Shimura datum $(\mathbf{H}, X_{\mathbf{H}})$, the degree of the reflex field $E(\mathbf{H}, X_{\mathbf{H}})$ over $E(\mathbf{G}, X)$ is bounded by R.*

Proof. This is a direct consequence of the definition of the reflex field. □

Proposition 7.2.2. *Let $(\mathbf{G}, X)$ be a Shimura datum, $K \subset \mathbf{G}(\mathbf{A}_{\mathrm{f}})$ a neat open compact subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$.*

There is a positive integer h such that the following holds.

Let V be a non-strongly special subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ and l be a prime splitting $\mathbf{T}_V$. For any m in $\mathbf{T}_V(\mathbb{Q}_l)$, m^h satisfies the condition that for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F)$

$$\sigma(V) \subset T_{m^h}(V).$$

Proof. Let V be as above. For simplicity of notations we write $\mathbf{T}$ for $\mathbf{T}_V$, $\mathbf{H}$ for $\mathbf{H}_V$ and $\mathbf{C}$ for $\mathbf{C}_V$. By definition a constant is called uniform if it is independent of V.

To show the existence of an element h as in the statement, we will prove several lemmas. The first one is the following.

Lemma 7.2.3. *There is a uniform integer n_1 such that for any $m \in \mathbf{T}(\mathbb{Q}_l)$, the power m^{n_1} is in the preimage of $r_{(\mathbf{C},\{x\})}((\mathbb{Q}_l \otimes F)^*)$ in $\mathbf{T}(\mathbb{Q}_l)$ by the natural map $\mathbf{T}(\mathbb{Q}_l) \longrightarrow \mathbf{C}(\mathbb{Q}_l)$.*

Proof. Let $\mathbf{L}$ be the torus $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G_m}_F$. The element x gives a cocharacter $\mu_{\mathbf{C}}\colon \mathbf{G_m}_{\mathbb{C}} \longrightarrow \mathbf{C}_{\mathbb{C}}$ defined by $\mu_{\mathbf{C}}(z) = x_{\mathbb{C}}(z,1)$. The morphism $r_{(\mathbf{C},\{x\})}\colon \mathbf{L} \longrightarrow \mathbf{C}$ corresponds to the morphism on cocharacter groups $X_*(\mathbf{L}) \longrightarrow X_*(\mathbf{C})$ which sends the cocharacter $\mu_\sigma \in X_*(\mathbf{L})$ (induced by $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$) to $\sigma(\mu_{\mathbf{C}})$. The lemma 4.4 of [33] says that there is a basis (χ_i) of characters of $\mathbf{C}$ such that the $< \chi_i, \sigma(\mu_{\mathbf{C}}) >$ are uniformly bounded. It follows that the index of $r_{(\mathbf{C},\{x\})}((\mathbb{Q}_l \otimes F)^*)$ in $\mathbf{C}(\mathbb{Q}_l)$ is finite (this is the consequence of the fact that $r_{(\mathbf{C},\{x\})}$ is surjective) and uniformly bounded. Let n_1 be a uniform bound on this index. It follows that for any $m \in \mathbf{T}(\mathbb{Q}_l)$, the power m^{n_1} is in the preimage of $r_{(\mathbf{C},\{x\})}((\mathbb{Q}_l \otimes F)^*)$ in $\mathbf{T}(\mathbb{Q}_l)$. □

Recall that we have an exact sequence

$$\mathrm{Gal}(\overline{\mathbb{Q}}/F) \overset{r_{(\mathbf{H},X_{\mathbf{H}})}}{\longrightarrow} \pi(\mathbf{H}) \overset{p}{\longrightarrow} \pi(\mathbf{C}) \ .$$

We know that $p(m^{n_1})$ is in $p \circ r_{(\mathbf{H},X_{\mathbf{H}})}(\mathrm{Gal}(\overline{\mathbb{Q}}/F)) = r_{(\mathbf{C},\{x\})}(\mathrm{Gal}(\overline{\mathbb{Q}}/F))$. Hence, there is an element σ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that

$$p(m^{n_1}) = (p \circ r_{(\mathbf{H},X_{\mathbf{H}})})(\sigma) \ .$$

It follows that there exists an element y in the kernel of p such that

$$\overline{m^{n_1}} = y r_{(\mathbf{H},X_{\mathbf{H}})}(\sigma) \ ,$$

where $\overline{m^{n_1}}$ denotes the image of m^{n_1} in $\pi(\mathbf{H})$.

Our next aim is to show that a uniform power of m is actually in $r_{(\mathbf{H},X_{\mathbf{H}})}(\mathrm{Gal}(\overline{\mathbb{Q}}/F))$. This follows directly from the following lemma.

Lemma 7.2.4. *There exists a uniform integer n such that any element of the kernel of p is killed by n.*

Proof. Let y be an element of $\mathbf{H}(\mathbf{A}_{\mathrm{f}})$ whose image in $\pi(\mathbf{H})$ belongs to the kernel of p. Using that $\mathbf{H} = \mathbf{T}\mathbf{H}^{\mathrm{der}}$ and that $\mathbf{T} \cap \mathbf{H}^{\mathrm{der}}$ is finite of uniformly bounded order we see that there is a uniform integer n_2, an element t in $\mathbf{T}(\mathbf{A}_{\mathrm{f}})$ and α in $\mathbf{H}^{\mathrm{der}}(\mathbf{A}_{\mathrm{f}})$ such that

$$y^{n_2} = t \cdot \alpha \ .$$

As $\mathbf{H}^{\mathrm{der}}(\mathbf{A}_{\mathrm{f}})/\rho\tilde{\mathbf{H}}^{\mathrm{der}}(\mathbf{A}_{\mathrm{f}})$ is killed by a uniform integer n_3, we have in $\pi(\mathbf{H})$,

$$\overline{y^{n_2 n_3}} = \overline{t^{n_3}} \in \overline{\mathbf{T}(\mathbf{A}_{\mathrm{f}})} \ ,$$

where the bar denotes "image in $\pi(\mathbf{H})$". As y (and hence $y^{n_2n_3}$) is in the kernel of p, the image of t^{n_3} in $\mathbf{C}(\mathbf{A}_{\mathrm{f}})$ is in $\mathbf{C}(\mathbb{Q})$. Using the exact sequence

$$W \longrightarrow \mathbf{T} \stackrel{\nu}{\longrightarrow} \mathbf{C}$$

where W is finite of uniformly bounded order, say n_4, we see that n_4-th power of any element of $\mathbf{C}(\mathbb{Q})$ is in the image of $\mathbf{T}(\mathbb{Q})$ hence there exists a q in $\mathbf{T}(\mathbb{Q})$ such that

$$\nu(t^{n_3n_4}) = \nu(q) \ .$$

It follows that

$$t^{n_3n_4} = qw \ ,$$

where w is in $W(\mathbf{A}_{\mathrm{f}})$ (the kernel of ν on adelic points). As $W(\mathbb{Q})$ is killed by n_4, we see that $t^{n_3n_4^2} = q^{n_4} \in \mathbf{T}(\mathbb{Q})$.

