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# Pitt's and Boas' inequalities for Fourier and Hankel transforms

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# PITT'S AND BOAS' INEQUALITIES FOR FOURIER AND HANKEL TRANSFORMS

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ABSTRACT. We prove Pitt and Boas' type inequalities for products of radial functions and spherical harmonics in $\mathbb{R}^n$. In the process, we obtain upper and lower estimates of the operator norm of the Hankel transform with power weights. Our inequalities are sharp in some specific cases.

## 1. INTRODUCTION

Weighted norm inequalities for the Fourier transform provide a natural way to describe the balance between the relative size of a function and its Fourier transform at infinity. A classical example is Pitt's inequality with special radial weights

$$\| |y|^{-s}\widehat{F}\|_{L^q(\mathbb{R}^n)} \le C\| |x|^t F\|_{L^p(\mathbb{R}^n)}, \qquad F \in \mathcal{S}(\mathbb{R}^n). \tag{1.1}$$

Here $1 < p \le q < \infty$, $\widehat{F}(y) = \int_{\mathbb{R}^n} F(x)e^{-ixy}dx$ denotes the Fourier transform of $F$, and

$$0 \le s < \frac{n}{q}, \quad 0 \le t < \frac{n}{p'}, \quad \text{and} \quad s = t + n\left(\frac{1}{q} - \frac{1}{p'}\right). \tag{1.2}$$

$p'$ denotes the dual exponent of $p$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. A proof of this inequality is in [Be3, BH]. See also [Be4] and [Be2].

It is worth mentioning that, when $q = 2$, Pitt's inequality has been characterized as a Hardy-Rellich inequality in recent literature, with alternative proofs and extensions. See, e.g., [Ei], [BT], and [TZ].

The range of the parameters in (1.2) cannot be improved in general (see, e.g., [SaWh]), but we can greatly extend this range, and also show that inequality

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(1.1) can be reverted, i.e,

$$A\| \, |y|^{-s}\widehat{F}\|_{L^q(\mathbb{R}^n)} \le \| \, |x|^t F\|_{L^p(\mathbb{R}^n)} \le B\| \, |y|^{-s}\widehat{F}\|_{L^q(\mathbb{R}^n)} \tag{1.3}$$

for the case $p = q$, if we restrict $F$ to certain subspaces of $\mathcal{S}(\mathbb{R}^n)$.

When $n = 1$ and $1 < p = q < \infty$, and $f$ is even and non increasing in $(0, \infty)$ and vanishes at infinity, the Hardy–Littlewood theorem states that there exist constants $A$, $B > 0$ for which

$$A\|\widehat{f}\|_p \le \| \, |x|^{1-\frac{2}{p}} f\|_p \le B\|\widehat{f}\|_p, \tag{1.4}$$

see, e.g., [Ti, Ch. IV]. Here and in the rest of the paper, we let $\| \cdot \|_p = \| \cdot \|_{L^p(0,\infty)}$ for the sake of simplicity.

Boas conjectured in [Bo] that weighted versions of (1.4) are also true, i.e., under the same assumptions on $f$ and $p$,

$$A\| \, |y|^{-s}\widehat{f}\|_p \le \| \, |x|^{s+1-\frac{2}{p}} f\|_p \le B\| \, |y|^{-s}\widehat{f}\|_p. \tag{1.5}$$

provided that $s \in \left(-\frac{1}{p'}, \frac{1}{p}\right)$, $1 < p < \infty$. This conjecture has been proved by Y. Sagher in [Sa].

More recently, a subclass of the class of functions of bounded variation on $[\varepsilon, \infty)$, for any $\varepsilon > 0$, that strictly includes the class of monotonic functions was introduced in [LT1] (see also [LT2]). These functions vanish at infinity, and there exist $C > 0$ and $c > 1$ so that, for every $r > 0$,

$$\int_r^\infty |df(u)| \le C \int_{r/c}^\infty |f(u)| \, \frac{du}{u},$$

where $\int_a^b g(u) \, |df(u)|$ is the Riemann–Stieltjes integral. These functions are called General Monotonic (GM).

In [GLT], the authors proved that inequality (1.5) is valid for GM functions and for $s \in \left(-\frac{1}{p'}, \frac{1}{p}\right)$, thus improving Sagher's theorem considerably. Boas-type inequalities with general weights (i.e., not necessarily power) are proved in [LT3].

In this paper we consider functions in $\mathbb{R}^n$ that are products of radial functions and spherical harmonics. We prove a sharp version of Pitt's inequality (1.1), and we prove Boas' inequality (1.3) when the radial components are GM functions.

1.1. **Pitt's inequality and spherical harmonics.** Here and throughout the paper, $x = (x_1, \ldots, x_n)$, $r = |x|$, and $\omega = x/|x|$. We let $f(|x|) = f(r)$. Following, e.g., [StWe], Ch. IV, we say that a solid harmonic of degree $k$ in $\mathbb{R}^n$ is a homogeneous harmonic polynomial $H_k(x)$ of degree $k$. We can write $H_k(x) = r^k Y_k(\omega)$, where $Y_k$ is a spherical harmonic of degree $k$. We denote by $\Sigma_k$ the set of the spherical harmonics of degree $k$. By Bochner's identity,

$$\widehat{fY_k}(\rho\sigma) = (2\pi)^{n/2} i^k Y_k(\sigma) \rho^{1-\frac{n}{2}} \int_0^\infty r^{\frac{n}{2}} J_{n/2+k-1}(r\rho) f(r) \, dr, \tag{1.6}$$

where $\rho = |y|, \sigma = y/|y|$ and $J_s(r)$ is the standard Bessel function of the first kind.

We prove the following

**Theorem 1.1.** *For every $Y_k \in \Sigma_k$, and every radial $f \in \mathcal{S}(\mathbb{R}^n)$, Pitt's inequality*

$$\| \, |y|^{-s} \widehat{fY_k} \|_{L^q(\mathbb{R}^n)} \le C \| \, |x|^t f Y_k \|_{L^p(\mathbb{R}^n)} \tag{1.7}$$

*holds with $1 \le p \le q \le \infty$, and if and only if*

$$s = t + n\left(\frac{1}{q} - \frac{1}{p'}\right) \quad \text{and} \quad (n-1)\left(\frac{1}{2} - \frac{1}{p}\right) + \max\left\{\frac{1}{p'} - \frac{1}{q}, 0\right\} \le t < \frac{n}{p'} + k. \tag{1.8}$$

*If $p \le 2$ and $q = p'$, and $s = \bar{t} = (n-1)\left(\frac{1}{2} - \frac{1}{p}\right)$, (1.7) holds with*

$$C = (2\pi)^{\frac{n}{2}} C_{k,p} \sup_{Y_k \in \Sigma_k} \frac{\|Y_k\|_{L^q(\mathbb{S}^{n-1})}}{\|Y_k\|_{L^p(\mathbb{S}^{n-1})}} \tag{1.9}$$

*where*

$$C_{k,p} = 2^{\frac{1}{2} - \frac{1}{p'}} \frac{p^{\frac{(2k+n-1)p+2}{4p}} \Gamma\left(\frac{(2k+n-1)p'+2}{4}\right)^{\frac{1}{p'}}}{(p')^{\frac{(2k+n-1)p'+2}{4p'}} \Gamma\left(\frac{(2k+n-1)p+2}{4}\right)^{\frac{1}{p}}}. \tag{1.10}$$

*The constant $C$ cannot be replaced by any smaller constant.*

We also show that when $C$ is as in (1.9), the equality is attained in (1.7).

