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SECONDARY CHARACTERISTIC CLASSES OF TRANSVERSELY HOMOGENEOUS FOLIATIONS

JESÚS A. ÁLVAREZ LÓPEZ AND HIRAKU NOZAWA

Abstract. Let G be a simple Lie group of real rank one, and S^q_∞ the ideal boundary of the corresponding hyperbolic symmetric space of noncompact type ($\mathbf{H}^n_{\mathbb{R}}$, $\mathbf{H}^n_{\mathbb{C}}$, $\mathbf{H}^n_{\mathbb{H}}$ or $\mathbf{H}^2_{\mathbb{O}}$). We show the finiteness of the possible values of the secondary characteristic classes of transversely homogeneous foliations on a fixed manifold whose transverse structures are modeled on the G-action on S^q_∞, except the case of transversely conformally flat foliations of even codimension q. For this exceptional case, we construct examples of foliations on a manifold which break the finiteness and show a weaker form of the finiteness result. These are generalizations of a finiteness theorem of secondary characteristic classes of transversely projective foliations on a fixed manifold by Brooks-Goldman and Heitsch to other transverse structures. We also show Bott-Thurston-Heitsch type formulas to compute the secondary characteristic classes of certain foliated bundles, and then obtain a rigidity result on transversely homogeneous foliations on the unit tangent sphere bundles of hyperbolic manifolds.

Contents

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1. Introduction

1.1. **Secondary characteristic classes of foliations and a theorem of Brooks-Goldman-Heitsch.** For a codimension q smooth foliation $\mathcal{F}$ of a smooth manifold M, we have the characteristic homomorphism $\Delta_{\mathcal{F}}\colon H^\bullet(WO_q) \to H^\bullet(M;\mathbb{R})$ (see Section 2.1). The cohomology classes in the image of $\Delta_{\mathcal{F}}$ are called the secondary characteristic classes of $\mathcal{F}$. These are cobordism invariants of foliations, which come from the continuous cohomology of the Haefliger's classifying space $B\Gamma^q$ [Hae79]. The relation between the dynamics or geometry of foliations and secondary characteristic classes has been one of the main themes in the study of foliations (see the review article [Hur02] by Hurder or [CC03, Chapter 7] by Candel-Conlon). Main examples of foliations with nontrivial secondary characteristic classes are quotient of homogeneous foliations on homogeneous spaces by lattices, which have been extensively studied [KT75b, Yam75, Bak78, Hei78, Pit79, Pel83, Asu10]. Transversely homogeneous foliations are generalizations of these foliations, whose secondary characteristic classes can be computed in a similar way. These foliations were used in the construction of families of foliations whose characteristic classes nontrivially and continuously vary by Thurston [Thu72b, Bot78] and Rasmussen [Ras80]. Other families with this property, constructed by Heitsch [Hei78], are quotient of homogeneous foliations on homogeneous spaces by lattices. Their constructions imply that there are uncountably many foliations which are not mutually cobordant, and certain homology groups with integer coefficients of the classifying space $B\Gamma^q$ are uncountable [Hei78, Section 6].

In spite of the role played by transversely homogeneous foliations in the construction of these examples, Brooks-Goldman and Heitsch showed that transversely projective foliations, a class of transversely homogeneous foliations, satisfy the following remarkable finiteness property of the secondary characteristic classes. Let G be a Lie group and P a closed subgroup of G. A $(G, G/P)$-foliation is a foliation whose transverse structure is modeled on the G-action on G/P (see Definition 3.1). When $G = \mathrm{SL}(q+1;\mathbb{R})$ and $G/P = S^q$, a $(G, G/P)$-foliation is called a *transversely projective foliation*. Fix a smooth manifold M with finitely presented fundamental group. Let $\mathrm{Fol}(G, G/P)$ be the set of $(G, G/P)$-foliations on M, and let

$$\Sigma(G, G/P) = \#\{\, \Delta_{\mathcal{F}} \mid \mathcal{F} \in \mathrm{Fol}(G, G/P) \,\} ,$$

where $q = \dim G/P$.

Theorem 1.1 (Brooks-Goldman [BG84] in the case of $q = 1$ and Heitsch [Hei86] for $q > 1$)**.** $\Sigma(\mathrm{SL}(q+1;\mathbb{R}), S^q) < \infty$.

In this article, we will generalize Theorem 1.1 for other cases of $(G, G/P)$. We also prove Bott-Thurston-Heitsch type formulas to compute secondary characteristic classes and apply such formulas to obtain certain rigidity of foliations.

1.2. **A sufficient condition for the finiteness of secondary characteristic classes.** We assume that G is linear algebraic and semisimple. Let $G_{\mathbb{C}}$ be a complex semisimple Lie group such that $\mathrm{Lie}(G_{\mathbb{C}}) = \mathrm{Lie}(G) \otimes \mathbb{C}$ as a Lie algebra over $\mathbb{R}$. Our first result is the following.

Theorem 1.2. *If $H^\bullet(G_{\mathbb{C}}/P;\mathbb{R}) \to H^\bullet(G/P;\mathbb{R})$ is trivial on positive degrees, then $\Sigma(G, G/P) < \infty$.*

When $(G, G/P) = (\mathrm{SL}(q+1;\mathbb{R}), S^q)$ for odd q, the assumption of Theorem 1.2 on (G, P) is satisfied (see Section 6.2). So Theorem 1.2 implies Theorem 1.1 for odd q. The following cases are our examples of $(G, G/P)$:

$$\left(\mathrm{SO}\,(n+1,1)\,, S^n_\infty\right)\ , \qquad \left(\mathrm{SU}\,(n+1,1)\,, S^{2n+1}_\infty\right)\ ,$$

$$\left(\mathrm{Sp}\,(n+1,1)\,,S_\infty^{4n+3}\right)\ ,\quad \left(F_{4(-20)},S_\infty^{15}\right)\ ,$$

where S_∞^n, S_∞^{2n+1}, S_∞^{4n+3} and S_∞^{15} are the ideal boundaries of the corresponding noncompact symmetric spaces $\mathbf{H}_\mathbb{R}^n$, $\mathbf{H}_\mathbb{C}^n$, $\mathbf{H}_\mathbb{H}^n$ and $\mathbf{H}_\mathbb{O}^2$, respectively. According to the case of manifolds, $(\mathrm{SO}\,(n+1,1)\,,S_\infty^n)$-foliations are called *transversely conformally flat* foliations and $(\mathrm{SU}\,(n+1,1)\,,S_\infty^{2n+1})$-foliations are called *transversely spherical CR* foliations. The unit tangent sphere bundles of hyperbolic manifolds have typical examples of these $(G,G/P)$-foliations (see Example 2.3). The map $H^\bullet(G_\mathbb{C}/P;\mathbb{R})\to H^\bullet(G/P;\mathbb{R})$ is trivial on positive degrees except in the case of transversely conformally flat foliations of even codimension (see Section 6). Thus we get the following.

Corollary 1.3. *If* $(G,G/P)$ *is* $(\mathrm{SO}(n+1,1),S_\infty^n)$ *for odd* n*,* $(\mathrm{SU}(n+1,1),S_\infty^{2n+1})$*,* $(\mathrm{Sp}(n+1,1),S_\infty^{4n+3})$ *or* $\left(F_{4(-20)},S_\infty^{15}\right)$*, then* $\Sigma(G,G/P)<\infty$*.*

Remark 1.4. Since $\mathrm{SU}(1,1)\cong\mathrm{SL}(2;\mathbb{R})$ and $\mathrm{SO}_0(2,1)\cong\mathrm{PSL}(2;\mathbb{R})$, where $\mathrm{SO}_0(2,1)$ is the identity component of $\mathrm{SO}(2,1)$, Corollary 1.3 for $(G,G/P)=(\mathrm{SU}(1,1),S_\infty^1)$ or $(\mathrm{SO}(2,1),S_\infty^1)$ is essentially contained in Theorem 1.1. Hantout [Han88] also investigated this type of finiteness results, but his result does not imply this corollary.

Remark 1.5. Note that the actions of $\mathrm{SU}(n+1,1)$ and $\mathrm{Sp}(n+1,1)$ on spheres may not be effective, depending on n, because their stabilizers are equal to the centers. But, by a slight modification of the proof of Theorem 1.2, we can show the finiteness for the case where $(G,G/P)$ is $(\mathrm{PSU}(n+1,1),S_\infty^{2n+1})$ or $(\mathrm{PSp}(n+1,1),S_\infty^{4n+3})$ (see Section 6.7).

Remark 1.6. It is not difficult to see that every nontrivial secondary characteristic class of $(G,G/P)$-foliations is a multiple of the Godbillon-Vey class for these cases (see Proposition 7.4).

Theorem 1.2 will be proved in Section 5 by using the complexification of characteristic classes and an observation on certain spectral sequences.

1.3. **Bott-Thurston-Heitsch type formulas.** The Godbillon-Vey class $\mathrm{GV}(\mathcal{F})$ of a foliation $\mathcal{F}$ is the secondary characteristic class first discovered in [GV71], and it is specially important for transversely homogeneous foliations as suggested by results of Pittie [Pit79]. In the standard notation, $\mathrm{GV}(\mathcal{F})=(2\pi)^{q+1}\Delta_\mathcal{F}(h_1c_1^q)$ for a codimension q foliation [KT75a, Theorem 7.20]. A typical example of transversely projective foliations is suspension foliations; namely, for a manifold N and a homomorphism $\pi_1N\to\mathrm{SL}(q+1;\mathbb{R})$, we get an S^q-bundle $p\colon\widetilde{N}\times_{\pi_1N}S^q\to N$ foliated by a transversely projective foliation transverse to the fibers of p (Example 3.4). The Bott-Thurston-Heitsch formula for the Godbillon-Vey class of transversely projective foliations computes the Godbillon-Vey class of such foliations.

Theorem 1.7 ([Thu72b] and [Bot78, Appendix by Brooks] for $q = 1$ and Heitsch [Hei78, Theorem 4.2] and [Hei83, Theorem 2.3] for $q > 1$)**.** *Let N be a manifold and* $\mathrm{hol}\colon \pi_1 N \to \mathrm{SL}(q+1;\mathbb{R})$ *a homomorphism. Let $p_M\colon M \to N$ be the S^q-bundle over N with the suspension foliation $\mathcal{F}$ obtained from* hol. *Then, for any orientation on the fibers of p_M, we have*

$$\frac{1}{(2\pi)^{q+1}} \int_{p_M} \mathrm{GV}(\mathcal{F}) = e(p_M) \tag{1}$$

in $H^{q+1}(N;\mathbb{R})$, where $e(p_M)$ is the Euler class of the S^q-bundle p_M.

Remark 1.8. The case of $q = 1$ is special because there are different choices of $\mathrm{SL}(2;\mathbb{R})$-actions on S^1. To get (1), the $\mathrm{SL}(2;\mathbb{R})$-action on the homogeneous space $\mathrm{SL}(2;\mathbb{R})/\mathrm{Aff}(1;\mathbb{R}) \approx S^1$ should be used in the construction of the suspension foliation $\mathcal{F}$, where

$$\mathrm{Aff}(1;\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \;\middle|\; a \in \mathbb{R}^\times,\ b \in \mathbb{R} \right\} .$$

This formula is important as one of few methods to calculate the Godbillon-Vey class explicitly. Heitsch obtained a similar formula for other secondary characteristic classes of transversely projective foliations ([Hei78, Theorem 4.2] and [Hei83, Theorem 2.3]).

We generalize this formula. Note that, for a manifold N and a homomorphism $\pi_1 N \to G$, we have a suspension foliation of the total space of a G/P-bundle over N, which naturally admits a structure of a $(G, G/P)$-foliation (Example 3.4). Let $\mathrm{SO}_0(n+1,1)$ be the identity component of $\mathrm{SO}(n+1,1)$.

Theorem 1.9. *Let $(G, G/P)$ denote one of $(\mathrm{SO}_0(n+1,1), S^n_\infty)$ for odd $n > 1$, $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$ for $n > 0$, $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ or $(F_{4(-20)}, S^{15}_\infty)$. Let $q = \dim G/P$ (the codimension of $(G, G/P)$-foliations), N a manifold and* $\mathrm{hol}\colon$ *$\pi_1 N \to G$ a homomorphism. Let $p_M\colon M \to N$ be the G/P-bundle over N with the suspension foliation $\mathcal{F}$ obtained from* hol. *Then, for any orientation on the fibers of p_M, we have*

$$\frac{1}{(2\pi)^{q+1}} \int_{p_M} \mathrm{GV}(\mathcal{F}) = r_G\, e(p_M) \tag{2}$$

in $H^{q+1}(N;\mathbb{R})$, where $e(p_M)$ is the Euler class of the S^q-bundle p_M, and r_G is the constant, depending on $(G, G/P)$, given in the following table:

$(G, G/P)$	r_G
$(\mathrm{SO}_0(n+1,1), S^n_\infty)$	n^{n+1}
$(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$	$\frac{2(n+1)^{2n+2}}{n+2} \cdot \frac{(2n+1)!}{n!(n+1)!}$
$(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$	$\frac{2^{3/2}(2n+3)^{4n+3}}{(n+2)(n+3)^{n+1}} \cdot \frac{(4n+3)!}{(2n+1)!(2n+2)!}$
$(F_{4(-20)}, S^{15}_\infty)$	$2^{19} \cdot 3^{69/2} \cdot 7^4 \cdot 11^{16} \cdot 13$

Remark 1.10. Rasmussen [Ras80, Theorem 5.1] also obtained a similar formula for the case of $(\mathrm{SO}_0(3,1), S^2_\infty)$. The codimension one case excluded from Theorem 1.9, where $(G, G/P)$ is either of $(\mathrm{SO}_0(2,1), S^1_\infty)$ or $(\mathrm{SU}(1,1), S^1_\infty)$, corresponds to the original Bott-Thurston formula (Theorem 1.7 for $q = 1$).

We will prove Theorem 1.9 by a direct calculation on Lie algebra cohomology with the application of the Hirzebruch's proportionality principle in Section 7. Note that it is not difficult to see that both sides of (2) are equal up to a nonzero constant factor like in the case of the original Bott-Thurston formula for codimension one case (see [BG84, Section 3]). This relation was already pointed out in the case of $(\mathrm{SO}(n+1,1), S^n_\infty)$ by Reznikov [Rez96, Section 5.16].

Remark 1.11. Note that, in the case of $(\mathrm{SO}_0(n+1,1), S^n_\infty)$ for even n, the Euler classes of S^n-bundles are trivial with real coefficients. So this type of formulas is not true in that case. But we will show a similar formula with the volume of the holonomy homomorphism (see Proposition 7.14).

Remark 1.12. Theorem 1.7 for $q = 1$ was used by Brooks-Goldman [BG84] to prove Theorem 1.1 for $q = 1$. Heitsch [Hei86] used Theorem 1.7 and its generalization to other secondary characteristic classes to prove Theorem 1.1. Based on a calculation similar to the proof of Theorem 1.9, we can give an alternative proof of Theorem 1.1 for even q (see Remarks 8.5 and 8.7). This alternative proof is slightly simpler than the original proof due to Heitsch [Hei86].

1.4. **The case of $G/P = S^q$ for even q.** In this case, it is easy to see that the assumption of Theorem 1.2 on the triviality of $H^\bullet(G_{\mathbb{C}}/P; \mathbb{R}) \to H^\bullet(G/P; \mathbb{R})$ for positive degrees is never satisfied (see Proposition 6.1). In fact, by using a Bott-Thurston-Heitsch type formula in Proposition 7.14 for the Godbillon-Vey class of transversely conformally flat foliation of even codimension, we get the following infiniteness result.

Theorem 1.13. *For each even q, there exists a connected noncompact smooth manifold X with finitely presented fundamental group and a family $\{\mathcal{F}_m\}_{m\in\mathbb{Z}}$ of codimension q transversely conformally flat foliations of X such that $\mathrm{GV}(\mathcal{F}_m) \neq \mathrm{GV}(\mathcal{F}_{m'})$ if $m \neq m'$.*

As far as we know, this is the first example of a family of transversely conformal foliations on a connected manifold whose Godbillon-Vey classes take infinitely many different values. We do not know compact examples. Asuke [Asu10] constructed finite families of transversely holomorphic foliations on compact homogeneous spaces whose Godbillon-Vey classes take different values. (Note that complex codimension one transversely holomorphic foliations are real codimension two transversely conformal foliations.)

Remark 1.14. Asuke [Asu10] proved that the Godbillon-Vey class does not change nontrivially for smooth families of transversely holomorphic foliations. As pointed out by Morita [Mor79], it is not known if there exist a smooth family of transversely conformal foliations of codimension greater than two whose Godbillon-Vey classes continuously and nontrivially vary.

We will show the finiteness of secondary characteristic classes in a weaker form in this case. Let $\chi(\nu\mathcal{F})$ be the Euler class of the normal bundle $\nu\mathcal{F}$ of $\mathcal{F}$. Let $\Sigma(G, G/P; \mathbb{R}/\mathbb{Z})$ denote the number of homomorphisms $H^\bullet(WO_q) \to H^\bullet(M; \mathbb{R}/\mathbb{Z})$ induced by the homomorphisms $\Delta_\mathcal{F}$ with $\mathcal{F} \in \mathrm{Fol}(G, G/P)$, and let

$$\Sigma(G, G/P, z) = \#\{\, \Delta_\mathcal{F} \mid \mathcal{F} \in \mathrm{Fol}(G, G/P),\ \chi(\nu\mathcal{F}) = z \,\}$$

for any fixed $z \in H^q(M; \mathbb{R})$. We get the following.

Theorem 1.15. *If $G/P = S^q$ for even q, then*

$$\Sigma(G, G/P; \mathbb{R}/\mathbb{Z}) < \infty\,, \tag{3}$$

$$\Sigma(G, G/P, z) < \infty\,, \tag{4}$$

for each $z \in H^q(M; \mathbb{R})$.

The proof of Theorem 1.15 is based on simple arguments with Lie algebra cohomology. Theorems 1.13 and 1.15 will be proved in Section 8.

1.5. **Transversely conformal foliations.** In this section, we assume that the fixed manifold M is compact. By a theorem of Tarquini [Tar04, Théorème 0.0.1], a transversely real analytic conformal foliation of codimension $q > 2$ is Riemannian or $(\mathrm{PSO}(q+1,1), S^q_\infty)$ on each connected component of M. Let $\mathrm{Fol}^{q,\omega}_\mathcal{C}$ be the set of codimension q transversely real analytic conformal foliations on M. Let

$$\Sigma^q_\mathcal{C} = \#\{\, \Delta_\mathcal{F} \mid \mathcal{F} \in \mathrm{Fol}^{q,\omega}_\mathcal{C} \,\}\,,$$
$$\Sigma^q_\mathcal{C}(z) = \#\{\, \Delta_\mathcal{F} \mid \mathcal{F} \in \mathrm{Fol}^{q,\omega}_\mathcal{C},\ z = \chi(\nu\mathcal{F}) \,\}$$

for z in $H^q(M; \mathbb{R})$, and let $\Sigma^q_\mathcal{C}(\mathbb{R}/\mathbb{Z})$ be the number of homomorphisms $H^\bullet(WO_q) \to H^\bullet(M; \mathbb{R}/\mathbb{Z})$ induced by the homomorphisms $\Delta_\mathcal{F}$ with $\mathcal{F} \in \mathrm{Fol}^{q,\omega}_\mathcal{C}$.

Since the secondary characteristic classes of Riemannian foliations are trivial (see [KT75a, Section 4.48 and Theorem 4.52]), we get the following corollary.

Corollary 1.16. (i) $\Sigma_{\mathcal{C}}^q < \infty$ *for odd* $q > 1$.

(ii) $\Sigma_{\mathcal{C}}^q(\mathbb{R}/\mathbb{Z}) < \infty$ *and* $\Sigma_{\mathcal{C}}^q(z) < \infty$ *for each* $z \in H^q(M;\mathbb{R})$ *and even* $q > 2$.

1.6. **Rigidity of transversely homogeneous foliations with nontrivial secondary invariants.** Let $(G, G/P)$ be $(\mathrm{SO}_0(n+1,1), S^n_\infty)$, $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ or $\big(F_{4(-20)}, S^{15}_\infty\big)$. Let $\mathcal{F}_\Gamma$ be the standard homogeneous $(G, G/P)$-foliation on $M = \Gamma\backslash G/K_P$, where Γ is a torsion-free uniform lattice of G and K_P is a maximal compact subgroup of P (Example 2.3). Here $\mathrm{GV}(\mathcal{F}_\Gamma)$ is nontrivial as computed in Corollary 7.12. Note that $\dim M = \deg \mathrm{GV}(\mathcal{F}_\Gamma)$. Fix an orientation of M so that $\int_M \mathrm{GV}(\mathcal{F}_\Gamma) > 0$. Then we show the following.

Theorem 1.17. (i) *If* $(G, G/P)$ *is one of* $(\mathrm{SO}_0(n+1,1), S^n_\infty)$ *for odd* $n > 1$, $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$ *for* $n \geq 1$, $(\mathrm{Sp}(n+1,1), S^q_\infty)$ *or* $(F_{4(-20)}, S^{15}_\infty)$, *then* $\mathcal{F}$ *is smoothly conjugate to* $\mathcal{F}_\Gamma$.

(ii) *If* $(G, G/P)$ *is* $(\mathrm{SO}_0(n+1,1), S^n_\infty)$ *for even* n, *then any* $(G, G/P)$*-foliation* $\mathcal{F}$ *of* M *satisfies* $\int_M \mathrm{GV}(\mathcal{F}) \leq \int_M \mathrm{GV}(\mathcal{F}_\Gamma)$. *Moreover the equality holds if and only if* $\mathcal{F}$ *is smoothly conjugate to* $\mathcal{F}_\Gamma$.

The essential part of the proof is to generalize the Bott-Thurston-Heitsch type formulas to foliations which may not be transverse to fibers (Lemma 9.1). It allows us to apply the rigidity theory of representations of lattices; in particular, the generalized Mostow rigidity [Cor91, Dun99, FK06] for lattices of $\mathrm{PSO}(n+1,1)$ or $\mathrm{PSU}(n+1,1)$ and the superrigidity [Cor92] of lattices of $\mathrm{Sp}(n+1,1)$ or $F_{4(-20)}$.

In the codimension one case, we will show the following.

Theorem 1.18. *If* $(G, G/P)$ *is one of* $(\mathrm{SO}_0(2,1), S^1_\infty)$ *or* $(\mathrm{SU}(1,1), S^1_\infty)$, *then any* $(G, G/P)$*-foliation* $\mathcal{F}$ *of* M *satisfies* $\mathrm{GV}(\mathcal{F}) = \mathrm{GV}(\mathcal{F}_\Gamma)$ *or* $\mathrm{GV}(\mathcal{F}) = 0$. *Moreover the former case holds if and only if* $\mathcal{F}$ *is smoothly conjugate to* $\mathcal{F}_\Gamma$.

To prove Theorem 1.18, we will apply a minimality theorem of Chihi-ben Ramdane [CbR08] and theorems of Thurston [Thu72a] and Levitt [Lev78] to isotope $(G, G/P)$-foliations with nontrivial Godbillon-Vey classes so that they are transverse to the fibers of $\Gamma\backslash G/K_P \to \Gamma\backslash G/K_G$, where K_G is a maximal compact subgroup of G. Then we can apply generalized Mostow rigidity [Gol88] for surface group representations.

Theorems 1.17 and 1.18 will be proved in Section 9.

Remark 1.19. Theorem 1.18 improves a result of Brooks-Goldman [BG84, Theorem 5]. Theorem 1.18 is also related to Mitsumatsu defect formula [Mit85] for the C^2 stable foliations of the geodesic flows of hyperbolic surfaces, and its generalization with weaker regularity assumption by Hurder-Katok [HK90, Theorem 3.11].

Organization of the article. Sections 2 and 3 are devoted to recall fundamental notions in this article, as indicated in the table of the contents. In Section 4, the complexification of secondary characteristic classes of transversely homogeneous foliations is explained, which will be used in Section 5 to prove Theorem 1.2. Section 6 is devoted to present the examples of the application of Theorem 1.2. In Section 7, first, the characteristic classes of homogeneous foliations on homogeneous spaces are calculated in terms of Lie algebra cohomology, and then the Bott-Thurston-Heitsch type formulas of Theorem 1.9 are deduced. Theorems 1.15 and 1.13 are proved in Section 8. (Note that the computation in Section 7 is used in Section 8, but it is not necessary for the proof of Theorems 1.15 and 1.13.) In Section 9, Theorems 1.17 and 1.18 are proved by applying the modification of the Bott-Thurston-Heitsch type formulas of Theorem 1.9.