The image of $t^{n_3n_4^2}$ in $\pi(\mathbf{H})$ equals the image of $y^{n_2n_3n_4^2}$ hence we can take n to be $n_2n_3n_4^2$. □

We have proved the following:

Lemma 7.2.5. *There is a uniform integer h such that the image of m^h in $\pi(\mathbf{H})$ is in $r_{(\mathbf{H},X_{\mathbf{H}})}(\mathrm{Gal}(\overline{\mathbb{Q}}/F))$.*

Proof. Take $h = n_1n_2n_3n_4^2$. □

It remains to see that some Galois conjugate (and therefore the whole of the Galois orbit) of V is in $T_{m^h}V$. The variety V is the image of $(X_{\mathbf{H}}^+, 1)$ in $\mathrm{Sh}_K(\mathbf{G}, X)$. Let σ be the element of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ as above. By definition of the Galois action on the set of connected components of a Shimura variety, we get

$$\sigma(V) = (X_{\mathbf{H}}^+, m^h) \subset T_{m^h}V$$

□

7.3. **Proof of theorem 7.1.** As V is non-strongly special, the torus $\mathbf{T}_V^{\mathrm{ad}} := \lambda(\mathbf{T}_V)$ is a non-trivial torus in $\mathbf{G}^{\mathrm{ad}}$, where $\lambda : \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ denotes the natural morphism. Let $\mathbf{A}^{\mathrm{ad}}$ be a maximal split torus of $\mathbf{G}^{\mathrm{ad}}_{\mathbb{Q}_l}$ containing $\mathbf{T}^{\mathrm{ad}}_{V,\mathbb{Q}_l}$. Let C be the unique chamber of the Bruhat-Tits building $\mathcal{X}$ of $\mathbf{G}^{\mathrm{ad}}_{\mathbb{Q}_l}$ fixed by I_l and x_0 a special vertex in the closure of C such that the intersection of its stabilizer with $\mathbf{T}_V^{\mathrm{ad}}(\mathbb{Q}_l)$ is maximal compact in $\mathbf{T}_V^{\mathrm{ad}}(\mathbb{Q}_l)$. Choose a minimal parabolic subgroup $\mathbf{P}^{\mathrm{ad}}$ of $\mathbf{G}^{\mathrm{ad}}_{\mathbb{Q}_l}$ whose Levi subgroup is the centralizer $\mathbf{M}^{\mathrm{ad}}$ of $\mathbf{A}^{\mathrm{ad}}$.

We use the notations of section 7.1 applied to $\mathbf{G}^{\mathrm{ad}}_{\mathbb{Q}_l}$. By lemma 2.4 of [33] and the proposition 7.4.3 of [17] there exists a uniform constant k_1 and an element $m \in \mathbf{T}_V(\mathbb{Q}_l)$ such that $\lambda(m)$ has a non-trivial image in $\Lambda^+ \subset X_*(\mathbf{M}^{\mathrm{ad}})$ and $[K_l : K_l \cap mK_lm^{-1}] < l^{k_1}$.

By proposition 7.2.2, there is a uniform constant h such that for some $\sigma \in \mathrm{Gal}(\overline{F}/F)$, one has $\sigma(V) \subset T_{m^h}V$.

The uniform constant $k = k_1 h$ and the element m^h satisfies the conditions of the theorem :

From $\sigma(V) \subset T_{m^h}V \subset T_{m^h}Z$ and as $T_{m^h}Z$ is defined over F, we deduce $V \subset T_{m^h}Z$. As $V \subset Z$ we obtain condition (1).

As $\lambda(m)$ has a non-trivial image in $\Lambda^+ \subset X_*(\mathbf{M}^{\mathrm{ad}})$, $\lambda(m^h)$ too. By proposition 7.1.4, for any k_1, k_2 in K_l, the image of $k_1 \cdot m \cdot k_2$ in $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$ generates an unbounded subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_l)$: this is condition (2).

As $\deg T_m = [K_l : K_l \cap mK_l m^{-1}] < l^{k_1}$ and $T_{m^h} \subset (T_m)^h$ as algebraic correspondences, $[K_l : K_l \cap mK_l m^{-1}] = \deg T_{m^h} \leq (\deg T_m)^h \leq l^k$: this is condition (3).

This finishes the proof of theorem 7.1.

8. Condition on the prime l

In this section, we use theorem 2.2.2, theorem 6.1, theorem 7.1 to show (under one of the assumption of theorem 3.1.1) that the existence of a prime number l satisfying certain conditions forces a subvariety Z of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ containing a non-strongly special subvariety V to contain a special subvariety V' containing V properly.

8.1. **Passing to an Iwahori subgroup.** In the process of creating V' we will encounter one minor technical difficulty : we will have to lift the situation to an Iwahori level in order to apply theorem 7.1. The following lemma introduces an absolute constant f which controls this phenomenon.

Lemma 8.1.1. *Let $\mathbf{G}$ be a reductive $\mathbb{Q}$-group.*

a) For any prime l, any split torus $\mathbf{T} \subset \mathbf{G}$ and any special maximal compact subgroup $K_l \subset \mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}$, there exists an Iwahori subgroup I_l of K_l in good position with respect to $\mathbf{T}$.

b) There exists an integer f such that for any prime l and any special maximal compact subgroup K_l of $\mathbf{G}(\mathbb{Q}_l)$, any Iwahori subgroup $I_l \subset K_l$ is of index $|K_l/I_l|$ smaller than l^f.