To the best of our knowledge, the supremum of the ratio of the norms of spherical harmonics in (1.9) is not known. Reverse Hölder inequalities for spherical harmonics have been discussed in [So] and [Du]. In [Du], it is proved that

$$\frac{\|Y_k\|_{L^{p'}(\mathbb{S}^{n-1})}}{\|Y_k\|_{L^p(\mathbb{S}^{n-1})}} \le \left(\frac{p'}{p}\right)^{k/2} \frac{\Gamma\left(\frac{kp+n}{2}\right)^{1/p} \Gamma\left(\frac{n}{2}\right)^{1/p'}}{\Gamma\left(\frac{kp'+n}{2}\right)^{1/p'} \Gamma\left(\frac{n}{2}\right)^{1/p}} = O(k^{(n-1)(1/p-1/2)})$$

whenever $1 < p \le 2$.

1.2. **Best constants in Pitt's inequalities.** To the best of our knowledge, the sharp constant in Pitt's inequality (1.1) has been evaluated only when $p \le 2$, $q = p'$ and $s = t = 0$ (i.e., when it reduces to the Hausdorff–Young inequality) and also when $p = q = 2$ and $s = t < \frac{n}{2}$. See [Be1], [Ya] and [Ei].

The best constant in inequality (1.7) is known only in the aforementioned cases, and when $p$, $q$, $t$ and $s$ are as in Theorem 1.1. When $p = q = 2$ and $s = t < \frac{n}{2}$, it is proved in [Ya] that

$$\| \, |y|^{-t} \widehat{fY_k} \|_{L^2(\mathbb{R}^n)} < c_k(t) \| \, |x|^t f Y_k \|_{L^2(\mathbb{R}^n)} \tag{1.11}$$

with

$$c_k(t) = (2\pi)^{\frac{n}{2}} 2^{-t} \frac{\Gamma\left(\frac{1}{2}\left(\frac{n}{2} - t + k\right)\right)}{\Gamma\left(\frac{1}{2}\left(\frac{n}{2} + t + k\right)\right)}. \tag{1.12}$$

Inequality (1.11) is sharp, and equality cannot be attained when $t > 0$ (see [Ya, L. 3.8]).

The author also observes (see [Ya, L. 2.1]) that the best constant in inequality (1.1) is $c_0(t) = \sup_k c_k(t)$. That means that when $p = q = 2$, the best constant in (1.1) is attained in the space of radial functions. This fact easily follows from the orthogonality of the spherical harmonics in $L^2(\mathbb{S}^{n-1})$.

We use (1.11), the sharp inequality proved in Theorem 1.1, and a sharp inequality in [Be3], to estimate the constant in (1.7) when $q = p'$ and $t > \bar{t} = (n-1)\left(\frac{1}{2} - \frac{1}{p}\right)$.

**Theorem 1.2.** *Let* $1 < p \le 2$ *and let* $0 \le d < \frac{n}{2} + \frac{1}{2} - \frac{1}{p}$*. Pitt's inequality*

$$\| \, |y|^{-\bar{t}-d} \widehat{fY_k} \|_{L^{p'}(\mathbb{R}^n)} \le C \| \, |x|^{\bar{t}+d} f Y_k \|_{L^p(\mathbb{R}^n)}$$

*holds with* $C = BC_{k,p} \sup_{Y_k \in \Sigma_k} \frac{\|Y_k\|_{L^{p'}(\mathbb{S}^{n-1})}}{\|Y_k\|_{L^p(\mathbb{S}^{n-1})}}$*, where* $C_{k,p}$ *is as in* (1.10)*, and*

$$B = (2\pi)^{\frac{n}{2}} \frac{2^{-d}\Gamma\left(\frac{(n-1)p+2}{4p}\right)\Gamma\left(\frac{(-2d+n+1)p-2}{4p}\right)}{\Gamma\left(\frac{(n+1)p-2}{4p}\right)\Gamma\left(\frac{(2d+n-1)p+2}{4p}\right)}. \tag{1.13}$$

The constant on the right hand side of(1.13) may not be sharp for all values of $p$ and $d$, but it is easy to verify that it equals $c_k(\bar{t}) = c_k(0)$ in (1.11) when $p = 2$, and it reduces to $C_{k,p}$ in Theorem 1.1 when $d = 0$.

1.3. **Boas' conjecture and the Hankel transform.** The Hankel transform is a natural generalization of the Fourier transform of radial functions. There are several definitions of Hankel transform in the literature, which are roughly equivalent to one another (see, e.g., [CCTV] and also [De]).

In this paper we use the following definition: for every $\alpha \ge -\frac{1}{2}$ and for every $k \ge 0$, we let



$$\widetilde{f}_k(\rho) = \widetilde{f}_{k,\alpha}(\rho) = \rho^{-\alpha} \int_0^\infty r^{\alpha+1} J_{\alpha+k}(\rho r) f(r)\, dr, \qquad f \in \mathcal{S}(0,\infty) \tag{1.14}$$

be the Fourier–Hankel transform of order $k$ of $f$.

Then, by Bochner's identity (1.6), the Fourier transform of the product of a radial function $f(r)$ and a spherical harmonic $Y_k(\omega)$ can be written as

$$\widehat{fY_k}(\rho\sigma) = (2\pi)^{n/2} i^k Y_k(\sigma) \widetilde{f}_{k,\frac{n}{2}-1}(\rho). \tag{1.15}$$

In polar coordinates,

$$\big\| \, |x|^t f Y_k \big\|_{L^p(\mathbb{R}^n)} = \|Y_k\|_{L^p(\mathbb{S}^{n-1})} \left\| r^{t+(n-1)/p} f \right\|_p$$

and Pitt's inequality (1.7), with $C = c\,(2\pi)^{\frac{n}{2}} \sup\limits_{Y_k\in\Sigma_k} \dfrac{\|Y_k\|_{L^q(\mathbb{S}^{n-1})}}{\|Y_k\|_{L^p(\mathbb{S}^{n-1})}}$ for some $c > 0$, follows from

$$\left\|\rho^{-s+(n-1)/q}\widetilde{f}_k\right\|_q \le c\left\|r^{t+(n-1)/p}f\right\|_p, \qquad \alpha = \frac{n}{2}-1, \quad f\in\mathcal{S}(0,\infty). \tag{1.16}$$

Similarly, Boas' inequality (1.3), with $F = Y_k(\omega)f(r)$, follows from

$$a\left\|\rho^{-s+(n-1)/q}\widetilde{f}_k\right\|_q \le \left\|r^{t+(n-1)/p}f\right\|_p \le b\left\|\rho^{-s+(n-1)/q}\widetilde{f}_k\right\|_q, \tag{1.17}$$

where $p = q$ and $a, b > 0$.

The proof of Theorem 1.1 (see also Remark 2.1) shows that when $s$, $t$ and $k$ are as in (1.8), inequality (1.16) holds for every $f \in \mathcal{S}(0,\infty)$. We show that, when $f$ is a GM function, (1.16) is valid for a wider range of parameters than (1.8), and (1.17) holds. Our results improve a result in [GLT], where the authors have proved inequalities (1.16) and (1.17) for radial functions in GM (i.e., for $k = 0$).