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2. Secondary characteristic classes of foliations

2.1. **Fundamentals of secondary characteristic classes.** Consider the Weil algebra $W(\mathfrak{gl}(q;\mathbb{R})) = \bigwedge \mathfrak{gl}(q;\mathbb{R})^* \otimes S\mathfrak{gl}(q;\mathbb{R})^*$ of $\mathfrak{gl}(q;\mathbb{R})$, and its $O(q)$-basic subalgebra,

$$W(\mathfrak{gl}(q;\mathbb{R}))_{O(q)} = \{\, \beta \in W(\mathfrak{gl}(q;\mathbb{R})) \mid \iota_X\beta = 0 \ \forall X \in \mathfrak{o}(q),\ \mathrm{Ad}(g)^*\beta = \beta \ \forall g \in O(q) \,\} .$$

For a principal $\mathrm{GL}(q;\mathbb{R})$-bundle E over a smooth manifold M with a $\mathrm{GL}(q;\mathbb{R})$-connection ∇^E, the Chern-Weil construction yields a homomorphism of differential graded algebras, $\widehat{\Delta}_E : W(\mathfrak{gl}(q;\mathbb{R})) \to \Omega^\bullet(E)$. Since the image of $W(\mathfrak{gl}(q;\mathbb{R}))_{O(q)}$ under $\widehat{\Delta}_E$ is contained in the image of the pull-back map $\pi^* \colon \Omega^\bullet(E/O(q)) \to \Omega^\bullet(E)$ by the $O(q)$-basicness, we get a differential map

$$\Delta_E \colon W(\mathfrak{gl}(q;\mathbb{R}))_{O(q)} \longrightarrow \Omega^\bullet(E/O(q)) .$$

By the contractibility of the fibers of $E/O(q) \to M$, there exists a section $s\colon M \to E/O(q)$. Thus we get a differential map given by the composite

$$W(\mathfrak{gl}(q;\mathbb{R}))_{O(q)} \xrightarrow{\Delta_E} \Omega^\bullet(E/O(q)) \xrightarrow{s^*} \Omega^\bullet(M) .$$

It is known that

$$W(\mathfrak{gl}(q;\mathbb{R}))_{O(q)} = \bigwedge[h_1, h_3, \dots, h_{[q]}] \otimes \mathbb{R}[c_1, c_2, \dots, c_q]$$

as a differential graded algebra, where $[q]$ is the maximal odd number less than $q+1$. Its grading is given by $\deg h_i = 2i-1$ and $\deg c_i = 2i$, and its differential map is determined by $dh_i = c_i$ and $dc_i = 0$. Here, c_i is the i-th Chern polynomial given by $\det(I_q + \frac{t}{2\pi}A) = \sum_{j=0}^{q} c_j(A)t^j$ [KT75a, p. 138 and 139]. (Note that these

Chern polynomials differ from the usual one by $\sqrt{-1}$-factors.) This construction yields nothing for a general $\mathrm{GL}(q;\mathbb{R})$-connection because $H^\bullet(W(\mathfrak{gl}(q;\mathbb{R}))_{O(q)}) = 0$. The normal bundle $\nu\mathcal{F} = TM/T\mathcal{F}$ of a foliated manifold $(M,\mathcal{F})$ has a special $\mathfrak{gl}(q;\mathbb{R})$-connection called a *Bott connection* [Bot72]. For a Bott connection ∇ on $\nu\mathcal{F}$, the frame bundle $\mathcal{P}(\nu\mathcal{F})$ with the principal $\mathrm{GL}(q;\mathbb{R})$-connection associated to ∇ satisfies $\Delta_{\mathcal{P}(\nu\mathcal{F})}(c_i) = 0$ for $i > q$ by Bott vanishing theorem. Thus, letting

$$WO_q = \bigwedge[h_1, h_3, \dots, h_{[q]}] \otimes \mathbb{R}[c_1, c_2, \dots, c_q]/\mathcal{I}_q ,$$

where $\mathcal{I}_q$ is the ideal of $\mathbb{R}[c_1, c_2, \dots, c_q]$ generated by the elements of degree greater than $2q$, we get a differential map $\Delta_{\mathcal{F}} \colon WO_q \to \Omega^\bullet(M)$. The map induced on cohomology,

$$\Delta_{\mathcal{F}} \colon H^\bullet(WO_q) \longrightarrow H^\bullet(M;\mathbb{R}) ,$$

depends only on $\mathcal{F}$ and is denoted with the same symbol. The cohomology $H^\bullet(WO_q)$ is nontrivial, $\Delta_{\mathcal{F}}$ is called the *characteristic homomorphism* of $\mathcal{F}$, and the elements of its image are the *secondary characteristic classes* of $\mathcal{F}$. For $I = \{i_1, \dots, i_k\} \subseteq \{1, 3, \cdots, [q]\}$ and $J = \{j_1, \dots, j_l\}$, where $1 \le j_m \le q$, let $h_I c_J = h_{i_1} \cdots h_{i_k} c_{j_1} \cdots c_{j_l}$. Vey showed that the union of

$$\{\, c_J \mid j \text{ is even } \forall j \in J \,\} \tag{5}$$

and

$$\{\, h_I c_J \mid i_1 + |J| \ge q+1,\ i_1 \le j \text{ for any odd } j \in J \,\} \tag{6}$$

is a basis of $H^\bullet(WO_q)$ as an $\mathbb{R}$-vector space, where $i_1 = \min I$ [Hei73, Theorem 2]. The characteristic classes in (5) are the Pontryagin classes of $\nu\mathcal{F}$. The characteristic classes in (6) are called *exotic*.

Example 2.1. Let $\mathcal{F}$ be a codimention q foliation on M defined by the kernel of a q-form ω. By the Frobenius theorem, we have $d\omega = \eta\wedge\omega$ for some 1-form η. Then $\eta\wedge(d\eta)^q$ is a closed $(2q+1)$-form on M, which is equal to $(2\pi)^{q+1}[\Delta_{\mathcal{F}}(h_1c_1^q)]$ [KT75a, Theorem 7.20]. This characteristic class $(2\pi)^{q+1}[\Delta_{\mathcal{F}}(h_1c_1^q)]$ is called the *Godbillon-Vey class* of $\mathcal{F}$ [GV71]. The notation $\mathrm{GV}(\mathcal{F}) = (2\pi)^{q+1}[\Delta_{\mathcal{F}}(h_1c_1^q)]$ is standard.

2.2. **Examples of foliations with nontrivial characteristic classes.** Quotient of homogeneous foliations on homogeneous spaces by lattices are the main examples of foliations with nontrivial secondary characteristic classes.

Example 2.2 (Roussarie's example [GV71])**.** Let Γ be a torsion-free uniform lattice of $\mathrm{SL}(2;\mathbb{R})$. Let $\pi\colon \mathrm{SL}(2;\mathbb{R}) \to \mathrm{SL}(2;\mathbb{R})/\operatorname{Aff}(1;\mathbb{R})$ be the canonical projection, where $\operatorname{Aff}(1;\mathbb{R})$ is the subgroup of $\mathrm{SL}(2;\mathbb{R})$ given in Remark 1.8. Then the fibers of π induce a codimension one foliation on $M = \Gamma\backslash\mathrm{SL}(2;\mathbb{R})$. Let $\{\omega, \eta, \theta\}$ be a basis of $\mathfrak{sl}(2;\mathbb{R})^*$ so that the fibers of π are defined by $\ker\omega$ and

$$d\omega = \eta\wedge\omega , \quad d\eta = \omega\wedge\theta , \quad d\theta = -\eta\wedge\theta .$$

By their left invariance, the 1-forms ω, η and θ on $\mathrm{SL}(2;\mathbb{R})$ induce 1-forms on M, which are denoted with the same symbols. Let $\mathcal{F}$ be the foliation on M defined by the kernel of ω. By the definition of $\mathrm{GV}(\mathcal{F})$, we get

$$\mathrm{GV}(\mathcal{F}) = [\eta \wedge d\eta] = [\eta \wedge \omega \wedge \theta] \ .$$

Since $\eta \wedge \omega \wedge \theta$ is a volume form on M, it follows that $\mathrm{GV}(\mathcal{F}) \neq 0$. In fact, by the Bott-Thurston formula (Theorem 1.9 for $q = 1$), we get

$$\int_M \mathrm{GV}(\mathcal{F}) = 4\pi^2 e \ ,$$

where e is the Euler number of the surface $\Gamma\backslash \mathrm{SL}(2;\mathbb{R})/\mathrm{SO}(2)$.

Example 2.3. The following example is a generalization of the last example to higher dimensions. Let G be $\mathrm{SO}(n+1,1)$, $\mathrm{SU}(n+1,1)$, $\mathrm{Sp}(n+1,1)$ or $F_{4(-20)}$, and consider G/P as the ideal boundary of the corresponding hyperbolic symmetric space G/K_G:

$$\begin{aligned}
\mathbf{H}^n_{\mathbb{R}} &= \mathrm{SO}(n+1,1)/\mathrm{S}(\mathrm{O}(n+1) \times \{\pm 1\}) \ ,\\
\mathbf{H}^n_{\mathbb{C}} &= \mathrm{SU}(n+1,1)/\mathrm{S}(\mathrm{U}(n+1)\,\mathrm{U}(1)) \ ,\\
\mathbf{H}^n_{\mathbb{H}} &= \mathrm{Sp}(n+1,1)/\mathrm{Sp}(n+1)\,\mathrm{Sp}(1) \ ,\\
\mathbf{H}^2_{\mathbb{O}} &= F_{4(-20)}/\mathrm{Spin}(9) \ .
\end{aligned}$$

Let K_G be a maximal compact subgroup of G, and take a maximal compact subgroup K_P of P as $K_P = K_G \cap P$. The ideal boundary of G/K_G is a sphere of real dimension n, $2n+1$, $4n+3$ and 15, respectively. $\Gamma\backslash G/K_P$ admits a foliation $\mathcal{F}_\Gamma$ whose lift to G/K_P is defined by the fibers of $G/K_P \to G/P$. Here, $\Gamma\backslash G/K_G$ is a real, complex, quaternionic or octonionic hyperbolic manifold, and $\Gamma\backslash G/K_P \to \Gamma\backslash G/K_G$ is the total space of its unit tangent sphere bundle (see Section 6), depending on the choice of G. Later, we will compute $\mathrm{GV}(\mathcal{F}_\Gamma)$ (Proposition 7.9), and this Godbillon-Vey class is essentially the unique nontrivial secondary characteristic class for these foliations (Section 7.3). Yamato [Yam75] studied the secondary characteristic classes of $\mathcal{F}_\Gamma$ in the case where $G = \mathrm{SO}(n+1,1)$.

Example 2.4. The following example is a further generalization of the last example. Let G be a Lie group and P a closed subgroup of G. Let K be a closed subgroup of P. Let Γ be a torsion-free uniform lattice of G. Then the fibers of the canonical projection $G/K \to G/P$ define a foliation $\mathcal{F}_\Gamma$ on a closed manifold $\Gamma\backslash G/K$. The characteristic classes of this type of foliations were extensively studied and calculated by Kamber-Tondeur [KT75b], Baker [Bak78], Heitsch [Hei78], Pittie [Pit79], Pelletier [Pel83] and Asuke [Asu10].

3. Transversely homogeneous foliations

3.1. **Definition of $(G, G/P)$-foliations.** Let $(M, \mathcal{F})$ be a foliated manifold. Let G be a Lie group and P a closed subgroup of G. When the group G is endowed with the discrete topology, it is denoted by G^δ. We denote the G-action on G/P by $(g, xP) \mapsto g \cdot xP$.

Definition 3.1. A (*Haefliger*) *cocycle* with values in $(G, G/P)$, defining $\mathcal{F}$, is a triple $(\{U_i\}, \{\pi_i\}, \{\gamma_{ij}\})$, where:

(1) $\{U_i\}$ is an open covering of M,
(2) each π_i is a submersion $U_i \to G/P$ such that the leaves of $\mathcal{F}|_{U_i}$ are the fibers of π_i, and
(3) each γ_{ij} is a continuous map $U_i \cap U_j \to G^\delta$ such that $\pi_i(x) = \gamma_{ij}(x) \cdot \pi_j(x)$ for any $x \in U_i \cap U_j$.

Two cocycles with values in $(G, G/P)$, defining $\mathcal{F}$, are called *equivalent* when their union is contained in some cocycle with values in $(G, G/P)$, defining $\mathcal{F}$. When $\mathcal{F}$ is endowed with an equivalence class of cocycles with values in $(G, G/P)$, defining $\mathcal{F}$, it is called a $(G, G/P)$-*foliation*.

Cocycles valued in $(G, G/P)$ are examples of 1-cocycles valued in groupoids defined by Haefliger [Hae58]. Transversely homogeneous foliations are natural generalizations of quotient of homogeneous foliations on homogeneous spaces in terms of 1-cocycles valued in groupoids.

Remark 3.2. When G preserves a metric on G/P, any $(G, G/P)$-foliation is Riemannian. In this case, the secondary characteristic classes are well known to be trivial (for example, see [KT75a, Section 4.48 and Theorem 4.52]).

Example 3.3. Example 2.2 is an $(\mathrm{SL}(2;\mathbb{R}), S^1)$-foliation, and Example 2.4 a $(G, G/P)$-foliation. Example 2.3 is a special case of Example 2.4, where $(G, G/P)$ is $(\mathrm{SO}(n+1,1), S^n_\infty)$, $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ or $(F_{4(-20)}, S^{15}_\infty)$, and where S^n_∞, S^{2n+1}_∞, S^{4n+3}_∞ or S^{15}_∞ are the ideal boundaries of the corresponding hyperbolic symmetric spaces.

Example 3.4 (Suspension foliations)**.** Let N be a smooth manifold and $h\colon \pi_1 N \to G$ a homomorphism. A $\pi_1 N$-action on G/P is defined by $\pi_1 N \to G \to \mathrm{Diff}(G/P)$, where the second homomorphism is the G-action on G/P. Then the quotient space $\widetilde{N} \times_{\pi_1 N} G/P$ of the diagonal $\pi_1 N$-action on $\widetilde{N} \times G/P$ has a foliation $\mathcal{F}$ induced by the horizontal foliation $\widetilde{N} \times G/P = \bigsqcup_{x \in G/P} \widetilde{N} \times \{x\}$. Here, it is easy to see that $\mathcal{F}$ naturally admits a structure of $(G, G/P)$-foliation by definition. (One can also apply Proposition 3.8 below.)

Example 3.5. Let $(M_i, \mathcal{F}_i)$ be a smooth manifold with a $(G, G/P)$-foliation for $i \in \{0, 1\}$. Assume that we have a closed transversal S_i of $(M_i, \mathcal{F}_i)$ such that S_0 is diffeomorphic to S_1 as $(G, G/P)$-manifolds. Let U_i be an open tubular neighborhood of S_i such that the leaves of $\mathcal{F}_i|_{U_i}$ are the fibers of a normal bundle of S_i. We can paste $U_0 \setminus S_0$ and $U_1 \setminus S_1$ to construct another manifold with a $(G, G/P)$-foliation. Chihi and ben Ramdane [CbR08] used this method to construct manifolds with $(\mathrm{SL}(2;\mathbb{R}), S^1)$-foliations with nontrivial Godbillon-Vey classes and dense holonomy groups in $\mathrm{SL}(2;\mathbb{R})$.

Example 3.6. Let $(M,\mathcal{F})$ be a smooth manifold with a $(G,G/P)$-foliation. If we have a smooth map $f\colon M'\to M$ which is transverse to $\mathcal{F}$, we can pull back $\mathcal{F}$ to M' as a $(G,G/P)$-foliation. This construction can be used when f is a branched covering whose branch locus is transverse to $\mathcal{F}$.

Example 3.7. Thurston [Thu72b] constructed examples of codimension one foliations on Seifert fibered 3-manifolds whose Godbillon-Vey class varies nontrivially by making surgery to Example 2.2. Rasmussen [Ras80] generalized this construction to the case of codimension two. Thurston also constructed families of suspension foliations on the total spaces of S^1-bundles over closed surfaces of genus two whose characteristic classes vary nontrivially. These examples are constructed by pasting two transversely projective foliations of the total space of S^1-bundles over punctured tori [Bot78, Section 4]. Heitsch [Hei78] constructed families of $(\prod_{i=1}^k \mathrm{SL}(n_i;\mathbb{R}), S^{(\sum_i n_i)-1})$-foliations whose characteristic classes vary by deforming $\prod_{i=1}^k \mathrm{SL}(n_i;\mathbb{R})$-actions on $S^{(\sum_i n_i)-1}$.

3.2. Haefliger type description of transversely homogeneous foliations.

3.2.1. *Flat principal G-bundle associated to $\mathcal{F}$ and the holonomy homomorphism.* Let $(M,\mathcal{F})$ be a $(G,G/P)$-foliation defined by a cocycle $(\{U_i\},\{\pi_i\},\{\gamma_{ij}\})$ valued in $(G,G/P)$. The condition $\pi_i=\gamma_{ij}\cdot\pi_j$ implies the 1-cocycle condition $\gamma_{ik}=\gamma_{ij}\cdot\gamma_{jk}$. Thus $\{\gamma_{ij}\}$ is a 1-cocycle valued in G^δ, which defines a flat principal G-bundle $\pi_G\colon \mathcal{X}_G(\mathcal{F})\to M$. Recall that

$$\mathcal{X}_G(\mathcal{F})=\left(\bigsqcup_i U_i\times G\right)\Big/(x,y)\sim(x,\gamma_{ij}(x)(y))\ ,$$

and the projection π_G is induced by the first factor projections $U_i\times G\to U_i$. The holonomy homomorphism $\pi_1 M\to G$ of this flat G-bundle is called the *holonomy homomorphism* of $\mathcal{F}$ and denoted by $\mathrm{hol}(\mathcal{F})$.

3.2.2. *The Haefliger structure of $\mathcal{F}$.* We recall the description of $(G,G/P)$-foliations in terms of a G/P-bundle over M, which is a special case of the Haefliger structures of general foliations. It was studied by Blumenthal [Blu79] and used by Brooks-Goldman [BG84] and Heitsch [Hei86] to prove Theorem 1.1.

Proposition 3.8. *A $(G,G/P)$-foliation $\mathcal{F}$ on M is determined by one of the following data:*

- (i) *A flat principal G-bundle $\mathcal{X}_G\to M$ and a section s of $\mathcal{X}_G/P\to M$ such that s is transverse to the foliation $\mathcal{E}$ of $\mathcal{X}_G/P$ defined by the flat G-connection.*
- (ii) *A homomorphism* $\mathrm{hol}\colon\pi_1M\to G$ *and a submersion* $\mathrm{dev}\colon\widetilde{M}\to G/P$ *such that* $\mathrm{dev}(\gamma\cdot x)=\mathrm{hol}(\gamma)\cdot\mathrm{dev}(x)$ *for any $x\in\widetilde{M}$ and any $\gamma\in\pi_1M$.*

Let $\overline{\gamma}_{ij}(x)\colon G/P\to G/P$ be the diffeomorphism induced by the left product of $\gamma_{ij}(x)$. Here, $\{\overline{\gamma}_{ij}\}$ is a 1-cocycle valued in $\mathrm{Diff}(G/P)^\delta$, which defines a G/P-bundle $\pi_{G/P}\colon \mathcal{X}_{G/P}(\mathcal{F}) \to M$ with a flat G-connection whose holonomy homomorphism is equal to $\operatorname{hol}(\mathcal{F})$. Recall that

$$\mathcal{X}_{G/P}(\mathcal{F}) = \Big(\bigsqcup_i U_i \times G/P\Big)\Big/(x,y) \sim (x, \overline{\gamma}_{ij}(x)(y)) = \mathcal{X}_G(\mathcal{F})/P\ ,$$

and the projection $\pi_{G/P}$ is induced by the first factor projections $U_i \times G/P \to U_i$. The graphs of the maps π_i,

$$\operatorname{Graph}(\pi_i) = \{\, (x, \pi_i(x)) \mid x \in U_i \,\} \subset U_i \times G/P\ ,$$

define a subset of $\mathcal{X}_{G/P}(\mathcal{F})$, which gives a global section s of $\mathcal{X}_{G/P}(\mathcal{F}) \to M$. By construction, $\mathcal{F}$ is obtained as the pull-back by s of the foliation of $\mathcal{X}_{G/P}(\mathcal{F})$ defined by the flat connection. Summarizing, $\mathcal{F}$ determines a flat G/P-bundle $\pi_{G/P}\colon \mathcal{X}_{G/P}(\mathcal{F}) \to M$ with a section s, which in turn determines $\mathcal{F}$.

Let $\widetilde{M}$ be the universal cover of M. The pull-back of $\mathcal{X}_G(\mathcal{F})/P \to M$ to $\widetilde{M}$ is a trivial flat G/P-bundle. A section s of $\mathcal{X}_G(\mathcal{F})/P \to M$ yields a section $\widetilde{s}$ of this trivial G/P-bundle over $\widetilde{M}$ by pull-back. In an obvious way, giving $\widetilde{s}$ is equivalent to giving a submersion $\operatorname{dev}\colon \widetilde{M} \to G/P$ that is $\pi_1 M$-equivariant with respect to $\operatorname{hol}(\mathcal{F})\colon \pi_1 M \to G$; i.e., $\operatorname{dev}(\gamma\cdot x) = \operatorname{hol}(\mathcal{F})(\gamma)\cdot \operatorname{dev}(x)$ for $x \in \widetilde{M}$ and $\gamma \in \pi_1 M$.

3.2.3. *Enlargement of the Haefliger structure of $\mathcal{F}$.* We will use a bundle larger than the one described in the last section, which was used by Benson-Ellis [BE85]. Let K_P be a maximal compact subgroup of P. We consider a G/K_P-bundle $\pi_{G/K_P}\colon \mathcal{X}_G(\mathcal{F})/K_P \to M$ with a flat G-connection constructed by a 1-cocycle valued in $\operatorname{Diff}(G/K_P)^\delta$ in a way analogous to $\pi_{G/P}$ in the last section. There is also a P/K_P-bundle $p\colon \mathcal{X}_G(\mathcal{F})/K_P \to \mathcal{X}_G(\mathcal{F})/P$. Since P/K_P is contractible, there is a section s' of p, which is unique up to homotopy. We get a section $\hat{s}$ of π_{G/K_P} defined by the composite

$$M \xrightarrow{\ s\ } \mathcal{X}_G(\mathcal{F})/P \xrightarrow{\ s'\ } \mathcal{X}_G(\mathcal{F})/K_P\ .$$

Clearly, $\hat{s}$ is transverse to the foliation $p^*\mathcal{E}_{\operatorname{hol}(\mathcal{F})}$ of $\mathcal{X}_G(\mathcal{F})/K_P$, where $\mathcal{E}_{\operatorname{hol}(\mathcal{F})}$ is the foliation of $\mathcal{X}_G(\mathcal{F})/P$ defined by the flat G-connection. Thus we get the following.

Proposition 3.9. *A $(G, G/P)$-foliation $\mathcal{F}$ on M is determined by one of the following data:*

(i) *A flat principal G-bundle $\mathcal{X}_G \to M$ and a section $\hat{s}$ of $\mathcal{X}_G/K_P \to M$ such that $\hat{s}$ is transverse to the foliation $p^*\mathcal{E}$ of $\mathcal{X}_G/K_P$, where $p\colon \mathcal{X}_G/K_P \to \mathcal{X}_G/P$ is the canonical projection and $\mathcal{E}$ is the foliation of $\mathcal{X}_G/P$ defined by the flat G-connection.*

(ii) *A homomorphism $\operatorname{hol}\colon \pi_1 M \to G$ and a smooth map $\widehat{\operatorname{dev}}\colon \widetilde{M} \to G/K_P$ such that $\widehat{\operatorname{dev}}$ is transverse to the foliation defined by the fibers of $G/K_P \to G/P$ and $\widehat{\operatorname{dev}}(\gamma\cdot x) = \operatorname{hol}(\gamma)\cdot\widehat{\operatorname{dev}}(x)$ for any $x \in \widetilde{M}$ and $\gamma \in \pi_1 M$.*

4. Characteristic classes of transversely homogeneous foliations

4.1. **Bott connections on the P/K_P-coset foliation of G/K_P.** Assume that G is semisimple and P is a closed subgroup of G. Recall that K_P is a maximal compact subgroup of P. In this section, we will recall the well known construction of a left invariant Bott connection on the normal bundle of the right P/K_P-coset foliation $\mathcal{F}_P$ on G/K_P, originally due to Kamber-Tondeur [KT75b, Theorem 3.7] (announced in [KT74]).

Let $\sigma \colon \mathfrak{g}/\mathfrak{p} \to \mathfrak{g}$ be a splitting of the exact sequence

$$0 \longrightarrow \mathfrak{p} \longrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\mathfrak{p} \longrightarrow 0 .$$

Then consider the connection $\widetilde{\nabla}$ on the normal bundle $\nu\mathcal{G}_P$ of the right P-coset foliation $\mathcal{G}_P$ on G determined by

$$\widetilde{\nabla}_X Y = \pi\big([(\mathrm{id}_{\mathfrak{g}} - \sigma\pi)X, \sigma(Y)]\big)$$

for $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}/\mathfrak{p}$. Observe that $\widetilde{\nabla}$ is left invariant. For $X \in \mathfrak{p}$, we get $\widetilde{\nabla}_X Y = \mathrm{ad}(X)(Y)$. This fact implies that $\widetilde{\nabla}$ is a Bott connection on $\nu\mathcal{G}_P$. If we take an $\mathrm{ad}\, K_P$-equivariant section σ, then $\widetilde{\nabla}$ induces a left invariant Bott connection ∇ on $\nu\mathcal{F}_P$.

Let $(\bigwedge \mathfrak{g}^*)_{K_P}$ be the K_P-basic subalgebra of $\bigwedge \mathfrak{g}^*$; namely,

$$\left(\bigwedge \mathfrak{g}^*\right)_{K_P} = \left\{ \beta \in \bigwedge \mathfrak{g}^* \;\middle|\; \iota_X\beta = 0 \ \forall X \in \mathrm{Lie}(K_P),\ \mathrm{Ad}(g)^*\beta = \beta \ \forall g \in K_P \right\} ,$$

which is identified to the algebra of left invariant differential forms on G/K_P. By the left invariance of ∇, we get $\Delta_{\mathcal{F}_P} \colon WO_q \to (\bigwedge \mathfrak{g}^*)_{K_P}$. Let $\mathfrak{g}^*_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g} \otimes \mathbb{C}, \mathbb{C})$. Let $P_{\mathbb{C}}$ be the connected Lie subgroup of $G_{\mathbb{C}}$ such that $\mathrm{Lie}(P_{\mathbb{C}}) = \mathrm{Lie}(P) \otimes \mathbb{C}$. By complexifying ∇, we get a complex connection $\nabla^{\mathbb{C}}$ on the complexified normal bundle of the right $P_{\mathbb{C}}$-coset foliation $\mathcal{F}_{P_{\mathbb{C}}}$ on $G_{\mathbb{C}}/(K_P)_{\mathbb{C}}$, obtaining the characteristic homomorphism $\Delta_{\mathcal{F}_{P_{\mathbb{C}}}} \colon WO_q \otimes \mathbb{C} \to (\bigwedge \mathfrak{g}^*_{\mathbb{C}})_{(K_P)_{\mathbb{C}}}$. Thus we get that the following diagram commutes:

(7)
$$\begin{array}{ccc} & & (\bigwedge \mathfrak{g}^*_{\mathbb{C}})_{(K_P)_{\mathbb{C}}} \\ & \overset{\Delta_{\mathcal{F}_{P_{\mathbb{C}}}}}{\nearrow} & \downarrow \\ WO_q \otimes \mathbb{C} & \xrightarrow[\Delta_{\mathcal{F}_P}]{} & (\bigwedge \mathfrak{g}^*)_{K_P} \otimes \mathbb{C} , \end{array}$$

where the vertical arrow is canonical.

4.2. **Complexification of the enlargement of Haefliger structures.** Let $\mathcal{F}$ be a $(G, G/P)$-foliation of a manifold M. Let $G_{\mathbb{C}}$ be the connected and simply connected complex Lie group with $\mathrm{Lie}(G_{\mathbb{C}}) = \mathrm{Lie}(G) \otimes \mathbb{C}$. Let K_P be the maximal compact subgroup of P. Let $\pi_{G/K_P} \colon \mathcal{X}_G(\mathcal{F})/K_P \to M$ be the enlargement of the Haefliger structure considered in Proposition 3.9.

We construct the fiberwise complexification of π_{G/K_P} as follows. Let $\mathrm{hol}(\mathcal{F})_{\mathbb{C}}$ denote the composite

$$\pi_1 M \xrightarrow{\mathrm{hol}(\mathcal{F})} G \longrightarrow G_{\mathbb{C}} .$$

Let $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})$ be the quotient of $\widetilde{M} \times G_{\mathbb{C}}$ by the diagonal action of $\pi_1 M$, obtaining a flat principal $G_{\mathbb{C}}$-bundle $\pi_{G_{\mathbb{C}}} \colon \mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F}) \to M$ whose holonomy homomorphism is $\mathrm{hol}(\mathcal{F})_{\mathbb{C}}$. Then we get a canonical map $\mathcal{X}_G(\mathcal{F})/K_P \to \mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/(K_P)_{\mathbb{C}}$, which is a complexification map $G/K_P \to G_{\mathbb{C}}/(K_P)_{\mathbb{C}}$ on each fiber. Thus a section s of $\mathcal{X}_G(\mathcal{F})/K_P \to M$ gives a section $s_{\mathbb{C}}$ of $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/(K_P)_{\mathbb{C}} \to M$.