Proof. To prove *a*) let l, $\mathbf{T}$ and K_l as in the statement. Choose $\mathbf{A}$ a maximal split torus of $\mathbf{G}_{\mathbb{Q}_l}$ containing $\mathbf{T}_{\mathbb{Q}_l}$, denote by $\mathbf{M}$ the centralizer of $\mathbf{A}$ in $\mathbf{G}_{\mathbb{Q}_l}$ and choose any minimal parabolic $\mathbf{P}$ with Levi $\mathbf{M}$. By construction the Iwahori subgroup I_l of K_l defined by $\mathbf{P}$ (c.f. section 7.1) satisfies that $I_l \cap \mathbf{A}(\mathbb{Q}_l)$ is the maximal compact open subgroup of $\mathbf{A}(\mathbb{Q}_l)$. In particular $I_l \cap \mathbf{T}(\mathbb{Q}_l)$ is the maximal compact open subgroup of $\mathbf{T}(\mathbb{Q}_l)$.

To prove *b*) notice that the set K_l/I_l naturally identifies with the $\mathbb{F}_l$-points of some flag variety over $\mathbb{F}_l$. The dimension of these flag varieties is bounded independently of l. This implies the result. □

8.2. **The criterion.** We can now state the main result of this section :

Theorem 8.2.1. *Assume the GRH.*

Let $(\mathbf{G}, X)$ *be a Shimura datum,* $K = \prod_{p \text{ prime}} K_p$ *a neat compact open subgroup of* $\mathbf{G}(\mathbf{A}_\mathrm{f})$*, and* F *a number field containing the reflex field* $E(\mathbf{G}, X)$*. Let* N *be a positive integer,* k *the constant defined in theorem 7.1, and* f *the constant defined in lemma 8.1.1.*

Let $V \subset S_K(\mathbf{G}, X)_\mathbb{C}$ *be a non-strongly special subvariety. Let* l *be a prime splitting* $\mathbf{T}_V$ *such that* K_l *is contained in a special maximal compact subgroup* $K_l^{\max}$ *of* $\mathbf{G}(\mathbb{Q}_l)$ *in good position with respect to* $\mathbf{T}_V$*.*

Let Z *be a Hodge-generic* F*-irreducible* F*-subvariety* $S_K(\mathbf{G}, X)_\mathbb{C}$ *containing* V *and satisfying*

$$l^{(k+2f)\cdot 2^{a(r+1)}} \cdot (\deg_{L_K} Z)^{2^{a(r)}} < C(N)\alpha_V \beta_V^N \quad , \tag{8.1}$$

where r *denotes* $\dim Z - \dim V$ *and* $a : \mathbb{N} \longrightarrow \mathbb{N}$ *is the function defined by* $a(n) = \frac{n(n+1)}{2}$*.*

Then Z *contains a special subvariety* V' *that contains* V *properly.*

Moreover if one considers only the subvarieties V *such that the associated tori* $\mathbf{T}_V$ *lie in one* $\mathbf{GL}_n(\mathbb{Q})$*-conjugacy class, then the assumption of the GRH can be dropped.*

8.3. **An auxiliary proposition.** In addition to theorem 2.2.2, theorem 6.1, and theorem 7.1, the main ingredient for the proof of theorem 8.2.1 is the following :

Proposition 8.3.1. *Assume the GRH.*

Let $(\mathbf{G}, X)$ *be a Shimura datum,* $K = \prod_{p \text{ prime}} K_p$ *a neat compact open subgroup of* $\mathbf{G}(\mathbf{A}_\mathrm{f})$*, and* F *a number field containing the reflex field* $E(\mathbf{G}, X)$*. Let* N *be a positive integer. Let* $V \subset S_K(\mathbf{G}, X)_\mathbb{C}$ *be a non-strongly special subvariety.*

Let l *be a prime splitting* $\mathbf{T}_V$ *such that* K_l *is contained in an Iwahori subgroup* I_l *of* $\mathbf{G}(\mathbb{Q}_l)$ *in good position with respect to* $\mathbf{T}_V$*.*

Let Z *be a Hodge-generic* F*-irreducible* F*-subvariety of* $S_K(\mathbf{G}, X)_\mathbb{C}$ *containing* V *and satisfying*

$$l^{k\cdot 2^{r-1}} (\deg_{L_K} Z)^{2^r} < C(N)\alpha_V \beta_V^N \tag{8.2}$$

for $r = \dim Z - \dim V$*.*

Let m *be an element of* $\mathbf{T}_V(\mathbb{Q}_l)$ *satisfying the conclusion of theorem 7.1 with respect to* Z*. Then one of the following holds :*

(a) $Z \subset T_m Z$.

(b) *there exists an F-irreducible subvariety Y of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ satisfying the following properties :*
  - $\mathrm{Gal}(\overline{F}/F) \cdot V \subsetneq Y \subset Z \cap T_m Z \subsetneq Z$.
  - $\deg_{L_K} Y \leq l^{k \cdot 2^{r-1}} \cdot (\deg_{L_K} Z)^{2^r}$.
  - *V is not strongly special in $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_{\mathbf{G}_Y})_{\mathbb{C}}$, where $\mathbf{G}_Y \subset \mathbf{G}$ denotes the generic Mumford-Tate group of a component Y_1 of Y containing V, $(\mathbf{G}_Y, X_{\mathbf{G}_Y})$ is the corresponding Shimura sub-datum of $(\mathbf{G}, X)$ and K_Y denotes the intersection $K \cap \mathbf{G}_Y(\mathbf{A}_{\mathrm{f}})$.*

Moreover if one considers only the subvarieties V such that the associated tori $\mathbf{T}_V$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class, then the assumption of the GRH can be dropped.

8.3.1. We start with the following auxiliary lemma :

Lemma 8.3.2. *Assume the GRH.*

Let $(\mathbf{G}, X)$ be a Shimura datum, $K = \prod_{p \text{ prime}} K_p$ a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$, and F a number field containing the reflex field $E(\mathbf{G}, X)$. Let N be a positive integer. Let $V \subset S_K(\mathbf{G}, X)_{\mathbb{C}}$ be a non-strongly special subvariety.

Let Y_1 be a geometrically irreducible subvariety of $S_K(\mathbf{G}, X)_{\mathbb{C}}$ which satisfies the following conditions :

(a) $V \subsetneq Y_1$.
(b) $r_{\mathbf{T}_V}(\mathrm{Gal}(\overline{F}/F)) \cdot Y_1 \subset Y := \mathrm{Gal}(\overline{F}/F) \cdot Y_1$.
(c) $\deg_{L_K} Y \leq C(N) \alpha_V \beta_V^N$.