Let $f$ be a GM function such that

$$I := \int_0^1 r^{2\alpha+k+1}|f(r)|\,dr + \int_1^\infty r^{\alpha+1/2}|df(r)| < \infty. \tag{1.18}$$

Then Lemma 3.1 below implies that its Fourier–Hankel transform is defined in the improper sense (i.e., as $\lim\limits_{\substack{a\to 0\\ b\to\infty}} \int_a^b$) and $\widetilde{f}_k \in C(0,\infty)$.

We prove the following

**Theorem 1.3.** *Let* $1 < p \le q < \infty$ *and* $f \in GM$. *Then, inequality* (1.16) *holds if and only if*

$$s = t + n\left(\frac{1}{q} - \frac{1}{p'}\right) \tag{1.19}$$

*and*

$$(n-1)\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{p} < t < \frac{n}{p'} + k. \tag{1.20}$$

Clearly, (1.20) is less restrictive than the right hand side inequality in (1.8). Note that condition (1.18) follows automatically from the finiteness of the right hand side of (1.16) (see Remark 3.1) which is satisfied when $p$, $t$ and $k$ are as in (1.19) and (1.20).

The next theorem reverses Theorem 1.3.

**Theorem 1.4.** *Let* $1 < q \le p < \infty$, *and let* $f$ *be GM; assume that* $f \ge 0$ *and also that* (1.18) *is satisfied. If* (1.19) *and*

$$-\frac{n-1}{p} - \frac{1}{p} < t < \infty,$$

*hold, then the reverse Pitt's inequality*

$$\left\|\rho^{-s+(n-1)/q}\widetilde{f}_k\right\|_q \gtrsim \left\|r^{t+(n-1)/p}f\right\|_p$$

*holds.*

Here and in the sequel the expressions $f \lesssim g$, $f \gtrsim g$, $f \asymp g$ mean the inequalities $f \le Cg$, $f \ge Cg$, $Cg \le f \le C'g$, respectively, where $C, C', \dots$ denote positive constants.

Taking $p = q$ in Theorems 1.3 and 1.4, we get

**Corollary 1.1.** *Let $1 < p < \infty$. If $f \in GM$, $f \ge 0$ and (1.18) holds, then*

$$\left\|\rho^{-s+(n-1)/p}\widetilde{f}_k\right\|_p \asymp \left\|r^{t+(n-1)/p}f\right\|_p$$

*if and only if*

$$\frac{n}{p'} - \frac{n+1}{2} < t < \frac{n}{p'} + k.$$

*Remark* 1.1. Theorems 1.3 and 1.4 and Corollary 1.1 hold for any $n = 2\alpha + 2 \ge 1$ and $k \ge 0$, not necessarily integers.

We prove these theorems in Section 3.

## 2. Proof of Theorems 1.1 and 1.2

In [De], the author considered the $L^p(0,\infty)$–$L^q(0,\infty)$ mapping properties of operators in the class

$$\mathcal{L} = \left\{L^\eta_{\nu,\mu}\colon\ L^\eta_{\nu,\mu}f(\rho) = \rho^\mu \int_0^\infty (r\rho)^\nu J_\eta(r\rho) f(r)\,dr, \quad \eta \ge -\frac{1}{2} \ \text{ and } \ \mu, \nu \in \mathbb{R}\right\}$$

and proved the following

**Theorem 2.1.** (i) *$L^\eta_{\nu,\mu}$ is bounded from $L^p(0,\infty)$ to $L^q(0,\infty)$ whenever $\eta \ge -\frac{1}{2}$, $1 \le p \le q \le \infty$, and if and only if*

$$\mu = \frac{1}{p'} - \frac{1}{q} \quad \text{and} \quad -\eta - \frac{1}{p'} < \nu \le \frac{1}{2} - \max\left\{\frac{1}{p'} - \frac{1}{q},\ 0\right\}. \tag{2.1}$$

(ii) *When $q = p'$ and $\nu = \frac{1}{2}$, the following inequality holds for every $1 < p \le 2$, $\eta \ge -\frac{1}{2}$ and $f \in \mathcal{S}(0,\infty)$:*

$$\frac{\|L^\eta_{\frac{1}{2},0}f\|_{L^{p'}(0,\infty)}}{\|f\|_{L^p(0,\infty)}} \le 2^{\frac{1}{p}-\frac{1}{2}} \frac{p^{\frac{1}{2}\left(\eta+\frac{1}{2}+\frac{1}{p}\right)}}{(p')^{\frac{1}{2}\left(\eta+\frac{1}{2}+\frac{1}{p'}\right)}} \frac{\Gamma\left((\eta+\frac{1}{2})\frac{p'}{2}+\frac{1}{2}\right)^{\frac{1}{p'}}}{\Gamma\left((\eta+\frac{1}{2})\frac{p}{2}+\frac{1}{2}\right)^{\frac{1}{p}}}. \tag{2.2}$$

*The constant on the right-hand side of (2.2) is best possible and is attained by the functions $f_\lambda(x) = x^{\eta+\frac{1}{2}}e^{-\lambda x^2}$, $\lambda > 0$.*

We show that proving Theorem 1.1 is equivalent to estimating the $L^p(0,\infty)$–$L^q(0,\infty)$ norm of an operator in the class $\mathcal{L}$.

2.1. **Proof of Theorem 1.1.** We have observed in Section 1.3 that when $F = f(r)Y_k(\omega)$, where $f$ is radial and $Y_k$ is a spherical harmonic of degree $k$, Pitt's inequality (1.1), with $C = c\,(2\pi)^{\frac{n}{2}} \sup\limits_{Y_k\in\Sigma_k} \frac{\|Y_k\|_{L^q(\mathbb{S}^{n-1})}}{\|Y_k\|_{L^p(\mathbb{S}^{n-1})}}$, follows if we prove

$$\left\|x^{-s+(n-1)/q}\widetilde{f_k}\right\|_q \le c\left\|y^{t+(n-1)/p}f\right\|_p, \quad f\in\mathcal{S}(0,\infty),$$

where $\widetilde{f_k}$ is given by (1.14) and $\alpha = \frac{n}{2}-1$.

We let $r^{t+(n-1)/p}f = g$; thus, $\|r^{t+(n-1)/p}f\|_p = \|g\|_p$, and

$$\left\|\rho^{-s+(n-1)/q}\widetilde{f_k}\right\|_q^q = \int_0^\infty \rho^{q(-s+(n-1)/q-\alpha)}\left|\int_0^\infty r^{\alpha+1}J_{\alpha+k}(\rho r)f(r)\,dr\right|^q d\rho$$

$$= \int_0^\infty \rho^{q(-s+(n-1)/q-\alpha)}\left|\int_0^\infty J_{\alpha+k}(\rho r)r^{-t-(n-1)/p+\alpha+1}g(r)\,dr\right|^q d\rho$$

$$= \int_0^\infty \rho^{q(-s+(n-1)/q-2\alpha+t+(n-1)/p-1)}\left|\int_0^\infty J_{\alpha+k}(\rho r)(\rho r)^{-t-(n-1)/p+\alpha+1}g(r)\,dr\right|^q d\rho$$

and recalling that $\alpha = \frac{n}{2}-1$, we obtain

$$\left\|\rho^{-s+(n-1)/q}\widetilde{f_k}\right\|_q^q =$$

$$= \int_0^\infty \rho^{q(t-s+(n-1)(1/q-1/p'))}\left|\int_0^\infty J_{\alpha+k}(\rho r)(\rho r)^{-t-(n-1)/p+\alpha+1}g(r)\,dr\right|^q d\rho$$

$$= \|L^{\alpha+k}_{\nu,\mu}g\|_q^q$$

where $\mu = t-s+(n-1)\,(1/q-1/p')$ and $\nu = \alpha+1-t-\frac{n-1}{p} = \frac{n}{2}-t-\frac{n-1}{p}$.