The universal covers of $\mathcal{X}_G(\mathcal{F})/K_P$ and $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/(K_P)_{\mathbb{C}}$ are the products $\widetilde{M} \times G/K_P$ and $\widetilde{M} \times G_{\mathbb{C}}/(K_P)_{\mathbb{C}}$, respectively. Consider the diagram

$$\begin{array}{ccc}
(\bigwedge \mathfrak{g}_{\mathbb{C}}^*)_{(K_P)_{\mathbb{C}}} & \longrightarrow & \Omega^\bullet(\widetilde{M} \times G_{\mathbb{C}}/(K_P)_{\mathbb{C}}; \mathbb{C}) \\
\downarrow & & \downarrow \\
(\bigwedge \mathfrak{g}^*)_{K_P} \otimes \mathbb{C} & \longrightarrow & \Omega^\bullet(\widetilde{M} \times G/K_P; \mathbb{C}) ,
\end{array}$$

where the horizontal arrows are the pull-back by the second projections and the vertical arrows are the canonical maps defined by complexification. Since $\pi_1 M$ acts on G/K_P and $G_{\mathbb{C}}/(K_P)_{\mathbb{C}}$ by the left product of G, left invariant forms on G/K_P and $G_{\mathbb{C}}/(K_P)_{\mathbb{C}}$ descend to $\mathcal{X}_G(\mathcal{F})/K_P$ and $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/(K_P)_{\mathbb{C}}$. Thus we get the commutative diagram

$$\begin{array}{ccc}
(\bigwedge \mathfrak{g}_{\mathbb{C}}^*)_{(K_P)_{\mathbb{C}}} & \longrightarrow & \Omega^\bullet(\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/(K_P)_{\mathbb{C}}; \mathbb{C}) \\
\downarrow & & \downarrow \\
(\bigwedge \mathfrak{g}^*)_{K_P} \otimes \mathbb{C} & \longrightarrow & \Omega^\bullet(\mathcal{X}_G(\mathcal{F})/K_P; \mathbb{C}) .
\end{array} \tag{8}$$

Recall that $P_{\mathbb{C}}$ is the connected Lie subgroup of $G_{\mathbb{C}}$ with $\mathrm{Lie}(P_{\mathbb{C}}) = \mathrm{Lie}(P) \otimes \mathbb{C}$. Combining the diagrams (7) and (8), we get the following.

Proposition 4.1. *The following diagram is commutative:*

$$\begin{array}{ccc}
 & & H^\bullet(\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/(K_P)_{\mathbb{C}}; \mathbb{C}) \\
 & \nearrow^{\Delta_{\widehat{\mathcal{E}}_{\mathbb{C}}}} & \downarrow \\
H^\bullet(WO_q \otimes \mathbb{C}) & \xrightarrow[\Delta_{\widehat{\mathcal{E}}}]{} & H^\bullet(\mathcal{X}_G(\mathcal{F})/K_P; \mathbb{C}) ,
\end{array}$$

where $\widehat{\mathcal{E}}$ is the pull-back of the $(G, G/P)$-foliation of $\mathcal{X}_G(\mathcal{F})/P$ by the projection $\mathcal{X}_G(\mathcal{F})/K_P \to \mathcal{X}_G(\mathcal{F})/P$, and $\widehat{\mathcal{E}}_{\mathbb{C}}$ is the pull-back of the $(G_{\mathbb{C}}, G_{\mathbb{C}}/P_{\mathbb{C}})$-foliation of $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/P_{\mathbb{C}}$ by the projection $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/(K_P)_{\mathbb{C}} \to \mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/P_{\mathbb{C}}$.

The following simple observation is the unique new idea in our proof of Theorem 1.2.

Proposition 4.2. *Assume that* $H^\bullet(G_{\mathbb{C}}/P;\mathbb{R}) \to H^\bullet(G/P;\mathbb{R})$ *is trivial on positive degrees. Then the image of* $\Delta_{\mathcal{E}_{\mathrm{hol}(\mathcal{F})}} : H^\bullet(WO_q) \to H^\bullet(\mathcal{X}_G(\mathcal{F})/K_P;\mathbb{R})$ *is contained in the image of* $\pi^*_{G/K_P} : H^\bullet(M;\mathbb{R}) \to H^\bullet(\mathcal{X}_G(\mathcal{F})/K_P;\mathbb{R})$.

Proof. By Proposition 4.1, the image of $\Delta_{\mathcal{E}_{\mathrm{hol}(\mathcal{F})}}$ is contained in the image of

$$H^\bullet(\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/(K_P)_{\mathbb{C}};\mathbb{R}) \longrightarrow H^\bullet(\mathcal{X}_G(\mathcal{F})/K_P;\mathbb{R}) .$$

Consider the Leray-Serre spectral sequences associated to the fiber bundles

$$\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/(K_P)_{\mathbb{C}} \to M , \quad \mathcal{X}_G(\mathcal{F})/K_P \to M .$$

Since $(K_P)_{\mathbb{C}}$ and K_P are homotopically equivalent to P, it follows that $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/(K_P)_{\mathbb{C}}$ and $\mathcal{X}_G(\mathcal{F})/K_P$ are homotopically equivalent to $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/P$ and $\mathcal{X}_G(\mathcal{F})/P$, respectively. Thus the restriction map between E_2-terms is given by

$$r\colon H^\bullet\big(M,\mathcal{H}^\bullet(G_{\mathbb{C}}/P)\big) \longrightarrow H^\bullet\big(M,\mathcal{H}^\bullet(G/P)\big) ,$$

where $\mathcal{H}^\bullet(G_{\mathbb{C}}/P)$ and $\mathcal{H}^\bullet(G/P)$ are the corresponding local systems associated to $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/P$ and $\mathcal{X}_G(\mathcal{F})/P$, respectively. By the assumption of the triviality of $H^\bullet(G_{\mathbb{C}}/P;\mathbb{R}) \to H^\bullet(G/P;\mathbb{R})$ on positive degrees, it follows that the image of r is contained in $H^\bullet(M;\mathbb{R})$. □

4.3. **Two results of Benson-Ellis.** Let $H^\bullet(\mathfrak{g},K_P) = H^\bullet((\bigwedge \mathfrak{g}^*)_{K_P})$. Let $\mathcal{F}$ be a $(G,G/P)$-foliation of a manifold M. Assume that G is semisimple.

Theorem 4.3 (Benson-Ellis [BE85])**.** *The following diagram commutes:*

$$\begin{array}{ccccc} & & H^\bullet(\mathfrak{g},K_P) & & \\ & \overset{\Delta_{\mathcal{F}_P}}{\nearrow} & & \searrow & \\ H^\bullet(WO_q) & & \xrightarrow[\Delta_{\mathcal{F}}]{\hspace{6em}} & & H^\bullet(M;\mathbb{R}) , \end{array}$$

Note that the argument in the last section gives an alternative proof of Theorem 4.3.

Let U be an open subset of $\mathbb{R}^\ell$.

Theorem 4.4 (Benson-Ellis [BE85], see also Haefliger [Hei86, Theorem in Section 6])**.** *For a smooth family* $\{\mathcal{F}_t\}_{t\in U}$ *of* $(G,G/P)$*-foliations of* M*, the family* $\{\Delta_{\mathcal{F}_t}\}_{t\in U}$ *is locally constant in* $\mathrm{Hom}(H^\bullet(WO_q),H^\bullet(M;\mathbb{R}))$.

This rigidity comes from the vanishing results of cohomology of representations of semisimple Lie algebras.

5. Proof of Theorem 1.2

Like in the proof of Theorem 1.1 by Brooks-Goldman and Heitsch, the unique essential part of the proof of Theorem 1.2 is the following proposition.

Proposition 5.1. *If the holonomy homomorphisms of two* $(G,G/P)$*-foliations,* $\mathcal{F}_0$ *and* $\mathcal{F}_1$*, on* M *are homotopic, then* $\Delta_{\mathcal{F}_0} = \Delta_{\mathcal{F}_1}$.

Now, Theorem 1.2 follows from Proposition 5.1 with the arguments of Brooks-Goldman [BG84, Lemma 2].

Proof of Theorem 1.2 by using Proposition 5.1. Recall that we assume that $\pi_1 M$ is finitely presented. It is well known that $\pi_0(\mathrm{Hom}(\pi_1 M, G))$ is finite (see Remark 5.2 at the end of this section). Thus there exist a finite number of $(G, G/P)$-foliations $\mathcal{F}_1, \ldots, \mathcal{F}_m$ of M such that, for any $(G, G/P)$-foliation $\mathcal{F}$ of M, its holonomy homomorphism is in the same connected component of $\mathrm{Hom}(\pi_1 M, G)$ as the holonomy homomorphism of some $\mathcal{F}_i$. Thus Proposition 5.1 implies Theorem 1.2. □

Proposition 5.1 directly follows from Theorem 4.4 and Proposition 4.2.

Proof of Proposition 5.1. Let $\mathcal{X}_G(\mathcal{F}_i)/K_P \to M$ be the enlargement of the Haefliger structure of $\mathcal{F}_i$ considered in Proposition 3.9 for $i \in \{0, 1\}$. Recall that a section $s_i \colon M \to \mathcal{X}_G(\mathcal{F}_i)/K_P$ is associated to $\mathcal{F}_i$. Consider the foliation $\widehat{\mathcal{E}}_i = p_i^* \mathcal{E}_{\mathrm{hol}(\mathcal{F}_i)}$ of $\mathcal{X}_G(\mathcal{F}_i)/K_P$, where $p_i \colon \mathcal{X}_G(\mathcal{F}_i)/K_P \to \mathcal{X}_G(\mathcal{F}_i)/P$ is the canonical projection and $\mathcal{E}_{\mathrm{hol}(\mathcal{F}_i)}$ is the foliation of $\mathcal{X}_G(\mathcal{F}_i)/P$ defined by the flat G-connection.

The homotopy class of $(\mathcal{X}_G(\mathcal{F}_i)/K_P, \widehat{\mathcal{E}}_i)$ as a $(G, G/P)$-foliation is determined by the homotopy class of the holonomy homomorphism of $\mathcal{F}_i$. Thus, by assumption and Theorem 4.4, we get $\Delta_{\widehat{\mathcal{E}}_0} = \Delta_{\widehat{\mathcal{E}}_1}$.

By Proposition 4.2, the image of $\Delta_{\widehat{\mathcal{E}}_0}$ is contained in the image of $p^* \colon H^\bullet(M; \mathbb{R}) \to H^\bullet(\mathcal{X}_G(\mathcal{F}_0)/K_P; \mathbb{R})$. Thus $(s^0)^* \Delta_{\widehat{\mathcal{E}}_0} = (s^1)^* \Delta_{\widehat{\mathcal{E}}_0}$ on $H^\bullet(WO_q)$, and therefore

$$\Delta_{\mathcal{F}_0} = (s^0)^* \Delta_{\widehat{\mathcal{E}}_0} = (s^1)^* \Delta_{\widehat{\mathcal{E}}_0} = (s^1)^* \Delta_{\widehat{\mathcal{E}}_1} = \Delta_{\mathcal{F}_1} \; . \qquad \square$$

Remark 5.2. For a finitely presented group S with k generators, we can give $\mathrm{Hom}(S, \mathrm{GL}(n; \mathbb{R}))$ the structure of a real algebraic variety via a tautological embedding $j \colon \mathrm{Hom}(S, \mathrm{GL}(n; \mathbb{R})) \to \mathrm{GL}(n; \mathbb{R})^k$ (this is an observation of Lusztig as written in [Sul76, Footnote of p. 186]). For an algebraic group G of $\mathrm{GL}(n; \mathbb{R})$, we see that

$$\mathrm{Hom}(S, G) = j\Big(\mathrm{Hom}\big(S, \mathrm{GL}(n; \mathbb{R})\big)\Big) \cap G^k$$

is also a real algebraic variety. Thus $\pi_0\big(\mathrm{Hom}(S, G)\big)$ is finite by a theorem of Whitney [Whi57].

Remark 5.3. We indicate an alternative way to prove the finiteness of the Godbillon-Vey class by using the complexification of the Haefliger structure of $\mathcal{F}$ under the assumption of the triviality of $H^\bullet(G_{\mathbb{C}}/P_{\mathbb{C}}; \mathbb{R}) \to H^\bullet(G/P; \mathbb{R})$ on positive degrees. Note that this assumption is weaker than the assumption of the triviality of $H^\bullet(G_{\mathbb{C}}/P; \mathbb{R}) \to H^\bullet(G/P; \mathbb{R})$ on positive degrees. Consider a $G_{\mathbb{C}}/P_{\mathbb{C}}$-bundle $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/P_{\mathbb{C}} \to M$, which is regarded as the complexification of the Haefliger structure $\mathcal{X}_G(\mathcal{F})/P \to M$ of $\mathcal{F}$. Assume that $c_1(\mathcal{E}^{\mathbb{C}}_{\mathrm{hol}(\mathcal{F})})$ is trivial if $\dim G/P$ is even. By results of Asuke [Asu03, Corollary 1.9 and Proposition 2.2], the Godbillon-Vey class extends to $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/P_{\mathbb{C}}$. So, if $H^\bullet(G_{\mathbb{C}}/P_{\mathbb{C}}; \mathbb{R}) \to H^\bullet(G/P; \mathbb{R})$

is trivial on positive degrees, then we get the finiteness of the Godbillon-Vey class like in the above proof of Theorem 1.2.

Remark 5.4. We can show the triviality of $H^\bullet(G_\mathbb{C}/P_\mathbb{C};\mathbb{R}) \to H^\bullet(G/P;\mathbb{R})$ on positive degrees by using the Schubert cell decomposition of $G_\mathbb{C}/P_\mathbb{C}$ if $G_\mathbb{C}/P_\mathbb{C}$ is a generalized Bott tower; namely, the total space of consecutive complex projective space bundles and G/P is the total space of the corresponding consecutive real projective space bundles. The Schubert cell decomposition of $G_\mathbb{C}/P_\mathbb{C}$ is a cell decomposition whose cells are orbits of the action of a Borel subgroup of $G_\mathbb{C}$. This cell decomposition induces a cell decomposition of G/P. In the case of generalized Bott towers, we can contract the inclusion $G/P \to G_\mathbb{C}/P_\mathbb{C}$ cell by cell to a constant map.

6. Examples

6.1. **The Euler class of the bundle $G_\mathbb{C}/P \to G_\mathbb{C}/G$.** Let us consider the case of $G/P = S^q$. We characterize the assumption of Theorem 1.2 by the nontriviality of the Euler class of the sphere bundle

$$G/P \longrightarrow G_\mathbb{C}/P \longrightarrow G_\mathbb{C}/G\ ,$$

which is homotopy equivalent to

$$K_G/K_P \longrightarrow K_{G_\mathbb{C}}/K_P \xrightarrow{\varphi} K_{G_\mathbb{C}}/K_G\ . \tag{9}$$

Proposition 6.1. *$H^\bullet(G_\mathbb{C}/P;\mathbb{R}) \to H^\bullet(G/P;\mathbb{R})$ is trivial on positive degrees if and only if the Euler class e of φ is nontrivial in $H^{q+1}(G_\mathbb{C}/G;\mathbb{R})$.*

Proof. From the Gysin sequence of φ, we get an exact sequence

$$H^q(G_\mathbb{C}/P;\mathbb{R}) \xrightarrow{\int_\varphi} H^0(G_\mathbb{C}/G;\mathbb{R}) \xrightarrow{\wedge e} H^{q+1}(G_\mathbb{C}/G;\mathbb{R})\ .$$

Thus e is nontrivial if and only if the image of $\int_\varphi$ is nontrivial. In turn, the image of $\int_\varphi$ is nontrivial if and only if the restriction map $H^q(G_\mathbb{C}/P) \to H^q(G/P)$ is nontrivial. □

6.2. **The case of transversely projective foliations of odd codimension.** In this case, $(G, G/P) = (\mathrm{SL}(q+1;\mathbb{R}), S^q)$ for odd q. Let $q = 2k-1$ and $Y_\ell = \mathrm{SU}(\ell)/\mathrm{SO}(\ell)$. Now, the sphere bundle (9) is

$$\mathrm{SO}(2k)/\mathrm{SO}(2k-1) \longrightarrow \mathrm{SU}(2k)/\mathrm{SO}(2k-1) \xrightarrow{p_k} Y_{2k}\ . \tag{10}$$

We show that the nontriviality of the Euler class of (10) follows from the Borel's computation of the Betti numbers of homogeneous spaces [Bor53].

Lemma 6.2. *The Euler class of* (10) *is nontrivial in $H^{2k}(Y_{2k})$.*

Proof. According to the computation of $H^\bullet(Y_\ell)$ by Borel [Bor53, Proposition 31.4], we get that

$$H^\bullet(Y_{2k}) \longrightarrow H^\bullet(Y_{2k-1}) \tag{11}$$

is surjective and

$$\dim H^\bullet(Y_{2k}) = 2 \dim H^\bullet(Y_{2k-1}) \;. \tag{12}$$

Consider also the fibration

$$Y_{2k-1} \xrightarrow{\iota} \mathrm{SU}(2k)/\,\mathrm{SO}(2k-1) \longrightarrow \mathrm{SU}(2k)/\,\mathrm{SU}(2k-1) \cong S^{4k-1} \;.$$

Assume that the Euler class of p_k is trivial. Then

$$\dim H^\bullet(\mathrm{SU}(2k)/\,\mathrm{SO}(2k-1)) = \dim H^\bullet(S^{2k-1}) \cdot \dim H^\bullet(Y_{2k}) \;; \tag{13}$$

in particular, $p_k^*\colon H^\bullet(Y_{2k}) \to H^\bullet(\mathrm{SU}(2k)/\,\mathrm{SO}(2k-1))$ is injective. By the surjectivity of (11), we get the surjectivity of $\iota^*\colon H^\bullet(\mathrm{SU}(2k)/\,\mathrm{SO}(2k-1)) \to H^\bullet(Y_{2k-1})$. Thus, by the Leray-Hirsch theorem, we obtain

$$\dim H^\bullet(\mathrm{SU}(2k)/\,\mathrm{SO}(2k-1)) = \dim H^\bullet(Y_{2k-1}) \cdot \dim H^\bullet(S^{4k-1}) \;. \tag{14}$$

But (13) and (14) contradict (12). Thus the Euler class of p_k is nontrivial. □

So $H^\bullet(K_{G_\mathbb{C}}/K_P;\mathbb{R}) \to H^\bullet(K_G/K_P;\mathbb{R})$ is trivial on positive degrees. Thus Theorem 1.2 gives an alternative proof of Theorem 1.1 for the case of odd codimension.

6.3. **The case of transversely conformally flat foliations.** Now, $(G, G/P) = (\mathrm{SO}(n+1,1), S^n_\infty)$. So $G_\mathbb{C} = \mathrm{SO}(n+2;\mathbb{C})$, and

$$K_{G_\mathbb{C}} = \mathrm{SO}(n+2) \;, \quad K_G = \mathrm{S}(\mathrm{O}(n+1)\times\{\pm1\}) \;, \quad K_P = \mathrm{S}(\mathrm{O}(n)\times\{\pm1\}) \;.$$

Thus the sphere bundle (9) is

$$\begin{array}{ccc} \mathrm{S}(\mathrm{O}(n+1)\times\{\pm1\})/\,\mathrm{S}(\mathrm{O}(n)\times\{\pm1\}) & \longrightarrow & \mathrm{SO}(n+2)/\,\mathrm{S}(\mathrm{O}(n)\times\{\pm1\}) \\ & & \downarrow{\scriptstyle \zeta_{\mathrm{SO}}} \\ & & \mathrm{SO}(n+2)/\,\mathrm{S}(\mathrm{O}(n+1)\times\{\pm1\}) \;. \end{array}$$

The isotropy group of the $\mathrm{SO}(n+2)$-action on the unit tangent sphere bundle of $\mathrm{SO}(n+2)/\,\mathrm{S}(\mathrm{O}(n+1)\times\{\pm1\})$ is $\mathrm{S}(\mathrm{O}(n)\times\{\pm1\})$. So ζ_{SO} is the unit tangent sphere bundle of $\mathrm{SO}(n+2)/\,\mathrm{S}(\mathrm{O}(n+1)\times\{\pm1\}) \cong \mathbb{R}P^{n+1}$. Hence the Euler class of ζ_{SO} is equal to the fundamental class of $\mathbb{R}P^{n+1}$ if n is odd. Thus, by Proposition 6.1, the assumption of Theorem 1.2 is satisfied in this case.

6.4. **The case of transversely spherical CR foliations.** Now, $(G, G/P) = (\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, where the codimension $q = 2n+1$ is odd. In this case, $G_\mathbb{C} = \mathrm{SL}(n+2;\mathbb{C})$ and

$$K_{G_\mathbb{C}} = \mathrm{SU}(n+2)\ , \quad K_G = \mathrm{S}(\mathrm{U}(n+1)\,\mathrm{U}(1))\ , \quad K_P = \mathrm{S}(\mathrm{U}(n)\,\mathrm{U}(1))\ .$$

Thus the sphere bundle (9) is

$$\begin{array}{ccc} \mathrm{S}(\mathrm{U}(n+1)\,\mathrm{U}(1))/\,\mathrm{S}(\mathrm{U}(n)\,\mathrm{U}(1)) & \longrightarrow & \mathrm{SU}(n+2)/\,\mathrm{S}(\mathrm{U}(n)\,\mathrm{U}(1)) \\ & & \downarrow{\scriptstyle \zeta_{\mathrm{SU}}} \\ & & \mathrm{SU}(n+2)/\,\mathrm{S}(\mathrm{U}(n+1)\,\mathrm{U}(1))\ . \end{array}$$

The isotropy group of the $\mathrm{SU}(n+2)$-action on the unit tangent sphere bundle of $\mathrm{SU}(n+2)/\,\mathrm{S}(\mathrm{U}(n+1)\,\mathrm{U}(1))$ is $\mathrm{S}(\mathrm{U}(n)\,\mathrm{U}(1))$. So ζ_{SU} is the unit tangent sphere bundle of $\mathrm{SU}(n+2)/\,\mathrm{S}(\mathrm{U}(n+1)\,\mathrm{U}(1)) \cong \mathbb{C}P^{n+1}$. Thus the Euler class of ζ_{SU} is equal to $n+2$ times the fundamental class of $\mathbb{C}P^{n+1}$. By Proposition 6.1, the assumption of Theorem 1.2 is satisfied in this case.

6.5. **The case** $(G, G/P) = (\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$**.** Note that the codimension is always odd in this case. We get $G_\mathbb{C} = \mathrm{Sp}(n+2;\mathbb{C})$ and

$$K_{G_\mathbb{C}} = \mathrm{Sp}(n+2)\ , \quad K_G = \mathrm{Sp}(n+1)\,\mathrm{Sp}(1)\ , \quad K_P = \mathrm{Sp}(n)\,\mathrm{Sp}(1)\ .$$

Thus the sphere bundle (9) is

$$\begin{array}{ccc} \mathrm{Sp}(n+1)\,\mathrm{Sp}(1)/\,\mathrm{Sp}(n)\,\mathrm{Sp}(1) & \longrightarrow & \mathrm{Sp}(n+2)/\,\mathrm{Sp}(n)\,\mathrm{Sp}(1) \\ & & \downarrow{\scriptstyle \zeta_{\mathrm{Sp}}} \\ & & \mathrm{Sp}(n+2)/\,\mathrm{Sp}(n+1)\,\mathrm{Sp}(1)\ . \end{array}$$

The isotropy group of the $\mathrm{Sp}(n+2)$-action on the unit tangent sphere bundle of $\mathrm{Sp}(n+2)/\,\mathrm{Sp}(n+1)\,\mathrm{Sp}(1)$ is $\mathrm{Sp}(n)$. Thus ζ_{Sp} is the unit tangent sphere bundle of $\mathrm{Sp}(n+2)/\,\mathrm{Sp}(n+1)\,\mathrm{Sp}(1) \cong \mathbb{H}P^{n+1}$. Hence the Euler class of ζ_{Sp} is equal to $n+2$ times the fundamental class of $\mathbb{H}P^{n+1}$. By Proposition 6.1, the assumption of Theorem 1.2 is satisfied in this case.

6.6. **The case** $(G, G/P) = (F_{4(-20)}, S^{15}_\infty)$**.** We recall the explicit presentation of $F_{4(-20)}$, F_4 and $F_4^\mathbb{C}$ as automorphism groups of Jordan algebras due to Freudenthal [Fre85] and Yokota [Yok75]. We follow Yokota [Yok09]. Let $\mathbb{O}$ be the Cayley algebra over $\mathbb{R}$. Let $M(3;\mathbb{O})$ be the 3×3 matrix group with coefficients in $\mathbb{O}$. Let $X^* = {}^t\overline{X}$, where the bar denotes conjugation in $\mathbb{O}$. Let $I_1'' = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$,

$$\mathcal{J}(1,2) = \{X \in M(3;\mathbb{O}) \mid I_1'' X^* I_1'' = X\}\ ,$$
$$\mathcal{J} = \{X \in M(3;\mathbb{O}) \mid X^* = X\}\ ,$$

and $\mathcal{J}^{\mathbb{C}} = \mathcal{J} \otimes \mathbb{C}$. A product $\circ$ is defined on these $\mathbb{R}$-vector spaces by $X \circ Y = \frac{1}{2}(XY + YX)$. Endowed with this product, $\mathcal{J}(1,2)$, $\mathcal{J}$ and $\mathcal{J}^{\mathbb{C}}$ are called *Jordan algebras.* $\mathcal{J}$ can be written as follows:

$$\mathcal{J} = \left\{ \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} \in M(3;\mathbb{O}) \,\middle|\, \xi_i \in \mathbb{R},\ x_i \in \mathbb{O} \right\} .$$

Here, $F_{4(-20)}$, F_4 and $F_4^{\mathbb{C}}$ are defined as the automorphism groups of these Jordan algebras:

$$\begin{aligned} F_{4(-20)} &= \{\, \sigma \in \operatorname{Aut}_{\mathbb{R}}(\mathcal{J}(1,2)) \mid \sigma(x \circ y) = \sigma(x) \circ \sigma(y) \,\} , \\ F_4 &= \{\, \sigma \in \operatorname{Aut}_{\mathbb{R}}(\mathcal{J}) \mid \sigma(x \circ y) = \sigma(x) \circ \sigma(y) \,\} , \\ F_4^{\mathbb{C}} &= \{\, \sigma \in \operatorname{Aut}_{\mathbb{C}}(\mathcal{J}^{\mathbb{C}}) \mid \sigma(x \circ y) = \sigma(x) \circ \sigma(y) \,\} . \end{aligned}$$

It is well known that $G_{\mathbb{C}} = F_4^{\mathbb{C}}$ and $K_{G_{\mathbb{C}}} = F_4$. We will get an explicit form of the parabolic subgroup P.