Then V is a non-strongly special subvariety of $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}}$, where $\mathbf{G}_Y \subset \mathbf{G}$ denotes the generic Mumford-Tate group of Y_1, $(\mathbf{G}_Y, X_Y) \subset (\mathbf{G}, X)$ is the corresponding Shimura sub-datum and K_Y denotes the intersection $K \cap \mathbf{G}_Y(\mathbf{A}_{\mathrm{f}})$.

Moreover if one considers only the subvarieties V such that the associated tori $\mathbf{T}_V$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class, then the assumption of the GRH can be dropped.

Proof. Suppose by contradiction that V is strongly special in $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}}$. Thus the connected center $\mathbf{T}_V$ of $\mathbf{H}_V$ is contained in the connected center $Z(\mathbf{G}_Y)^0$ of $\mathbf{G}_Y$. By lemma 5.2, one obtains the equality :

$$\mathbf{T}_V = Z(\mathbf{G}_Y)^0 \quad .$$

Let's define the compact open subgroup $K_Y^m \subset \mathbf{G}_Y(\mathbf{A}_{\mathrm{f}})$ as the product $K_{\mathbf{T}_V}^m \cdot K_Y$ and consider the diagram deduced from the inclusion $K_Y \subset K_Y^m$:

$$\begin{array}{ccccc} V & \hookrightarrow & Y_1 & \hookrightarrow & \mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}} \\ \downarrow & & \downarrow & & \downarrow \pi \\ V^m & \hookrightarrow & Y_1^m & \hookrightarrow & \mathrm{Sh}_{K_Y^m}(\mathbf{G}_Y, X_Y)_{\mathbb{C}} \end{array} .$$

The morphism $\pi : \mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}} \longrightarrow Sh_{K_Y^m}(\mathbf{G}_Y, X_Y)_{\mathbb{C}}$ is a Galois étale cover of group K_Y^m/K_Y. Exactly as in [33] we obtain $\deg_{L_K} Y \geq \deg(Y \cap \pi^{-1}(Y_1^m)) \times \#(\mathrm{Gal}(\overline{F}/F) \cdot Y_1^m)$. The first factor is larger than $C(N)\alpha_V$, the second is larger than β_V^N. This contradicts the assumption (c) : $\deg_{L_K} Y \leq C(N)\alpha_V\beta_V^N$. □

8.3.2. *Proof of proposition 8.3.1.*

Proof. Suppose we are not in case (a).

Step 1 : As $V \subset Z \cap T_m Z$, there exists a geometric irreducible component Y_1 of $Z \cap T_m Z$ containing V. Notice that Z and $T_m Z$ do not have any geometric irreducible component in common as Z and $T_m Z$ are defined over F, Z is F-irreducible and $Z \not\subset T_m Z$. In particular $\dim Y_1 < \dim Z$.

Lemma 8.3.3. $V \subsetneq Y_1$

Proof. Otherwise $V = Y_1$ and $\mathrm{Gal}(\overline{F}/F) \cdot V$ is a union of geometrically irreducible components of $Z \cap T_m Z$. Thus

$$\deg_{L_K}(\mathrm{Gal}(\overline{F}/F) \cdot V) \leq \deg_{L_K}(Z \cap T_m Z) \leq (\deg_{L_K} Z)^2 [K_l : K_l \cap m K_l m^{-1}] .$$

As m satisfies the conclusion of theorem 7.1, $[K_l : K_l \cap m K_l m^{-1}] < l^k$.

As $\deg_{L_K}(\mathrm{Gal}(\overline{F}/F) \cdot V) \geq C(N)\alpha_V\beta_V^N$ by theorem 2.2.2, we finally obtain the inequality :

$$C(N)\alpha_V\beta_V^N \leq (\deg_{L_K} Z)^2 l^k .$$

Contradiction to inequality 8.2 on page 38. □

Let Y be the $\mathrm{Gal}(\overline{F}/F)$-orbit of Y_1. We obtain $\mathrm{Gal}(\overline{F}/F) \cdot V \subsetneq Y \subset Z \cap T_m Z \subsetneq Z$. Moreover $\deg_{L_K} Y \leq (\deg_{L_K} Z)^2 l^k < C(N)\alpha_V\beta_V^N$.

Step 2 : Let $\mathbf{G}_1$ be the generic Mumford-Tate group of Y_1, $(\mathbf{G}_1, X_1) \subset (\mathbf{G}, X)$ the sub-Shimura datum it induces, K_{Y_1} the compact open subgroup $K \cap \mathbf{G}_{Y_1}(\mathbf{A}_\mathrm{f})$ of $\mathbf{G}_1(\mathbf{A}_\mathrm{f})$.

If V is non-strongly special in $\mathrm{Sh}_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}}$ then Y satisfies the condition (b) of proposition 8.3.1 and we are done.

Thus we can assume that V is strongly special in $\mathrm{Sh}_{K_1}(\mathbf{G}_1, X_1)_{\mathbb{C}}$. As $V \subsetneq Y_1$ and $\deg_{L_K} Y \leq C(N)\alpha_V\beta_V^N$, by lemma 8.3.2 we know there exists $\sigma \in \mathrm{Gal}(\overline{F}/F)$ such that $r_{\mathbf{T}_V}(\sigma) \cdot Y_1 \not\subset \mathrm{Gal}(\overline{F}/F) \cdot Y_1$.

As $\sigma(V) = r_{\mathbf{T}_V}(\sigma)V$, we have $\sigma(V) \subset \sigma(Y_1) \cap r_{\mathbf{T}_V}(\sigma) \cdot Y_1$. Thus

$$\mathrm{Gal}(\overline{F}/F) \cdot V \subset Y \cap r_{\mathbf{T}_V}(\sigma)(Y) \quad .$$

Let Y_2 be a geometric irreducible component of $Y \cap r_{\mathbf{T}_V}(\sigma)(Y)$ containing V. We obtain

$$\mathrm{Gal}(\overline{F}/F) \cdot V \subset \mathrm{Gal}(\overline{F}/F) \cdot Y_2 \subsetneq Y \quad .$$

Moreover $\deg_{L_K}(\mathrm{Gal}(\overline{F}/F) \cdot Y_2) \leq \deg_{L_K}(Y \cap r_{\mathbf{T}_V}(\sigma)(Y)) \leq ((\deg_{L_K} Z)^2 l^k)^2$. Once more the inequality 8.2 on page 38 implies that V is a proper subvariety of Y_2.