Let us show that condition (1.8) is equivalent to condition (2.1).

First, note that by (2.1), $\mu = \frac{1}{p'}-\frac{1}{q}$, which implies $t-s = n\left(\frac{1}{p'}-\frac{1}{q}\right)$ as in (1.2). Also, the condition $-k-\alpha-\frac{1}{p'} < \nu = \alpha+1-t-\frac{n-1}{p}$ yields that $t < \frac{n}{p'}+k$.

We also need $\nu \le \frac{1}{2}-\max\left\{\frac{1}{p'}-\frac{1}{q},0\right\}$; assume that $\max\left\{\frac{1}{p'}-\frac{1}{q},0\right\}=0$, since the other case is similar. So, $\nu = \alpha+1-t-\frac{n-1}{p} = \frac{n}{2}-t-\frac{n-1}{p} \le \frac{1}{2}$ implies

$$t \ge \bar{t} = (n-1)\left(\frac{1}{2}-\frac{1}{p}\right).$$

This concludes the first part of the theorem.

By Theorem 2.1, the $L^p$–$L^q$ norm of $L^{\alpha+k}_{\nu,\mu}$ can be explicitly evaluated, and is as in (2.2), if $1 < p \le 2$, $q = p'$, $\nu = \frac{1}{2}$ and $\mu = 0$. In particular we can take, $s = t = \bar{t}$. Then $C_{k,p}$ in (1.9) equals the right-hand side of (2.2) with $\eta = k+\frac{n}{2}-1$, which is (1.10). □

The proof of Theorem 1.1 yields the following

*Remark* 2.1. Under the assumptions of Theorem 1.1, and with the notation of Theorem 2.1, Pitt's inequality (1.16) holds whenever $s$, $t$ and $k$ are as in (2.1), with the sharp constant

$$c = \|L^{\alpha+k}_{\nu,\mu}\|_{L_p\to L_q},$$

where $\mu = t - s + (n-1)\left(\frac{1}{q} - \frac{1}{p'}\right)$ and $\nu = \frac{n}{2} - t - \frac{n-1}{p}$.

*Remark* 2.2. In [De] it is conjectured that the $L^p$–$L^q$ norm of $L^{\eta}_{\nu,\mu}$ is

$$C = C^{\eta}_{\nu,\,p,\,q} = 2^{\nu-\frac{1}{q}}\,\frac{p^{\frac{1}{2}(1+\eta-\nu+\frac{1}{p})}}{q^{\frac{1}{2}(\eta+\nu+\frac{1}{p'})}}\,\frac{\Gamma\left(\frac{\eta+\nu+mu}{2}q+\frac{1}{2}\right)^{\frac{1}{q}}}{\Gamma\left(\frac{1+\eta-\nu}{2}p+\frac{1}{2}\right)^{\frac{1}{p}}}$$

for all admissible values of the parameters, but unfortunately the conjecture does not hold in general.

We have observed in Section 1.2 that when $q = p = 2$ and $t = s < \frac{n}{2}$, the best constant in Pitt's inequality (1.7) is $c_k(t)$ in (1.12); by Remark 2.1 and the discussion in Section 1.3, $c_k(t) = (2\pi)^{\frac{n}{2}}||L^{k+\frac{n}{2}-1}_{\frac{1}{2}-t,0}||_{L^2\to L^2}$. If the conjecture was true, $c_k(t)$ would then equal $(2\pi)^{\frac{n}{2}}C^{k+\frac{n}{2}-1}_{\frac{1}{2}-t,\,2,2} = (2\pi)^{\frac{n}{2}}\frac{\sqrt{\Gamma\left(k+\frac{n}{2}-t\right)}}{\sqrt{\Gamma\left(k+\frac{n}{2}+t\right)}}$, but it is not too difficult to see that when $k,\ t>0$, this constant is strictly smaller than the one in (1.12).

2.2. **Proof of Theorem 1.2.** We use the well-known formula

$$\frac{\pi^{\frac{n+z}{2}}}{\Gamma\left(\frac{n+z}{2}\right)}\widehat{|x|^z} = (2\pi)^{n+z}\frac{\pi^{\frac{-z}{2}}}{\Gamma\left(\frac{-z}{2}\right)}|x|^{-n-z}, \qquad -n<z<0,$$

and, by letting $z = -n+d$, we can see at once that $|x|^{-d}$ is the Fourier transform of $c_d|x|^{-n+d}$, with

$$c_d = 2^{-d}\pi^{-\frac{n}{2}}\frac{\Gamma\left(\frac{n-d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}. \tag{2.3}$$

We recall that $\bar{t} = (n-1)\left(\frac{1}{2}-\frac{1}{p}\right)$. Thus, if we let $f = g|x|^{-d}$ and we denote with $\mathcal{F}$ the Fourier transform for ease of notation, we obtain

$$\begin{aligned}\|\,|x|^{-\bar{t}-d}\mathcal{F}(Y_kg|x|^{-d})\|_{L^{p'}(\mathbb{R}^n)} &= c_d\|\,|x|^{-\bar{t}}\mathcal{F}(|x|^{-n+d})\mathcal{F}(Y_kg|x|^{-d})\|_{L^{p'}(\mathbb{R}^n)}\\ &= c_d\|\,|x|^{-\bar{t}}\mathcal{F}\left(|x|^{-n+d}*Y_kg|x|^{-d}\right)\|_{L^{p'}(\mathbb{R}^n)}.\end{aligned}$$

The function $|x|^{-n+d}*(Y_kg|x|^{-d})$ is a product of a radial function times $Y_k$. By Theorem 1.1,

$$\begin{aligned}\|\,|x|^{-\bar{t}-d}\mathcal{F}(Y_kg|x|^{-d})\|_{L^{p'}(\mathbb{R}^n)} &= c_d\|\,|x|^{-\bar{t}}\mathcal{F}\left(|x|^{-n+d}*Y_kg|x|^{-d}\right)\|_{L^{p'}(\mathbb{R}^n)}\\ &\le (2\pi)^{\frac{n}{2}}C_{k,p}\,c_d\|\,|x|^{\bar{t}}\left(|x|^{-n+d}*(Y_kg|x|^{-d})\right)\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

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where $C_{k,p}$ is as in (1.9).

To conclude the proof of the Theorem, we apply the following Lemma (see Theorem 2 in [Be3]).

**Lemma 2.1.** *For every $a < n/p$ and $b < n/p'$, $a + b > 0$, and for $c = n - a - b$, the following sharp inequality holds*

$$\| \, |x|^{-a} \left(|x|^{-c} * (|x|^{-b}h)\right) \|_{L^p(\mathbb{R}^n)} \le D_{a,b} \|h\|_{L^p(\mathbb{R}^n)} \tag{2.4}$$

*with*

$$D_{a,b} = \frac{\pi^{n/2}\Gamma\left(\frac{a+b}{2}\right)\Gamma\left(\frac{n}{2p} - \frac{a}{2}\right)\Gamma\left(\frac{n}{2p'} - \frac{b}{2}\right)}{\Gamma\left(\frac{1}{2}(-a-b+n)\right)\Gamma\left(\frac{a}{2} + \frac{n}{2p'}\right)\Gamma\left(\frac{b}{2} + \frac{n}{2p}\right)}.$$

So, the inequality

$$\| \, |x|^{\bar{t}} \left(|x|^{-n+d} * (g|x|^{-d}Y_k)\right) \|_{L^p(\mathbb{R}^n)} \le B \| \, |x|^{\bar{t}} g Y_k \|_{L^p(\mathbb{R}^n)} \tag{2.5}$$

is equivalent to (2.4) if $h = g|x|^{\bar{t}}Y_k$, $a = -\bar{t}$, $b = d + \bar{t}$ and $c = n - d$.

