THE GENERATOR PROBLEM FOR $\mathcal{Z}$-STABLE C^*-ALGEBRAS

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ABSTRACT. The generator problem was posed by Kadison in 1967, and it remains open until today. We provide a solution for the class of C^*-algebras absorbing the Jiang-Su algebra $\mathcal{Z}$ tensorially. More precisely, we show that every unital, separable, $\mathcal{Z}$-stable C^*-algebra A is singly generated, which means that there exists an element $x \in A$ that is not contained in any proper sub-C^*-algebra of A.

To give applications of our result, we observe that $\mathcal{Z}$ can be embedded into the reduced group C^*-algebra of a discrete group that contains a non-cyclic, free subgroup. It follows that certain tensor products with reduced group C^*-algebras are singly generated. In particular, $C_r^*(F_\infty) \otimes C_r^*(F_\infty)$ is singly generated.

1. INTRODUCTION

By an operator algebra we mean a *-subalgebra of $B(H)$ that is either closed in the norm topology (a concrete C^*-algebra) or the weak operator topology (a von Neumann algebra). One way of realizing an operator algebra is to take a subset of $B(H)$ and consider the smallest operator algebra containing it.

In a trivial way, every operator algebra can be obtained this way. The situation becomes interesting if one imposes restrictions on the generating set, and one natural possibility is to require that it consists of only one element, i.e., to consider operator algebras that are generated by a single operator. It is an old problem to determine which operator algebras arise this way.

More generally, one tries to compute the minimal number of elements that generate a given operator algebra, see 2.1. It is often convenient to consider self-adjoint generators. Note that two self-adjoint elements a, b generate the same operator algebra as the element $a + ib$. Thus, if we ask whether an operator algebra is singly generated, it is equivalent to ask whether it is generated by two self-adjoint elements.

In the case of von Neumann algebras, the generator problem was included in Kadison's famous 'Problems on von Neumann algebras', [Kad67]. This

problem list has turned out to be very influential, yet its original form remains unpublished. It is indirectly available in an article by Ge, [Ge03], where a brief summary of the developments around Kadison's famous problems is given.

Question 1.1 (Kadison, [Kad67, Problem 14], see also [Ge03]). Is every separably-acting[1] von Neumann algebra singly generated?

As noted in [She09], there exist singly generated von Neumann algebras that are not separably-acting. However, the separably-acting von Neumann algebras are the natural class for which one might expect single generation. The answer to Question 1.1 is still open in general, but many authors have contributed to show that large classes of separably-acting von Neumann algebras are singly generated.

We just mention an incomplete list of results. It starts with von Neumann, [vN31], who showed that the abelian operator algebras named after him are generated by a single self-adjoint element, thus implicitly raising the generator problem. Some thirty years later, this was extended by Pearcy, [Pea62], who showed that all von Neumann algebras of type I are singly generated. Then Wogen, [Wog69, Theorem 2], proved that all properly infinite von Neumann algebras are singly generated, thus reducing the generator problem to the type II_1 case.

Later, this was reduced to the case of a II_1-factor by Willig, [Wil74], and then to the case of a finitely-generated II_1-factor by Sherman, [She09, Theorem 3.8]. This means that Question 1.1 has a positive answer if every separably-acting, finitely generated II_1-factor is singly generated.

There are many properties known to imply that a II_1-factor is singly generated. We just mention that Ge and Popa, [GP98, Theorem 6.2], show that every tensorially non-prime[2] II_1-factor is singly generated. Our main result Theorem 3.5 can be considered as a partial C^*-algebraic analog of this result.

Let us also mention that the free group factors $W^*(F_k)$ are the outstanding examples of separably-acting von Neumann algebra for which it is not known whether they are singly generated.

In the case of C^*-algebras, the generator problem is more subtle. There is already no obvious class of C^*-algebras for which one conjectures that they are singly generated. Every singly generated C^*-algebra is separable[3]. However, the converse is false, and counterexamples can be found among the commutative C^*-algebras.

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[1] A von Neumann algebra is called 'separably-acting', or just 'separable', if it is a subalgebra of $B(l^2\mathbb{N})$, or equivalently if it has a separable predual.

[2] A II_1-factor M is called tensorially non-prime if it is isomorphic to a tensor product, $M_1 \bar{\otimes} M_2$, of two II_1-factors M_1, M_2.

[3] A C^*-algebra is called 'separable' if it contains a countable, norm-dense subset

In fact, the C^*-algebra $C_0(X)$ is generated by n self-adjoint elements if and only if X can be embedded into $\mathbb{R}^n$. Thus, $C_0(X)$ is singly generated if and only if X is planar, i.e., can be embedded into the plane $\mathbb{R}^2$.

It is easy to see that a C^*-algebra A is generated by n self-adjoint elements if and only if its minimal unitization $\widetilde{A}$ is generated by n self-adjoint elements. Therefore, we will mostly consider the generator problem for separable, unital C^*-algebra. In that case, taking the tensor product with a matrix algebra has the effect of reducing the necessary number of generators. If A is generated by $n^2 + 1$ self-adjoint elements, then $A \otimes M_n$ is singly generated, see e.g. [Nag04, Theorem 3].

One derives the principle that a C^*-algebra needs less generators if it is 'more non-commutative'. Consequently, one might expect a (separable) C^*-algebra to be singly generated if it is 'maximally non-commutative'. As a non-unital instance of this principle, we note that the stabilization, $A \otimes \mathbb{K}$, of a separable unital C^*-algebra A is singly generated, [OZ76, Theorem 8]. In the unital case, there are at least three natural cases when one considers a C^*-algebra A to be 'maximally non-commutative', which are the following:

(1) A contains a simple, unital, nonelementary sub-C^*-algebra,
(2) A contains a sequence of pairwise orthogonal, full elements,
(3) A has no finite-dimensional irreducible representations.

In general, the implications $(1) \Rightarrow (2) \Rightarrow (3)$ hold; it is not known if the converses are true.

Conditions (2) and (3) can also be considered for possibly non-unital C^*-algebras, and we let (2^*) be the weaker statement that A contains *two* orthogonal, full elements. The implication '$(3) \Rightarrow (2)$' holds exactly if the implication '$(3) \Rightarrow (2^*)$' holds.

The Global Glimm halving problem asks the following: Given a (possibly non-unital) C^*-algebra A that satisfies condition (3), does there exist a full map from the cone over M_2 to A? It is not known whether the Global Glimm halving problem has a positive answer, but if it does then it shows that implication '$(3) \Rightarrow (2)$' holds, since the cone over M_2 contains two orthogonal, full elements.

Let us remark that the analogs of conditions $(1) - (3)$ for von Neumann algebras are all equivalent. In fact, if a von Neumann algebra M has no finite-dimensional representations, then the hyperfinite II_1-factor $\mathcal{R}$ unitally embeds into M.