Lemma 6.3 (Announced by Borel [Bor50] and proved by Matsushima [Mat52])**.** *The isotropy group of the F_4-action on $\mathcal{J}$ at $E_{11} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ is* $\operatorname{Spin}(9)$. *Thus the orbit of E_{11} under the F_4-action is the octonionic projective plane $\mathbb{O}P^2 = F_4/\operatorname{Spin}(9)$.*

Here, $\mathbb{O}P^2$ is given by the following formula [Yok75]:

$$\mathbb{O}P^2 = \{\, X \in M(3;\mathbb{O}) \mid X^2 = X,\ \operatorname{tr} X = 1 \,\} .$$

There is a left G-action on $\mathbb{O}P^2$ defined by $(g, X) \mapsto \frac{gX}{\operatorname{tr}(gX)}$. The orbit of E_{11} under this G-action is the octonionic hyperbolic plane $\mathbf{H}^2_{\mathbb{O}} = F_{4(-20)}/\operatorname{Spin}(9)$, and the boundary $\partial \mathbf{H}^2_{\mathbb{O}}$ of $\mathbf{H}^2_{\mathbb{O}}$ in $\mathbb{O}P^2$ is given by

$$\partial \mathbf{H}^2_{\mathbb{O}} = \{\, X \in \mathbb{O}P^2 \mid \operatorname{tr}(X \circ I_1'' X) = 0 \,\} .$$

Since $\mathbb{O}P^2$ consists of the matrices

$$X = \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} \in \mathcal{J}$$

such that

$$\begin{gathered} \xi_2\xi_3 = |x_1|^2 , \quad \xi_3\xi_1 = |x_2|^2 , \quad \xi_1\xi_2 = |x_3|^2 , \\ x_2x_3 = \xi_1\overline{x}_1 , \quad x_3x_1 = \xi_2\overline{x}_2 , \quad x_1x_2 = \xi_3\overline{x}_3 , \\ \xi_1 + \xi_2 + \xi_3 = 1 , \end{gathered}$$

a simple calculation shows that $\operatorname{tr}(X \circ I_1'' X) = 0$ is equivalent to $\xi_1 = \frac{1}{2}$ for points $X \in \mathbb{O}P^2$ as above, obtaining a diffeomorphism

$$\partial \mathbf{H}^2_{\mathbb{O}} \approx \{\, (x_2, x_3) \in \mathbb{O}^2 \mid |x_2|^2 + |x_3|^2 = 1/4 \,\} ;$$

in particular, $\partial\mathbf{H}^2_{\mathbb{O}} \approx S^{15}$. Then P is the isotropy group of the G-action on $\partial\mathbf{H}^2_{\mathbb{O}}$ at $X_0 = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$.

We determine the sphere bundle (9) in this case. Let K_G denote the isotropy group of the F_4-action at $E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which is a maximal compact subgroup of G isomorphic to $\mathrm{Spin}(9)$ by Lemma 6.3. A maximal compact subgroup K_P of P is given by $K_P = K_G \cap P$. Since the F_4-action on $\mathcal{J}$ fixes the identity matrix [Yok09, Lemma 2.2.4] or [Yok75, Lemma 2.3-(1)], K_P is equal to the isotropy group of the $\mathrm{Spin}(9)$-action on $\mathcal{J}$ at $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, which is isomorphic to $\mathrm{Spin}(7)$ [Yok09, Theorem 2.7.5] or [Yok75, Remark 6.3]. Thus the sphere bundle (9) is

$$S^{15} \cong \mathrm{Spin}(9)/\mathrm{Spin}(7) \longrightarrow F_4/\mathrm{Spin}(7) \xrightarrow{\zeta_{F_4}} F_4/\mathrm{Spin}(9)\ .$$

We will show the following.

Lemma 6.4. *ζ_{F_4} is diffeomorphic to the unit tangent sphere bundle of $F_4/\mathrm{Spin}(9)$.* ∎

The orbit $\mathcal{K}$ of E_{11} under the F_4-action on $\mathcal{J}$ is $\mathbb{O}P^2 = F_4/\mathrm{Spin}(9)$ by Lemma 6.3. Let us describe the tangent space $T_{E_{11}}\mathcal{K}$ of $\mathcal{K}$ at E_{11}.

Lemma 6.5. *We have*

$$T_{E_{11}}\mathcal{K} = \left\{ \begin{pmatrix} 0 & x_3 & \overline{x}_2 \\ \overline{x}_3 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix} \in M(3;\mathbb{O}) \;\middle|\; x_2, x_3 \in \mathbb{O} \right\}\ . \tag{15}$$

Proof. Let $\mathfrak{f}_4 = \mathrm{Lie}(F_4)$. Consider the infinitesimal $\mathfrak{f}_4$-action $\rho\colon \mathfrak{f}_4 \to T_{E_{11}}\mathcal{K}$ at E_{11}. We get $\rho(\mathfrak{f}_4) = T_{E_{11}}\mathcal{K}$ by definition. Let $\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Since $\sigma^2 = 1$, we obtain an involution $\sigma\colon \mathfrak{f}_4 \to \mathfrak{f}_4$ given by $\sigma(X) = \sigma X \sigma$. Then we get a decomposition $\mathfrak{f}_4 = (\mathfrak{f}_4)_\sigma \oplus (\mathfrak{f}_4)_{-\sigma}$, where $(\mathfrak{f}_4)_\sigma$ is the σ-invariant part and $(\mathfrak{f}_4)_{-\sigma}$ is the σ-antiinvariant part. By [Yok09, Theorem 2.9.1] or [Yok90, Theorem 2.4.4], we get $\mathrm{Spin}(9) = (F_4)^\sigma$. By Lemma 6.3, it follows that $\rho((\mathfrak{f}_4)_\sigma) = 0$. On the other hand, for $X \in (\mathfrak{f}_4)_{-\sigma}$, we get $\sigma(X)E_{11} = \sigma X \sigma E_{11} = -E_{11}$. Thus $\rho(\mathfrak{f}_4) = T_{E_{11}}\mathcal{K}$ is contained in the σ-antiinvariant part $(\mathcal{J})_{-\sigma}$ of $\mathcal{J}$. Since it is easy to see that $(\mathcal{J})_{-\sigma}$ is equal to the right hand side of (15) and $\dim(\mathcal{J})_{-\sigma} = \dim\mathcal{K}$, we get the equality (15). □

We saw that K_P is the isotropy group of the adjoint K_G-action on

$$(\mathcal{J})_{-\sigma} = \left\{ \begin{pmatrix} 0 & x_3 & \overline{x}_2 \\ \overline{x}_3 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix} \;\middle|\; x_2, x_3 \in \mathbb{O} \right\}$$

at $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Thus Lemma 6.5 implies that K_P is the isotropy group of the K_G-action on $T_{E_{11}}\mathcal{K}$. This proves Lemma 6.4. Hence, according to [Hir49] or [Yok55], the Euler class of ζ_{F_4} is equal to 3 times the fundamental class of $\mathbb{O}P^2$ by the cell decomposition of $\mathbb{O}P^2$. So the assumption of Theorem 1.2 is satisfied in this case.

6.7. **A remark on the center.** The G-actions on G/P are not effective for some of the pairs $(G, G/P)$ considered in Corollary 1.3. In fact, in the case where G is either $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$ or $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ for even n, the stabilizers of the G-action on G/P are given by $\{\, cI_{n+2} \mid c \in \mathbb{C}^\times, c^{n+2} = 1\}$ and $\{\pm I_{n+2}\}$, respectively, where they are equal to the centers $\mathrm{Z}(G)$ of G. In the other cases considered in Corollary 1.3, the G-actions on G/P are effective. The quotient of $\mathrm{SU}(n+1,1)$ and $\mathrm{Sp}(n+1,1)$ by the centers are denoted by $\mathrm{PSU}(n+1,1)$ and $\mathrm{PSp}(n+1,1)$.

The finiteness of $\Sigma(\mathrm{PSU}(n+1,1), S^{2n+1}_\infty)$ and $\Sigma(\mathrm{PSp}(n+1,1), S^{4n+3}_\infty)$ is proved like in the cases $\Sigma(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$ and $\Sigma(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ of Theorem 1.2. We only need to notice the following two facts. By the discreteness of $\mathrm{Z}(G)$, there is no difference when we consider their Lie algebras. Since $\mathrm{Z}(G)$ is contained in $\mathrm{Z}(G_\mathbb{C})$ and K_P in both cases, the canonical embedding $G/K_P \to G_\mathbb{C}/(K_P)_\mathbb{C}$ is not changed by taking quotient by $\mathrm{Z}(G)$.

7. Bott-Thurston-Heitsch type formulas

7.1. **Pittie's Bott connections.** The purpose of Section 7 is to prove Bott-Thurston-Heitsch type formulas (Theorem 1.9). Section 7.1 is devoted to recall the Pittie's construction of a Bott connection for the P/K_P-coset foliation $\mathcal{F}_P$ of G/K_P, where G is semisimple and P is parabolic. It will be used to caluculate the Godbillon-Vey class of $\mathcal{F}_P$ in Lie algebra cohomology in Section 7.2. Since $(G, G/P)$-foliations are classified by $\mathcal{F}_P$ in the sense of Proposition 3.9-(ii), this computation can be applied to $(G, G/P)$-foliations (Section 7.4). By using the computation in Section 7.2, we will also show that the Godbillon-Vey class is the essentially unique nontrivial secondary characteristic class for $(G, G/P)$-foliations in Section 7.3.

First we recall the decompositions of the semisimple $\mathfrak{g}_\mathbb{C}$ and parabolic $\mathfrak{p}_\mathbb{C}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$ contained in $\mathfrak{p}_\mathbb{C}$. Let

$$\mathfrak{g}_\mathbb{C} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Upsilon} (\mathfrak{g}_\mathbb{C})_\alpha$$

be the root-space decomposition of $\mathfrak{g}_\mathbb{C}$, where Υ is the set of roots. Fix a set Π of simple roots which additively generate Υ, and let Υ^+ be the set of corresponding positive roots. Since a Borel subalgebra contained in $\mathfrak{p}_\mathbb{C}$ is conjugate to the standard Borel subalgebra $\bigoplus_{\alpha\in\Upsilon^+} (\mathfrak{g}_\mathbb{C})_\alpha$, we can assume that $\mathfrak{p}_\mathbb{C}$ contains $\bigoplus_{\alpha\in\Upsilon^+} (\mathfrak{g}_\mathbb{C})_\alpha$. Then there exists a subset Φ of Υ^+ such that

$$\mathfrak{p}_\mathbb{C} = \bigoplus_{\alpha\in-\Phi} (\mathfrak{g}_\mathbb{C})_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Upsilon^+} (\mathfrak{g}_\mathbb{C})_\alpha\,. \tag{16}$$

Thus, with

$$\mathfrak{r} = \bigoplus_{\alpha\in\Phi\cup(-\Phi)} (\mathfrak{g}_{\mathbb{C}})_\alpha \oplus \mathfrak{h}\ , \quad \mathfrak{u} = \bigoplus_{\alpha\in\Upsilon^+\setminus\Phi} (\mathfrak{g}_{\mathbb{C}})_\alpha\ , \quad \mathfrak{v} = \bigoplus_{\alpha\in\Upsilon^+\setminus\Phi} (\mathfrak{g}_{\mathbb{C}})_{-\alpha}\ ,$$

we get a decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_{\mathbb{C}} + \mathfrak{v} = \mathfrak{r} + \mathfrak{u} + \mathfrak{v}\ . \tag{17}$$

Here, $\mathfrak{r}$ is a reductive subalgebra of $\mathfrak{g}_{\mathbb{C}}$ called the *Levi part* of $\mathfrak{p}_{\mathbb{C}}$. Note that $\mathfrak{u}$ and $\mathfrak{v}$ are $\operatorname{ad}\mathfrak{r}$-invariant and nilpotent.

Let $\widehat{\mathcal{F}}_{P_{\mathbb{C}}}$ the right $P_{\mathbb{C}}$-coset foliation of $G_{\mathbb{C}}$. Left invariant complex connections on the normal bundle $\nu\widehat{\mathcal{F}}_{P_{\mathbb{C}}}$ of $\widehat{\mathcal{F}}_{P_{\mathbb{C}}}$ are in one-to-one correspondence with $\mathbb{C}$-linear maps $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_{\mathbb{C}};\mathbb{C})$. Let $\sigma\colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{p}_{\mathbb{C}}$ be the projection with respect to the decomposition (17). Consider the connection $\widetilde{\nabla}$ on $\nu\widehat{\mathcal{F}}_{P_{\mathbb{C}}}$ determined by

$$\widetilde{\nabla}_X Y = \pi\big([(\mathrm{id}_{\mathfrak{g}} - \sigma\pi)X, \sigma(Y)]\big) \tag{18}$$

for $X \in \mathfrak{g}_{\mathbb{C}}$ and $Y \in \mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_{\mathbb{C}}$. The connection form Θ of $\widetilde{\nabla}^{\mathbb{C}}$ is regarded as an element of $\mathfrak{g}_{\mathbb{C}}^* \otimes \mathfrak{gl}(\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_{\mathbb{C}};\mathbb{C})$. Pittie observed that, if we identify $\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_{\mathbb{C}}$ to $\mathfrak{v}$ via the canonical projection, then the connection form Θ of the connection given by (18) is the Maurer-Cartan form of the adjoint action of $\mathfrak{p}_{\mathbb{C}}$ on $\mathfrak{v}$, which is given by

$$\theta_{ij}(X) = \eta_i([X, Y_j]) \tag{19}$$

for $1 \leq i \leq q$, $1 \leq j \leq q$, where $\{Y_j\}$ is a basis of $\mathfrak{v}$ and $\{\eta_j\}$ is the basis of $\mathfrak{v}^*$ dual to $\{Y_j\}$. Let $p_{\mathfrak{u}^*\wedge\mathfrak{v}^*}$ denote the composite

$$\textstyle\bigwedge^2 \mathfrak{g}_{\mathbb{C}}^* = \bigwedge^2 \mathfrak{p}_{\mathbb{C}}^* \oplus \mathfrak{p}_{\mathbb{C}}^* \wedge \mathfrak{v}^* \oplus \bigwedge^2 \mathfrak{v}^* \longrightarrow \mathfrak{p}_{\mathbb{C}}^* \wedge \mathfrak{v}^* \longrightarrow \mathfrak{u}^* \wedge \mathfrak{v}^*$$

of the projections with respect to the decompositions (16) and (17). Let us denote the composite

$$\mathfrak{g}_{\mathbb{C}}^* \xrightarrow{d} \textstyle\bigwedge^2 \mathfrak{g}_{\mathbb{C}}^* \xrightarrow{p_{\mathfrak{u}^*\wedge\mathfrak{v}^*}} \mathfrak{u}^* \wedge \mathfrak{v}^* \tag{20}$$

by $\hat{d}$. The curvature form Ω of Θ is the element of $\bigwedge^2 \mathfrak{g}_{\mathbb{C}}^* \otimes \mathfrak{gl}(\mathfrak{g}_{\mathbb{C}}/\mathfrak{p}_{\mathbb{C}};\mathbb{C})$ given by $\Omega = d\Theta - \Theta\wedge\Theta$. We will use the following observation of Pittie.

Proposition 7.1 ([Pit79, Proposition 2.1])**.** $\hat{d}\Theta = \Omega$.

This formula is a consequence of the $(\operatorname{ad}\mathfrak{r})$-invariance of $\mathfrak{u}$ and $\mathfrak{v}$.

Let $\Upsilon^+ \setminus \Phi = \{\alpha_i\}_{1\leq i\leq q}$. We will use the following observation of Pittie, which is a direct consequence of the formula (19).

Proposition 7.2 ([Pit79, Theorem 2.3])**.** $\Delta_{\mathcal{F}_P}(h_1) = -\frac{1}{2\pi}\sum_{i=1}^q \alpha_i$.

Pittie observed that $-\Delta_{\mathcal{F}_P}(c_1)$ is a Kähler form of $G_{\mathbb{C}}/P_{\mathbb{C}}$ under the identification of $\bigwedge\mathfrak{u}^* \otimes \bigwedge\mathfrak{v}^*$ with the left invariant de Rham complex of $G_{\mathbb{C}}/P_{\mathbb{C}}$ in a standard way. By using the Lefschetz decomposition of cohomology of Kähler manifolds, Pittie showed the following.

Theorem 7.3 ([Pit79, Theorem 3.1])**.** *$\Delta_{\mathcal{F}_P}(H^\bullet(WO_q))$ is linearly spanned by the Pontryagin classes and $\left\{\Delta_{\mathcal{F}_P}(h_1h_Ic_1^q) \mid I \subseteq \{3,5,\dots,[q]\}\right\}$, where $[q]$ is the maximal odd number less than $q+1$.*

7.2. **Computation in Lie algebra cohomology.**

7.2.1. *The case $(G, G/P) = (\mathrm{SL}(q+1;\mathbb{R}), S^q)$.* Let $q' = q+1$. In this case, $\mathfrak{p}_{\mathbb{C}}$ and $\mathfrak{v}$ are the subalgebras of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(q';\mathbb{C})$ consisting of the matrices of the form

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & \cdots & 0 \\ * & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 0 \end{pmatrix} ,$$

respectively. Let E_{ij} be the element of $\mathfrak{g}_{\mathbb{C}}$ with 1 at the (i,j)-th entry and 0 at the other entries. Let $E_{ij}^\vee$ be the dual of E_{ij}. In this case, $\{E_{1j}\}_{2\le j\le q'}$ is a basis of $\mathfrak{v}$. Let $\Theta = (\theta_{ij})_{2\le i,j\le q'}$ be the matrix presentation of Θ with respect to $\{E_{1j}\}_{2\le j\le q'}$. From $[E_{kh}, E_{lj}] = \delta_{hl}E_{kj} - \delta_{jk}E_{lh}$ and $E_{ji}^\vee(E_{kh}) = \delta_{jk}\delta_{ih}$, we get

$$\theta_{ij}(E_{kh}) = E_{1i}^\vee([E_{kh}, E_{1j}]) = \delta_{h1}\delta_{1k}\delta_{ij} - \delta_{jk}\delta_{ih} .$$

Then

$$\Theta = (\theta_{ij})_{2\le i,j\le q'} = \begin{pmatrix} E_{22}^\vee - E_{11}^\vee & E_{32}^\vee & \cdots & E_{q'2}^\vee \\ E_{23}^\vee & E_{33}^\vee - E_{11}^\vee & \cdots & E_{q'3}^\vee \\ \vdots & \vdots & \ddots & \vdots \\ E_{2q'}^\vee & E_{3q'}^\vee & \cdots & E_{q'q'}^\vee - E_{11}^\vee \end{pmatrix} .$$

By observing that $\sum_{i=1}^{q'} E_{ii}^\vee = 0$ on $\mathfrak{g}_{\mathbb{C}}^*$, we get

$$\Delta_{\mathcal{F}_P}(h_1) = \frac{1}{2\pi}\operatorname{tr}\Theta = \frac{1}{2\pi}\sum_{i=2}^{q'}(E_{ii}^\vee - E_{11}^\vee) = -\frac{q'}{2\pi}E_{11}^\vee ,$$

$$\Delta_{\mathcal{F}_P}(c_1) = d\Delta_{\mathcal{F}_P}(h_1) = \frac{q'}{2\pi}\sum_{k=2}^{q'} E_{1k}^\vee \wedge E_{k1}^\vee .$$

Note that Θ equals the Maurer-Cartan form $\Theta_{MC} = (E_{ij}^\vee)_{2\le i,j\le q'}$ of $\mathfrak{sl}(q;\mathbb{C})$ modulo $\Delta_{\mathcal{F}_P}(h_1)$. Thus

$$\Delta_{\mathcal{F}_P}(h_1c_1^q) = -\frac{(q')^{q+1}\,q'!}{(2\pi)^{q+1}}E_{11}^\vee \wedge \bigwedge_{k=2}^{q'} E_{1k}^\vee \wedge E_{k1}^\vee , \tag{21}$$

$$\Delta_{\mathcal{F}_P}(h_1h_Ic_1^q) = \Delta_{\mathcal{F}_P}(h_1c_1^q)\, h_I(\Theta_{MC}) . \tag{22}$$

We will use these formulas to give an alternative proof of Theorem 1.1. Heitsch obtained more general formulas of this type for secondary characteristic classes of the form h_Ic_J by the application of his residues formulas ([Hei78, Theorem 4.2] and [Hei83, Theorem 2.3]).

7.2.2. *The case* $(G, G/P) = (\mathrm{SO}(n+1,1), S^n_\infty)$. Note that Yamato [Yam75] also made computation of characteristic classes of this case in a different way. Let $n' = n+1$ and $n'' = n+2$. Let

$$I'_{n''} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(n''; \mathbb{R}) \ , \tag{23}$$

where I_n is the $n \times n$ identity matrix. We use the following description of $\mathfrak{g} = \mathfrak{so}(n+1,1)$:

$$\begin{aligned}\mathfrak{g} &= \left\{ X \in \mathfrak{gl}(n''; \mathbb{R}) \mid {}^tXI'_{n''} + I'_{n''}X = 0 \right\} \\ &= \left\{ \begin{pmatrix} a & u & 0 \\ {}^tv & A & {}^tu \\ 0 & v & -a \end{pmatrix} \in \mathfrak{gl}(n''; \mathbb{R}) \,\middle|\, a \in \mathbb{R},\ A \in \mathfrak{so}(n; \mathbb{R}),\ u, v \in \mathbb{R}^n \right\} .\end{aligned}$$

Since $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a vector in the light cone, we get

$$\begin{aligned}\mathfrak{p} &= \{ X \in \mathfrak{g} \mid \exists a \in \mathbb{R} \text{ so that } Xe_1 = ae_1 \} \\ &= \left\{ \begin{pmatrix} a & u & 0 \\ 0 & A & {}^tu \\ 0 & 0 & -a \end{pmatrix} \in \mathfrak{gl}(n''; \mathbb{R}) \,\middle|\, a \in \mathbb{R},\ A \in \mathfrak{so}(n; \mathbb{R}),\ u \in \mathbb{R}^n \right\} .\end{aligned}$$

Then

$$\mathfrak{g}_{\mathbb{C}} = \left\{ \begin{pmatrix} a & u & 0 \\ {}^tv & A & {}^tu \\ 0 & v & -a \end{pmatrix} \in \mathfrak{gl}(n''; \mathbb{C}) \,\middle|\, a \in \mathbb{C},\ A \in \mathfrak{so}(n; \mathbb{C}),\ u, v \in \mathbb{C}^n \right\} ,$$

$$\mathfrak{p}_{\mathbb{C}} = \left\{ \begin{pmatrix} a & u & 0 \\ 0 & A & {}^tu \\ 0 & 0 & -a \end{pmatrix} \in \mathfrak{gl}(n''; \mathbb{C}) \,\middle|\, a \in \mathbb{C},\ A \in \mathfrak{so}(n; \mathbb{C}),\ u \in \mathbb{C}^n \right\} ,$$

$$\mathfrak{v} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ {}^tv & 0 & 0 \\ 0 & v & 0 \end{pmatrix} \in \mathfrak{gl}(n''; \mathbb{C}) \,\middle|\, v \in \mathbb{C}^n \right\} .$$

Let

$$a = E_{11} - E_{n''n''} \ , \quad v_j = E_{j1} + E_{n''j} \ , \quad A_{kh} = E_{kh} - E_{hk} \ .$$

Then we get a basis

$$\{v_j\}_{2 \le j \le n'} \cup \{a\} \cup \{{}^tv_j\}_{2 \le j \le n'} \cup \{A_{kh}\}_{2 \le k < h \le n'} \tag{24}$$

of $\mathfrak{g}_{\mathbb{C}}$. Here $\{v_j\}_{2 \le j \le n'}$ is a basis of $\mathfrak{v}$ and $\{a\} \cup \{{}^tv_j\}_{2 \le j \le n'} \cup \{A_{kh}\}_{2 \le k < h \le n'}$ is a basis of $\mathfrak{p}_{\mathbb{C}}$. We get

$$[a, v_j] = -v_j \ , \quad [{}^tv_i, v_j] = \delta_{ij} a \ , \quad [A_{kh}, v_j] = \delta_{jh} v_k - \delta_{jk} v_h \ .$$

For $z \in \mathfrak{g}_{\mathbb{C}}$, let $z^\vee \in \mathfrak{g}_{\mathbb{C}}^*$ denote the dual of z with respect to the basis (24). Since $\theta_{ij}(X) = v_i^\vee([X, v_j])$ for $X \in \mathfrak{p}_{\mathbb{C}}$, it follows that

$$\theta_{ij}(a) = -\delta_{ij} \ , \quad \theta_{ij}({}^t v_l) = 0 \ , \quad \theta_{ij}(A_{kh}) = \delta_{jh}\delta_{ik} - \delta_{jk}\delta_{ih} \ .$$

Thus

$$\Theta = (\theta_{ij}) = \begin{pmatrix} -a^\vee & A_{32}^\vee & \cdots & A_{n'2}^\vee \\ A_{23}^\vee & -a^\vee & & \vdots \\ \vdots & & \ddots & A_{n'n}^\vee \\ A_{2n'}^\vee & \cdots & A_{nn'}^\vee & -a^\vee \end{pmatrix} \ .$$

Since $\hat{d}(a^\vee) = -\sum_{k=2}^{n} {}^t v_k^\vee \wedge v_k^\vee$ and $\hat{d}A_{kh}^\vee = 0$ (see (20) for the definition of $\hat{d}$), Proposition 7.1 implies

$$\Omega = \begin{pmatrix} \sum_{k=2}^{n'} {}^t v_k^\vee \wedge v_k^\vee & 0 & \cdots & 0 \\ 0 & \sum_{k=2}^{n'} {}^t v_k^\vee \wedge v_k^\vee & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sum_{k=2}^{n'} {}^t v_k^\vee \wedge v_k^\vee \end{pmatrix} \ .$$

We get

$$\Delta_{\mathcal{F}_P}(h_1 c_1^n) = -\frac{n^{n+1} n!}{(2\pi)^{n+1}} \, a^\vee \wedge \bigwedge_{k=2}^{n+1} {}^t v_k^\vee \wedge v_k^\vee \ . \tag{25}$$

The Godbillon-Vey class of $\mathcal{F}_P$ is given by this formula and the well known relation $\mathrm{GV}(\mathcal{F}_P) = (2\pi)^{n+1}\Delta_{\mathcal{F}_P}(h_1 c_1^n)$ [KT75a, Theorem 7.20]. Later, in Proposition 7.4, we will show that any other nontrivial secondary characteristic class is a multiple of the Godbillon-Vey class by using (25). To be used later in the proof of Theorem 1.9, we state also the following equation:

$$\Delta_{\mathcal{F}_P}(h_1 c_1^n) = \frac{(-1)^{\frac{n(n-1)}{2}+1} n^{n+1} n!}{2^{2n+1}\pi^{n+1}} a^\vee \wedge \bigwedge_{k=2}^{n+1} ({}^t v_k^\vee + v_k^\vee) \wedge \bigwedge_{k=2}^{n+1} (v_k^\vee - {}^t v_k^\vee) \ . \tag{26}$$