For these values of the parameters, the constant in (2.5) is then

$$D_{-\bar{t},d+\bar{t}} = \frac{\pi^{n/2}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{(n-1)p+2}{4p}\right)\Gamma\left(\frac{(-2d+n+1)p-2}{4p}\right)}{\Gamma\left(\frac{n-d}{2}\right)\Gamma\left(\frac{(n+1)p-2}{4p}\right)\Gamma\left(\frac{(2d+n-1)p+2}{4p}\right)}$$

and Pitt's inequality in Theorem 1.2 holds with $C = (2\pi)^{\frac{n}{2}} D_{-\bar{t},d+\bar{t}}\, c_d C_{k,p}$. But

$$D_{-\bar{t},d+\bar{t}}\, c_d = \frac{2^{-d}\Gamma\left(\frac{(n-1)p+2}{4p}\right)\Gamma\left(\frac{(-2d+n+1)p-2}{4p}\right)}{\Gamma\left(\frac{(n+1)p-2}{4p}\right)\Gamma\left(\frac{(2d+n-1)p+2}{4p}\right)}$$

which is the same as $B$ in (1.13). □

## 3. Proof of Theorems 1.3 and 1.4

We call a function *admissible* if it is of bounded variation on $(\varepsilon, \infty)$, for every $\varepsilon > 0$, and vanishes at infinity. Before proving the theorems, we need to prove two technical Lemmas.

**Lemma 3.1.** *For an admissible function $f$ such that* (1.18) *holds, we have $\widetilde{f}_k \in C(0,\infty)$ and for $\rho > 0$*

$$|\widetilde{f}_k(\rho)| \lesssim \rho^k \int_0^{1/\rho} r^{2\alpha+k+1}|f(r)|\,dr + \rho^{-\alpha-3/2}\int_{1/\rho}^{\infty} r^{\alpha+1/2}\,|df(r)|. \tag{3.1}$$

*Proof.* Define

$$\psi(t) = \int_0^t u^{\alpha+1} J_{\alpha+k}(u)\,du. \tag{3.2}$$

Let us estimate $|\psi(t)|$ for $t > 1$, $\alpha \geq -\frac{1}{2}$ and $k \geq 0$. We have $\psi(t) = \int_0^1 + \int_1^t$. Since $J_s(u) \asymp u^{\alpha+k}$ for small values of $u$, then

$$\int_0^1 \asymp \int_0^1 u^{2\alpha+k+1}\,du \asymp 1.$$

It is well known that $u^{1/2}J_{\alpha+k}(u) = C_1\cos(u-c) + C_2\frac{\sin(u-c)}{u} + O(u^{-2})$ for $u > 1$. Therefore,

$$\int_1^t = \int_1^t u^{\alpha+1/2}\left(C_1\cos(u-c) + C_2\frac{\sin(u-c)}{u} + O(u^{-2})\right)du.$$

Using second mean value theorem, for some $\tau \in (1,t)$ we have

$$\int_1^t = \int_1^\tau \left(C_1\cos(u-c) + C_2\frac{\sin(u-c)}{u}\right)du$$
$$+ t^{\alpha+1/2}\int_\tau^t \left(C_1\cos(u-c) + C_1\frac{\sin(u-c)}{u}\right)du + t^{\alpha+1/2}O\left(\int_1^t u^{-2}\,du\right).$$

Here all integrals are bounded and therefore,

$$|\psi(t)| \lesssim t^{\alpha+1/2}, \qquad t > 1.$$

To get (3.1), we first use (1.14):

$$\widetilde{f}_k(\rho) = \rho^{-\alpha}\left(\int_0^{1/\rho} r^{\alpha+1}J_{\alpha+k}(\rho r)f(r)\,dr + \int_{1/\rho}^\infty r^{\alpha+1}J_{\alpha+k}(\rho r)f(r)\,dr\right) =: I_1 + I_2.$$

Let us estimate $I_1$. Since $J_{\alpha+k}(t) \asymp t^{\alpha+k}$ when $0 < t < 1$, then

$$\rho^{-\alpha}\left|\int_0^{1/\rho} r^{\alpha+1}J_{\alpha+k}(\rho r)f(r)\,dr\right| \lesssim \rho^k \int_0^{1/\rho} r^{2\alpha+k+1}|f(r)|\,dr.$$

which is the first integral in (3.1).

Estimating $I_2$ is more complicated. Using (3.2) we have

$$[\psi(\rho r)]_r' = \rho(\rho r)^{\alpha+1}J_{\alpha+k}(\rho r),$$

and hence

$$I_2 = \rho^{-2\alpha-2}\int_{1/\rho}^\infty [\psi(\rho r)]_r'\,f(r)\,dr.$$

Integrating by parts,

(3.3)

$$I_2 = \rho^{-2\alpha-2}\left(\psi(\rho r)f(r)\Big|_{1/\rho}^\infty - \int_{1/\rho}^\infty \psi(\rho r)\,df(r)\right)$$
$$= \rho^{-2\alpha-2}\Big(\lim_{r\to\infty}\psi(\rho r)f(r) - \psi(1)f(1/\rho) - \rho^{\alpha+\frac{1}{2}}\int_{1/\rho}^\infty \Big[(\rho r)^{-\alpha-\frac{1}{2}}\psi(\rho r)\Big]\,r^{\alpha+\frac{1}{2}}\,df(r)\Big).$$

We have shown that $|\psi(t)| \lesssim t^{\alpha+\frac{1}{2}}$. So, $(\rho r)^{-\alpha-\frac{1}{2}}|\psi(\rho r)| \lesssim 1$, and the integral in (3.3) is

$$\rho^{\alpha+\frac{1}{2}} \left| \int_{1/\rho}^{\infty} \left[ (\rho r)^{-\alpha-\frac{1}{2}} \psi(\rho r) \right] r^{\alpha+\frac{1}{2}}\, df(r) \right| \lesssim \rho^{\alpha+\frac{1}{2}} \int_{1/\rho}^{\infty} r^{\alpha+\frac{1}{2}}\, |df(r)|. \tag{3.4}$$

Also, when $r$ is much larger that $\frac{1}{\rho}$,

$$|\psi(\rho r) f(r)| \lesssim (\rho r)^{\alpha+\frac{1}{2}} |f(r)| \lesssim r^{\alpha+\frac{1}{2}} |f(r)|.$$

By (1.18), $r^{\alpha+\frac{1}{2}}|f(r)|$ is integrable in $[1, \infty)$, and so $\lim\limits_{r\to\infty} r^{\alpha+\frac{1}{2}} f(r) = 0$. Clearly, also $\lim\limits_{r\to\infty} f(r) = 0$. On the other hand,

$$|\psi(1) f(1/\rho)| = \left| \psi(1) \int_{1/\rho}^{\infty} df(t) \right| \lesssim \rho^{\alpha+\frac{1}{2}} \int_{1/\rho}^{\infty} t^{\alpha+\frac{1}{2}}\, |df(t)|. \tag{3.5}$$