Historically, the generator problem for C^*-algebras is mostly asked for C^*-algebras that are simple ore more generally have no finite-dimensional representations:

Question 1.2. Is every simple, separable, unital C^*-algebra singly generated?

Question 1.3. Is a separable, unital C^*-algebra singly generated provided it has no finite-dimensional irreducible representations?

The answers to both questions are open. We note that a positive answer to Question 1.3 implies a positive answer to Question 1.2. The converse is not clear.

Let us mention some results that solve the generator problem for particular classes of separable C^*-algebras. It was shown by Topping, [Top68], that every UHF-algebra is singly generated. This was generalized by Olsen and Zame, [OZ76, Theorem 9], who showed that the tensor product, $A \otimes B$, of any separable, unital C^*-algebra A with a UHF-algebra B is singly generated.

Later, it was shown by Li and Shen, [LS10, Theorem 3.1], that every unital, approximately divisible[4] C^*-algebra is singly generated. This generalizes the result of Olsen and Zame, since the tensor product with a UHF-algebra is always approximately divisible.

In this article we prove that every separable, unital, $\mathcal{Z}$-stable C^*-algebra is singly generated, see Theorem 3.7. This generalizes the result of Li and Shen, since every approximately divisible C^*-algebra is $\mathcal{Z}$-stable, see [TW08, Theorem 2.3]. The notion of $\mathcal{Z}$-stability has proven to be very important in the classification program of nuclear C^*-algebras, see e.g. [Win07] or [ET08], and it is has been shown that many nuclear, simple C^*-algebras are $\mathcal{Z}$-stable, see e.g. [Win10]. $\mathcal{Z}$-stability is also relevant in the non-nuclear context; for example, unital $\mathcal{Z}$-stable C^*-algebras satisfy Kadison's similarity property, see [JW11].

This paper proceeds as follows:

In Section 2 we set up our notation and give some basic facts about the generator rank, see 2.1, and $C_0(X)$-algebras, see 2.4.

Section 3 contains the proof of our main result, which states that the tensor product $A \otimes_{\max} B$ of two separable, unital C^*-algebras is singly generated, if A satisfies condition (2) from above (e.g. A is simple and non-elementary) and B admits a unital embedding of the Jiang-Su algebra $\mathcal{Z}$, see Theorem 3.5.

We derive that every separable, unital, $\mathcal{Z}$-stable C^*-algebra is singly generated, see Theorem 3.7. Our main result can be considered as a (partial) C^*-algebraic analog of a theorem of Ge and Popa, [GP98, Theorem 6.2], which shows that a tensor product, $M \bar{\otimes} N$, of two II$_1$-factors M, N is singly generated. In fact, we can reprove their theorem with our methods, see Corollary 3.11.

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[4]A unital C^*-algebra A is 'approximately divisible' if for every $\varepsilon > 0$ and finite subset $F \subset A$ there exists a finite-dimensional, unital sub-C^*-algebra $B \subset A$ such that B has no characters and $\|xb - bx\| \leq \varepsilon \|b\|$ for all $x \in F, b \in B$.

In Section 4 we give further applications of our main theorem to tensor products with reduced group C^*-algebras. We first observe that $\mathcal{Z}$ embeds unitally into $C_r^*(F_\infty)$, the reduced group C^*-algebra of the free group on infinitely many generators, see Lemma 4.1. Consequently, if a discrete group Γ contains a non-cyclic free subgroup, then $\mathcal{Z}$ embeds unitally into $C_r^*(\Gamma)$, see Proposition 4.2.

We deduce that tensor products of the form $A \otimes_{\max} C_r^*(\Gamma)$ are singly generated if A is a separable, unital C^*-algebra satisfying condition (2) from above, and Γ is a group containing a non-cyclic free subgroup, see Corollary 4.4. For example, $C_r^*(F_\infty) \otimes C_r^*(F_\infty)$ is singly generated, although this C^*-algebra is not $\mathcal{Z}$-stable, see Example 4.5.

2. Preliminaries

By a morphism between C^*-algebras we mean a $*$-homomorphism, and by an ideal of a C^*-algebra we understand a closed, two-sided ideal. If A is a C^*-algebra, then we denote by $\widetilde{A}$ its minimal unitization. Often, we write M_k for the C^*-algebra of k-by-k matrices $M_k(\mathbb{C})$.

2.1. Let A be a C^*-algebra, and $A_{\mathrm{sa}} \subset A$ the subset of self-adjoint elements. We say that a set $S \subset A_{\mathrm{sa}}$ generates A, denoted $A = C^*(S)$, if the smallest sub-C^*-algebra of A containing S is A itself. We denote by $\mathrm{gen}(A)$ the smallest number $n \in \{1, 2, 3, \dots, \infty\}$ such that A contains a generating subset $S \subset A_{\mathrm{sa}}$ of cardinality n, and we call $\mathrm{gen}(A)$ the **generating rank** of A.

We stress that for the definition of $\mathrm{gen}(A)$, the generators are assumed to be self-adjoint. Two self-adjoint elements a, b generate the same C^*-algebra as the (non-self-adjoint) element $a + ib$. Therefore, a C^*-algebra A is said to be singly generated if $\mathrm{gen}(A) \leq 2$.

For more details on the generating rank we refer the reader to Nagisa, [Nag04], where also the following simple facts are noted for C^*-algebras A and B:

(1) $\mathrm{gen}(\widetilde{A}) = \mathrm{gen}(A)$,
(2) $\mathrm{gen}(C^*(A, B)) \leq \mathrm{gen}(A) + \mathrm{gen}(B)$, if A, B are sub-C^*-algebras of a common C^*-algebra, and where $C^*(A, B)$ denotes the sub-C^*-algebra they generate together,
(3) $\mathrm{gen}(A \oplus B) = \max\{\mathrm{gen}(A), \mathrm{gen}(B)\}$ if at least one of the algebras is unital.

Let $I \lhd A$ be an ideal in a C^*-algebra A. It is easy to see that the generating rank of the quotient A/I is not bigger than the generating rank of A, i.e., $\mathrm{gen}(A/I) \leq \mathrm{gen}(A)$, and the generating rank of A can be estimated as $\mathrm{gen}(A) \leq \mathrm{gen}(I) + \mathrm{gen}(A/I)$. The following result gives an estimate for $\mathrm{gen}(I)$, and it is probably well-known to experts; since we could not locate it in the literature, we include a short proof.

Proposition 2.2. *Let A be a C^*-algebra, and $I \lhd A$ an ideal. Then $\mathrm{gen}(I) \leq \mathrm{gen}(A) + 1$.*

Proof. We may assume $\mathrm{gen}(A)$ is finite. So let $a_1, \ldots, a_k$ be a set of self-adjoint generators for A. Then A and I are separable, and so I contains a strictly positive element h. It follows that $C^*(h)$ contains a quasi-central approximate unit, see [AP77, Corollary 3.3] and [Arv77]. It is straightforward to show that I is generated by the $k + 1$ elements $h, ha_1h, \ldots, ha_kh$. $\qquad\square$

The following result is attributed to Kirchberg in [Nag04].