To derive (26) from (25), we note that

$$\operatorname{sign} \begin{pmatrix} 1 & 2 & 3 & \cdots & m & m+1 & m+2 & \cdots & 2m-1 & 2m \\ 1 & 3 & 5 & \cdots & 2m-1 & 2 & 4 & \cdots & 2m-2 & 2m \end{pmatrix} = \frac{m(m-1)}{2} \ . \tag{27}$$

7.2.3. *The case* $(G, G/P) = (\mathrm{SU}(n+1,1), S_\infty^{2n+1})$. Let $n' = n+1$ and $n'' = n+2$. Let $I'_{n''}$ be the matrix given by (23). We use the following description of $\mathfrak{g} = \mathfrak{su}(n', 1)$:

$$\mathfrak{g} = \left\{\, X \in \mathfrak{sl}(n''; \mathbb{C}) \ \middle|\ {}^t\overline{X} I'_{n''} + I'_{n''} X = 0 \,\right\}$$

$$= \left\{ \begin{pmatrix} a & u & \sqrt{-1}c \\ {}^t\overline{v} & A & {}^t\overline{u} \\ \sqrt{-1}g & v & -\overline{a} \end{pmatrix} \in \mathfrak{sl}(n'';\mathbb{C}) \;\middle|\; \begin{matrix} a \in \mathbb{C},\ c, g \in \mathbb{R}, \\ A \in \mathfrak{u}(n),\ u, v \in \mathbb{C}^n \end{matrix} \right\} .$$

Since $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a vector in the light cone, we get

$$\begin{aligned} \mathfrak{p} &= \{ X \in \mathfrak{g} \mid \exists a \in \mathbb{C} \text{ so that } Xe_1 = ae_1 \} \\ &= \left\{ \begin{pmatrix} a & u & \sqrt{-1}c \\ 0 & A & {}^t\overline{u} \\ 0 & 0 & -\overline{a} \end{pmatrix} \in \mathfrak{sl}(n'';\mathbb{C}) \;\middle|\; \begin{matrix} a \in \mathbb{C},\ c \in \mathbb{R}, \\ A \in \mathfrak{u}(n),\ u \in \mathbb{C}^n \end{matrix} \right\} . \end{aligned}$$

Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n'';\mathbb{C})$, and

$$\mathfrak{p}_{\mathbb{C}} = \left\{ \begin{pmatrix} a_1 & u_1 & c \\ 0 & A & {}^tu_2 \\ 0 & 0 & a_2 \end{pmatrix} \in \mathfrak{sl}(n'';\mathbb{C}) \;\middle|\; \begin{matrix} a_1, a_2, c \in \mathbb{C},\ u_1, u_2 \in \mathbb{C}^n, \\ A \in \mathfrak{gl}(n;\mathbb{C}) \end{matrix} \right\} ,$$

$$\mathfrak{v} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ {}^tv_1 & 0 & 0 \\ g & v_2 & 0 \end{pmatrix} \in \mathfrak{sl}(n'';\mathbb{C}) \;\middle|\; g \in \mathbb{C},\ v_1, v_2 \in \mathbb{C}^n \right\} .$$

We can compute Θ and Ω like in the last case. But here we compute only the Godbillon-Vey class of $\mathcal{F}_P$. By using the computation, we will see that any other nontrivial secondary characteristic classes are multiples of the Godbillon-Vey class (Proposition 7.4). We will apply Proposition 7.2 to compute $\Delta_{\mathcal{F}_P}(h_1)$. As a Cartan subalgebra $\mathfrak{h}$, we take the subalgebra of $\mathfrak{g}_{\mathbb{C}}$ consisting of diagonal matrices. As a basis of $\mathfrak{v}$ consisting of root vectors, we can take $\{E_{k1}\}_{2 \le k \le n'} \cup \{E_{n''1}\} \cup \{E_{n''k}\}_{2 \le k \le n'}$. For a root vector $z \in \mathfrak{g}_{\mathbb{C}}$, let $z^\vee \in \mathfrak{g}_{\mathbb{C}}^*$ be the element such that $z^\vee(z) = 1$ and $z^\vee(z') = 0$ for any $z' \in \mathfrak{h}$ and any root vector z' which is linearly independent of z. The root of E_{ij} is given by $E_{ii}^\vee - E_{jj}^\vee$. Thus Proposition 7.2 implies

$$\begin{aligned} \Delta_{\mathcal{F}_P}(h_1) &= \frac{1}{2\pi}\Big(E_{n''n''}^\vee - E_{11}^\vee + \sum_{k=2}^{n'}(E_{kk}^\vee - E_{11}^\vee) + \sum_{k=2}^{n'}(E_{n''n''}^\vee - E_{kk}^\vee)\Big) \\ &= -\frac{n'}{2\pi}(E_{11}^\vee - E_{n''n''}^\vee) . \end{aligned}$$

So

$$\Delta_{\mathcal{F}_P}(c_1) = d\big(\Delta_{\mathcal{F}_P}(h_1)\big) = \frac{n'}{2\pi}\left(\sum_{k=2}^{n''} E_{1k}^\vee \wedge E_{k1}^\vee + \sum_{k=1}^{n'} E_{kn''}^\vee \wedge E_{n''k}^\vee\right) .$$

Thus we get the following formula:

$$\Delta_{\mathcal{F}_P}(h_1 c_1^{2n+1}) = -\frac{2(n')^{2n+2}(2n+1)!}{(2\pi)^{2n+2}} \tag{28}$$

$$\times\,(E_{11}^\vee - E_{n''n''}^\vee)\wedge\bigwedge_{k=2}^{n''}(E_{1k}^\vee\wedge E_{k1}^\vee)\wedge\bigwedge_{k=2}^{n'}(E_{kn''}^\vee\wedge E_{n''k}^\vee)\;.$$

The Godbillon-Vey class of $\mathcal{F}_P$ is given by this formula and the well known relation $\mathrm{GV}(\mathcal{F}_P) = (2\pi)^{2n+2}\Delta_{\mathcal{F}_P}(h_1c_1^{2n+1})$ [KT75a, Theorem 7.20]. Later, in Proposition 7.4, we will show that any other nontrivial secondary characteristic class of transversely spherical CR foliations is a multiple of the Godbillon-Vey class by using (28). To use later for the proof of Theorem 1.9, we state also the following direct consequence of (28) and (27):

$$\begin{aligned}(29)\quad \Delta_{\mathcal{F}_P}(h_1c_1^{2n+1}) &= (-1)^{n+1}\frac{2(n')^{2n+2}(2n+1)!}{2^{2n+1}(2\pi)^{2n+2}}\\ &\quad\times(E_{11}^\vee - E_{n''n''}^\vee)\wedge\bigwedge_{k=2}^{n''}(E_{1k}^\vee + E_{k1}^\vee)\wedge\bigwedge_{k=2}^{n'}(E_{kn''}^\vee + E_{n''k}^\vee)\\ &\quad\wedge\bigwedge_{k=2}^{n''}(E_{k1}^\vee - E_{1k}^\vee)\wedge\bigwedge_{k=2}^{n'}(E_{n''k}^\vee - E_{kn''}^\vee)\;.\end{aligned}$$

7.2.4. *The case* $(G, G/P) = (\mathrm{Sp}(n+1,1), S_\infty^{4n+3})$. Let $n' = n+1$ and $n'' = n+2$. Let

$$J' = \begin{pmatrix} 0 & I'_{n''}\\ -I'_{n''} & 0\end{pmatrix}\;,$$

where $I'_{n''}$ is the matrix given by (23). We use the following description of $\mathfrak{g} = \mathfrak{sp}(n',1)$:

$$\mathfrak{g} = \left\{ X = \begin{pmatrix} Z_1 & Z_2\\ Z_3 & Z_4\end{pmatrix}\in\mathfrak{gl}(2n'';\mathbb{C})\;\middle|\; \begin{array}{c} {}^tXJ' + J'X = 0,\\ Z_4 = \overline{Z}_1,\ Z_2 = -\overline{Z}_3\end{array}\right\}\;.$$

Here,

$$\mathfrak{p} = \{\, X\in\mathfrak{g} \mid \exists s,t\in\mathbb{C} \text{ so that } Xe_1 = se_1 + te_{n''+1}\,\}\;,$$

where e_i is the i-th standard unit vector of $\mathbb{C}^{2n''}$. Thus $\mathfrak{p}$ consists of the matrices of the form:

$$\begin{pmatrix} a & b & \sqrt{-1}c & d & f & g\\ 0 & A & {}^t\overline{b} & 0 & B & -{}^tf\\ 0 & 0 & -\overline{a} & 0 & 0 & d\\ -\overline{d} & -\overline{f} & -\overline{g} & \overline{a} & \overline{b} & -\sqrt{-1}\overline{c}\\ 0 & -\overline{B} & {}^t\overline{f} & 0 & \overline{A} & {}^tb\\ 0 & 0 & -\overline{d} & 0 & 0 & -a\end{pmatrix}\;,$$

where $c\in\mathbb{R}$, a, $e\in\mathbb{C}$, b, $d\in\mathbb{C}^n$, $A\in\mathfrak{sl}(n;\mathbb{C})$ with $A = {}^tA$, and $B\in\mathfrak{u}(n)$. We get

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(n'';\mathbb{C}) = \left\{X\in\mathfrak{gl}(2n'';\mathbb{C})\ \mid\ {}^tXJ' + J'X = 0\right\}\;,$$

which consists of the matrices of the form

$$X = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} \in \mathfrak{gl}(2n''; \mathbb{C})$$

such that

$${}^tZ_1 I'_{n''} + I'_{n''} Z_4 = -{}^tZ_3 I'_{n''} + I'_{n''} Z_3 = -{}^tZ_2 I'_{n''} + I'_{n''} Z_2 = 0 \ .$$

Then $\mathfrak{p}_{\mathbb{C}}$ is the subalgebra of $\mathfrak{g}_{\mathbb{C}}$ consisting of the matrices

$$\begin{pmatrix} a_1 & b_1 & c & d_1 & f_1 & g_1 \\ 0 & A & {}^tb_2 & 0 & B_1 & -{}^t\overline{f}_1 \\ 0 & 0 & a_2 & 0 & 0 & d_1 \\ d_2 & f_2 & g_2 & -a_2 & b_2 & -{}^tc \\ 0 & B_2 & -{}^t\overline{f}_2 & 0 & -{}^tA & {}^tb_1 \\ 0 & 0 & d_2 & 0 & 0 & -a_1 \end{pmatrix} ,$$

where $a_1, a_2, c, f_1, f_2, g_1, g_2 \in \mathbb{C}$, $b_1, b_2, d_1, d_2 \in \mathbb{C}^n$, $A \in \mathfrak{sl}(n; \mathbb{C})$ and $B_1, B_2 \in \mathfrak{u}(n)$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with the following basis:

$$(30) \qquad \{E_{11} - E_{2n''\,2n''}\} \cup \{E_{kk} - E_{n''+k\,n''+k}\}_{2 \leq k \leq n'} \cup \{E_{n''\,n''} - E_{n''+1\,n''+1}\} \ .$$

Thus $\mathfrak{v}$ consists of the matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ {}^tu_1 & 0 & 0 & -{}^t\overline{x}_1 & 0 & 0 \\ v & u_2 & 0 & y_1 & x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -{}^t\overline{x}_2 & 0 & 0 & {}^tu_2 & 0 & 0 \\ y_2 & x_2 & 0 & -v & u_1 & 0 \end{pmatrix} ,$$

where $v, y_1, y_2 \in \mathbb{C}$ and $u_1, u_2, x_1, x_2 \in \mathbb{C}^n$.

Here, we compute the Godbillon-Vey class like in the last example by using Proposition 7.2. Later, by using the computation, we will see any other non-trivial secondary characteristic class is a multiple of the Godbillon-Vey class (see Proposition 7.4). As a basis of $\mathfrak{v}$ consisting of root vectors, take

$$\begin{aligned} u_{1,k} &= E_{k\,1} + E_{2n''\,n''+k} \ , & 2 \leq k \leq n' \ , \\ u_{2,k} &= E_{n''\,k} + E_{n''+k\,n''+1} \ , & 2 \leq k \leq n' \ , \\ v &= E_{n''\,1} - E_{2n''\,n''+1} \ , & \\ x_{1,k} &= -E_{k\,n''+1} + E_{n''\,n''+k} \ , & 2 \leq k \leq n' \ , \\ y_1 &= E_{n''\,n''+1} \ , & \\ x_{2,k} &= -E_{n''+k\,1} + E_{2n''\,k} \ , & 2 \leq k \leq n' \ , \\ y_2 &= E_{2n''\,1} \ . & \end{aligned}$$

Let $\{\gamma_i\}_{1\le i\le n''}$ be the basis of $\mathfrak{h}^*$ dual to (30). The roots corresponding to these vectors are given as follows:

$u_{1,k}$	$u_{2,k}$	v	$x_{1,k}$	y_1	$x_{2,k}$	y_2
$-\gamma_1+\gamma_k$	$-\gamma_k+\gamma_{n''}$	$-\gamma_1+\gamma_{n''}$	$\gamma_k+\gamma_{n''}$	$2\gamma_{n''}$	$-\gamma_1-\gamma_k$	$-2\gamma_1$

Here, $2\le k\le n'$. Thus, by Proposition 7.2,

$$\Delta_{\mathcal{F}_P}(h_1) = -\frac{2n+3}{2\pi}\,(\gamma_1-\gamma_{n''})\ . \tag{31}$$

For a root vector $z\in\mathfrak{g}_{\mathbb{C}}$, let $z^\vee\in\mathfrak{g}_{\mathbb{C}}^*$ be determined by $z^\vee(z)=1$ and $z^\vee(z')=0$ for any $z'\in\mathfrak{h}$ and any root vector z' which is linearly independent of z. We have

$$\hat{d}\gamma_1 = -\sum_{k=2}^{n'}({}^tu_{1,k}^\vee\wedge u_{1,k}^\vee) - ({}^tv^\vee\wedge v^\vee) - \sum_{k=2}^{n'}({}^tx_{2,k}^\vee\wedge x_{2,k}^\vee) - ({}^ty_2^\vee\wedge y_2^\vee)\ ,$$

$$\hat{d}\gamma_{n''} = ({}^tv^\vee\wedge v^\vee) + \sum_{k=2}^{n'}({}^tu_{2,k}^\vee\wedge u_{2,k}^\vee) + ({}^ty_1^\vee\wedge y_1^\vee) + \sum_{k=2}^{n'}({}^tx_{1,k}^\vee\wedge x_{1,k}^\vee)$$

(see (20) for the definition of $\hat{d}$). Let ζ be the standard symplectic form on $\mathfrak{u}\oplus\mathfrak{v}$ defined by

$$\begin{aligned}\zeta = \sum_{k=2}^{n'}({}^tu_{1,k}^\vee\wedge u_{1,k}^\vee) + 2({}^tv^\vee\wedge v^\vee) + \sum_{k=2}^{n'}({}^tx_{2,k}^\vee\wedge x_{2,k}^\vee) + ({}^ty_2^\vee\wedge y_2^\vee)\\ + \sum_{k=2}^{n'}({}^tu_{2,k}^\vee\wedge u_{2,k}^\vee) + ({}^ty_1^\vee\wedge y_1^\vee) + \sum_{k=2}^{n'}({}^tx_{1,k}^\vee\wedge x_{1,k}^\vee)\ .\end{aligned}$$

Then

$$\Delta_{\mathcal{F}_P}(c_1) = d\big(\Delta_{\mathcal{F}_P}(h_1)\big) = \frac{2n+3}{2\pi}\,\zeta\ . \tag{32}$$

By (31) and (32), we obtain the following formula of the Godbillon-Vey class:

$$\Delta_{\mathcal{F}_P}(h_1c_1^{4n+3}) = -\frac{(2n+3)^{4n+4}}{2^{4n+4}\pi^{4n+4}}\,(\gamma_1-\gamma_{n''})\wedge\zeta^{4n+3}\ . \tag{33}$$

This formula gives the Godbillon-Vey class of $\mathcal{F}_P$ by the well known relation $\mathrm{GV}(\mathcal{F}_P) = (2\pi)^{4n+4}\Delta_{\mathcal{F}_P}(h_1c_1^{4n+3})$ [KT75a, Theorem 7.20]. Later, in Proposition 7.4, we will show that any other nontrivial secondary characteristic class of $(\mathrm{Sp}(n+1,1),S^{4n+3})$-foliations is a multiple of the Godbillon-Vey class by using (33). To use later for the proof of Theorem 1.9, we also state the following direct consequence of (33) and (27):

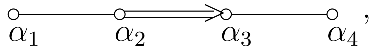
$$\begin{aligned}&\Delta_{\mathcal{F}_P}(h_1c_1^{4n+3})\\ &\quad= \frac{(2n+3)^{4n+4}(4n+3)!}{2^{8n+6}\pi^{4n+4}}\,(\gamma_1-\gamma_{n''})\wedge\bigwedge_z({}^tz^\vee+z^\vee)\wedge\bigwedge_z(z^\vee-{}^tz^\vee)\ ,\end{aligned} \tag{34}$$

where z runs in

$$\{u_{1,k}\}_{2\le k\le n'} \cup \{u_{2,k}\}_{2\le k\le n'} \cup \{x_{1,k}\}_{2\le k\le n'} \cup \{x_{2,k}\}_{2\le k\le n'} \cup \{y_1, y_2, v\}$$

in this order.

7.2.5. *The case* $(G, G/P) = (F_{4(-20)}, S^{15}_\infty)$. Here, we refer to [Yok09, Section 2.6] for the structure of $\mathfrak{f}_4^{\mathbb{C}} = \mathrm{Lie}(F_4^{\mathbb{C}})$. The Dynkin diagram of the Lie algebra $\mathfrak{f}_4^{\mathbb{C}}$ is:

(35) 

,

where the roots $\{\alpha_i\}_{1\le i\le 4}$, for a standard choice of a Cartan subalgebra $\mathfrak{h} = \bigoplus_{i=0}^{3} \mathbb{C}H_i$, are given by

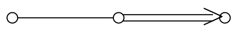
$$\alpha_1 = \lambda_0 - \lambda_1 , \quad \alpha_2 = \lambda_1 - \lambda_2 , \quad \alpha_3 = \lambda_2 , \quad \alpha_4 = \frac{1}{2}(-\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) ,$$

where $\lambda_i = B(\cdot, H_i)$ with respect to the Killing form B of $\mathfrak{f}_4^{\mathbb{C}}$. The list of positive roots of $\mathfrak{f}_4^{\mathbb{C}}$ for this simple root system is given by

$$\begin{aligned}
\lambda_0 &= \alpha_1 + \alpha_2 + \alpha_3 , & \lambda_1 &= \alpha_2 + \alpha_3 ,\\
\lambda_2 &= \alpha_3 , & \lambda_3 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 ,\\
\lambda_0 - \lambda_1 &= \alpha_1 , & \lambda_0 - \lambda_2 &= \alpha_1 + \alpha_2 ,\\
-\lambda_0 + \lambda_3 &= \alpha_2 + 2\alpha_3 + 2\alpha_4 , & \lambda_1 - \lambda_2 &= \alpha_2 ,\\
-\lambda_1 + \lambda_3 &= \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 , & -\lambda_2 + \lambda_3 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 ,\\
\lambda_0 + \lambda_1 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 , & \lambda_0 + \lambda_2 &= \alpha_1 + \alpha_2 + 2\alpha_3 ,\\
\lambda_0 + \lambda_3 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 , & \lambda_1 + \lambda_2 &= \alpha_2 + 2\alpha_3 ,\\
\lambda_1 + \lambda_3 &= \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 , & \lambda_2 + \lambda_3 &= \alpha_1 + 2\alpha_2 + 4\alpha_3 + 4\alpha_4 ,
\end{aligned}$$

$$\begin{aligned}
\tfrac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 ,\\
\tfrac{1}{2}(-\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) &= + \alpha_4 ,\\
\tfrac{1}{2}(\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 ,\\
\tfrac{1}{2}(\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) &= \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 ,\\
\tfrac{1}{2}(-\lambda_0 - \lambda_1 + \lambda_2 + \lambda_3) &= \alpha_3 + \alpha_4 ,\\
\tfrac{1}{2}(-\lambda_0 + \lambda_1 - \lambda_2 + \lambda_3) &= \alpha_2 + \alpha_3 + \alpha_4 ,\\
\tfrac{1}{2}(\lambda_0 - \lambda_1 - \lambda_2 + \lambda_3) &= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 ,\\
\tfrac{1}{2}(-\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3) &= \alpha_2 + 2\alpha_3 + \alpha_4 .
\end{aligned}$$

As mentioned in Section 6.6, the semisimple part of the Levi part of $\mathfrak{p}_{\mathbb{C}}$ is $\mathfrak{so}(7; \mathbb{C})$, whose Dynkin diagram is:

 .

According to the Dynkin diagram (35) of $\mathfrak{f}_4^{\mathbb{C}}$, the unique possibility of $\Phi \cap \Pi$ in (16) is $\{\alpha_1, \alpha_2, \alpha_3\}$. Then $\mathfrak{v}$ is spanned by the 15 negative roots that are not generated by $\{-\alpha_1, -\alpha_2, -\alpha_3\}$, whose sum is $-11\lambda_3$, as can be computed by using the above list of positive roots. By Proposition 7.2, we get $\Delta_{\mathcal{F}_P}(h_1) = -\frac{11}{2\pi}\lambda_3$. Take root vectors $\{E_\alpha\}_{\alpha\in\Upsilon}$ so that $B(E_\alpha, E_{-\alpha}) = 1$ for the Killing form B of $\mathfrak{f}_4^{\mathbb{C}}$. For $\alpha \in \Upsilon$, let H_α be the element of $\mathfrak{h}$ such that $B(H, H_\alpha) = \alpha(H)$ for any $H \in \mathfrak{h}$. By Proposition 7.1 and since

$$d\lambda_3(E_\alpha, E_{-\alpha}) = -B(E_\alpha, E_{-\alpha})\,\lambda_3(H_\alpha) = -\lambda_3(H_\alpha)\ ,$$

we get

$$\Delta_{\mathcal{F}_P}(c_1) = d\Delta_{\mathcal{F}_P}(h_1) = \frac{11}{2\pi}\sum_{\alpha\in\Upsilon^+\setminus\Phi}\lambda_3(H_\alpha)\,E_\alpha^\vee \wedge E_{-\alpha}^\vee\ .$$

By using $B(\sum_{i=1}^4 \lambda_i H_i, \sum_{j=1}^4 \lambda'_j H_j) = 18\sum_{i=1}^4 \lambda_i\lambda'_i$, we can compute $\lambda_3(H_\alpha)$ in terms of the above list of positive roots of $\mathfrak{f}_4^{\mathbb{C}}$. Then we get the following formula of the Godbillon-Vey class:

$$\Delta_{\mathcal{F}_P}(h_1c_1^{15}) = -\frac{11^{16}\,18^{15}\,15!}{2^{24}\pi^{16}}\lambda_3 \wedge \bigwedge_{\alpha\in\Upsilon^+\setminus\Phi} E_\alpha^\vee \wedge E_{-\alpha}^\vee\ . \tag{36}$$

This formula gives $\mathrm{GV}(\mathcal{F}_P)$ by the well known relation $\mathrm{GV}(\mathcal{F}_P)=(2\pi)^{16}\Delta_{\mathcal{F}_P}(h_1c_1^{15})$ [KT75a, Theorem 7.20]. In Proposition 7.4, we will show that any other nontrivial secondary characteristic class of $(F_{4(-20)}, S^{15}_\infty)$-foliations is a multiple of the Godbillon-Vey class by using (36). To be used later in the proof of Theorem 1.9, we also state the following direct consequence of (36) and (27):

$$\begin{aligned}\Delta_{\mathcal{F}_P}(h_1c_1^{15}) \\ = \frac{11^{16}\,3^{30}\,15!}{2^{24}\pi^{16}}\lambda_3 \wedge \bigwedge_{\alpha\in\Upsilon^+\setminus\Phi}(E_\alpha^\vee + E_{-\alpha}^\vee) \wedge \bigwedge_{\alpha\in\Upsilon^+\setminus\Phi}(E_{-\alpha}^\vee - E_\alpha^\vee)\ .\end{aligned} \tag{37}$$

7.3. **The Godbillon-Vey class spans the secondary characteristic classes.** We assume that $(G, G/P)$ is equal to $(\mathrm{SO}(n+1,1), S^n_\infty)$, $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ or $(F_{4(-20)}, S^{15}_\infty)$. In the last section, we saw that the Godbillon-Vey class of $\mathcal{F}_P$ is nontrivial, being given by a volume form on G/K_P. By using the computation, we will prove the following result in this section.

Proposition 7.4. *$\Delta_{\mathcal{F}}(H^\bullet(WO_q))$ is spanned by the Godbillon-Vey class $\Delta_{\mathcal{F}}(h_1c_1^q)$ for any $(G, G/P)$-foliation $\mathcal{F}$ of M.*

Recall that the secondary characteristic classes of the form $\Delta_{\mathcal{F}_P}(h_Ic_J)$ with nonempty I are called *exotic*. First, we observe the following.

Lemma 7.5. *Every nontrivial exotic secondary characteristic class of $\mathcal{F}_P$ is a multiple of the Godbillon-Vey class $\Delta_{\mathcal{F}_P}(h_1c_1^q)$ in $H^\bullet(\mathfrak{g}, K_P)$.*

Proof. Note that $\deg h_Ic_J \geq 2q+1$ for any h_Ic_J in WO_q with nonempty I. Since $(G, G/P)$ is $(\mathrm{SO}(n+1,1), S^n_\infty)$, $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ or $(F_{4(-20)}, S^{15}_\infty)$, we have $G/K_P = 1 + 2\dim G/P$. Then $\Delta_{\mathcal{F}_P}(h_Ic_J) = 0$ for any

$h_I c_J$ in WO_q with $\deg h_I c_J > 2q+1$, and $\Delta_{\mathcal{F}}(h_I c_J)$ is a multiple of a volume form on G/K_P for any $h_I c_J$ in WO_q with $\deg h_I c_J = 2q+1$. Since the Godbillon-Vey class is represented by a volume form on G/K_P by (25), (28), (33) and (36), the proof is concluded. □

For the Pontryagin classes, an argument similar to Heitsch [Hei86, Section 4] for transversely projective foliations implies the following.

Lemma 7.6. *For any $(G, G/P)$-foliation $\mathcal{F}$ of M, the Pontryagin classes of $\nu\mathcal{F}$ are zero in $H^\bullet(M)$.*

Proof. Let $T_0(G/K_G)$ be the complement of the zero section of the total space of the tangent bundle of G/K_G. Since G/K_P is G-equivariantly diffeomorphic to the total space of the unit tangent bundle of G/K_G in these cases as mentioned in Section 6, we identify G/K_P as a submanifold of $T_0(G/K_G)$. We have a G-equivariant contraction $\gamma\colon T_0(G/K_G) \to G/K_P$. Let $\rho\colon T_0(G/K_G) \to G/K_G$ be the projection. Consider the vector bundle [$\ker \rho_*$ on $T_0(G/K_G)$ consisting of vertical vectors. Let $E = (\ker \rho_*)|_{G/K_P}$. We have $\nu\mathcal{F}_P \oplus \mathbb{R}_\gamma = E$, where $\mathbb{R}_\gamma$ is the trivial vector bundle of rank one over G/K_P spanned by vectors tangent to the fibers of γ. Here, E has a G-invariant flat connection ∇' induced from the vector bundle structure of $\ker \rho_*$. Thus, the total Pontryagin form $p(E, \nabla')$ of (E, ∇') is zero.