We now iterate step 2, replacing Y_1 by Y_2. As $\dim V < \dim Y_2 < \dim Y_1 < \dim Z$, in at most $r = \dim Z - \dim V$ iterations we obtain the variety Y of case (b).

□

8.4. **Proof of theorem 8.2.1.** We prove theorem 8.2.1 by induction on $r = \dim Z - \dim V$.

8.4.1. *Case* $r = 1$. Let $(\mathbf{G}, X)$ be a Shimura datum, $K = \prod_{p \text{ prime}} K_p$ a neat compact open subgroup of $\mathbf{G}(\mathbf{A}_\mathrm{f})$, and F a number field containing the reflex field $E(\mathbf{G}, X)$. Let l be a prime splitting $\mathbf{T}_V$ such that K_l is contained in a special maximal compact subgroup $K_l^{\max}$ of $\mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}_V$. Let $V \subset S_K(\mathbf{G}, X)_\mathbb{C}$ be a non-strongly special subvariety contained as an hypersurface in a Hodge-generic F-irreducible subvariety Z of $S_K(\mathbf{G}, X)_\mathbb{C}$.

We denote $d_Z := \deg_{L_K} Z$ and we suppose the condition 8.1 on page 38 for $r = 1$:

$$l^{3(k+2f)} \cdot (\deg_{L_K} Z)^2 < C(N)\alpha_V \beta_V^N \quad . \tag{8.3}$$

In order to apply theorem 7.1 for producing V', we first lift the situation to an Iwahori-level at the prime l.

Let $I \subset K$ be the compact open subgroup $K^l I_l$ of $\mathbf{G}(\mathbf{A}_\mathrm{f})$ where I_l denotes the intersection of K_l and an Iwahori subgroup of $K_l^{\max}$ as in the lemma 8.1.1. As K is neat its subgroup I is also neat. We get a finite morphism of Shimura varieties

$$\pi_F \colon \mathrm{Sh}_I(\mathbf{G}, X)_F \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_F \quad ,$$

of degree bounded above by l^f by lemma 8.1.1,b).

Let $\tilde{Z}_F$ be an irreducible component of $\pi_F^{-1} Z_F$. Its base change $\tilde{Z} := \tilde{Z}_F \times_F \mathbb{C}$ is the union of the $\mathrm{Gal}(\overline{F}/F)$-conjugates of an irreducible component of $\pi^{-1}(Z)$. The image of $\widetilde{Z}$ in $\mathrm{Sh}_K(\mathbf{G}, X)_\mathbb{C}$ is Z and

$$\deg_{L_I} \widetilde{Z} \leq l^f \cdot \deg_{L_K} Z \quad .$$

Let $\widetilde{V}$ be an irreducible component of the preimage of V in $\widetilde{Z}$, this is a non-strongly special subvariety of $\mathrm{Sh}_I(\mathbf{G}, X)_{\mathbb{C}}$ contained in $\widetilde{Z}$. We have the inequality

$$\deg_{L_I}(\mathrm{Gal}(\overline{F}/F)\cdot\widetilde{V}) \geq \deg_{L_K}(\mathrm{Gal}(\overline{F}/F)\cdot V) \quad .$$

As the morphism $\pi\colon \mathrm{Sh}_I(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is finite and preserves the property of a subvariety of being special, exhibiting a special subvariety V' such that $V \subsetneq V' \subset Z$ is equivalent to exhibiting a special subvariety $\widetilde{V'}$ such that $\widetilde{V} \subsetneq \widetilde{V'} \subset \widetilde{Z}$.

Thus by replacing K by I, Z by $\widetilde{Z}$, V by $\widetilde{V}$, we can (and we will from now on) assume that K_l is contained in an Iwahori-subgroup of $\mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}_V$ up to the modification $\deg_K Z \leq d_Z \cdot l^f$.

As K_l is contained in an Iwahori-subgroup of $\mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}_V$, we can apply theorem 7.1. Let m satisfying the conclusion of theorem 7.1. By condition (1) of theorem 7.1, $\mathrm{Gal}(\overline{F}/F)\cdot V \subset Z \cap T_m Z$. If Z and $T_m Z$ have no common (geometric) irreducible component, then any $\sigma(V)$, $\sigma \in \mathrm{Gal}(\overline{F}/F)$ is an irreducible component of $Z \cap T_m Z$ for dimension reasons. By Bezout theorem, we get

$$\begin{aligned} C(N)\alpha_V\beta_V^N \leq \deg_{L_K}(\mathrm{Gal}(\overline{F}/F)\cdot V) &\leq \deg_{L_K}(Z\cap T_m Z) \\ &\leq (\deg_{L_K} Z)^2[K_l : K_l \cap mK_l m^{-1}] < l^{k+2f}\cdot d_Z^2 \quad . \end{aligned}$$

Contradiction to the inequality (8.3). Thus we are in case (a) of proposition 8.3.1 : $Z \subset T_m Z$. As m also satisfies condition (2) of theorem 7.1, we can apply theorem 6.1 to this m : there exists V' special subvariety of Z containing V properly.

8.4.2. *The induction.* Fix $r > 1$ an integer and suppose by induction that theorem 8.2.1 holds for $\dim Z - \dim V < r$. Let $\mathbf{G}$, X, K, V, l as in the statement of theorem 8.2.1 and let Z be a Hodge-generic F-irreducible F-subvariety of $S_K(\mathbf{G}, X)$, containing V with $\dim Z - \dim V = r$. Let $d_Z := \deg_{L_K} Z$ and suppose the inequality 8.1 on page 38 is satisfied :

$$l^{(k+2f)\cdot 2^{a(r+1)}}\cdot (d_Z)^{2^{a(r)}} < C(N)\alpha_V\beta_V^N \quad .$$

As in the case $r = 1$, we can assume that K_l is contained in an Iwahori-subgroup of $\mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}_V$ up to the modification : $\deg_K Z \leq d_Z \cdot l^f$. Choose $m \in \mathbf{G}(\mathbb{Q}_l)$ satisfying the conclusion of theorem 7.1. As condition 8.1 on page 38 implies condition 8.2 on page 38, one can apply proposition 8.3.1.