Theorem 2.3 (Kirchberg). *Every separable, unital, properly infinite C^*-algebra is singly generated.*

Proof. We sketch a proof based on the proof of [OZ76, Theorem 9]. Let A be a separable, unital, properly infinite C^*-algebra. Then there exist isometries $s_1, s_2, \ldots \in A$ with pairwise orthogonal ranges (i.e., A contains a unital copy of the Cuntz algebra $\mathcal{O}_\infty$).

Let $a_1, a_2, \ldots \in A$ be a sequence of (positive) generators for A such that their spectra satisfy $\sigma(a_k) \subset [1/2 \cdot 1/4^k, 1/4^k]$. A generator for A is given by:

$$x := \sum_{k \geq 1} (s_k a_k s_k^* + 1/2^k s_k).$$

As in in the proof of [OZ76, Theorem 9], one can show that $\sigma(x) \subset \{0\} \cup \bigcup_{k \geq 1} [1/2 \cdot 1/4^k, 1/4^k]$. Let $B := C^*(x) \subset A$. Proceeding inductively, one shows that $a_k, s_k \in B$. We only sketch this for $k = 1$. Set $p := s_1 s_1^*$. Let f_n be a sequence of polynomials converging uniformly to 1 on $[1/8, 1/4]$ and to 0 on $[0, 1/16]$. Then $f_n(x)$ converges to an element $y \in B$ of the form $y = p + pb(1 - p)$ for some $b \in A$. We compute $yy^* = p(1_A + b(1 - p)b^*)p$. Then for a continuous function $f \colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and $f(t) = 1$ for $t \geq 1$, we get $f(yy^*) = p \in B$. Then $s_1 a_1 s_1^* = pxp \in B$ and $s_1 = 2 \cdot px(1 - p) \in B$, and then also $a_1 \in B$. $\qquad\square$

2.4. Let X be a locally compact σ-compact Hausdorff space. A $C_0(X)$-algebra is a C^*-algebra A together with a morphism $\eta \colon C_0(X) \to Z(M(A))$, from the commutative C^*-algebra $C_0(X)$ to the center of the multiplier algebra of A, such that for any approximate unit $(u_\lambda)_\Lambda$ of $C_0(X)$, $\eta(u_\lambda)a \to a$ for any $a \in A$, or equivalently, the closure of $\eta(C_0(X))A$ is all of A. Thus, if X is compact, then η is necessarily unital. We will usually suppress reference to the structure map, and simply write fa or $f \cdot a$ instead of $\eta(f)a$ for the product of a function $f \in C_0(X)$ and an element $a \in A$.

Let $Y \subset X$ be a closed subset, and $U := X \setminus Y$ its complement (an open subset). Then $C_0(U) \cdot A$ is an ideal of A, denoted by $A(U)$. The quotient $A/A(U)$ is denoted by $A(Y)$.

Given a point $x \in X$, we write $A(x)$ for $A(\{x\})$, and we call this C^*-algebra the fiber of A at x. For an element $a \in A$, we denote by $a(x)$ the image of a in the fiber $A(x)$. For each $a \in A$, we may consider the map $\check{a} \colon x \mapsto \|a(x)\|$. This is a real-valued, upper-semicontinuous function on X, vanishing at infinity.

The $C_0(X)$-algebra A is called continuous if $\check{a}$ is a continuous function for each $a \in A$.

For more information on $C_0(X)$-algebras we refer the reader to [Kas88, §1] or the more recent [Dad09, §2].

2.5. The Jiang-Su algebra $\mathcal{Z}$ was constructed in [JS99]; it may be regarded as a C^*-algebraic analog of the hyperfinite II_1-factor. It can be obtained as an inductive limit of prime dimension drop algebras $\mathcal{Z}_{p,q} := \{f\colon [0,1] \to M_p \otimes M_q \mid f(0) \in 1_p \otimes M_q, f(1) \in M_p \otimes 1_q\}$.

For more details, we refer the reader to [Win11], where $\mathcal{Z}$ is characterized in an entirely abstract manner, and to [Rør04] and [RW10], where it is shown that the generalized dimension drop algebra $\mathcal{Z}_{2^\infty,3^\infty} := \{f\colon [0,1] \to M_{2^\infty} \otimes M_{3^\infty} \mid f(0) \in 1 \otimes M_{3^\infty}, f(1) \in M_{2^\infty} \otimes 1\}$ embeds unitally into $\mathcal{Z}$; in fact, $\mathcal{Z}$ can be written as a stationary inductive limit of $\mathcal{Z}_{2^\infty,3^\infty}$.

3. RESULTS

Lemma 3.1. *Let A be a separable, unital C^*-algebra. Then $\operatorname{gen}(A \otimes \mathcal{Z}_{2^\infty,3^\infty}) \leq 5$.*

Proof. Consider the ideal $I := C_0(0,1) \otimes M_{6^\infty}$ in $B := A \otimes \mathcal{Z}_{2^\infty,3^\infty}$. The quotient B/I is isomorphic to $(A \otimes M_{2^\infty}) \oplus (A \otimes M_{3^\infty})$. Thus, we have a short exact sequence:

$$A \otimes C_0(0,1) \otimes M_{6^\infty} \longrightarrow A \otimes \mathcal{Z}_{2^\infty,3^\infty} \longrightarrow (A \otimes M_{2^\infty}) \oplus (A \otimes M_{3^\infty})$$

It follows from [OZ76] that the tensor product of a unital, separable C^*-algebra with a UHF-algebra is singly generated. In particular, $\operatorname{gen}(A \otimes M_{2^\infty})$, $\operatorname{gen}(A \otimes M_{3^\infty}) \leq 2$. Therefore, the quotient satisfies $\operatorname{gen}(B/I) = \max\{\operatorname{gen}(A \otimes M_{2^\infty}), \operatorname{gen}(A \otimes M_{3^\infty})\} \leq 2$, see 2.1.

Note that I is an ideal in the C^*-algebra $C := A \otimes C(S^1) \otimes M_{2^\infty}$. We have $\operatorname{gen}(C) \leq 2$, and then $\operatorname{gen}(I) \leq \operatorname{gen}(C) + 1 \leq 3$, by Proposition 2.2. Then, the extension is generated by at most $2 + 3 = 5$ self-adjoint elements. $\square$

The following is a Stone-Weierstrass type result. We prove it using the factorial Stone-Weierstrass conjecture, which states that a sub-C^*-algebra $B \subset A$ exhausts A if it separates the factorial states of A. The factorial Stone-Weierstrass conjecture was proved for separable C^*-algebras independently by Longo, [Lon84], and Popa, [Pop84].

See 2.4 for a short introduction to $C_0(X)$-algebras.