By (3.3), (3.4), and (3.5),

$$|I_2| \lesssim \rho^{-\alpha-\frac{3}{2}} \int_{1/\rho}^{\infty} t^{\alpha+\frac{1}{2}}\, |df(t)|$$

as required. This concludes the proof of the Lemma. □

*Remark* 3.1. First let us recall the following inequality: for any $f \in GM$ there is $c > 1$ such that

$$\int_{r}^{\infty} u^{\sigma} |df(u)| \lesssim \int_{r/c}^{\infty} u^{\sigma-1} |f(u)|\, du, \qquad \sigma \geq 0, \tag{3.6}$$

which follows from the definition of $GM$ functions, see also [GLT, p. 111]. Let us show that $I$ defined by (1.18) is bounded by $\left\| r^{t+(2\alpha+1)/p} f \right\|_p$ under conditions of Theorem 1.3. Indeed, using (3.6),

$$I \lesssim \int_0^1 r^{2\alpha+k+1} |f(r)|\, dr + \int_1^{\infty} r^{\alpha-1/2} |f(r)|\, dr \lesssim \int_0^1 \frac{r^{2\alpha+k+1}}{(1+r)^{\alpha+k+3/2}}\, |f(r)|\, dr.$$

Then using Hölder's inequality, we estimate

$$I \lesssim \left\| r^{t+(2\alpha+1)/p} f \right\|_p \left\| r^{-t-(2\alpha+1)/p} \frac{r^{2\alpha+k+1}}{(1+r)^{\alpha+k+3/2}} \right\|_{p'} \lesssim \left\| r^{t+(2\alpha+1)/p} f \right\|_p,$$

provided that $(n-1)\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{p} < t < \frac{n}{p'} + k$, i.e., (1.20). □

**Lemma 3.2.** *For an admissible non-negative function $f$ such that* (1.18) *holds and*

$$\int_0^1 |\widetilde{f}_k(\rho)| \rho^{2\alpha+k+1}\, d\rho < \infty;$$

*we have for any* $z>0$ *and any* $1<b<2\pi$

$$z^k\int_0^{1/z}|\widetilde{f}_k(\rho)|\rho^{2\alpha+k+1}\,d\rho\gtrsim\int_{z/b}^{zb}\frac{f(r)}{r}\,dr. \tag{3.7}$$

*Proof.* The proof is in two steps: first, we construct non-negative kernel with finite support similar to $K$ from [GLT]. Then, we use its properties.

To construct our kernel, we apply the Bessel–Hankel transform and the generalized Bessel convolution; see, e.g., [Le].

Let $j_\nu(t)=2^\nu\Gamma(\nu+1)t^{-\nu}J_\nu(t)$ be the normalized Bessel function, with $\nu\ge -1/2$. Recall that $j_\nu(0)=1$. We let

$$B_\nu f(s)=B_\nu^{-1}f(s)=\frac{1}{2^\nu\Gamma(\nu+1)}\int_0^\infty f(t)j_\nu(st)t^{2\nu+1}\,dt$$

be the Bessel–Hankel transform. The Bessel generalized translation operator is given by

$$T_sf(t)=\frac{f(t+s)+f(|t-s|)}{2},\qquad \nu=-1/2,$$

$$T_sf(t)=\frac{1}{\int_0^\pi\sin^{2\nu}\xi\,d\xi}\int_0^\pi f(\sqrt{t^2+s^2-2ts\cos\xi})\sin^{2\nu}\xi\,d\xi,\qquad \nu>-1/2.$$

Using this, we define the generalized Bessel convolution as follows

$$(f*g)_\nu(s)=\frac{1}{2^\nu\Gamma(\nu+1)}\int_0^\infty T_sf(t)g(t)t^{2\nu+1}\,dt.$$

Note that the Fourier transform and convolution of radial functions are related to the Bessel translation and convolution operators.

We recall some properties of the generalized Bessel convolution. Let

$$F(u)=(f*f)_\nu(s),\qquad f\in L^1_{\mathrm{loc}}(\mathbb{R}_+),\quad \operatorname{supp}f\subset[0,a],\quad a>0.$$

Then

(i) $F\in C(\mathbb{R}_+)$ and $\operatorname{supp}F\subset[0,2a]$;
(ii) for every $s>0$, $0\le F(s)\le F(0)=(B_\nu f^2)(0)$;
(iii) $B_\nu F=(B_\nu f)^2$.

Let $\chi_a=\chi_{[0,a]}$. Then

$$B_\nu\chi_a(s)=\int_0^a\frac{J_\nu(st)}{(st)^\nu}t^{2\nu+1}\,dt=\frac{a^{\nu+2}}{s^\nu}\int_0^1 J_\nu(ast)t^{\nu+1}\,dt.$$

Taking into account that

$$\int_0^1 J_\nu(ut)t^{\nu+1}\,dt=u^{-1}J_{\nu+1}(u),$$

we get

$$B_\nu\chi_a(s)=C_aj_{\nu+1}(as),\qquad\text{with}\quad C_a=\frac{a^{2\nu+2}}{2^{\nu+1}\Gamma(\nu+2)}.$$

Define $K(\rho) = C_a^{-1}(\chi_a * \chi_a)_\nu(\rho)$. Using properties of convolution, we get the following properties of $K$:

(i) $K \in C(\mathbb{R}_+)$ and $\operatorname{supp} K \subset [0, 2a]$;
(ii) $0 \le K(\rho) \le K(0)$, $\rho > 0$, where

$$K(0) = C_a^{-1}(\chi_a * \chi_a)_\nu(0) = \frac{C_a^{-1}}{2^\nu \Gamma(\nu+1)} \int_0^a t^{2\nu+1}\,dt = 1;$$

(iii) $B_\nu K(r) = C_a^{-1}(B_\nu \chi_a(r))^2 = C_a j_{\nu+1}^2(ar)$.

We now let $f$ be a non-negative function satisfying the assumptions of the lemma. We recall that

$$\widetilde{f}_k(\rho) = \rho^{-\alpha} \int_0^\infty r^{\alpha+1} J_{\alpha+k}(\rho r) f(r)\,dr, \qquad \rho > 0.$$

Multiplying both sides by $K(\rho)\rho^{2\alpha+k+1}$ and integrating we get, on the left-hand side

$$I = \int_0^{2a} \widetilde{f}_k(\rho) K(\rho) \rho^{2\alpha+k+1}\,d\rho \le \int_0^{2a} |\widetilde{f}_k(\rho)| \rho^{2\alpha+k+1}\,d\rho \tag{3.8}$$

by (ii). The left-hand side is

$$I = \int_0^{2a} K(\rho)\rho^{\alpha+k+1} \left( \int_0^\infty r^{\alpha+1} J_{\alpha+k}(\rho r) f(r)\,dr \right) d\rho.$$

Changing the order of integration (we can justify this using(1.18) and the properties on $K$, as in [GLT]), we obtain

$$I = \int_0^\infty r^{\alpha+1} f(r) \left( \int_0^{2a} K(\rho)\rho^{\alpha+k+1} J_{\alpha+k}(\rho r)\,d\rho \right) dr.$$