Let $\widetilde{M}$ be the universal cover of M and $\widehat{\mathrm{dev}}\colon \widetilde{M} \to G/K_P$ be a $\pi_1 M$-equivariant map such that $\widetilde{\mathcal{F}} = \widehat{\mathrm{dev}}^* \mathcal{F}_P$, where $\widetilde{\mathcal{F}}$ is the lift of $\mathcal{F}$ to $\widetilde{M}$ (see Proposition 3.9). By the $\pi_1 M$-equivariance of $\widehat{\mathrm{dev}}$, the vector bundles $\widehat{\mathrm{dev}}^* \mathbb{R}_\gamma$ and $\widehat{\mathrm{dev}}^* E$ over $\widetilde{M}$ descend to vector bundles over M, which are denoted by $\mathbb{R}_M$ and E_M, respectively. Since E_M admits a flat connection by construction, the total Pontryagin class $p(E_M)$ of E_M is 0. By $\nu\mathcal{F} \oplus \mathbb{R}_M = E_M$ and the product formula of total Pontryagin classes, we get $p(\nu\mathcal{F}) = p(E_M) = 0$. □

Proposition 7.4 is a consequence of Lemmas 7.5 and 7.6 and Theorem 4.3.

7.4. **Proof of Bott-Thurston-Heitsch type formulas.**

7.4.1. *The volume of flat G/K_G-bundles.* Here, we recall the definition of the characteristic classes of G/K_G-bundles with flat G-connections. For a G/K_G-bundle $p_Q\colon Q \to N$ with a flat G-connection whose holonomy homomorphism is $h\colon \pi_1 N \to G$, we have the Chern-Weil homomorphism $H^\bullet(\mathfrak{g}, K_G) \to H^\bullet(Q; \mathbb{R})$. The sections s of p_Q are unique up to isotopy because of the contractibility of G/K_G. By composing the pull-back by s with the Chern-Weil homomorphism, we get a map $H^\bullet(\mathfrak{g}, K_G) \to H^\bullet(N; \mathbb{R})$. Since this map depends only on h, we denote it by Ξ_h.

We fix an orientation on G/K_G. Let ω_{G/K_G} be the corresponding left invariant volume form on G/K_G of norm 1 with respect to the metric obtained from the

Killing metric on $\mathfrak{g}$. Let $\mathrm{vol}_{G/K_G} = [\omega_{G/K_G}]$ and

$$\mathrm{vol}(h) = \Xi_h(\mathrm{vol}_{G/K_G}) \in H^m(N;\mathbb{R}) ,$$

where $m = \dim G/K_G$. The class $\mathrm{vol}(h)$ is called the *volume* of Q or of the holonomy presentation h.

Example 7.7. For the case where $N = \Gamma\backslash G/K_G$ for a torsion-free uniform lattice Γ of G, the volume of $\Gamma \hookrightarrow G$ is denoted by $\mathrm{vol}(\Gamma)$, which is represented by the volume form on N induced from ω_{G/K_G}.

Remark 7.8. Ξ_h is called the *Borel regulator map* by algebraic geometers. For the importance of the volume in algebraic geometry, see [Rez96] and the references therein.

7.4.2. *Bott-Thurston-Heitsch type formulas for homogeneous foliations.* We apply the computation of the last section to calculate the Godbillon-Vey classes of homogeneous foliations $\mathcal{F}_\Gamma$ in Example 2.3. We consider the K_G/K_P-bundle $\phi_{K_G}\colon \Gamma\backslash G/K_P \to \Gamma\backslash G/K_G$. In the next proposition, we will need orientations of the fibers of ϕ_{K_G} and of G/K_G to define the fiber integration along ϕ_{K_G} and to determine a volume form ω_{G/K_G} on $\Gamma\backslash G/K_G$. In the proof, we will take these orientations by using the decomposition of the volume form of G/K_P into a volume form of G/K_G and a fiberwise volume form of ϕ_{K_G}.

Proposition 7.9. *Let* $(G, G/P)$ *be one of* $(\mathrm{SO}_0(n+1,1), S^n_\infty)$, $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ *or* $(F_{4(-20)}, S^{15}_\infty)$. *Let* $q = \dim G/P$ *(the codimension of* $\mathcal{F}_\Gamma$*). We have*

$$\fint_{\phi_{K_G}} \Delta_{\mathcal{F}_\Gamma}(h_1 c_1^q) = c_G\, \omega_{G/K_G} \tag{38}$$

in $\Omega^{q+1}(\Gamma\backslash G/K_G)$ *for some orientations of* G/K_G *and of the fibers of* ϕ_{K_G}, *where* c_G *is the constant depending on* $(G, G/P)$ *given by the following table:*

$(G, G/P)$	c_G
$(\mathrm{SO}_0(n+1,1), S^n_\infty)$	$\dfrac{(-1)^{\frac{n(n-1)}{2}+1} n^{\frac{n+1}{2}}\, n!\,\mathrm{vol}(S^n)}{2^{\frac{3n+3}{2}} \pi^{n+1}}$
$(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$	$\dfrac{(-1)^{n+1}(n+1)^{2n+2}(2n+1)!\,\mathrm{vol}(S^{2n+1})}{2^{n+1}\pi^{2n+2}(n+2)^{n+1}}$
$(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$	$\dfrac{(2n+3)^{4n+4}(4n+3)!\,\mathrm{vol}(S^{4n+3})}{2^{6n+\frac{11}{2}}\pi^{4n+4}(n+3)^{2n+2}}$
$(F_{4(-20)}, S^{15}_\infty)$	$\dfrac{3^{\frac{35}{2}}\, 7^4\, 11^{16}\, 15!\,\mathrm{vol}(S^{15})}{2^6\pi^{16}}$

Here, $\mathrm{vol}(S^q)$ *is the volume of the unit sphere in* $\mathbb{R}^{q+1}$*, given by* $\mathrm{vol}(S^q) = \frac{(2\pi)^{(q+1)/2}}{2\cdot 4\cdots(q-1)}$ *for odd* q *and* $\mathrm{vol}(S^q) = \frac{2(2\pi)^{q/2}}{1\cdot 3\cdots(q-1)}$ *for even* q*.*

Proof. Consider the case where $(G, G/P) = (\mathrm{SO}_0(n+1,1), S^n_\infty)$. We will use the notation of Sections 7.2.2. Let $\mathfrak{k}_G = \mathfrak{k}_P \oplus \mathfrak{m}$ be the orthogonal decomposition with respect to the Killing metric. Regarding $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}(n+2;\mathbb{R})$ like in Section 7.2.2, $\mathfrak{k}_G$ and $\mathfrak{m}$ are realized as

$$\mathfrak{k}_G = \{A \in \mathfrak{g} \mid A = -A^*\}\ , \tag{39}$$

$$\mathfrak{m} = \left\{ \left(\begin{smallmatrix} 0 & -x & 0 \\ {}^t x & 0 & -{}^t x \\ 0 & x & 0 \end{smallmatrix}\right) \in \mathfrak{gl}(n+2;\mathbb{R}) \,\middle|\, x \in \mathbb{R}^n \right\}\ . \tag{40}$$

By (26), $\mathrm{GV}(\mathcal{F}_P)$ is a wedge product of two components; the first component is a wedge product of Hermitian matrices and the second is a wedge product of skew-Hermitian matrices. By using (39), it is easy to see that the first part $a^\vee \wedge \bigwedge_{k=2}^{n+1}({}^t v_k^\vee + v_k^\vee)$ is K_G-basic; namely, it is the pull-back of a volume form on G/K_G by the projection $\phi_{K_G}\colon G/K_P \to G/K_G$. We orient G/K_G with this volume form. Since the Killing metric B_θ of $\mathfrak{g}_\mathbb{C}$ is given by $B_\theta(X,Y) = n\,\mathrm{tr}(XY^*)$, the norm of $a^\vee$ and ${}^t v_k^\vee + v_k^\vee$ are $\frac{1}{\sqrt{2n}}$ and $\frac{1}{\sqrt{n}}$, respectively. Thus, letting ω_{G/K_G} be the volume form on G/K_G defining the same orientation and of norm 1 with respect to the Killing metric, we get

$$a^\vee \wedge \bigwedge_{k=2}^{n+1}({}^t v_k^\vee + v_k^\vee) = \frac{1}{\sqrt{2}n^{\frac{n+1}{2}}}\, \phi_{K_G}^* \omega_{G/K_G}\ . \tag{41}$$

Recall that $K_G \cong \mathrm{SO}(n+1)$. We consider the standard $\mathrm{SO}(n+1)$-action on $\mathbb{R}^{n+1}$ so that the orbit of the first fundamental vector e_1 is S^n. We can identify $\mathfrak{m}$ with $T_{e_1}S^n$ by the infinitesimal action. Under this identification, the second part $\bigwedge_{k=2}^{n+1}({}^t v_k^\vee - v_k^\vee)$ of the right hand side of (26) gives the invariant volume form on S^n of norm $2^{n/2}$ with respect to the standard metric on $\mathbb{R}^{n+1}$. We orient the S^n-fibers of ϕ_{K_G} with this volume form. Then, by (41), we get

$$\fint a^\vee \wedge \bigwedge_{k=2}^{n+1}({}^t v_k^\vee + v_k^\vee) \wedge \bigwedge_{k=2}^{n+1}({}^t v_k^\vee - v_k^\vee) = \frac{2^{\frac{n-1}{2}}\,\mathrm{vol}(S^n)}{n^{\frac{n+1}{2}}}\, \omega_{G/K_G}\ . \tag{42}$$

Here, (38) in the case where $(G, G/P) = (\mathrm{SO}_0(n+1,1), S^n_\infty)$ follows from (26) and (42).

In the case where $(G, G/P) = (\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$ or $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$, Equation (38) is proved in a way similar to the last case of $(G, G/P) = (\mathrm{SO}_0(n+1,1), S^n_\infty)$ by using (29) and (34). We will use the notation in Sections 7.2.3 and 7.2.4. The right hand sides of (29) and (34) are wedge products of two parts; the first one is a wedge product of Hermitian matrices and the second one is a wedge product of skew-Hermitian matrices. Regarding $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}(n+2;\mathbb{C})$ (resp., $\mathfrak{gl}(2n+4;\mathbb{C})$) as in Section 7.2.3 (resp., 7.2.4), (39) is true. Then, we can easily see that the first part is K_G-basic. So we orient G/K_G with the corresponding volume form on G/K_G like in the last case. The Killing metric B_θ of $\mathfrak{g}_\mathbb{C}$ is given by $B_\theta(X,Y) = 2(n+2)\operatorname{tr}(XY^*)$ (resp., $4(n+3)\operatorname{tr}(XY^*)$) for the case where $(G, G/P)$ is $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$ (resp., $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$). Thus, letting ω_{G/K_G} be the volume form on G/K_G of compatible orientation and of norm 1 with respect to the Killing metric of $\mathfrak{g}$, we get the equation corresponding to (41):

$$(E^\vee_{11} - E^\vee_{n''n''}) \wedge \bigwedge_{k=2}^{n''} (E^\vee_{1k} + E^\vee_{k1}) \wedge \bigwedge_{k=2}^{n'} (E^\vee_{kn''} + E^\vee_{n''k}) = \frac{1}{(n+2)^{n+1}}\, \phi^*_{K_G}\omega_{G/K_G} \tag{43}$$

for the case where $(G, G/P) = (\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$ and

$$(\gamma_1 - \gamma_{n''}) \wedge \bigwedge_z ({}^t z^\vee + z^\vee) = \frac{1}{2^{4n+3}(n+3)^{2n+2}}\, \phi^*_{K_G}\omega_{G/K_G} \tag{44}$$

for the case where $(G, G/P) = (\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$, where z runs in

$$\{u_{1,k}\}_{2\le k\le n'} \cup \{u_{2,k}\}_{2\le k\le n'} \cup \{x_{1,k}\}_{2\le k\le n'} \cup \{x_{2,k}\}_{2\le k\le n'} \cup \{y_1, y_2, v\} \tag{45}$$

in this order. We embed K_G/K_P into $\mathbb{C}^{n+1}$ (resp., $\mathbb{H}^{n+1}$) as the standard unit sphere. The orthogonal complement $\mathfrak{m}$ of $\mathfrak{k}_P$ in $\mathfrak{k}_G$ is also described in a way similar to (40). Like in the case of $(G, G/P) = (\mathrm{SO}_0(n+1,1), S^n_\infty)$, the second part of the right hand side of (29) (resp., (34)) is a volume form on S^{2n+1} (resp., S^{4n+3}). So we orient the fibers of G/K_G with this volume form. Taking into account the structure of the Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ (resp., $S^3 \to S^{4n+3} \to \mathbb{H}P^n$), we see that, under the identification of $\mathfrak{m}$ and the tangent space of S^{2n+1} (resp., S^{4n+3}), the norm of the invariant multivector fields

$$\bigwedge_{k=2}^{n''} (E_{1k} - E_{k1}) \wedge \bigwedge_{k=2}^{n'} (E_{kn''} - E_{n''k})$$

(resp., $\bigwedge_z ({}^t z - z)$, where z runs in (45)) with respect to the standard metric on the standard unit sphere in $\mathbb{C}^{n+1}$ (resp., $\mathbb{H}^{n+1}$) is 2^n (resp., $2^{2n+\frac{1}{2}}$). By using the pairing of invariant volume forms on K_G/K_P with the above multivector fields, we see that

$$\bigwedge_{k=2}^{n''} (E^\vee_{1k} - E^\vee_{k1}) \wedge \bigwedge_{k=2}^{n'} (E^\vee_{kn''} - E^\vee_{n''k})$$

(resp., $\bigwedge_z ({}^t z^\vee - z^\vee)$, where z runs in (45)) is the invariant volume form on K_G/K_P with norm with respect to the standard metric is 2^{3n+1} (resp., $2^{6n+\frac{7}{2}}$). Thus, by (43) or (44), we get the equation corresponding to (41) in each case:

$$(46)\quad \fint (E_{11}^\vee - E_{n''n''}^\vee) \wedge \bigwedge_{k=2}^{n''} (E_{1k}^\vee + E_{k1}^\vee) \wedge \bigwedge_{k=2}^{n'} (E_{kn''}^\vee + E_{n''k}^\vee) \wedge \bigwedge_{k=2}^{n''} (E_{1k}^\vee - E_{k1}^\vee) \wedge \bigwedge_{k=2}^{n'} (E_{kn''}^\vee - E_{n''k}^\vee) = \frac{2^{3n+1}\operatorname{vol}(S^{2n+1})}{(n+2)^{n+1}}\,\omega_{G/K_G}$$

for the case where $(G, G/P) = (\mathrm{SU}(n+1,1), S_\infty^{2n+1})$ and

$$(47)\quad \fint (\gamma_1 - \gamma_{n''}) \wedge \bigwedge_z ({}^t z^\vee + z^\vee) \wedge \bigwedge_z ({}^t z^\vee - z^\vee) = \frac{2^{2n+\frac{1}{2}}\operatorname{vol}(S^{4n+3})}{(n+3)^{2n+2}}\,\omega_{G/K_G}$$

for the case where $(G, G/P) = (\mathrm{Sp}(n+1,1), S_\infty^{4n+3})$, where z runs in (45) in the given order. Then (38) for the case where $(G, G/P) = (\mathrm{SU}(n+1,1), S_\infty^{2n+1})$ (resp., $(\mathrm{Sp}(n+1,1), S_\infty^{4n+3})$) follows from (46) and (29) (resp., (47) and (34)).

In the case where $(G, G/P) = (F_{4(-20)}, S_\infty^{15})$, $\mathrm{GV}(\mathcal{F}_P)$ is divided into two parts in a similar way to the other cases. We will use the notation of Section 7.2.5. We orient G/K_G and the fibers of ϕ_{K_G} in a way similar to the other cases using the first and second components of (37). By $B_\theta(H_3, H_3) = \sqrt{18}$ and $B_\theta(E_\alpha, E_\alpha) = 1$, letting ω_{G/K_G} be a volume form on G/K_G of compatible orientation and of norm 1 with respect to the Killing metric, we get the equation corresponding to (41):

$$(48)\quad \lambda_3 \wedge \bigwedge_{\alpha\in\Upsilon^+\setminus\Phi} (E_\alpha^\vee + E_{-\alpha}^\vee) = \frac{2^7}{3}\,\phi_{K_G}^* \omega_{G/K_G}\,.$$

The computation of the norm of $\bigwedge_{\alpha\in\Upsilon^+\setminus\Phi}(E_\alpha^\vee - E_{-\alpha}^\vee)$ as an invariant volume form on S_∞^{15} is more complicated, reflecting the structure of the K_G-action on S_∞^{15}. Recall that $K_G = \mathrm{Spin}(9)$ and K_P is a subgroup of $\mathrm{Spin}(9)$ isomorphic to $\mathrm{Spin}(7)$ (Section 6.6). Let $\mathfrak{so}(9)_\mathbb{C} = \mathfrak{so}(8)_\mathbb{C} \oplus \mathfrak{m}_1$ (resp., $\mathfrak{so}(8)_\mathbb{C} = (\mathfrak{k}_P)_\mathbb{C} \oplus \mathfrak{m}_2$) be a decompositions as an $\mathfrak{so}(8)_\mathbb{C}$-module (resp., $(\mathfrak{k}_P)_\mathbb{C}$-module). Here $\mathfrak{m}_1 \oplus \mathfrak{m}_2$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are identified with the tangent space of $K_G/K_P \approx S^{15}$, $\mathrm{Spin}(9)/\mathrm{Spin}(8) \approx S^8$ and $\mathrm{Spin}(8)/K_P \approx S^7$ at a point, respectively. Here $\mathfrak{so}(9)_\mathbb{C}$ and $\mathfrak{so}(8)_\mathbb{C}$ are spanned by the root vectors E_α of $\mathfrak{f}_4^\mathbb{C}$ used in Section 7.2.5, because the Cartan subalgebra $\mathfrak{h} = \oplus_{i=1}^4 \mathbb{C}H_i$ of $\mathfrak{f}_4^\mathbb{C}$ used in Section 7.2.5 is contained in $\mathfrak{so}(9)_\mathbb{C}$ and $\mathfrak{so}(8)_\mathbb{C}$. The Killing form $B_{\mathfrak{so}(n)_\mathbb{C}}$ of $\mathfrak{so}(n)_\mathbb{C}$ is given by $B_{\mathfrak{so}(n)_\mathbb{C}}(X, Y) = (n-2)\operatorname{tr}(XY)$. Since

$$B_{\mathfrak{so}(n)_\mathbb{C}}\left(\sum_{i=1}^4 \lambda_i H_i, \sum_{i=1}^4 \lambda'_i H_i\right) = 2(n-2)\sum_{i=1}^4 \lambda_i \lambda'_i = \frac{n-2}{9}\, B\left(\sum_{i=1}^4 \lambda_i H_i, \sum_{i=1}^4 \lambda'_i H_i\right)$$

and

$$B_{\mathfrak{so}(n)_\mathbb{C}}(E_\alpha, E_{-\alpha})\, H'_\alpha = [E_\alpha, E_{-\alpha}] = B(E_\alpha, E_{-\alpha})\, H_\alpha\,,$$

where H'_α is the element of $\mathfrak{h}$ determined by $B_{\mathfrak{so}(n)_\mathbb{C}}(H'_\alpha, H) = \alpha(H)$ for any $H \in \mathfrak{h}$, we get

$$B_{\mathfrak{so}(n)_\mathbb{C}}(E_\alpha, E_{-\alpha}) = \frac{n-2}{9}\, B(E_\alpha, E_{-\alpha}) = \frac{n-2}{9} \tag{49}$$

for $n \in \{8, 9\}$. Let $\Upsilon^+ \setminus \Phi = \{\alpha_j\}_{j=1}^{15}$ so that

$$\mathfrak{m}_1 = \bigoplus_{j=1}^{7} \mathbb{C}(E_{\alpha_j} - E_{-\alpha_j})\ , \quad \mathfrak{m}_2 = \bigoplus_{j=8}^{15} \mathbb{C}(E_{\alpha_j} - E_{-\alpha_j})\ .$$

By (49) and the fact that $\mathfrak{k}_P$ is conjugate to $\mathfrak{so}(7)$ in $\mathrm{O}(8)$ [Yok09, Remark after Theorem 2.7.5], it follows that $(\frac{3}{2})^{7/2} \bigwedge_{j=1}^{7}(E^\vee_{\alpha_j} - E^\vee_{-\alpha_j})$ corresponds to the invariant volume form on S^7 of norm $2^{7/2}$ and $(\frac{3}{\sqrt{7}})^8 \bigwedge_{j=8}^{15}(E^\vee_{\alpha_j} - E^\vee_{-\alpha_j})$ corresponds to the invariant volume form of S^8 of norm 2^4. Thus, by (48), we get

$$\fint \lambda_3 \wedge \bigwedge_{\alpha\in\Upsilon^+\setminus\Phi} (E^\vee_\alpha + E^\vee_{-\alpha}) \wedge \bigwedge_{\alpha\in\Upsilon^+\setminus\Phi} (E^\vee_\alpha - E^\vee_{-\alpha}) = \frac{2^{18}\, 7^4\, \mathrm{vol}(S^{15})}{3^{25/2}}\, \omega_{G/K_G}\ .$$

Thus, combining this equation with (37), we get (38) for this case. □

The same computation gives the following relation of the Godbillon-Vey class and the volume in the level of Lie algebra cohomology.

Proposition 7.10. *Let $(G, G/P)$ be one of $(\mathrm{SO}_0(n+1,1), S^n_\infty)$, $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ or $(F_{4(-20)}, S^{15}_\infty)$. Let $q = \dim G/P$ (the codimension of $\mathcal{F}_P$). We have*

$$\fint_{\phi_{K_P}} \Delta_{\mathcal{F}_P}(h_1 c_1^q) = c_G\, \omega_{G/K_G} \tag{50}$$

in $\left(\bigwedge^{q+1}\mathfrak{g}^*\right)_{K_G}$ for some orientations of G/K_G and the fibers of $\phi_{K_G} \colon G/K_P \to G/K_G$, where c_G is the constant depending on $(G, G/P)$ given in Proposition 7.9.

Remark 7.11. By [KT75a, Theorem 7.83], the following diagram commutes:

$$\begin{array}{ccc} H^\bullet(\mathfrak{g}, K_P) & \longrightarrow & H^\bullet(\Gamma\backslash G/K_P) \\ {\scriptstyle \fint}\downarrow & & \downarrow{\scriptstyle \fint} \\ H^\bullet(\mathfrak{g}, K_G) & \xrightarrow[\kappa]{} & H^\bullet(\Gamma\backslash G/K_G)\ . \end{array} \tag{51}$$

The homomorphism κ is well known to be injective. The commutativity describes the relation between Propositions 7.9 and 7.10.

By the well known relation $\mathrm{GV}(\mathcal{F}_P) = (2\pi)^{q+1}\,[\Delta_{\mathcal{F}_P}(h_1 c_1^q)]$ [KT75a, Theorem 7.20], Proposition 7.9 or 7.10 implies the following.

Corollary 7.12. *Under the assumption of Proposition 7.9, we have*

$$\frac{1}{(2\pi)^{q+1}} \int_{\Gamma\backslash G/K_P} \mathrm{GV}(\mathcal{F}_P) = c_G \ \mathrm{vol}(\Gamma\backslash G/K_P) \ ,$$

where $\mathrm{vol}(\Gamma\backslash G/K_P)$ *is the volume of* $\Gamma\backslash G/K_P$ *with the metric induced from the Killing metric of* $\mathfrak{g}$.

7.4.3. *Bott-Thurston-Heitsch type formulas for suspension foliations.* The homogeneous foliations are suspension foliations over locally symmetric spaces whose holonomy homomorphisms are the canonical embeddings of lattices. We will show Bott-Thurston-Heitsch type formulas (Theorem 1.9) which can be applied to more general suspension foliations.

Suspension foliations $\mathcal{F}$ in the statement of Theorem 1.9 are $(G, G/P)$-foliations on the total spaces of G/P-bundles over manifolds N which are transverse to the G/P-fibers by construction. In the case where $\dim G/P > 1$, it is easy to see that, conversely, any $(G, G/P)$-foliation on the total space of a G/P-bundle over a manifold N which is transverse to G/P-fibers is a suspension foliation in the statement of Theorem 1.9. In this section, we prove Theorem 1.9 for $(G, G/P)$-foliations on the total spaces of G/P-bundles over manifolds N which are transverse to the G/P-fibers. Part of the argument will be used later in a more general situation.

Let $(G, G/P)$ be $(\mathrm{SO}_0(n+1,1), S^n_\infty)$, $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ or $(F_{4(-20)}, S^{15}_\infty)$. Let $q = \dim G/P$ (the codimension of $(G, G/P)$-foliations). Consider the case of codimension $q > 1$; namely, all cases except $(\mathrm{SO}_0(2,1), S^1_\infty)$ and $(\mathrm{SU}(1,1), S^1_\infty)$. Let N be a smooth manifold, and $p_M \colon M \to N$ an S^q-bundle over N. Let $\mathcal{F}$ be a $(G, G/P)$-foliation of M which is transverse to the fibers of p_M. Since G preserves an orientation of G/P, it follows that p_M is orientable.

We have two G-equivariant fibrations on G/K_P:

$$\begin{array}{ccc} G/P & \xleftarrow{\ \phi_P\ } & G/K_P \\ & & \big\downarrow \phi_{K_G} \\ & & G/K_G \ . \end{array}$$

Now, it is easy to see that the fibers of ϕ_P and ϕ_{K_G} are of complementary dimension and transverse to each other. This observation implies the following.

Lemma 7.13. *Let* $\mathrm{dev} \colon \widetilde{M} \to G/P$ *be the developing map of* $\mathcal{F}$*. For any* $\pi_1 M$*-equivariant map* $s \colon \widetilde{M} \to G/K_G$*, there exists a unique map* $\widehat{\mathrm{dev}} \colon \widetilde{M} \to G/K_P$ *which is* $\pi_1 M$*-equivariant, satisfies* $\widehat{\mathcal{F}} = \widehat{\mathrm{dev}}^* \mathcal{F}_P$ *and makes the following diagram* *commutative:*

$$\begin{array}{ccc} G/P & \xleftarrow{\ \phi_P\ } & G/K_P \\ {\scriptstyle \mathrm{dev}}\uparrow & \nearrow{\scriptstyle \widehat{\mathrm{dev}}} & \downarrow{\scriptstyle \phi_{K_G}} \\ \widetilde{M} & \xrightarrow[\ s\ ]{} & G/K_G\,. \end{array}$$

Moreover, if s is submersive at a point $x \in \widetilde{M}$, then $\widehat{\mathrm{dev}}$ is submersive at x.