Lemma 3.2. *Let A be a separable, continuous $C_0(X)$-algebra, and $B \subset A$ a sub-C^*-algebra such that the following two conditions are satisfied:*
  (i) For each $x \in X$, B exhausts the fiber $A(x)$,
  (ii) B separates the points of X by full elements, i.e., for each distinct pair of points $x_0, x_1 \in X$ there exists some $b \in B$ such that $b(x_1)$ is full in $B(x_1) = A(x_1)$ and $b(x_0) = 0$.

Then $A = B$.

Condition (ii) is for instance satisfied if B contains the image of the structure map $\eta \colon C_0(X) \to Z(M(A))$.

Proof. Set $Y := \mathrm{Prim}(Z(M(A)))$, and identify $Z(M(A))$ with $C(Y)$. Let $\pi \colon A \to B(H)$ be a non-degenerate factor representation. Then π extends to a representation $\tilde{\pi} \colon M(A) \to B(H)$. It is straightforward to show $\pi(A)'' = \tilde{\pi}(M(A))''$, so that $\tilde{\pi}$ is a factor representation of $M(A)$. For any element $c \in Z(M(A))$, we have $c \in \pi(A)' \cap \tilde{\pi}(M(A))'' = \mathbb{C} \cdot 1_H$. Thus, there exists a point $y \in Y$ such that $\tilde{\pi}(c) = c(y) \cdot 1_H$ for all $c \in Z(M(A))$. Since $\eta(C_0(X))$ contains an approximate unit for A, we have that $\tilde{\pi} \circ \eta$ is non-zero. Thus, there exists a point $x \in X$ such that $\tilde{\pi} \circ \eta(f) = f(x) \cdot 1_H$ for all $f \in C_0(X)$. This means that $\tilde{\pi} \circ \eta$ vanishes on the ideal $A(X \setminus \{x\})$, so that π factors through the fiber $A(x)$.

Let us show that $B \subset A$ separates the factors states of A. So let φ_1, φ_2 be two different, non-degenerate factors states of A. We have shown above that there are two points $x_1, x_2 \in X$ such that φ_i factors through $A(x_i)$, and we denote by $\bar{\varphi}_i \colon A(x_i) \to \mathbb{C}$ the induced factor state on $A(x_i)$, for $i = 1, 2$. We distinguish two cases:

Case 1: $x_1 = x_2$. In this case, since $\varphi_1 \neq \varphi_2$, there exists an element $a \in A$ such that $\varphi_1(a) \neq \varphi_2(a)$. By condition (i), there exists some element $b \in B$ such that $b(x_1) = a(x_1)$. Note that $\varphi_i(b) = \bar{\varphi}_i(b(x_1)) = \bar{\varphi}_i(a(x_1)) = \varphi_i(a)$, for $i = 1, 2$. Thus, b separates the two states.

Case 2: $x_1 \neq x_2$. In this case, by condition (ii), there exists an element $b \in B$ such that $b(x_2)$ is full in $A(x_2)$ and $b(x_1) = 0$. Since $\varphi_2 \neq 0$, there exists an element $a \in A$ such that $|\varphi_2(a)| = |\bar{\varphi}_2(a(x_2))| \geq 1$.

Since $b(x_2)$ is full, there exist finitely many elements $g_i, h_i \in A(x_2)$ such that $\|a(x_2) - \sum_i c_i b(x_2) d_i\| < 1$. By condition (i), there exist elements $\tilde{g}_i, \tilde{h}_i \in B$ such that $\tilde{g}_i(x_2) = g_i$ and $\tilde{h}_i(x_2) = h_i$. Set $b' := \sum_i \tilde{c}_i b \tilde{d}_i$. Then $|\varphi_2(b')| = |\bar{\varphi}_2(b'(x_2))| > 0$, while $b'(x_1) = 0$. This shows that b' separates the two states.

It follows that B separates the factor states of A, and therefore $B = A$ by the factorial Stone-Weierstrass conjecture, proved independently by Longo, [Lon84], and Popa, [Pop84]. □

Lemma 3.3. *Let A be a unital C^*-algebra with $\mathrm{gen}(A) \leq 3$. Then there exist a positive element $x \in A \otimes \mathcal{Z}_{2,3}$ and two positive, full elements $y', z' \in \mathcal{Z}_{2,3}$ such that $A \otimes \mathcal{Z}_{2,3}$ is generated by x and $1 \otimes y'$, and further y' and z' are orthogonal.*

Proof. We consider $\mathcal{Z}_{2,3}$ as the C^*-algebra of continuous functions from $[0,1]$ to M_6 with the boundary conditions

$$f(0) = \begin{pmatrix} Y & & \\ & Y & \\ & & Y \end{pmatrix} \quad f(1) = \begin{pmatrix} Z & \\ & QZQ^* \end{pmatrix},$$

where $Y \in M_2$ and $Z \in M_3$ are arbitrary matrices, and $Q \in M_3$ is the following fixed permutation matrix:

$$Q = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}.$$

This means that $f(0), f(1) \in M_6$ have the following form:

$$f(0) = \begin{pmatrix} \mu_{11} & \mu_{12} & & & & \\ \mu_{21} & \mu_{22} & & & & \\ & & \mu_{11} & \mu_{12} & & \\ & & \mu_{21} & \mu_{22} & & \\ & & & & \mu_{11} & \mu_{12} \\ & & & & \mu_{21} & \mu_{22} \end{pmatrix},$$

$$f(1) = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & & & \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & & & \\ \lambda_{31} & \lambda_{22} & \lambda_{33} & & & \\ & & & \lambda_{33} & \lambda_{31} & \lambda_{32} \\ & & & \lambda_{13} & \lambda_{11} & \lambda_{12} \\ & & & \lambda_{23} & \lambda_{21} & \lambda_{22} \end{pmatrix},$$

for numbers $\mu_{i,j}, \lambda_{i,j} \in \mathbb{C}$.

Note that $\mathcal{Z}_{2,3}$ is naturally a continuous $C([0,1])$-algebra, with fibers $\mathcal{Z}_{2,3}(0) \cong M_2$, $\mathcal{Z}_{2,3}(1) \cong M_3$, and $\mathcal{Z}_{2,3}(t) \cong M_6$ for points $t \in (0,1) \subset [0,1]$.