If we are in case (a) of proposition 8.3.1, once more as in the case $r = 1$ we are done by theorem 6.1.

Thus we can assume we are in case (b) : there exists an F-irreducible subvariety Y of $\mathrm{Sh}_K(G,X)$ satisfying the following properties :

- $\mathrm{Gal}(\overline{F}/F)\cdot V \subsetneq Y \subset Z \cap T_m Z \subsetneq Z$.
- $\deg_{L_K} Y \leq l^{(k+2f)\cdot 2^{r-1}} d_Z^{2r}$.
- V is not strongly special in $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, , X_{\mathbf{G}_Y})_{\mathbb{C}}$, where $\mathbf{G}_Y \subset \mathbf{G}$ denotes the generic Mumford-Tate group of a component Y_1 of Y containing V, $(\mathbf{G}_Y, X_{\mathbf{G}_Y}) \subset (\mathbf{G}, X)$ is the corresponding Shimura sub-datum and K_Y denotes the intersection $K \cap \mathbf{G}_Y(\mathbf{A}_{\mathrm{f}})$.

We obtain a finite morphism of Shimura varieties $\pi : \mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$, which is generically of degree one ([33]). Let $E(\mathbf{G}_Y, X_Y)$ be the reflex field of the Shimura datum $(\mathbf{G}_Y, X_Y)$ and let F' be the composite field

$$F' = F \cdot E(\mathbf{G}_Y, X_Y) \ .$$

The variety Y_1 contains the non-strongly special subvariety V. Let Y' be the $\mathrm{Gal}(\overline{F}/F')$-orbit of Y_1 in $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}}$, Y' is an F'-irreducible F'-subvariety of $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)_{\mathbb{C}}$.

Let us check that $\mathbf{G}_Y$, X_Y, K_Y, F', V, l and Y' satisfy the assumptions of theorem 8.2.1. The compact open subgroup $K_Y = K \cap \mathbf{G}_Y(\mathbf{A}_{\mathrm{f}})$ is a product $\prod_{p \text{ prime}} K_{Y,p}$, with $K_{Y,p} = K_p \cap \mathbf{G}_Y(\mathbb{Q}_p)$. As K_l is contained in a special maximal compact open subgroup $K_l^{\max}$ of $\mathbf{G}(\mathbb{Q}_l)$ in good position with respect to $\mathbf{T}_V$, $K_{Y,l}$ is contained in the compact open subgroup $K_l^{\max} \cap \mathbf{G}_Y(\mathbb{Q}_l)$, which is still in good position with respect to $\mathbf{T}_V$ as $\mathbf{T}_V \subset \mathbf{G}_Y$. It remains to check that

$$l^{(k+2f)\cdot 2^{a(r_Y+1)}} \cdot (\deg_{L_{K_{\mathbf{G}_Y}}} Y')^{2^{a(r_Y)}} < C(N)\alpha_V \beta_V^N \ ,$$

where $r_Y = \dim Y' - \dim V$.

As

$$\deg_{L_{K_{\mathbf{G}_Y}}} Y' \leq \deg_{L_K} Y' \leq \deg_{L_K} Y \leq l^{(k+2f)\cdot 2^r} \cdot d_Z^{2^{r+1}} \ ,$$

we are reduced to check the inequality

$$l^{(k+2f)\cdot(2^{a(r_Y+1)} + 2^{r-1+a(r_Y)})} \cdot d_Z^{2^{r+a(r_Y)}} < C(N)\alpha_V \beta_V^N \ .$$

As Z satisfies condition 8.1 on page 38, it is enough to check that

$$\begin{cases} 2^{a(r_Y+1)} + 2^{r-1+a(r_Y)} & \leq & 2^{a(r+1)} \\ 2^{r+a(r_Y)} & \leq & 2^{a(r)} \end{cases} .$$

The second equation is obviously satisfied because the function a is increasing, $r_Y \leq r-1$ and $r + a(r-1) = a(r)$.

For the first one, notice that $r-1+a(r_Y) \leq r+a(r-1) = a(r)$, thus :

$$2^{a(r_Y+1)} + 2^{r-1+a(r_Y)} \leq 2 \times 2^{a(r)} = 2^{a(r)+1} \leq 2^{a(r+1)}$$

and we are done.

As $\dim Y' - \dim V < \dim Z - \dim V = r$, we can by induction apply the theorem 8.2.1 to $\mathbf{G}_Y$, X_Y, K_Y, F', V, l and Y' : there exists a special subvariety V'_Y of $\mathrm{Sh}_{K_Y}(\mathbf{G}_Y, X_Y)$ such that $V \subsetneq V'_Y \subset Y'$. Let V' denote the special subvariety $\pi(V'_Y)$ of $\mathrm{Sh}_K(\mathbf{G}, X)$. As $\pi(Y') \subset Y \subset Z$ and π is finite, we obtain $V \subsetneq V' \subset Z$ and we are done. This finishes the induction and the proof of theorem 8.2.1.

9. The choice of a prime l

9.1. **Effective Chebotarev.** The choice of a prime l satisfying condition 8.1 on page 38 will be possible thanks to the effective Cebotarev theorem, which we now recall.

Definition 9.1.1. *Let L be a number field of degree n_L and absolute discriminant d_L. Let x be a positive real number. We denote by $\pi(x)$ the number of primes p such that p is split in L and $p \leq x$.*

Proposition 9.1.2. *Assume the Generalized Riemann Hypothesis (GRH). There exists a constant A such that the following holds. For any number field L and for any $x > \max(A, 2\log(d_L)^2(\log(\log(d_L)))^2)$ we have*

$$\pi(x) \geq \frac{x}{3n_L \log(x)} \quad .$$

Furthermore, if we consider number fields such that d_L is constant, then the assumption of the GRH can be dropped.

Proof. The first statement (assuming the GRH) is proved in the Appendix N of [16] and the second is a direct consequence of the classical Cebotarev theorem. □

9.2. **Proof of theorem 3.2.1.**

Proof. Let $(\mathbf{G}, X)$ be a Shimura datum and K a compact open subgroup of $\mathbf{G}(\mathbf{A}_f)$. Let F be a number field containing the reflex field $E(\mathbf{G}, X)$. Let Z be a Hodge-generic F-irreducible F-subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$. Suppose that Z contains a Zariski dense set $\mathbf{\Sigma}$, which is a union of special subvarieties V, $V \in \Sigma$, all of the same dimension $n(\Sigma)$ and such that for some positive integer N and for any modification Σ' of Σ the set $\{\alpha_V \beta_V,\ V \in \Sigma'\}$ is unbounded. We want to show, under one of the two assumptions of theorem 3.2.1, that for every V in Σ there exists a special subvariety V' such that $V \subsetneq V' \subset Z$ (possibly after replacing Σ by a modification).