The equality $\widehat{\mathcal{F}} = \widehat{\mathrm{dev}}^* \mathcal{F}_P$ is a trivial consequence of the construction like in Proposition 3.9. To prove the latter part of Lemma 7.13, note that dev is a submersion.

Regard $\mathrm{hol}(\mathcal{F})$ as a homomorphism $\pi_1 N \cong \pi_1 M \to G$. Given an orientation of G/K_G, the volume $\mathrm{vol}(\mathrm{hol}(\mathcal{F}))$ of $\mathrm{hol}(\mathcal{F})$ is defined in $H^{q+1}(N;\mathbb{R})$ as mentioned in Section 7.4.1.

Proposition 7.14. *We orient G/K_G and the fibers of ϕ_{K_G} like in Proposition 7.9. Then we have*

$$\frac{1}{(2\pi)^{q+1}} \fint_{p_M} \mathrm{GV}(\mathcal{F}) = c_G \,\mathrm{vol}(\mathrm{hol}_{\mathcal{F}}) \tag{52}$$

in $H^{q+1}(N;\mathbb{R})$ for an orientation of the fibers of p_M, where c_G is the function of $(G, G/P)$ mentioned in Proposition 7.9.

Proof. Take a $\pi_1 N$-equivariant map $\overline{s} \colon \widetilde{N} \to G/K_G$. We get a $\pi_1 M$-equivariant map $s = \overline{s} \circ p_{\widetilde{M}} \colon \widetilde{M} \to G/K_G$, where $p_{\widetilde{M}} \colon \widetilde{M} \to \widetilde{N}$ is the canonical projection. By Lemma 7.13, we get a $\pi_1 M$-equivariant map $\widehat{\mathrm{dev}} \colon \widetilde{M} \to G/K_P$ which makes the following diagram commutative:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\ \widehat{\mathrm{dev}}\ } & G/K_P \\ {\scriptstyle p_{\widetilde{M}}}\downarrow & & \downarrow{\scriptstyle \phi_{K_G}} \\ \widetilde{N} & \xrightarrow[\ \overline{s}\ ]{} & G/K_G\,. \end{array} \tag{53}$$

Since $\mathcal{F}$ is transverse to the fibers of p_M, the restriction of $\widehat{\mathrm{dev}}$ to each fiber of $p_{\widetilde{M}}$ is a covering map onto a fiber of ϕ_{K_G}. Since $p_{\widetilde{M}}$ and ϕ_{K_G} are S^q-bundles and $q > 1$, the restriction of $\widehat{\mathrm{dev}}$ to each fiber of $p_{\widetilde{M}}$ is a diffeomorphism. Thus the diagram (53) is the pull-back of fiber bundles. We fix an orientation of the fibers of p_M so that it is compatible with the orientation of the fibers of ϕ_{K_G} under $\widehat{\mathrm{dev}}^*$. Then $\fint_{p_{\widetilde{M}}} \widehat{\mathrm{dev}}^* \beta = \overline{s}^* \fint_{\phi_{K_G}} \beta$ for any $\beta \in \Omega^\bullet(G/K_P)$. We have $\fint_{\phi_{K_G}} \Delta_{\mathcal{F}_P}(h_1 c_1^q) = c_G \,\omega_{G/K_G}$ in $(\bigwedge^{q+1} \mathfrak{g}^*)_{K_P}$ by Proposition 7.10. Let $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\widetilde{M}$. Since $\widetilde{\mathcal{F}} = \widehat{\mathrm{dev}}^* \mathcal{F}_P$ by Lemma 7.13, we have

$$\int_{p_{\widetilde{M}}} \Delta_{\widetilde{\mathcal{F}}}(h_1 c_1^q) = \int_{p_{\widetilde{M}}} \widehat{\operatorname{dev}}^* \Delta_{\mathcal{F}_P}(h_1 c_1^q) = \overline{s}^* \int_{\phi_{K_G}} \Delta_{\mathcal{F}_P}(h_1 c_1^q) = c_G\, \overline{s}^* \omega_{G/K_G}$$

in $\Omega^\bullet(\widetilde{N})^{\pi_1 N}$. Eq. (52) follows from this equality and the well known relation $\mathrm{GV}(\mathcal{F}) = (2\pi)^{q+1}[\Delta_{\mathcal{F}}(h_1 c_1^q)]$ [KT75a, Theorem 7.20]. □

Proof of Theorem 1.9. Since the sign of both sides of (2) change when the orientation of the fibers of p_M changes, it suffices to prove (2) for any fixed orientation of the fibers of p_M. We orient G/K_G and the fibers of ϕ_{K_G} as in Proposition 7.9. Then we choose the orientation of the fibers of p_M like in the statement of Proposition 7.14.

By assumption, G/K_G is of even dimension $q+1$. Since G/K_G has a G-invariant metric, the Euler form e of the oriented tangent bundle of G/K_G is a left invariant volume form on G/K_G. Thus there exists a constant μ such that $e = \mu\, \mathrm{vol}_{G/K_G}$, where vol_{G/K_G} is the left invariant form of compatible orientation and of norm 1 with respect to the Killing metric on $\mathfrak{g}$. Let vol_Γ and e_Γ be the volume forms on $\Gamma\backslash G/K_G$ such that $p_N^* \mathrm{vol}_\Gamma = \mathrm{vol}_{G/K_G}$ and $p_N^* e_\Gamma = e$, where $p_N\colon G/K_G \to N$ is the universal covering of N. By the Hirzebruch proportionality principle [CGW76, Theorem 3.3] (see also [KO90]), we can compute the constant μ by using the compact dual $K_{G_\mathbb{C}}/K_G$ of G/K_G as follows:

$$\mu = \frac{\int_{\Gamma\backslash G/K_G} e_\Gamma}{\int_{\Gamma\backslash G/K_G} \mathrm{vol}_\Gamma} = (-1)^{(q+1)/2}\, \frac{e(K_{G_\mathbb{C}}/K_G)}{\mathrm{vol}(K_{G_\mathbb{C}}/K_G)}\ ,$$

where $e(K_{G_\mathbb{C}}/K_G)$ is the Euler number of $K_{G_\mathbb{C}}/K_G$ and $\mathrm{vol}(K_{G_\mathbb{C}}/K_G)$ is the volume of $K_{G_\mathbb{C}}/K_G$ with respect to the metric induced by the Killing form on $\mathfrak{g}_\mathbb{C}$. The volume $\mathrm{vol}(K_{G_\mathbb{C}}/K_G)$ was computed in [AY97], obtaining:

$K_{G_\mathbb{C}}/K_G$	e	vol
$\mathbb{R}P^{n+1}$	1	$2^{\frac{n-1}{2}} n^{\frac{n+1}{2}} \mathrm{vol}(S^{n+1})$
$\mathbb{C}P^{n+1}$	$n+2$	$\dfrac{2^{n+1}(n+2)^{n+1}\pi^{n+1}}{(n+1)!}$
$\mathbb{H}P^{n+1}$	$n+2$	$\dfrac{2^{6(n+1)}(n+3)^{n+1}\pi^{2(n+1)}}{(2n+3)!}$
$\mathbb{O}P^2$	3	$\dfrac{72^8 6\pi^8}{11!}$

Here, we also indicate the Euler number $e(K_{G_\mathbb{C}}/K_G)$ of $K_{G_\mathbb{C}}/K_G$. Thus Theorem 1.9 follows from Proposition 7.14, where the constant r_G in (2) is obtained from c_G in Proposition 7.14 by $r_G = (-1)^{(q+1)/2}\mu^{-1} c_G$. □

8. The case where $G/P = S^q$ for even q

8.1. Integration along the fibers of Haefliger structures. For transversely projective foliations, the integration of secondary invariants along the fibers of the Haefliger structures was computed by Brooks-Goldman [BG84, Lemma 2] and Heitsch [Hei86, Lemma in Section 5] to prove Proposition 5.1, which is an essential part of the proof of Theorem 1.1. In this section, we will see that such computation is reduced to a computation in Lie algebra cohomology in the case where G/P is a sphere. This observation enables us to state a sufficient condition, that implies Proposition 5.1, in terms of Lie algebra cohomology. We will also see that Proposition 5.1 is not true for transversely conformally flat foliations of even codimensions. In this section, the coefficient ring of cohomology is $\mathbb{C}$.

Let $\mathcal{X}_G(\mathcal{F})$ be the principal G-bundle over M associated to $\mathcal{F}$. Consider the diagram of bundle maps between fiber bundles over M,

$$
\begin{array}{ccc}
\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/K_P & \longleftarrow & \mathcal{X}_G(\mathcal{F})/K_P \\
\downarrow & & \downarrow \\
\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/K_G & \longleftarrow & \mathcal{X}_G(\mathcal{F})/K_G \ ,
\end{array}
\tag{54}
$$

where the horizontal maps are inclusions defined by fiberwise complexification and the vertical maps are canonical projections. Let $\mathcal{H}^\bullet(G_{\mathbb{C}}/K_P)$, $\mathcal{H}^\bullet(G/K_P)$ and $\mathcal{H}^\bullet(G_{\mathbb{C}}/K_G)$ be the local systems over M associated to the fiber bundles $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/K_P$, $\mathcal{X}_G(\mathcal{F})/K_P$ and $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/K_G$, respectively. Note that the local system associated to $\mathcal{X}_G(\mathcal{F})/K_G$ is trivial because the fiber G/K_G is contractible. By using integration along fibers of the vertical maps of (54), we get the commutative diagram

$$
\begin{array}{ccc}
H^\bullet(M;\mathcal{H}^\bullet(G_{\mathbb{C}}/K_P)) & \longrightarrow & H^\bullet(M;\mathcal{H}^\bullet(G/K_P)) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\
H^\bullet(M;\mathcal{H}^\bullet(G_{\mathbb{C}}/K_G)) & \longrightarrow & H^\bullet(M) \ .
\end{array}
\tag{55}
$$

Observe that we have natural isomorphisms

$$
H^\bullet(\mathfrak{g}, K_P) \otimes \mathbb{C} \cong H^\bullet(\mathfrak{k}_{G_{\mathbb{C}}}, K_P) \otimes \mathbb{C} \cong H^\bullet(K_{G_{\mathbb{C}}}/K_P) \cong H^\bullet(G_{\mathbb{C}}/K_P) \ , \tag{56}
$$

$$
H^\bullet(\mathfrak{g}, K_G) \otimes \mathbb{C} \cong H^\bullet(\mathfrak{k}_{G_{\mathbb{C}}}, K_G) \otimes \mathbb{C} \cong H^\bullet(K_{G_{\mathbb{C}}}/K_G) \cong H^\bullet(G_{\mathbb{C}}/K_G) \ , \tag{57}
$$

where the first isomorphisms in the two equations are the well known isomorphism in the Weyl's trick [KO90, Section 3]. We get the commutative diagram

$$
\begin{array}{ccccc}
H^\bullet(\mathfrak{g}, K_P) \otimes \mathbb{C} & \longrightarrow & H^\bullet(G_{\mathbb{C}}/K_P) & \longrightarrow & H^\bullet(M;\mathcal{H}^\bullet(G_{\mathbb{C}}/K_P)) \\
{\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow \\
H^\bullet(\mathfrak{g}, K_G) \otimes \mathbb{C} & \longrightarrow & H^\bullet(G_{\mathbb{C}}/K_G) & \longrightarrow & H^\bullet(M;\mathcal{H}^\bullet(G_{\mathbb{C}}/K_G)) \ .
\end{array}
\tag{58}
$$

Recall that $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/K_P$ has a $(G, G/P)$-foliation $p^*\mathcal{E}_{\mathrm{hol}(\mathcal{F})}$, which is obtained by pulling back the foliation $\mathcal{E}_{\mathrm{hol}(\mathcal{F})}$ on $\mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/P$ defined by the flat G-connection by the canonical projection $p : \mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/K_P \to \mathcal{X}_{G_{\mathbb{C}}}(\mathcal{F})/P$. By combining Theorem 4.3, diagrams (55) and (58), and the definition of the characteristic homomorphisms, we get the following.

Proposition 8.1. *The following diagram is commutative:*

(59)
$$\begin{array}{ccccc}
H^\bullet(WO_q) & & \xrightarrow{\quad\Delta'_{p^*\mathcal{E}_{\mathrm{hol}(\mathcal{F})}}\quad} & & \\
\downarrow{\scriptstyle \Delta_{\mathcal{F}_P}} & & & & \\
H^\bullet(\mathfrak{g}, K_P)\otimes\mathbb{C} & \longrightarrow & H^\bullet(M;\mathcal{H}^\bullet(G_{\mathbb{C}}/K_P)) & \longrightarrow & H^\bullet(M;\mathcal{H}^\bullet(G/K_P)) \\
\downarrow{\scriptstyle \int} & & \downarrow{\scriptstyle \int} & & \downarrow{\scriptstyle \int} \\
H^\bullet(\mathfrak{g}, K_G)\otimes\mathbb{C} & \longrightarrow & H^\bullet(M;\mathcal{H}^\bullet(G_{\mathbb{C}}/K_G)) & \longrightarrow & H^\bullet(M)\ ,
\end{array}$$
$$H^\bullet(\mathfrak{g}, K_G)\otimes\mathbb{C} \xrightarrow{\ \Xi_{\mathrm{hol}(\mathcal{F})}\ } H^\bullet(M)$$

where $\Delta'_{p^*\mathcal{E}_{\mathrm{hol}(\mathcal{F})}}$ *is the map induced by the characteristic homomorphism* $\Delta_{p^*\mathcal{E}_{\mathrm{hol}(\mathcal{F})}}$ *of* $p^*\mathcal{E}_{\mathrm{hol}(\mathcal{F})}$*, and* $\Xi_{\mathrm{hol}(\mathcal{F})} \colon H^\bullet(\mathfrak{g}, K_G) \to H^\bullet(M)$ *is the characteristic homomorphism of the flat* G/K_G*-bundle* $\mathcal{X}_G(\mathcal{F})/K_G \to M$ *mentioned in Section 7.4.1.*

This proposition is specially useful when G/P is a sphere because of the following.

Lemma 8.2. *Let* σ *be a cohomology class of* $\mathcal{X}_G(\mathcal{F})/K_P$*. Then* σ *belongs to the image of* $\pi^*_{G/K_P} \colon H^\bullet(M) \to H^\bullet(\mathcal{X}_G(\mathcal{F})/K_P)$ *if and only if* $\int \sigma = 0$*.*

Proof. Note that $\mathcal{X}_G(\mathcal{F})/K_P$ is homotopy equivalent to a sphere bundle $\mathcal{X}_G(\mathcal{F})/P$ over M. Since $\mathcal{X}_G(\mathcal{F})/P$ has a section, the Gysin sequence splits to give the exact sequence

$$0 \longrightarrow H^\bullet(M) \xrightarrow{\pi^*_{G/K_P}} H^\bullet(\mathcal{X}_G(\mathcal{F})/K_P) \xrightarrow{\int} H^\bullet(M) \longrightarrow 0\ . \qquad \square$$

The composite of the upper horizontal maps of (59) is induced on the E_2-terms of the Leray-Hirsch spectral sequence of $\mathcal{X}_G(\mathcal{F})/K_P \to M$ by the characteristic homomorphism $H^\bullet(\mathfrak{g}, K_P) \to H^\bullet(\mathcal{X}_G(\mathcal{F})/K_P)$ of the $(G, G/P)$-foliation $p^*\mathcal{E}_{\mathrm{hol}(\mathcal{F})}$ on $\mathcal{X}_G(\mathcal{F})/K_P$ mentioned in Proposition 3.9. Thus, as a consequence of Proposition 8.1 and Lemma 8.2, we get the following.

Proposition 8.3. *If* $\int \Delta_{\mathcal{F}_P}(\sigma) = 0$ *for* $\sigma \in H^\bullet(\mathfrak{g}, K_P)$*, then* $\Delta_{p^*\mathcal{E}_{\mathrm{hol}(\mathcal{F})}}(\sigma)$ *belongs to the image of* $\pi^*_{G/K_P} \colon H^\bullet(M) \to H^\bullet(\mathcal{X}_G(\mathcal{F})/K_P)$*.*

This proposition reduces the latter condition to the former condition, which involves only Lie algebra cohomology. Thus the following proposition gives an alternative proof of a consequence of the residue formulas of Heitsch.

Proposition 8.4 (Heitsch [Hei78, Theorem 4.2] and [Hei83, Theorem 2.3])**.** *In the case where* $(G, G/P) = (\mathrm{SL}(q+1;\mathbb{R}), S^q)$ *for even* q*, we have* $\fint \Delta_{\mathcal{F}_P}(\sigma) = 0$ *for any* $\sigma \in H^\bullet(WO_q)$*.*

Proof. We will use the notation of Example 7.2.1. First, we show $\fint \Delta_{\mathcal{F}_P}(h_1 c_1^q) = 0$. By (21) and (27), we get

$$\begin{aligned}\Delta_{\mathcal{F}_P}(h_1 c_1^q) &= -\frac{(q')^{q+1}\, q'!}{(2\pi)^{q+1}}\, E_{11}^\vee \wedge \bigwedge_{k=2}^{q'} E_{1k}^\vee \wedge E_{k1}^\vee \\ &= \frac{(-1)^{\frac{q(q-1)}{2}+1}(q')^{q+1}\, q'!}{2^q (2\pi)^{q+1}}\, E_{11}^\vee \wedge \bigwedge_{k=2}^{q'} (E_{1k}^\vee + E_{k1}^\vee) \wedge \bigwedge_{k=2}^{q'} (E_{1k}^\vee - E_{k1}^\vee)\ .\end{aligned}$$

Here, $\bigwedge_{k=2}^{q'}(E_{1k}^\vee - E_{k1}^\vee)$ is a volume form of $\mathrm{SO}(q')/\mathrm{SO}(q) \approx S^q$. Thus $\fint \Delta_{\mathcal{F}_P}(h_1 c_1^q)$ is obtained by integrating $E_{11}^\vee \wedge \bigwedge_{k=2}^{q'}(E_{1k}^\vee + E_{k1}^\vee)$ over S^q. But, since q is even, $E_{11}^\vee \wedge \bigwedge_{k=2}^{q'}(E_{1k}^\vee + E_{k1}^\vee)$ is an odd function on S^q; namely, we have

$$s^*\Big(E_{11}^\vee \wedge \bigwedge_{k=2}^{q'} (E_{1k}^\vee + E_{k1}^\vee)\Big) = -E_{11}^\vee \wedge \bigwedge_{k=2}^{q'} (E_{1k}^\vee + E_{k1}^\vee)\ ,$$

where s is the antipodal map of S^q. So the integration of $E_{11}^\vee \wedge \bigwedge_{k=2}^{q'}(E_{1k}^\vee + E_{k1}^\vee)$ over S^q is zero. This implies that $\fint \Delta_{\mathcal{F}_P}(h_1 c_1^q) = 0$.

Note that $h_I(\Theta_{MC})$ is K_G-basic; namely, $h_I(\Theta_{MC})$ is the pull-back of a differential form on $G_{\mathbb{C}}/K_G$. Thus, by (22),

$$\fint \Delta_{\mathcal{F}_P}(h_1 h_I c_1^q) = h_I(\Theta_{MC}) \fint \Delta_{\mathcal{F}_P}(h_1 c_1^q) = 0\ .$$

Since other secondary characteristic classes are generated by the classes of the form $h_1 h_I c_1^q$ by Theorem 7.3, the result follows. □

Remark 8.5. Heitsch [Hei86] applied consequences of his residue formulas, Theorem 1.7 and Proposition 8.4, to prove our Proposition 5.1 for the case where $(G, G/P) = (\mathrm{SL}(q+1;\mathbb{R}), S^q)$ for any q, and therefore Theorem 1.1. For even q, our proof of Proposition 8.4 is slightly simpler than the original proof of Heitsch [Hei86]. It is because we directly computed the map $H^\bullet(\mathfrak{g}, K_P) \to H^\bullet(\mathfrak{g}, K_G)$ in Section 7, while Heitsch applied his residue formulas ([Hei78, Theorem 4.2] and [Hei83, Theorem 2.3]). Thus we obtained a slightly simpler proof of Theorem 1.1 for even q. Note that we already gave an alternative proof of Theorem 1.1 for odd q in Section 6.2 by using Theorem 1.2.

In the case where $(G, G/P)$ is $(\mathrm{SO}(n+1,1), S^n_\infty)$ for odd n, $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ or $\big(F_{4(-20)}, S^{15}_\infty\big)$, our Bott-Thurston-Heitsch type formulas (Theorem 1.9) imply that the integration of $\mathrm{GV}(\mathcal{F}_P)$ along the fibers of the sphere bundle $G/K_P \to G/K_G$ is nonzero, but it is a constant multiple of the Euler class of the tangent sphere bundle of G/K_G. So we cannot apply Proposition 8.3

in this case to show Proposition 5.1. Nevertheless we get the following. Let $\varphi\colon \mathcal{X}_G(\mathcal{F})/K_P \to \mathcal{X}_G(\mathcal{F})/K_G$ be the canonical projection.

Proposition 8.6. *In the case where $(G, G/P)$ is equal to one of $(\mathrm{SO}(n{+}1,1), S^n_\infty)$ for odd n, $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ or $\big(F_{4(-20)}, S^{15}_\infty\big)$, we have $\fint_\varphi \mathrm{GV}(p^*\mathcal{E}_{\mathrm{hol}(\mathcal{F})}) = 0$ in $H^\bullet(M)$ for any $(G, G/P)$-foliation $\mathcal{F}$ of M.*

Proof. The sphere bundle φ has a section because it is homotopic to the Haefliger structure $\mathcal{X}_G(\mathcal{F})/P \to M$, which has a section (see Section 3.2.2). Thus its Euler class $e(\varphi)$ is zero. Since φ is a sphere bundle with a $(G, G/P)$-foliation transverse to fibers, we get $\fint \mathrm{GV}(p^*\mathcal{E}_{\mathrm{hol}(\mathcal{F})}) = r_G\, e(\varphi) = 0$ by the Bott-Thurston-Heitsch type formulas in Theorem 1.9. □

Remark 8.7. Note that the Godbillon-Vey class is essentially the unique nontrivial secondary class in this case by Proposition 7.4. Thus Lemma 8.2 gives us another proof of Proposition 5.1 for these $(G, G/P)$, and therefore another proof of Theorem 1.2.

On the other hand, the situation is different for transversely conformally flat foliations of even codimension. Let $(G, G/P)$ be $(\mathrm{SO}(n+1,1), S^n_\infty)$ for even n. Consider an S^n-bundle $M \to N$ and a $(G, G/P)$-foliation $\mathcal{F}$ of M transverse to the fibers with a nontrivial volume $\mathrm{vol}(\mathrm{hol}(\mathcal{F}))$. For example, we can take the fiber bundle $\Gamma\backslash G/K_P \to \Gamma\backslash G/K_G$ foliated by the homogeneous foliation for a torsion-free uniform lattice Γ of G. Recall that φ is the S^q-bundle $\mathcal{X}_G(\mathcal{F})/K_P \to \mathcal{X}_G(\mathcal{F})/K_G$ associated to $\mathcal{F}$ with the $(G, G/P)$-foliation $p^*\mathcal{E}_{\mathrm{hol}(\mathcal{F})}$ transverse to the fibers. We get the following.

Proposition 8.8. *$\fint_\varphi \mathrm{GV}(p^*\mathcal{E}_{\mathrm{hol}(\mathcal{F})})$ is nonzero.*

Proof. The volume of $p^*\mathcal{E}_{\mathrm{hol}(\mathcal{F})}$ is equal to $p^*_{K_G}\mathrm{vol}(\mathrm{hol}(\mathcal{F}))$, which is nontrivial by assumption. On the other hand, $\fint_\varphi \mathrm{GV}(p^*\mathcal{E}_{\mathrm{hol}(\mathcal{F})})$ is a nonzero constant multiple of the volume $p^*_{K_G}\mathrm{vol}(\mathrm{hol}(\mathcal{F}))$ by Proposition 7.14. □

8.2. **Finiteness with fixed Euler class.** Consider the case where $G/P = S^q$ for even q. In this section, we will show (4) in Theorem 1.15 (half of the weaker finiteness theorem for transversely conformally flat foliations). In this section, the coefficient ring of cohomology is $\mathbb{R}$. Since the Euler classes of even dimensional sphere bundles are trivial with real coefficients, the assumption of Theorem 1.2 is never satisfied by Proposition 6.1. Thus the Gysin sequence of the sphere bundle $\phi^{\mathbb{C}}\colon G_{\mathbb{C}}/K_P \to G_{\mathbb{C}}/K_G$ splits to give the exact sequence

$$0 \longrightarrow H^\bullet(G_{\mathbb{C}}/K_G) \xrightarrow{(\phi^{\mathbb{C}})^*} H^\bullet(G_{\mathbb{C}}/K_P) \xrightarrow{\fint_{\phi^{\mathbb{C}}}} H^{\bullet-q}(G_{\mathbb{C}}/K_G) \longrightarrow 0\,. \tag{60}$$

Let $\chi(\nu\mathcal{F}_P)$ be the Euler class of the normal bundle of the P/K_P-coset foliation $\mathcal{F}_P$ on G/K_P, which is of degree q.

Proposition 8.9. *$\fint_{\phi^{\mathbb{C}}} \chi(\nu\mathcal{F}_P) = 2$.*

Proof. Let $\phi_P\colon G/K_P \to G/P = S^q$ be the canonical projection. Consider the composite

$$K_G/K_P \longrightarrow G/K_P \xrightarrow{\phi_P} G/P \ .$$

Since $\phi_P^* TS^q = \nu\mathcal{F}_P$, we get

$$\int_{K_G/K_P} \chi(\nu\mathcal{F}) = \int_{S^q} \chi(TS^q) = 2 \ ,$$

which implies the equality of the statement. □

From (56), (57), (60) and Proposition 8.9, we get the following.

Proposition 8.10. *We have*

$$H^\bullet(\mathfrak{g}, K_P) \cong H^\bullet(\mathfrak{g}, K_G) \otimes \mathbb{R}[\chi]/(\chi^2)$$

as an $H^\bullet(\mathfrak{g}, K_G)$-module, where χ is the Euler class of the normal bundle of $\mathcal{F}$.

Consider the characteristic homomorphism $\Xi_{\mathrm{hol}(\mathcal{F})}\colon H^\bullet(\mathfrak{g}, K_G) \to H^\bullet(M)$, which depends only on $\mathrm{hol}(\mathcal{F})\colon \pi_1 M \to G$ (Section 7.4.1).