Let $a, b, c \in A$ be a set of invertible, positive generators for A. Denote by $e_{i,j}$ the matrix units in M_6. To shorten notation, for indices i, j set $f_{i,j} := e_{i,j} + e_{j,i}$. For $t \in [0,1]$ we define the following element of $A \otimes M_6$:

$$\begin{aligned} x_t :=& a \otimes (e_{1,1} + (1-t) \cdot e_{3,3} + e_{5,5}) \\ &+ b \otimes (f_{1,2} + (1-t) \cdot f_{3,4} + f_{5,6}) \\ &+ c \otimes (e_{2,2} + (1-t) \cdot e_{4,4} + e_{6,6}) \\ &+ 1_A \otimes (t \cdot f_{2,3} + t \cdot f_{4,5} + \delta(t) \cdot f_{1,3}) \end{aligned}$$

where $\delta \colon [0,1] \to [0,1]$ is a continuous function on $[0,1]$ that takes the value 0 at the endpoints 0 and 1, and is strictly positive at each point $t \in (0,1)$, e.g., δ could be given by $\delta(t) = 1/4 - (t - 1/2)^2$. We also define for $t \in [0,1]$ two elements of M_6:

$$\begin{aligned} y'_t :=& e_{1,1} + (1-t) \cdot e_{3,3} + e_{5,5} \\ z'_t :=& e_{2,2} + (1-t) \cdot e_{4,4} + e_{6,6} \end{aligned}$$

It is easy to check that the assignment $x\colon t \mapsto x_t$ defines an element $x \in A \otimes \mathcal{Z}_{2.3}$. Similarly, we get two elements $y', z' \in \mathcal{Z}_{2.3}$ defined via $t \mapsto y_t'$ and $t \mapsto z_t'$. In matrix form, these elements look as follows:

$$x_t := \left(\begin{array}{cc|cc|cc} a & b & \delta(t) & & & \\ b & c & t & & & \\ \hline \delta(t) & t & (1-t)a & (1-t)b & & \\ & & (1-t)b & (1-t)c & t & \\ \hline & & & t & a & b \\ & & & & b & c \end{array} \right)$$

$$y_t' := \left(\begin{array}{cc|cc|cc} 1 & & & \\ \hline & (1-t) & \\ \hline & & 1 \end{array} \right) \qquad z_t' := \left(\begin{array}{cc|cc|cc} 1 & & & \\ \hline & (1-t) & \\ \hline & & 1 \end{array} \right)$$

Set $y := 1 \otimes y'$, and let $D := C^*(x+1, y)$ be the sub-C^*-algebra of $E := A \otimes \mathcal{Z}_{2,3}$ generated by the two self-adjoint elements $x+1$ and y. Since $x \geq 0$, we get that both 1 and x lie in $C^*(x+1)$. It follows that $D = C^*(1, x, y)$, and we will show that $D = E$. Note that E has a natural continuous $C([0,1])$-algebra structure (induced by the one of $\mathcal{Z}_{2,3}$), with fibers $E(0) \cong A \otimes M_2$, $E(1) \cong A \otimes M_3$, and $E(t) \cong A \otimes M_6$ for points $t \in (0,1) \subset [0,1]$.

Let $J := E((0,1)) \lhd E$ be the natural ideal corresponding to the open set $(0,1) \subset [0,1]$. Note that $J \cong A \otimes C_0((0,1)) \otimes M_6$, and J is naturally a continuous $C_0((0,1))$-algebra. We will show in two steps that D exhausts the ideal J (i.e., $D \cap J = J$) and the quotient E/J (i.e., $D/(D \cap J) = E/J$).

Step 1: We want to apply Lemma 3.2 to the $C((0,1))$-algebra J with sub-C^*-algebra $D \cap J$. To verify condition (ii), note that the C^*-algebra generated by y' contains $C_0((0,1)) \otimes e_{3,3}$. Therefore, $D \cap J$ contains $1_A \otimes C_0((0,1)) \otimes e_{3,3}$, which separates the points of $(0,1)$. Since $1_A \otimes e_{3,3} \in E(t) \cong A \otimes M_6$ is full, condition (ii) of Lemma 3.2 holds and it remains to verify condition (i).

We need to show that $D \cap J$ exhausts all fibers of J. Fix some $t \in (0,1)$, and set $D_t := C^*(1, x_t, y_t) \subset A \otimes M_6$. To simplify notation, we write $\bar{e}_{i,j}$ for the matrix units $1_A \otimes e_{i,j} \in A \otimes M_6$. We need to show that D_t is all of $A \otimes M_6$. This will follow if D_t contains all $\bar{e}_{i,j}$, and for this it is enough to show that the off-diagonal matrix units $\bar{e}_{i,i+1}$ are in D_t, for $i = 1, \dots, 5$.

The spectrum of y_t is $\{0, 1-t, 1\}$. Applying functional calculus to y_t we obtain that the following three elements lie in D_t:

$$u := \bar{e}_{1,1} + \bar{e}_{5,5}$$
$$v := \bar{e}_{3,3}$$

$$w := 1 - v - u = \bar{e}_{2,2} + \bar{e}_{4,4} + \bar{e}_{6,6}$$

Then, we proceed as follows:

1. $\bar{e}_{1,3} = \delta(t)^{-1} u x_t v \in D_t$ and so $\bar{e}_{1,1}, \bar{e}_{5,5} \in D_t$.
2. $g := b \otimes e_{1,2} = \bar{e}_{1,1} x_t w \in D_t$. It follows $b \otimes e_{1,1} = (gg^*)^{1/2} \in D_t$, cf. [OZ76]. Then $b^{-1} \otimes e_{1,1} \in C^*(b \otimes e_{1,1}) \subset D_t$ and so $\bar{e}_{1,2} = (b^{-1} \otimes e_{1,1}) \cdot g \in D_t$ and $\bar{e}_{2,2} \in D_t$.
3. $b \otimes e_{3,4} = (1-t)^{-1} \bar{e}_{3,3} x_t (w - \bar{e}_{2,2}) \in D_t$. Arguing as above, it follows that $\bar{e}_{3,4} \in D_t$, and then $\bar{e}_{4,4}, \bar{e}_{6,6} \in D_t$.
4. $\bar{e}_{2,3} = t^{-1} \bar{e}_{2,2} x_t \bar{e}_{3,3} \in D_t$.
5. $\bar{e}_{4,5} = t^{-1} \bar{e}_{4,4} x_t \bar{e}_{5,5} \in D_t$.
6. $b \otimes e_{5,6} = \bar{e}_{5,5} x_t \bar{e}_{6,6} \in D_t$ and so $\bar{e}_{5,6} \in D_t$.

This shows that $D \cap J$ exhausts the fibers of J. We may apply Lemma 3.2 and deduce $D \cap J = J$, which finishes step 1.