Lemma 9.2.1. *Without loss of generality we can assume that :*

1. *The group K is a product of compact open subgroups K_p of the $\mathbf{G}(\mathbb{Q}_p)$, p prime.*
2. *There is a prime number p_0 such that K_{p_0} is sufficiently small so that the group K is neat.*
3. *Up to a modification of Σ, the subvarieties $V \in \Sigma$ are non-strongly special.*

Proof. To fulfill the first condition, let $\widetilde{K} \subset K$ be a compact open subgroup which is a product. Let $\widetilde{Z}$ be an F-irreducible component of the preimage of $f^{-1}(Z)$, where f : $\mathrm{Sh}_{\widetilde{K}}(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is the canonical finite morphism. The Hodge-generic F-irreducible F-subvariety $\widetilde{Z}$ of $\mathrm{Sh}_{\widetilde{K}}(\mathbf{G}, X)_{\mathbb{C}}$ contains a Zariski-dense set $\widetilde{\Sigma}$, which is a union of special subvarieties V, $V \in \widetilde{\Sigma}$, all of the same dimension $n(\Sigma)$: $\widetilde{\Sigma}$ is the set of all irreducible components $\widetilde{V}$ of $f^{-1}(V)$ contained in $\widetilde{Z}$ as V ranges through Σ. Notice that for any modification $\widetilde{\Sigma}'$ of $\widetilde{\Sigma}$ the set $\{\alpha_{V'(}N)\beta_{V'},\ V' \in \widetilde{\Sigma}'\}$ is unbounded : $\beta_{V'} = \beta_{f(V')}$ and $\alpha_{V'}(N)$ is equal to $\alpha_{f(V')}(N)$ up to a factor independent of V'. Thus $\widetilde{Z}$ satisfies the assumptions of theorem 3.2.1. As a subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is special if and only if some (equivalently any) irreducible component of its preimage by f is special, theorem 3.2.1 for $\widetilde{Z}$ implies theorem 3.2.1 for Z.

To fulfill the second condition, replace K_{p_0} by a smaller subgroup satisfying lemma 4.1.2. The same argument as above shows that it is safe to do this.

For the third condition : otherwise there exists a modification of Σ containing only strongly special subvarieties. But then Z is automatically special by the theorem 2.1.1 and we are done. □

From now on, we fix a faithful rational representation $\rho : \mathbf{G} \hookrightarrow \mathbf{GL}_n$ such that K is contained in $\mathbf{GL}_n(\hat{\mathbb{Z}})$. In the case of the assumption (2) in theorem 3.2.1, we take for ρ the representation which has the property that the centers $\mathbf{T}_V$ lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class (possibly replacing K by $K \cap \mathbf{GL}_n(\hat{\mathbb{Z}})$) as V ranges through Σ.

For almost all prime l, K_l is a special maximal compact open subgroup of $\mathbf{G}(\mathbb{Q}_l)$ and furthermore $K_l = \mathbf{G}(\mathbb{Z}_l)$, where the $\mathbb{Z}$-structure on $\mathbf{G}$ is given by taking the Zariski-closure in in $\mathbf{GL}_{n,\mathbb{Z}}$ via ρ. Moreover, if the group $\mathbf{T}_{V,\mathbb{F}_l}$ is a split torus then K_l is in good position with respect to $\mathbf{T}_V$.

By theorem 8.2.1, it is then enough for proving theorem 3.2.1 to show that for any V in Σ (up to a modification), there exists a prime l satisfying the following conditions :

1. the prime l splits $\mathbf{T}_V$.
2. $\mathbf{T}_{V,\mathbb{F}_l}$ is a split torus.
3. $l^{(k+2f)\cdot 2^{a(r+1)}} \cdot (\deg_{L_K} Z)^{2^{a(r)}} < C(N)\alpha_V\beta_V^N$.

Proposition 9.1. *For every $D > 0$, $\epsilon > 0$ and every integer $m \geq \max(\epsilon, 6)$, there exists an integer M such that (up to a modification of Σ) : for every V in Σ with $\alpha_V \beta_V$ larger than M there exists a prime l satisfying the following conditions*

(1) $l < D\alpha_V^{\epsilon}\beta_V^{m}$.

(2) $(\mathbf{T}_V)_{\mathbb{F}_l}$ is a split torus.

Moreover the number of such primes goes to infinity as $\alpha_V \beta_V$ goes to infinity.

Proof. For V in Σ recall that n_V is the degree of the splitting field L_V of $\mathbf{C}_V = \mathbf{H}_V/\mathbf{H}_V^{\mathrm{der}}$ over $\mathbb{Q}$. By [37, Lemma 4.2], there exists an integer n such that $n_V < n$ when V ranges through Σ.

Fix $D > 0$, $\epsilon > 0$ and $m \geq 6$. For V in Σ, let

$$x_V := D\alpha_V^{\epsilon}\beta_V^{m}$$

As we are assuming either GRH, or that the connected centers $\mathbf{T}_V$ of the generic Mumford-Tate groups $\mathbf{H}_V$ of V lie in one $\mathbf{GL}_n(\mathbb{Q})$-conjugacy class under ρ as V ranges through Σ, in which case d_{L_V} is independent of V, we can apply proposition 9.1.2 :

$$\pi(x_V) \geq \frac{x_V}{3n\log(x_V)}$$

provided that x_V is larger than some absolute constant and β_V^3.

If $x_V \geq 4$ (which is true if $\alpha_V\beta_V$ is large enough), then

$$\sqrt{x_V} \geq \log(x_V)$$

and it follows that

$$\pi(x_V) \geq \frac{\sqrt{x_V}}{3n} = \frac{(D\alpha_V^{\epsilon}\beta_V^{m})^{\frac{1}{2}}}{3n}$$

To prove the proposition we have to show that $\pi(x_V) > i(\mathbf{T}_V)$ if $\alpha_V\beta_V$ is large enough.