Proposition 8.11. *Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be two $(G, G/P)$-foliations of M with the same holonomy homomorphism. If $\chi(\nu\mathcal{F}_0) = \chi(\nu\mathcal{F}_1)$, then $\Delta_{\mathcal{F}_0}(\sigma) = \Delta_{\mathcal{F}_1}(\sigma)$ for any $\sigma \in H^\bullet(WO_q)$.*

Proof. By Theorem 4.3, it is sufficient to prove that $\Delta_{\mathcal{F}_0}(\sigma) = \Delta_{\mathcal{F}_1}(\sigma)$ for any $\sigma \in H^\bullet(\mathfrak{g}, K_P)$. For $\sigma \in H^\bullet(\mathfrak{g}, K_G)$, we get $\Delta_{\mathcal{F}_0}(\sigma) = \Delta_{\mathcal{F}_1}(\sigma)$ because $\Delta_{\mathcal{F}_i}(\sigma)$ is determined only by the holonomy homomorphism according to Proposition 8.1. Since $H^\bullet(\mathfrak{g}, K_P)$ is generated by χ and 1 as an $H^\bullet(\mathfrak{g}, K_G)$-module, we get $\Delta_{\mathcal{F}_0}(\sigma) = \Delta_{\mathcal{F}_1}(\sigma)$ for any $\sigma \in H^\bullet(\mathfrak{g}, K_P)$. □

Since $\pi_0(\mathrm{Hom}(\pi_1 M, G))$ is finite (see Remark 5.2), Theorem 4.4 and Proposition 8.11 imply (4) in Theorem 1.15.

8.3. **Finiteness over $\mathbb{R}/\mathbb{Z}$.** In this section, we will show (3) in Theorem 1.15 (the other half of the weaker finiteness theorem for transversely conformally flat foliations). Any $\sigma \in H^\bullet(WO_q)$ is said to be *divisible* by the Euler class χ if there exists some $\tau \in H^\bullet(\mathfrak{g}, K_P)$ such that $\Delta_{\mathcal{F}_P}(\sigma) = \tau \cdot \chi$. Note that such τ belongs to $H^\bullet(\mathfrak{g}, K_G)$ for any nontrivial divisible class σ by Proposition 8.10. Proposition 8.10 also implies the following.

Lemma 8.12. *If $\sigma \in H^\bullet(WO_q)$ is not divisible by the Euler class, then $⨍ \sigma = 0$.*

Thus Proposition 8.2 implies that there is a finite number of possibilities for $\Delta_{\mathcal{F}}(\sigma)$ when σ is not divisible by the Euler class.

On the other hand, we have the following.

Lemma 8.13. *If $\sigma \in H^\bullet(WO_q)$ is divisible by the Euler class, then $\Delta_{\mathcal{F}}(\sigma) = 0$ in $H^\bullet(M; \mathbb{R}/\mathbb{Z})$.*

Proof. Since $\chi(\nu\mathcal{F})$ belongs to the image of $H^\bullet(M;\mathbb{Z}) \to H^\bullet(M;\mathbb{R})$, we get $\Delta_{\mathcal{F}}(\sigma) = \Xi_{\mathrm{hol}(\mathcal{F})}(\tau) \cdot \chi(\nu\mathcal{F}) = 0$ in $H^\bullet(M;\mathbb{R}/\mathbb{Z})$. □

Since $\pi_0(\mathrm{Hom}(\pi_1 M, G))$ is finite (see Remark 5.2), Theorem 4.4 and Lemma 8.13 imply (3) in Theorem 1.15.

8.4. **Infiniteness of divisible classes.** In this section, we will show Theorem 1.13, an infiniteness result. Let $\sigma \in H^\bullet(WO_q)$ be a class divisible by the Euler class. More generally, we show the following result.

Theorem 8.14. *Assume that the restriction map $H^\bullet(\mathfrak{g}) \to H^\bullet(\mathfrak{k}_G)$ is surjective. Then there exists a connected manifold X with finitely presented fundamental group and an infinite family $\{\mathcal{F}_m\}_{m\in\mathbb{Z}}$ of $(G, G/P)$-foliations on X such that $\Delta_{\mathcal{F}_m}(\sigma) \neq \Delta_{\mathcal{F}_{m'}}(\sigma)$ if $m \neq m'$.*

To prove Theorem 8.14, we note the following fact.

Lemma 8.15. *Let $X \to Y$ be an S^q-bundle with a section. Then, for any $m \in \mathbb{Z}$, there exists a smooth bundle map $f_m\colon X \to X$ whose restriction to each S^q-fiber is of degree m.*

Proof. We fix a smooth fiberwise metric on $X \to Y$ so that each S^q-fiber is the standard round sphere. Let L be the image of a section of $X \to Y$. We can assume that L is a smooth submanifold of X. For $x \in X$, let F_x be the S^q-fiber of $X \to Y$ containing x, let $\{x_0\} = F_x \cap L$, and let c_x be a great circle of F through x and x_0. Under the identity $c_x \equiv \mathbb{R}/2\pi\mathbb{Z}$ with $x_0 \equiv 0$ given by the length parametrization, let $f_m(x) = mx$ for $m \in \mathbb{Z}$. This defines a smooth map $f_m\colon X \to X$ whose restriction to each fiber is of degree m. □

Proof of Theorem 8.14. Let Γ be a torsion-free uniform lattice of G. Note that Γ is finitely presented because it is the fundamental group of the closed manifold $\Gamma\backslash G/K_P$. Since q is even, the Euler class of the S^q-bundle $\Gamma\backslash G/K_P \to \Gamma\backslash G/K_G$ is zero. Hence it has a section. Then, by Lemma 8.15, we take a smooth map $f_m\colon \Gamma\backslash G/K_P \to \Gamma\backslash G/K_P$ of degree m for any $m \in \mathbb{Z}$. Let $\widetilde{f}_m\colon G/K_P \to G/K_P$ be the lift of f_m to the universal cover. Define $\Phi_m\colon G \times G/K_P \to G/K_P$ by $\Phi_m(g, x) = g\widetilde{f}_m(x)$. Since $\widetilde{f}_m$ is Γ-equivariant, we get

$$\Phi_m(g_1g_2, x) = g_1g_2\widetilde{f}_m(x) = g_1\widetilde{f}_m(g_2x) = \Phi_m(g_1, g_2x)$$

for $g_1 \in G$, $g_2 \in \Gamma$ and $x \in G/K_P$. Then Φ_m induces a smooth map $\Psi_m\colon X \to \Gamma\backslash G/K_P$, where X is the quotient of $G \times G/K_P$ by the Γ-action given by $g_2 \cdot (g_1, x) = (g_1g_2^{-1}, g_2x)$. This Ψ_m is a flat principal G-bundle over $\Gamma\backslash G/K_P$ by construction. Since $\pi_1 G$ is a finite group, $\pi_1 X$ is also finitely presented.

Let $\mathrm{ch}_m\colon H^\bullet(\mathfrak{g}) \to H^\bullet(X)$ be the characteristic homomorphism of Ψ_m as a flat principal G-bundle over $\Gamma\backslash G/K_P$. Let F be a fiber of Ψ_m, which is homotopy equivalent to K_G. By the assumption, the composite of

$$H^\bullet(\mathfrak{g}) \xrightarrow{\mathrm{ch}_m} H^\bullet(X) \longrightarrow H^\bullet(F) \cong H^\bullet(\mathfrak{k}_G)$$

is surjective, where the second arrow is the restriction map to F. Thus $\Psi_m^*\colon H^\bullet(\Gamma\backslash G/K_P) \to H^\bullet(X)$ is injective by the Leray-Hirsch theorem.

Consider the $(G, G/P)$-foliation $\mathcal{F}_m = \Psi_m^*\mathcal{F}_\Gamma$ on X, where $\mathcal{F}_\Gamma$ is the foliation of $\Gamma\backslash G/K_P$ whose lift to the universal cover G/K_P is the P/K_P-coset foliation $\mathcal{F}_P$. By assumption, there exists some $\tau \in H^\bullet(\mathfrak{g}, K_G)$ such that

$$\Delta_{\mathcal{F}_P}(\sigma) = \Xi_{\mathrm{hol}(\mathcal{F}_P)}(\tau) \cdot \chi(\nu\mathcal{F}_P) \ . \tag{61}$$

Since the map $\pi_1 X \to \pi_1(\Gamma\backslash G/K_P)$ induced by Ψ_m is independent of m, we get

$$\Xi_{\mathrm{hol}(\mathcal{F}_m)}(\tau) = \Xi_{\mathrm{hol}(\mathcal{F}_1)}(\tau) \tag{62}$$

for any m. On the other hand, since $\chi(\nu\mathcal{F}_\Gamma)$ is represented by the Poincaré dual of any S^q-fiber of $\Gamma\backslash G/K_P \to \Gamma\backslash G/K_G$, we get

$$\chi(\nu\mathcal{F}_m) = \Psi_m^*\chi(\nu\mathcal{F}_\Gamma) = m\Psi_1^*\chi(\nu\mathcal{F}_\Gamma) = m\chi(\nu\mathcal{F}_1) \tag{63}$$

by construction. By (61), (62) and (63), we get $\Delta_{\mathcal{F}_m}(\sigma) = m\Delta_{\mathcal{F}_1}(\sigma)$. By the injectivity of Ψ_1^*, $\Delta_{\mathcal{F}_1}(\sigma)$ is nontrivial of infinite order. Hence we get $\Delta_{\mathcal{F}_m}(\sigma) \neq \Delta_{\mathcal{F}_{m'}}(\sigma)$ for $m \neq m'$. □

Note that the manifolds X are noncompact in our construction. We get Theorem 1.13 as a corollary of Theorem 8.14 as follows.

Proof of Theorem 1.13. By Propositions 7.9, 8.1 and 8.10, there is some constant c so that $\mathrm{GV}(\mathcal{F}) = c\,\chi(\nu\mathcal{F})\,\mathrm{vol}(\mathrm{hol}(\mathcal{F}))$ for transversely conformally flat foliations $\mathcal{F}$ of even codimension. So the Godbillon-Vey class is divisible in this case. Moreover, the surjectivity of the restriction map $H^\bullet(\mathfrak{so}(n+1,1)) \to H^\bullet(\mathfrak{so}(n+1))$ follows from $H^\bullet(\mathfrak{so}(n+1,1)) \otimes \mathbb{C} \cong H^\bullet(\mathfrak{so}(n+2);\mathbb{C})$ and the surjectivity of $H^\bullet(\mathfrak{so}(n+2)) \to H^\bullet(\mathfrak{so}(n+1))$ (see, for example, [GHV76, Theorems VI and VII in Section 6.23]). Thus the assumption of Theorem 8.14 is satisfied, which implies Theorem 1.13. □

9. Rigidity of foliations on homogeneous spaces

9.1. **Generalization of Bott-Thurston-Heitsch type formulas.** Let $(G, G/P)$ be $(\mathrm{SO}_0(n+1,1), S^n_\infty)$, $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ or $(F_{4(-20)}, S^{15}_\infty)$. Let $q = \dim G/P$. Consider the case of codimension $q > 1$; namely, all cases except $(\mathrm{SO}_0(2,1), S^1_\infty)$ and $(\mathrm{SU}(1,1), S^1_\infty)$. Let $M = \Gamma\backslash G/K_P$ and $N = \Gamma\backslash G/K_G$. Let $\mathcal{F}$ be a $(G, G/P)$-foliation of $\Gamma\backslash G/K_P$ whose holonomy homomorphism is $\mathrm{hol}(\mathcal{F})\colon \pi_1 M \to G$. Since $\pi_1 M \cong \pi_1 N$, we regard $\mathrm{hol}(\mathcal{F})\colon \pi_1 N \to G$. We orient M and N with the orientation of G/K_P and the fibers of $\phi_{K_G}\colon G/K_P \to G/K_G$ in Proposition 7.9. The volume $\mathrm{vol}(\mathrm{hol}(\mathcal{F}))$ is defined in $H^{q+1}(N;\mathbb{R})$ with the orientation of G/K_G as mentioned in Section 7.4.1.

Lemma 9.1. *If* $(G, G/P)$ *is* $(\mathrm{SO}_0(n+1,1), S^n_\infty)$ *for n odd,* $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ *or* $(F_{4(-20)}, S^{15}_\infty)$*, then*

$$\frac{1}{(2\pi)^{q+1}} \int_M \mathrm{GV}(\mathcal{F}) = c_G \int_N \mathrm{vol}(\mathrm{hol}(\mathcal{F})) \ , \tag{64}$$

(65)
$$\frac{1}{(2\pi)^{q+1}}\int_M \mathrm{GV}(\mathcal{F}) = r_G \int_N e(p_M)\ ,$$

where $e(p_M)$ is the Euler class of $p_M\colon M \to N$, and r_G and c_G are the functions of $(G, G/P)$ mentioned in Theorem 1.9 and Proposition 7.9, respectively. If $(G, G/P)$ is $(\mathrm{SO}_0(n+1,1), S^n_\infty)$ for n even, then (64) is true.

Proof. First, we will prove (64) for all cases of $(G, G/P)$. The first part of this proof is like the proof of Proposition 7.14. Take a $\pi_1 N$-equivariant map $\overline{s}\colon \widetilde{N} \to G/K_G$ so that $\overline{s}$ is submersive at a point x. Let $p_{\widetilde{M}}\colon \widetilde{M} \to \widetilde{N}$ denote the canonical projection $\widetilde{M} \to \widetilde{N}$. We get a $\pi_1 M$-equivariant map $s = \overline{s} \circ p_{\widetilde{M}}\colon \widetilde{M} \to G/K_G$. By Lemma 7.13, we obtain a $\pi_1 M$-equivariant map $\widehat{\mathrm{dev}}\colon \widetilde{M} \to G/K_P$ which is submersive on $p_{\widetilde{M}}^{-1}(x)$ and makes the following diagram commutative:

$$\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\widehat{\mathrm{dev}}} & G/K_P \\
{\scriptstyle p_{\widetilde{M}}}\downarrow & & \downarrow{\scriptstyle \phi_{K_G}} \\
\widetilde{N} & \xrightarrow[\overline{s}]{} & G/K_G\ ,
\end{array}$$

where $\phi_{K_G}\colon G/K_P \to G/K_G$ is the canonical projection. Let $p_{\widetilde{\mathcal{Z}}}\colon \widetilde{\mathcal{Z}} \to N$ be the pull-back of the fiber bundle $\phi_{K_G}\colon G/K_P \to G/K_G$ by $\overline{s}$. We get the commutative diagram:

(66)
$$\begin{array}{ccccc}
\widetilde{M} & \xrightarrow{\widetilde{\psi}} & \widetilde{\mathcal{Z}} & \xrightarrow{\xi_{\widetilde{\mathcal{Z}}}} & G/K_P \\
 & {\scriptstyle p_{\widetilde{M}}}\searrow & \downarrow{\scriptstyle p_{\widetilde{\mathcal{Z}}}} & & \downarrow{\scriptstyle \phi_{K_G}} \\
 & & \widetilde{N} & \xrightarrow[\overline{s}]{} & G/K_G\ ,
\end{array}$$

where $\xi_{\widetilde{\mathcal{Z}}}$ is the canonical map and $\widetilde{\psi}$ is the map induced by the universality of the pull-back. By taking the quotient of the left triangle of (66) by Γ, we get the following diagram:

(67)
$$\begin{array}{ccc}
M & \xrightarrow{\psi} & \mathcal{Z} \\
 & {\scriptstyle p_M}\searrow & \downarrow{\scriptstyle p_{\mathcal{Z}}} \\
 & & N\ ,
\end{array}$$

where $\mathcal{Z}$ is the quotient of $\widetilde{\mathcal{Z}}$ by the induced Γ-action and ψ is the map induced by $\widetilde{\psi}$.

Let $\mathcal{F}_{\mathcal{Z}}$ be the foliation on $\mathcal{Z}$ whose lift to the universal cover $\widetilde{\mathcal{Z}}$ is $\xi^*_{\widetilde{\mathcal{Z}}}\mathcal{F}_P$. By applying Proposition 7.9 like in the proof of Proposition 7.14, we get

(68)
$$\frac{1}{(2\pi)^{q+1}} ⨍_{p_{\mathcal{Z}}} \mathrm{GV}(\mathcal{F}_{\mathcal{Z}}) = c_G\ \mathrm{vol}(\mathrm{hol}(\mathcal{F}))$$

in $H^{q+1}(N;\mathbb{R})$. Since $\mathcal{F} = \psi^*\mathcal{F}_{\mathcal{Z}}$, we obtain $\operatorname{GV}(\mathcal{F}) = \psi^* \operatorname{GV}(\mathcal{F}_{\mathcal{Z}})$. Hence

$$
\begin{aligned}
(69)\quad \frac{1}{(2\pi)^{q+1}} \int_M \operatorname{GV}(\mathcal{F}) &= \frac{\deg\psi}{(2\pi)^{q+1}} \int_{\mathcal{Z}} \psi^* \operatorname{GV}(\mathcal{F}_{\mathcal{Z}}) \\
&= (c_G \deg\psi) \int_N \operatorname{vol}(\operatorname{hol}(\mathcal{F})) ,
\end{aligned}
$$

where $\deg\psi$ is the degree of ψ as a continuous map. Since ψ is a bundle map that covers the identity map on N, we get

$$
(70)\qquad \deg\psi = \deg\left(\psi|_{p_M^{-1}(x)}\right) .
$$

Here, $\psi|_{p_M^{-1}(x)} : p_M^{-1}(x) \to p_{\mathcal{Z}}^{-1}(x)$ is a covering map because ψ is submersive on $p_M^{-1}(x)$. Since $\pi_1(p_M^{-1}(x)) \cong \pi_1(p_{\mathcal{Z}}^{-1}(x)) \cong \pi_1(S^q) = 1$ because $q > 1$, we obtain

$$
(71)\qquad \deg\left(\psi|_{p_M^{-1}(x)}\right) = 1 .
$$

By (69), (70) and (71), we get (64).

We get (65) by using Theorem 1.9 at (68) instead of Proposition 7.9. Note that $e(p_M) = e(p_{\mathcal{Z}})$, because ψ is a bundle map of degree one on each fiber. □

We obtain the following direct consequences.

Corollary 9.2. (i) *If* $(G, G/P)$ *is equal to* $(\mathrm{SO}_0(n+1,1), S^n_\infty)$ *for* n *odd,* $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, $(\mathrm{Sp}(n+1,1), S^{4n+3}_\infty)$ *or* $(F_{4(-20)}, S^{15}_\infty)$, *then any* $(G, G/P)$*-foliation* $\mathcal{F}$ *of* M *satisfies* $\operatorname{GV}(\mathcal{F}) = \operatorname{GV}(\mathcal{F}_\Gamma)$ *and* $\operatorname{hol}(\mathcal{F}) = \operatorname{hol}(\mathcal{F}_\Gamma)$.

(ii) *If* $(G, G/P)$ *is* $(\mathrm{SO}_0(n+1,1), S^n_\infty)$ *for* n *even, then* $\operatorname{GV}(\mathcal{F}) = \operatorname{GV}(\mathcal{F}_\Gamma)$ *if and only if* $\operatorname{vol}(\operatorname{hol}(\mathcal{F})) = \operatorname{vol}(\Gamma)$, *where* $\operatorname{vol}(\Gamma)$ *is the volume of* $\Gamma \hookrightarrow G$ (*see Example 7.7*).

Combining Lemma 9.1 with well known properties of the volume, we get the following consequences.

Proposition 9.3. *If* $\operatorname{GV}(\mathcal{F})$ *is nontrivial, then the image of the holonomy homomorphism* $\pi_1 M \to G$ *is Zariski dense in* G.

Proof. If $\operatorname{GV}(\mathcal{F})$ is nontrivial, then $\operatorname{vol}(\operatorname{hol}(\mathcal{F}))$ is also nontrivial by (64). Then the image of $\operatorname{hol}(\mathcal{F})$ is Zariski dense in G by [Cor91, Proposition 2.1]. □

Proposition 9.4. *If* $(G, G/P) = (\mathrm{SO}_0(n+1,1), S^n_\infty)$ *for even* n, *then* $\int_M \operatorname{GV}(\mathcal{F}) \le \int_M \operatorname{GV}(\mathcal{F}_\Gamma)$.

Proof. This is a consequence of (64) and the following generalized version of the Milnor-Wood inequality (see [FK06, Theorem 1.1]): For any homomorphism $h\colon \Gamma \to G$, we have

$$
(72)\qquad \int_N \operatorname{vol}(h) \le \int_N \operatorname{vol}(\Gamma) . \quad \square
$$

Remark 9.5. The inequality (72) is true also for any other simple Lie group G. In fact, it is a consequence of the positivity of the simplicial volume of locally symmetric spaces due to Lafont-Schmidt [LS06] (one applies the Hahn-Banach theorem [Gro82, Corollary in page 225] with [Buc08, Corollary 7]). But here we need only the case of $(G, G/P) = (\mathrm{SO}_0(n+1,1), S^n_\infty)$ for even n, where Corollary 9.2.i does not work.

9.2. **Rigidity of $(G, G/P)$-foliations of $\Gamma\backslash G/K_P$ of higher codimensions.** To prove Theorem 1.17-(i), we will apply the following generalized version of Mostow rigidity.

Theorem 9.6 (Goldman [Gol88] for the case where $G = \mathrm{PSO}(2,1)$, Dunfield [Dun99] for $G = \mathrm{PSO}(n+1,1)$, and Corlette [Cor91] for $G = \mathrm{PSU}(n+1,1)$)**.** *Let G denote* $\mathrm{PSO}(n+1,1)$ *or* $\mathrm{PSU}(n+1,1)$ *and* Γ *a torsion-free uniform lattice of G. Any homomorphism $h\colon \Gamma \to G$ with* $\mathrm{vol}(h) = \mathrm{vol}(\Gamma)$ *is conjugate to the canonical inclusion $\Gamma \to G$ by an inner automorphism of G.*

Remark 9.7. Francaviglia-Klaff [FK06] and Bucher-Burger-Iozzi [BBI12] generalized the definition of the volume of representations of uniform lattices to non-uniform lattices. (These two definitions do not coincide with each other.) It allows them to prove Theorem 9.6 in a way similar to [Dun99], including the case where Γ is a nonuniform lattice of $\mathrm{SO}(n+1,1)$.

Remark 9.8. Note that the assumption of the above theorem of Goldman is the equality $e(h) = e(\Gamma)$ for the Euler classes. But, because of the proportionality of the Euler class and the volume, it is equivalent to the equality on the volume.

Remark 9.9. To prove Theorem 1.17 for the case where G is $\mathrm{Sp}(n+1,1)$ or $F_{4(-20)}$, we will apply the superrigidity theorem of Corlette [Cor92], which asserts that any homomorphism $\Gamma \to G$ from a uniform lattice Γ of G is conjugate to the canonical inclusion if its image is Zariski dense. This rigidity is stronger than the case of Theorem 9.6, so we do not need the equality on the volumes.

Proof of Theorem 1.17-(i) in the case $q > 1$. If $(G, G/P)$ is $(\mathrm{SO}_0(n+1,1), S^n_\infty)$ for n odd or $(\mathrm{SU}(n+1,1), S^{2n+1}_\infty)$, Corollary 9.2-(i) implies $\mathrm{vol}(\mathrm{hol}(\mathcal{F})) = \mathrm{vol}(\Gamma)$. If $(G, G/P)$ is $(\mathrm{SO}_0(n+1,1), S^n_\infty)$ for n even, then we get $\mathrm{vol}(\mathrm{hol}(\mathcal{F})) = \mathrm{vol}(\Gamma)$ by the assumption and Corollary 9.2-(ii). Thus Theorem 9.6 implies that $\mathrm{hol}(\mathcal{F})\colon \pi_1 N \to G$ is conjugate to $\pi_1 N = \Gamma \hookrightarrow G$ by an inner automorphism of G. Hence the standard map $\phi_{K_G}\colon G/K_P \to G/K_G$ is conjugate to a $\pi_1 M$-equivariant map $s\colon G/K_P \to G/K_G$, which is a submersion. Then we get a $\pi_1 M$-equivariant submersion $\widehat{\mathrm{dev}}\colon G/K_P \to G/K_P$ by Lemma 7.13. It induces a covering map $\overline{\mathrm{dev}}\colon \Gamma\backslash G/K_P \to \Gamma\backslash G/K_P$, which must be a diffeomorphism because $\mathrm{GV}(\mathcal{F}) = \mathrm{GV}(\mathcal{F}_\Gamma)$. □

Proof of Theorem 1.17-(ii). Corollary 9.2-(i) and Proposition 9.3-(i) imply that the image of $\mathrm{hol}(\mathcal{F})\colon \pi_1 M \to G$ is Zariski dense in G. Thus Corlette's superrigidity theorem [Cor92] for uniform lattices in $\mathrm{Sp}(n+1,1)$ or $F_{4(-20)}$ implies that

$\operatorname{hol}(\mathcal{F})\colon \pi_1 N \to G$ is conjugate to $\pi_1 N = \Gamma \hookrightarrow G$. The rest of the proof is the same as in the case (i). □

9.3. **Codimension one case.** In the case where $(G, G/P)$ is $(\mathrm{SO}_0(2,1), S^1_\infty)$ or $(\mathrm{SU}(1,1), S^1_\infty)$, Lemma 9.1 is not true in general because of $\pi_1 S^1 \cong \mathbb{Z}$. But the theory of codimension one foliations, due to Thurston and Levitt, resolves this problem. Note that, in this case, K_G is isomorphic to $\mathrm{SO}(2)$ or $\mathrm{U}(1)$, P is isomorphic to $\mathrm{Aff}_+(1;\mathbb{R})$ or $\mathrm{Aff}(1;\mathbb{R})$, and K_P is trivial or $\{\pm 1\}$. Let $\mathcal{F}$ be a $(G, G/P)$-foliation on $M = \Gamma\backslash G/K_P$. Here, $N = \Gamma\backslash G/K_G$ is a closed Riemann surface and the projection $p\colon \Gamma\backslash G/K_P \to \Gamma\backslash G/K_G$ is a principal S^1-bundle.

Theorem 1.17-(i) in the case where $q = 1$ will be deduced from the following two results:

Theorem 9.10 (Chihi-ben Ramdane [CbR08])**.** *If* $\mathrm{GV}(\mathcal{F})$ *is nontrivial, then the image of the holonomy homomorphism of* $\mathcal{F}$ *is a uniform lattice or a dense subgroup of* G*. In particular,* $\mathcal{F}$ *is minimal.*

Theorem 9.11 (Thurston [Thu72a] and Levitt [Lev78])**.** *A codimension one foliation* $\mathcal{F}$ *on* M *without compact leaves is isotopic to a foliation transverse to the fibers of* p*.*

Proof of Theorem 1.18. Assume that $\mathrm{GV}(\mathcal{F})$ is nontrivial. Then $\mathcal{F}$ is minimal by Theorem 9.10. By Theorem 9.11, we can isotope $\mathcal{F}$ to a foliation transverse to the fibers of p. Since the Euler number of p is equal to the Euler number of N by construction and the Euler class is propotional to the volume, we get $\operatorname{vol}(\operatorname{hol}(\mathcal{F})) = \operatorname{vol}(\Gamma)$, where $\operatorname{hol}(\mathcal{F})$ is the holonomy homomorphism of $\mathcal{F}$. According to Theorem 9.6, $\operatorname{hol}(\mathcal{F})$ is conjugate to $\operatorname{hol}(\mathcal{F}_\Gamma)$, which is the canonical inclusion $\Gamma \hookrightarrow G$. Since the conjugation class of suspension foliations are determined by the conjugation class of the holonomy homomorphisms, the proof is concluded. □

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