Step 2: We want to show that D/J exhausts $E/J = E(\{0,1\}) \cong A \otimes (M_2 \oplus M_3)$. Let us denote the matrix units in M_2 by $e_{i,j}^{(0)}$, $i = 1, 2$, and the matrix units in M_3 by $e_{i,j}^{(1)}$, $i = 1, 2, 3$. To simplify notation, we write $\bar{e}_{i,j}^{(k)}$ for the matrix units $1_A \otimes e_{i,j}^{(k)} \in A \otimes (M_2 \oplus M_3)$. Let us denote the image of x and y in D/J by v and w:

$$\begin{aligned} v &= a \otimes (e_{1,1}^{(0)} + e_{1,1}^{(1)}) + b \otimes (e_{1,2}^{(0)} + e_{2,1}^{(0)} + e_{1,2}^{(1)} + e_{2,1}^{(1)}) \\ &\quad + c \otimes (e_{2,2}^{(0)} + e_{2,2}^{(1)}) + \bar{e}_{2,3}^{(1)} + \bar{e}_{3,2}^{(1)} \\ &= \begin{pmatrix} a & b \\ b & c \end{pmatrix} \oplus \begin{pmatrix} a & b & \\ b & c & 1 \\ & 1 & \end{pmatrix}. \end{aligned}$$

$$w = \bar{e}_{1,1}^{(0)} + \bar{e}_{1,1}^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}.$$

As in step 1, it is enough to show that D/J contains the off-diagonal matrix units $\bar{e}_{1,2}^{(0)}$, $\bar{e}_{1,2}^{(1)}$ and $\bar{e}_{2,3}^{(1)}$. We argue as follows:

1. $g := wv(1 - w) = b \otimes (e_{1,2}^{(0)} + e_{1,2}^{(1)}) \in D/J$. As in step 1, it follows that $b \otimes (e_{1,1}^{(0)} + e_{1,1}^{(1)}) = (gg^*)^{1/2} \in D/J$. Then $b^{-1} \otimes (e_{1,1}^{(0)} + e_{1,1}^{(1)}) \in D/J$, and so $\bar{e}_{1,2}^{(0)} + \bar{e}_{1,2}^{(1)} = (b^{-1} \otimes (e_{1,1}^{(0)} + e_{1,1}^{(1)})) \cdot g \in D/J$. It follows that $\bar{e}_{2,2}^{(0)} + \bar{e}_{2,2}^{(1)} \in D/J$.
2. $\bar{e}_{3,3}^{(1)} = 1 - w - (\bar{e}_{2,2}^{(0)} + \bar{e}_{2,2}^{(1)}) \in D/J$.
3. $\bar{e}_{2,3}^{(1)} = v \bar{e}_{3,3}^{(1)} \in D/J$, and so $\bar{e}_{2,2}^{(1)} \in D/J$.
4. $b \otimes e_{1,2}^{(1)} = wv \bar{e}_{2,2}^{(1)} \in D/J$. Again, this implies $\bar{e}_{1,2}^{(1)} \in D/J$ and so $\bar{e}_{1,1}^{(1)} \in D/J$.
5. $\bar{e}_{1,1}^{(0)} = w - \bar{e}_{1,1}^{(1)} \in D/J$.
6. $\bar{e}_{2,2}^{(0)} = 1 - w - \bar{e}_{2,2}^{(1)} - \bar{e}_{3,3}^{(1)} \in D/J$.

7. $b \otimes e_{1,2}^{(0)} = \bar{e}_{1,1}^{(0)} v \bar{e}_{2,2}^{(0)} \in D/J$. Again, this implies $\bar{e}_{1,2}^{(0)} \in D/J$.

This finishes step 2.

We have seen that $A \otimes \mathcal{Z}_{2,3}$ is generated by $x + 1$ and y. Moreover, z' is full, positive and orthogonal to y'. □

Lemma 3.4. *Let A be a separable, unital C^*-algebra. Then there exist a positive element $x \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ and two positive, full elements $y', z' \in \mathcal{Z}_{2^\infty,3^\infty}$ such that $A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ is generated by x and $y := 1 \otimes y'$, and further y' and z' are orthogonal.*

Proof. Let $B := A \otimes \mathcal{Z}_{2^\infty,3^\infty}$. Note that $\mathcal{Z}_{2^\infty,3^\infty} \otimes \mathcal{Z}_{2,3}$ is naturally a $C([0,1] \times [0,1])$-algebra. Then, the quotient corresponding to the diagonal $\{(t,t) \mid t \in [0,1]\} \subset [0,1] \times [0,1]$ is isomorphic to $\mathcal{Z}_{2^\infty,3^\infty}$, and we denote the resulting surjective morphism by $\pi \colon \mathcal{Z}_{2^\infty,3^\infty} \otimes \mathcal{Z}_{2,3} \to \mathcal{Z}_{2^\infty,3^\infty}$. We proceed in two steps.

Step 1: We show that $\mathrm{gen}(B) \leq k+1$ implies $\mathrm{gen}(B) \leq k$ for $k \geq 2$. So assume B is generated by the self-adjoint, invertible elements $a_1, \ldots, a_{k+1}$. The sub-C^*-algebra $C := C^*(a_{k-1}, a_k, a_{k+1}) \subset B$ is unital and satisfies $\mathrm{gen}(C) \leq 3$. Consider the C^*-algebra $B \otimes \mathcal{Z}_{2,3}$. By Lemma 3.3, the sub-$C^*$-algebra $C \otimes \mathcal{Z}_{2,3}$ is generated by two self-adjoint elements, say b, c.

One readily checks that $B \otimes \mathcal{Z}_{2,3}$ is generated by the k self-adjoint elements $a_1 \otimes 1, \ldots, a_{k-2} \otimes 1, b, c$. Since $B = A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ is isomorphic to a quotient of $B \otimes \mathcal{Z}_{2,3} = A \otimes \mathcal{Z}_{2^\infty,3^\infty} \otimes \mathcal{Z}_{2,3}$, we obtain $\mathrm{gen}(B) \leq \mathrm{gen}(B \otimes \mathcal{Z}_{2,3}) \leq k$.

Step 2: By Lemma 3.1, we have $\mathrm{gen}(B) \leq 5$. Applying Step 1 several times, we obtain $\mathrm{gen}(B) \leq 3$.

It follows from Lemma 3.3 that there exists a positive element $\tilde{x} \in B \otimes \mathcal{Z}_{2,3}$ and two positive, full elements $\tilde{y}', \tilde{z}' \in \mathcal{Z}_{2,3}$ such that $B \otimes \mathcal{Z}_{2,3}$ is generated by $\tilde{x}$ and $1 \otimes \tilde{y}'$, and further $\tilde{y}'$ and $\tilde{z}'$ are orthogonal.

Consider the surjective morphism $\mathrm{id} \otimes \pi \colon A \otimes \mathcal{Z}_{2^\infty,3^\infty} \otimes \mathcal{Z}_{2,3} \to A \otimes \mathcal{Z}_{2^\infty,3^\infty}$. One checks that the elements $x := (\mathrm{id} \otimes \pi)(\tilde{x}) \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$, and $y' := \pi(\tilde{y}')$, $z' := \pi(\tilde{z}') \in \mathcal{Z}_{2^\infty,3^\infty}$ have the desired properties. □

Theorem 3.5. *Let A, B be two separable, unital C^*-algebras. Assume the following:*

(1) A contains a sequence $a_1, a_2, \ldots$ of full, positive elements that are pairwise orthogonal,

(2) B admits a unital embedding of the Jiang-Su algebra $\mathcal{Z}$.

Then $A \otimes_{max} B$ is singly generated. Every other tensor product $A \otimes_\lambda B$ is a quotient of $A \otimes_{max} B$, and therefore is also singly generated.