Here

$$\int_0^{2a} K(\rho)\rho^{\alpha+k+1} J_{\alpha+k}(\rho r)\,d\rho = r^{\alpha+k} B_{\alpha+k} K(r) = C_a r^{\alpha+k} j_{\nu+1}^2(ar), \qquad \nu = \alpha + k.$$

Thus,

$$I = C_a \int_0^\infty r^{2\alpha+k+1} f(r) j_{\nu+1}^2(ar)\,dr.$$

Let us show that $t^{2\alpha+k+2} j_{\nu+1}^2(t) \gtrsim 1$ for $(2b)^{-1} < t < b/2$, $1 < b < 2\pi$. Indeed, $2\alpha + k + 2 > 0$, therefore $t^{2\alpha+k+2} \gtrsim 1$ for $t > (2b)^{-1}$. Moreover, the Bessel function $j_{\nu+1}(t)$ is decreasing on $[0, q_{\nu+1}]$, where $q_{\nu+1}$ is its first positive zero. Then $j_{\nu+1}(t) \gtrsim 1$ for $0 < t < b/2$, where

$$1 < b < 2 \inf_{\nu+1 \ge 1/2} q_{\nu+1} = 2\pi.$$

Then using non-negativity of $f$ and $C_a \asymp a^{2\nu+2}$, the integral $I$ can be rewritten as follows:

$$I \asymp \frac{a^{2(\alpha+k)+2}}{a^{2\alpha+k+2}} \int_0^\infty \frac{f(r)}{r} (ar)^{2\alpha+k+2} j_{\nu+1}^2(ar)\, dr \gtrsim a^k \int_{1/(2ab)}^{b/(2a)} \frac{f(r)}{r}\, dr. \tag{3.9}$$

Combining estimates (3.8) and (3.9) and putting $2a = 1/z$, we complete the proof of the Lemma. □

We will also need the following weighted Hardy-type inequality (see [Br]).

**Lemma 3.3.** *Let $u$, $v$ be non-negative measurable functions which satisfy*

$$\sup_{a>0} \left(\int_a^\infty u(t)^q\, dt\right)^{\frac{1}{q}} \left(\int_0^a v(t)^{-p'}\, dt\right)^{\frac{1}{p'}} < \infty \tag{3.10}$$

*for some $1 \le p \le q \le \infty$. Then, there exists a constant $C > 0$ such that for every non-negative measurable $F$,*

$$\left(\int_0^\infty \left(u(t) \int_0^t F(s)\, ds\right)^q dt\right)^{\frac{1}{q}} \le C \left(\int_0^\infty \big(v(t)F(t)\big)^p dt\right)^{\frac{1}{p}}. \tag{3.11}$$

3.1. **Proof of Theorem 1.3.** We recall that inequality (1.16) is equivalent to

$$\left\|\rho^{-s+\frac{2\alpha+1}{q}} \widetilde{f}_k\right\|_q \lesssim \left\|r^{t+\frac{2\alpha+1}{p}} f\right\|_p$$

where $\widetilde{f}_k$ is defined by (1.14), $\alpha = \frac{n}{2} - 1 \ge -\frac{1}{2}$,

$$s = t + (2\alpha+2)\left(\frac{1}{q} - \frac{1}{p'}\right), \tag{3.12}$$

and

$$(2\alpha+1)\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{p} < t < k + \frac{2\alpha+2}{p'}. \tag{3.13}$$

By Lemma 3.1 and Remark 3.1,

(3.14)

$$\begin{aligned} |\widetilde{f}_k(\rho)| &\lesssim \rho^k \int_0^{1/\rho} r^{2\alpha+k+1} |f(r)|\, dr + \rho^{-\alpha-3/2} \int_{1/\rho}^\infty r^{\alpha+1/2}\, |df(r)| \\ &\lesssim \rho^k \int_0^{1/\rho} r^{2\alpha+k+1} |f(r)|\, dr + \rho^{-\alpha-3/2} \int_{1/\rho}^\infty r^{\alpha-1/2}\, |f(r)| dr =: I_1 + I_2. \end{aligned}$$

By (3.14), we can see at once that

$$\left\|\rho^{-s+\frac{2\alpha+1}{q}} \widetilde{f}_k\right\|_q \le \left\|r^{-s+\frac{2\alpha+1}{q}} I_1\right\|_q + \left\|r^{-s+\frac{2\alpha+1}{q}} I_2\right\|_q = K_1 + K_2.$$

So, with the change of variables $r \to r^{-1}$, and the substitution $s = t + (2\alpha+2)\left(\frac{1}{q} - \frac{1}{p'}\right)$

$$K_1 = \left(\int_0^\infty r^{-sq+kq+2\alpha+1}\left(\int_0^{\frac{1}{r}} z^{2\alpha+k+1}|f(z)|\,dz\right)^q dr\right)^{\frac{1}{q}}$$
$$= \left(\int_0^\infty r^{sq-kq-2\alpha-3}\left(\int_0^{r} z^{2\alpha+k+1}|f(z)|\,dz\right)^q dr\right)^{\frac{1}{q}}$$
$$= \left(\int_0^\infty \left(r^{t-\frac{1}{q}-k-\frac{2\alpha+2}{p'}}\int_0^{r} z^{2\alpha+k+1}|f(z)|\,dz\right)^q dr\right)^{\frac{1}{q}}.$$

We apply Hardy's inequality (Lemma 3.3) with $u(r) = r^{t-\frac{1}{q}-k-\frac{2\alpha+2}{p'}}$, $F(z) = z^{2\alpha+k+1}|f(z)|$, and $v(r) = r^{-\frac{2\alpha+1}{p'}-k+t}$. By the assumptions on $t$ in (3.13), $t - \frac{1}{q} - k - \frac{2\alpha+2}{p'} < -\frac{1}{q}$ and $-\frac{2\alpha+1}{p'} - k + t < \frac{1}{p'}$, so that both integrals in (3.10) are finite, and

$$\left(\int_a^\infty u(r)^q\,dr\right)^{\frac{1}{q}}\left(\int_0^a v(r)^{-p'}\,dr\right)^{\frac{1}{p'}} =$$
$$= \left(\int_a^\infty \left(r^{t-\frac{1}{q}-k-\frac{2\alpha+2}{p'}}\right)^q dr\right)^{\frac{1}{q}}\left(\int_0^a \left(r^{-\frac{2\alpha+1}{p'}-k+t}\right)^{-p'} dr\right)^{\frac{1}{p'}}$$
$$\lesssim a^{t-k-\frac{2\alpha+2}{p'}}\,a^{\frac{2\alpha+2}{p'}+k-t} = 1.$$

So, by (3.11),

$$K_1 \lesssim \left(\int_0^\infty r^{tp+2\alpha+1}|f(r)|^p\,dr\right)^{\frac{1}{p}}.$$

The norm $K_2$ can be estimated similarly. We use again the change of variables $r \to r^{-1}$, and the substitution $s = t + (2\alpha+2)\left(\frac{1}{q} - \frac{1}{p'}\right)$:

$$K_2 = \left(\int_0^\infty r^{-sq+2\alpha+1-\alpha q-\frac{3}{2}q}\left(\int_{\frac{1}{r}}^\infty z^{\alpha-\frac{1}{2}}|f(z)|\,dz\right)^q dr\right)^{\frac{1}{q}}$$
$$= \left(\int_0^\infty \left(r^{t+\frac{3}{2}+\alpha-\frac{2\alpha+2}{p'}-\frac{1}{q}}\int_r^\infty z^{\alpha-\frac{1}{2}}|f(z)|\,dz\right)^q dr\right)^{\frac{1}{q}}.$$