Definition 9.2.2. *Given a positive real number t we denote by Σ_t the set of V in Σ with $i(\mathbf{T}_V) > t$.*

We consider two cases.

- Suppose that for any t the set Σ_t is a modification of Σ. In particular the function $i_V := i(\mathbf{T}_V)$ is unbounded as V ranges through Σ. Recall (Proposition 4.3.9 of [17] and proof of the proposition 5.11 of [33]) that $|K^m_{\mathbf{T}_V}/K_{\mathbf{T}_V}| \gg \prod_p p^{n_p}$ where the product ranges over the i_V primes such that $\mathbf{T}_{V,\mathbb{F}_p}$ is not a torus and $n_p \geq 1$. As the pth prime is at least p we get the inequality

$$\alpha_V = B^{i_V}|K^m_{\mathbf{T}_V}/K_{\mathbf{T}_V}| > B^{i_V} i_V! \ .$$

Recall the well-known inequality: for every integer $n \geq 1$,

$$en^n e^{-n} < n! < en^{n+1}e^{-n} \quad .$$

That gives :

$$\alpha_V > e(\frac{Bi_V}{e})^{i_V} > (\frac{Bi_V}{e})^{i_V} \quad .$$

Hence :

$$\alpha_V^{\frac{\epsilon}{2}} > (\frac{Bi_V}{e})^{\frac{\epsilon i_V}{2}} \quad .$$

For $i_V > \frac{4}{\epsilon}$ we obtain :

$$\alpha_V^{\frac{\epsilon}{2}} > (\frac{Bi_V}{e})^2 \quad .$$

As moreover $\beta_V \geq 1$ we obtain :

$$\pi(x_V) > \frac{D^{1/2}B^2}{2e^2 n} \cdot i_V^2 \quad .$$

Hence, whenever $i_V > t = \frac{2e^2 n}{D^{1/2}B^2}$ we obtain $\pi(x_V) > i_V$. As the set Σ_t is a modification of Σ we get the proposition 9.1 (for any constant M).

- Otherwise there exists a positive number t such that $\Sigma - \Sigma_t$ is a modification of Σ. Replacing Σ by $\Sigma - \Sigma_t$ the function i_V is bounded as V ranges through Σ. Let H be an upper bound for i_V. Of course $\pi(x_V)$ will be larger than i_V when $\pi(x_V) \geq H$. The inequality we want to prove then is

$$\alpha_V^{\frac{\epsilon}{2}}\beta_V^{\frac{m}{2}} > \frac{3nH}{D^{\frac{1}{2}}} \quad .$$

The inequality $\alpha_V^{\epsilon/2}\beta_V^{m/2} \geq (\alpha_V\beta_V)^{\epsilon/2}$ shows that in this case M can be taken to be $(3nH/D^{1/2})^{2/\epsilon}$.

□

Let $r := \dim Z - n(\Sigma)$. Let N be a positive integer. Let $\varepsilon < \frac{1}{(k+2f)\cdot 2^{a(r+1)}}$, $D = (\frac{C(N)}{(\deg_{L_K} Z)^{2^{a(r)}}})^{\frac{1}{(k+2f)\cdot 2^{a(r+1)}}}$ and $n = \frac{N}{(k+2f)\cdot 2^{a(r+1)}}$. Let M be the integer provided by proposition 9.1.

We apply proposition 9.1 for ϵ, k and D : up to a modification of Σ, for every $V \in \Sigma$ we can choose a prime $l \neq p_0$ such that l splits $\mathbf{T}_V$, $\mathbf{T}_{V,\mathbb{F}_l}$ is a split torus and $l < D\alpha_V^{\epsilon}\beta_V^{k}$. This last inequality is exactly condition 8.1 on page 38 of theorem 8.2.1.

Finally for every V in Σ we can apply theorem 8.2.1 to Z, V and l and we are done. □

Appendix A

In this appendix we prove theorem 2.1.1 (and thus also theorem 2.1.1) assuming [33, theorem 3.8]. Let $(\mathbf{G}, X)$ be a Shimura datum, $K = \prod_{p \text{ premier}} K_p$ be a compact open subgroup of $\mathbf{G}(\mathbf{A}_{\mathrm{f}})$ and $\mathbf{T}$ be an $\mathbb{R}$-anisotropic $\mathbb{Q}$-subtorus of $\mathbf{G}^{\mathrm{ad}}$. Let $(V_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbf{T}$-special subvarieties of $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ and μ_{V_n}, $n \in \mathbb{N}$, be the canonical probability measure on $\mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ supported by V_n. Without loss of generality we can assume that all the V_n are contained in $S_K(\mathbf{G}, X)_{\mathbb{C}}$. Let $\lambda : \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}}$ be the adjoint morphism and $f : (\mathbf{G}, X) \longrightarrow (\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ be the associated morphism of Shimura data. Let K^{ad} be any compact open subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbf{A}_{\mathrm{f}})$ containing $\lambda(K)$. We choose K^{ad} such that the finite morphism $\pi : \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} \longrightarrow \mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$ is of degree one in restriction to $S_K(\mathbf{G}, X)_{\mathbb{C}}$. For $n \in \mathbb{N}$ let $\overline{V_n} \subset \mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ be the image $\pi(V_n)$. The sequence of special subvarieties $(\overline{V_n})_{n\in\mathbb{N}}$ is still $\mathbf{T}$-special in $\mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$. The canonical probability measure $\mu_{\overline{V_n}}$ supported by $\overline{V_n}$ identifies with $\pi_*\mu_{V_n}$. By [33, theorem 3.8], there exists a $\mathbf{T}$-special subvariety $\overline{V} \subset \mathrm{Sh}_{K^{\mathrm{ad}}}(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$ and a subsequence $(\mu_{\overline{V_{n_k}}})_{k\in\mathbb{N}}$ weakly converging to $\mu_{\overline{V}}$. Furthermore $\overline{V}$ contains $\overline{V_{n_k}}$ for all k sufficiently large. Let $V = \pi^{-1}(\overline{V} \cap S_K(\mathbf{G}, X)_{\mathbb{C}}$. As π is of degree one in restriction to $S_K(\mathbf{G}, X)_{\mathbb{C}}$, V is a special subvariety, T-special because $\overline{V}$ is. Moreover V contains all V_{n_k} for k sufficiently large and the sequence $\mu_{V_{n_k}}$ weakly converges to μ_V. This finishes the proof of the theorem.

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