Proof. There exists a unital embedding of $\mathcal{Z}_{2^\infty,3^\infty}$ in $\mathcal{Z}$, so we may assume that there is a unital embedding of $\mathcal{Z}_{2^\infty,3^\infty}$ in B. We may assume that the elements $a_1, a_2, \ldots \in A$ are contractive.

Choose a sequence $b_1, b_2, \ldots \in B$ of contractive, positive elements that is dense in the set of all contractive, positive elements of B.

Consider the sub-C^*-algebra $A \otimes \mathcal{Z}_{2^\infty,3^\infty} \subset A \otimes_{\max} B$. By Lemma 3.4, there exist a positive element $x \in A \otimes \mathcal{Z}_{2^\infty,3^\infty}$ and two full, positive elements $y', z' \in$

$\mathcal{Z}_{2\infty,3\infty}$ such that $A \otimes \mathcal{Z}_{2\infty,3\infty}$ is generated by x and $y := 1 \otimes y'$, and further y' and z' are orthogonal.

Define the following two elements of $A \otimes_{\max} B$:

$$v := x, \qquad w := 1 \otimes y' - \sum_{k \geq 1} 1/2^k \cdot a_k \otimes (z'b_k z').$$

Let $D := C^*(v, w)$ be the sub-C^*-algebra of $A \otimes_{\max} B$ generated by v and w. We claim that $D = A \otimes B$.

Step 1: We show $A \otimes \mathcal{Z}_{2\infty,3\infty} \subset D$. Note that the two elements $1 \otimes y'$ and $\sum_{k \geq 1} 1/2^k \cdot a_k \otimes (z'b_k z')$ are positive and orthogonal. It follows that $1 \otimes y'$ is the positive part of w, and therefore $1 \otimes y' \in D$. Therefore, $C^*(v, 1 \otimes y') = A \otimes \mathcal{Z}_{2\infty,3\infty} \subset D$.

Step 2: We show $1 \otimes B \subset D$. We have $g := \sum_{k \geq 1} 1/2^k \cdot a_k \otimes (z'b_k z') \in D$. It follows from Step 1 that $a_k \otimes 1 \in D$, and so $a_k^2 \otimes (z'b_k z') = 2^k \cdot (a_k \otimes 1)g \in D$. Since a_k^2 is full, there exist finitely many elements $c_i, d_i \in A$ such that $1_A = \sum_i c_i a_k^2 d_i$. By Step 1, we have $c_i \otimes 1, d_i \otimes 1 \in D$. Then $1 \otimes (z'b_k z') = \sum_i (c_i \otimes 1)(a_k^2 \otimes (z'b_k z'))(d_i \otimes 1) \in D$, for each k.

Let $b \in B$ be a contractive, positive element. Then $b = \lim_j b_{k(j)}$ for certain indices $k(j)$. Then $1 \otimes (z'bz') = \lim_j 1 \otimes (z'b_{k(j)}z') \in D$. It follows that the hereditary sub-C^*-algebra $1 \otimes \overline{z'Bz'}$ is contained in D. Since z' is full in $\mathcal{Z}_{2\infty,3\infty}$, there exist finitely many elements $c_i, d_i \in \mathcal{Z}_{2\infty,3\infty}$ such that $1_B = \sum_i c_i z' d_i$. We have seen that $1 \otimes z'bz' \in D$ for any $b \in B$. Then $1 \otimes bz' = \sum_i (1 \otimes c_i)(1 \otimes z'd_i bz') \in D$ for any $b \in B$. Similarly $1 \otimes b = \sum_i (1 \otimes bc_i z')(1 \otimes d_i) \in D$ for any $b \in B$, as desired.

It follows from Steps 1 and 2 that for each $a \in A$ and $b \in B$ the simple tensor $a \otimes b$ is contained in D. The conclusion follows since $A \otimes_{\max} B$ is the closure of the linear span of simple tensors. $\square$

Corollary 3.6. *Let A, B be two separable, unital C^*-algebras that both admit a unital embedding of the Jiang-Su algebra $\mathcal{Z}$. Then $A \otimes_{max} B$ is singly generated.*

Proof. It is easy to verify that condition (i) of Theorem 3.5 is fulfilled if A admits a unital embedding of $\mathcal{Z}$. $\square$

Theorem 3.7. *Let A be a unital, separable C^*-algebra. Then $A \otimes \mathcal{Z}$ is singly generated.*

Proof. Note that $A \otimes \mathcal{Z} \cong (A \otimes \mathcal{Z}) \otimes \mathcal{Z}$. It is clear that both $A \otimes \mathcal{Z}$ and $\mathcal{Z}$ admit unital embeddings of $\mathcal{Z}$. Then apply the above Corollary 3.6. $\square$

Corollary 3.8. *Let A be a separable C^*-algebra. Then $\mathrm{gen}(A \otimes \mathcal{Z}) \leq 3$.*

Proof. Let $\widetilde{A}$ be the minimal unitization of A. It follows from Theorem 3.7 that $\mathrm{gen}(\widetilde{A} \otimes \mathcal{Z}) \leq 2$. Since $A \otimes \mathcal{Z}$ is an ideal in $\widetilde{A} \otimes \mathcal{Z}$, we get $\mathrm{gen}(A \otimes \mathcal{Z}) \leq \mathrm{gen}(\widetilde{A} \otimes \mathcal{Z}) + 1 \leq 3$ from Proposition 2.2, as desired. $\square$

Our results allow us to give new proofs for results about single generation of certain von Neumann algebras.

Proposition 3.9. *Assume M, N are separably-acting von Neumann algebras that both admit a unital embedding of the hyperfinite II_1-factor. Then $M \bar{\otimes} N$ is singly generated.*

Proof. Consider the GNS-representation $\pi \colon \mathcal{Z} \to B(H)$ of the Jiang-Su algebra with respect to its tracial state. The weak closure, $\pi(\mathcal{Z})''$, is isomorphic to the hyperfinite II_1-factor $\mathcal{R}$. Thus, there exists a weakly dense, unital copy of $\mathcal{Z}$ inside $\mathcal{R}$.

Choose weakly dense, separable, unital C^*-algebras $A_0 \subset M$, and similarly $B_0 \subset N$. Consider $\mathcal{Z} \subset \mathcal{R} \subset M$ and set $A := C^*(A_0, \mathcal{Z}) \subset M$. Similarly set $B := C^*(B_0, \mathcal{Z}) \subset N$.

Then A and B are separable, unital C^*-algebras that both contain unital copies of the Jiang-Su algebra. By Corollary 3.6, $A \otimes_{\max} B$ is singly generated.

Consider the sub-C^*-algebra $C := C^*(A \bar{\otimes} 1, 1 \bar{\otimes} B) \subset M \bar{\otimes} N$. Then C is a quotient of $A \otimes_{\max} B$, and therefore singly generated. Since C is weakly dense in $M \bar{\otimes} N$, we obtain that $M \bar{\otimes} N$ is singly generated, as desired. $\square$

Remark 3.10. We note that a von Neumann algebra M admits a unital embedding of $\mathcal{R}$ if and only if M has no (non-zero) finite-dimensional representations.