To estimate the latter integral, we use the dual version of the Hardy's inequality (3.11) which is given as follows: if

$$\sup_{a>0}\left(\int_0^a u(t)^q\,dt\right)^{\frac{1}{q}}\left(\int_a^\infty v(t)^{-p'}\,dt\right)^{\frac{1}{p'}} < \infty \tag{3.15}$$

for some $1 \le p \le q \le \infty$, then

$$\left(\int_0^\infty \left(u(t)\int_t^\infty F(s)\,ds\right)^q dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty \big(v(t)F(t)\big)^p dt\right)^{\frac{1}{p}}. \tag{3.16}$$

We apply this with $u(r) = r^{t-\frac{1}{q}+\frac{3}{2}+\alpha-\frac{2\alpha+2}{p'}}$, $F(z) = z^{\alpha-\frac{1}{2}}|f(z)|$, and $v(r) = r^{-\frac{2\alpha+1}{p'}+\alpha+\frac{3}{2}+t}$. Then condition (3.15) holds provided $(2\alpha+1)\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{p} < t$ and we get

$$K_2 \le \left(\int_0^\infty r^{tp+2\alpha+1}|f(r)|\,dr\right)^{\frac{1}{p}}.$$

To summarize, we have proved that

$$\left\|\rho^{-s+\frac{2\alpha+1}{q}}\widetilde{f}_k\right\|_q \lesssim \left\|r^{t+\frac{2\alpha+1}{p}}f\right\|_p$$

and since $2\alpha+1 = n-1$, Theorem 1.3 is proved.

3.2. **Proof of Theorem 1.4.** Let $s = t + (2\alpha+2)\left(\frac{1}{q}-\frac{1}{p'}\right)$ and

$$t > -\frac{2\alpha+2}{p}. \tag{3.17}$$

Note that this assumption is less restrictive than the one of Theorem 1.3. Let us use (3.7) in Lemma 3.2

$$\int_{z/b}^{zb}\frac{f(r)}{r}\,dr \lesssim z^k\int_0^{1/z}|\widetilde{f}_k(\rho)|\rho^{2\alpha+k+1}\,d\rho, \qquad z>0,$$

and the following inequality, which is valid for $a>0$ and $b>1$

$$\int_a^\infty |\psi(x)|\,dx \lesssim \int_{a/b}^\infty\left(\int_{z/b}^{bz}|\psi(x)|\,dx\right)\frac{dz}{z}. \tag{3.18}$$

Since $f$ is GM and is non-negative, by (3.18)

$$f(r) \lesssim \int_r^\infty |df(x)| \lesssim \int_{r/c}^\infty \frac{f(x)}{x}\,dx \lesssim \int_{\frac{r}{bc}}^\infty \frac{1}{z}\left(\int_{z/b}^{bz}\frac{f(x)}{x}\,dx\right)dz.$$

By (3.7) in Lemma 3.2 and with the substitution $z^{-1}\to z$,

$$\begin{aligned} f(r) &\lesssim \int_{\frac{r}{bc}}^\infty z^{k-1}\left(\int_0^{1/z}|\widetilde{f}_k(\rho)|\rho^{2\alpha+k+1}\,d\rho\right)dz \\ &= \int_0^{\frac{bc}{r}} z^{-k-1}\left(\int_0^z |\widetilde{f}_k(\rho)|\rho^{2\alpha+k+1}\,d\rho\right)dz. \end{aligned}$$

We raise the left-hand side and the right-hand side of the last inequality to $p$-th power, we multiply by $r^{tp+2\alpha+1}$ and we integrate. We obtain

$$\left(\int_0^\infty r^{tp+2\alpha+1}|f(r)|^p\,dr\right)^{\frac{1}{p}} \lesssim$$
$$\lesssim \left(\int_0^\infty r^{tp+2\alpha+1}\left(\int_0^{\frac{bc}{r}} z^{-k-1}\left(\int_0^z |\widetilde{f}_k(\rho)|\rho^{2\alpha+k+1}\,d\rho\right)dz\right)^p dr\right)^{\frac{1}{p}}$$

and after the change of variables $r \to \frac{bc}{r}$,

$$\left(\int_0^\infty r^{tp+2\alpha+1}|f(r)|^p\,dr\right)^{\frac{1}{p}} \lesssim$$
$$\lesssim \left(\int_0^\infty \left(r^{-t-\frac{2\alpha+3}{p}}\int_0^r z^{-k-1}\left(\int_0^z |\widetilde{f}_k(\rho)|\rho^{2\alpha+k+1}\,d\rho\right)dz\right)^p dr\right)^{\frac{1}{p}}.$$

We use Hardy's inequality in Lemma 3.3 with $u(r) = r^{-t-\frac{2\alpha+3}{p}}$, $F(z) = z^{-k-1}\int_0^z |\widetilde{f}_k(\rho)|\rho^{2\alpha+k+1}\,d\rho$, and $v(r) = r^{-t-\frac{2\alpha+2}{p}+1-\frac{1}{q}}$.

Let us verify that $u$ and $v$ satisfy condition (3.10) in Lemma 3.3. Indeed, because of the definition of $t$, both integrals in (3.11) are finite, and

$$\left(\int_a^\infty u(r)^p\,dr\right)^{\frac{1}{p}}\left(\int_0^a v(r)^{-q'}\,dr\right)^{\frac{1}{q'}} =$$
$$= \left(\int_a^\infty \left(r^{-t-\frac{2\alpha+3}{p}}\right)^p dr\right)^{\frac{1}{p}}\left(\int_0^a \left(r^{-t-\frac{2\alpha+2}{p}+1-\frac{1}{q}}\right)^{-q'} dr\right)^{\frac{1}{q'}}$$
$$\lesssim a^{-t-\frac{2\alpha+3}{p}+\frac{1}{p}}\,a^{t+\frac{2\alpha+2}{p}-1+\frac{1}{q}+\frac{1}{q'}} = 1,$$

provided that condition (3.17) holds.

We get

$$\left(\int_0^\infty r^{tp+2\alpha+1}|f(r)|^p\,dr\right)^{\frac{1}{p}} \lesssim \left(\int_0^\infty \left(r^{-t-\frac{2\alpha+2}{p}-\frac{1}{q}-k}\int_0^r |\widetilde{f}_k(\rho)|\rho^{2\alpha+k+1}\,d\rho\right)^q\right)^{\frac{1}{q}}.$$

Using again Hardy's inequality (3.11) with $q = p$,

$$u(r) = r^{-t-\frac{2\alpha+2}{p}-\frac{1}{q}-k}, \qquad v(r) = r^{-t-(2\alpha+2)\left(\frac{1}{q}-\frac{1}{p}'\right)-k-\frac{2\alpha+1}{q'}} = r^{-s-k-\frac{2\alpha+1}{q'}},$$

and $F(\rho) = |\widetilde{f}_k(\rho)|\rho^{2\alpha+k+1}$, we obtain

$$\left(\int_0^\infty r^{tp+2\alpha+1}|f(r)|^p\,dr\right)^{\frac{1}{p}} \lesssim \left(\int_0^\infty \rho^{-s+\frac{2\alpha+1}{q}}|\widetilde{f}_k(\rho)|^q\,d\rho\right)^{\frac{1}{q}}$$

provided that $t > -\frac{2\alpha+2}{p} - k$. This concludes the proof.

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