The analogous statement for C^*-algebras would be that a C^*-algebra A admits a unital embedding of $\mathcal{Z}$ if and only if A has no (non-zero) finite-dimensional representations. It was shown by Elliott and Rørdam, [ER06], that this is true for C^*-algebras of real rank zero. However, in [DHTW09] a simple, separable, unital, non-elementary AH-algebra is constructed into which $\mathcal{Z}$ does not embed.

As a particular case of Proposition 3.9 we obtain the following result of Ge and Popa.

Corollary 3.11 (Ge, Popa, [GP98, Theorem 6.2]). *Assume M, N are separably-acting II_1-factors. Then $M \bar{\otimes} N$ is singly generated.*

4. APPLICATIONS

In this section we show that the Jiang-Su algebra $\mathcal{Z}$ embeds unitally into the reduced group C^*-algebras, $C_r^*(\Gamma)$, of groups Γ that contain a non-cyclic free subgroup, see Proposition 4.2. We only consider discrete groups, and we let F_k denote the free group with k generators ($k \in \{2, 3 \ldots, \infty\}$).

We can apply Theorem 3.5 to show that certain tensor products of the form $A \otimes_{\max} C_r^*(\Gamma)$ are singly generated, see Corollary 4.4. In particular, $C_r^*(F_\infty) \otimes C_r^*(F_\infty)$ is singly generated, although it is not $\mathcal{Z}$-stable, see Example 4.5.

4.1. It was shown by Robert, [Rob10], that the Jiang-Su algebra $\mathcal{Z}$ embeds unitally into $C_r^*(F_\infty)$. A key observation is that $C_r^*(F_\infty)$ has strict comparison of positive elements. This follows from the work of Dykema and Rørdam on reduced free product C^*-algebras, see [DR98] and [DR00].

Dykema and Rørdam study the comparison of projections, but this can be generalized to obtain results about the comparison of positive elements, as noted by Robert, [Rob10]. In particular, [DR98, Lemma 5.3] and [DR00, Theorem 2.1] can be generalized, and it follows that $C_r^*(F_\infty)$ has strict comparison of positive elements.

Proposition 4.2. *If Γ is a discrete group that contains F_∞ as a subgroup, then $\mathcal{Z}$ embeds unitally into $C_r^*(\Gamma)$.*

Proof. In general, for any subgroup Γ_1 of a discrete group Γ, we have a unital embedding $C_r^*(\Gamma_1) \subset C_r^*(\Gamma)$. Hence, if F_∞ is a subgroup of Γ, then $C_r^*(\Gamma)$ contains a unital copy of $C_r^*(F_\infty)$, which in turn contains a unital copy of $\mathcal{Z}$. $\qquad\square$

Remark 4.3. Every non-cyclic free group F_k ($k \geq 2$) contains F_∞ as a subgroup. In general, by the Nielsen-Schreier theorem, every subgroup of a free group is again free. Thus, if a, b are free elements, then the the elements $a^k b^k$ generate a subgroup $\Gamma = \langle a^k b^k, k \geq 1 \rangle$ that is free, and since none of the elements $a^k b^k$ is contained in the subgroup generated by the other elements, we have $\Gamma \cong F_\infty$.

Thus, when we ask which discrete groups contain F_∞ as a subgroup, we are equivalently asking which groups Γ contain a non-cyclic free subgroup. It is a necessary condition that Γ is non-amenable. The converse implication is known as the von Neumann conjecture, but this was disproved in 1980 by Ol'shanskij.

A counterexample are the so-called Tarski monster groups, in which every non-trivial proper subgroup is cyclic of some fixed prime order. Clearly, such a group cannot contain F_∞ as a subgroup, and it is Ol'shanskij's contribution to show that Tarski monster groups exist and are non-amenable.

On the other hand, every group with the weak Powers property, as defined in [BN88], has a non-cyclic free subgroup. A proof can be found in [dlH07], which also lists classes of groups that have the (weak) Powers property. We just mention that all free products $\Gamma_1 * \Gamma_2$ with $|\Gamma_1| \geq 2, |\Gamma_2| \geq 3$ have the Powers property, and therefore Proposition 4.2 applies.

We may derive the following from Theorem 3.5 and Proposition 4.2:

Corollary 4.4. *Let A be a separable, unital C^*-algebra that contains a countable sequence of pairwise orthogonal, full elements (e.g., A is simple and nonelementary), and let Γ be a group that contains a non-cyclic free subgroup. Then $A \otimes_{max} C_r^*(\Gamma)$ is singly generated.*

Example 4.5. Let Γ_1, Γ_2 be two groups that contain non-cyclic free subgroups. Then $C_r^*(\Gamma_1 \times \Gamma_2) \cong C_r^*(\Gamma_1) \otimes_{\max} C_r^*(\Gamma_2)$ is singly generated. For example, for

any $k, l \in \{2, 3, \ldots, \infty\}$, the C^*-algebra $C_r^*(F_k) \otimes_{\max} C_r^*(F_l)$ is singly generated. In particular, $C_r^*(F_\infty) \otimes_{\max} C_r^*(F_\infty)$ is singly generated.

It was pointed out to the authors by S. Wassermann that $C_r^*(F_k) \otimes C_r^*(F_l)$ is not $\mathcal{Z}$-stable, for any $k, l \in \{2, 3, \ldots, \infty\}$. In fact, if $C_r^*(F_k) \otimes C_r^*(F_l) \cong A \otimes B \otimes C$, then one of the three algebras A, B or C is isomorphic to $\mathbb{C}$. This is a generalization of the fact that $C_r^*(F_k)$ is tensorially prime, and it can be proved similarly.

We note that it is a difficult open problem whether $C_r^*(F_k)$ is singly generated itself.

Question 4.6. Given a non-amenable (discrete) group Γ. Does $C_r^*(\Gamma)$ admit a unital embedding of $\mathcal{Z}$?

For each group Γ, the trivial group-morphism $\Gamma \to \{1\}$ induces a surjective morphism $C^*(\Gamma) \to \mathbb{C}$. Thus, the Jiang-Su algebra can never unitally embed into a full group C^*-algebra. If Γ is amenable, then $C_r^*(\Gamma) \cong C^*(\Gamma)$, and consequently there is no unital embedding of $\mathcal{Z}$ into the reduced group C^*-algebra of an amenable group.

On the other hand, if Γ contains a non-cyclic free subgroup, then from Proposition 4.2 we get a positive answer to Question 4.6. Not every non-amenable group contains a non-cyclic free subgroup, see Remark 4.3. However, it is known that the reduced group C^*-algebra of a non-amenable group has no finite-dimensional representations, which is a necessary condition for the Jiang-Su algebra to embed.

ACKNOWLEDGMENTS

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