Agència
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Memòria justificativa de recerca de les convocatòries BCC, BE, BP, CTP-AIRE, DEBEQ, FI, INEFC, NANOS i PIV

La memòria justificativa consta de les dues parts que venen a continuació:
1.- Dades bàsiques i resums
2.- Memòria del treball (informe científic)

Tots els camps són obligatoris

1.- Dades bàsiques i resums

Nom de la convocatòria

BE

Llegenda per a les convocatòries:

BCC	Convocatòria de beques per a joves membres de comunitats catalanes a l'exterior
BE	Beques per a estades per a la recerca fora de Catalunya
BP	Convocatòria d'ajuts postdoctorals dins del programa Beatriu de Pinós
CTP-AIRE	Ajuts per accions de cooperació en el marc de la comunitat de treball dels Pirineus. Ajuts de mobilitat de personal investigador.
DEBEQ (Modalitat A3)	Beques de Cooperació Internacional i Desenvolupament
FI	Beques predoctorals per a la formació de personal investigador
INEFC	Beques predoctorals i de col·laboració, dins de l'àmbit de l'educació física i l'esport i les ciències aplicades a l'esport
NANOS	Beques de recerca per a la formació en el camp de les nanotecnologies
PIV	Beques de recerca per a professors i investigadors visitants a Catalunya

Títol del projecte: ha de sintetitzar la temàtica científica del vostre document.

Algebraic properties of isochronous centers of polynomial quadratic-like Hamiltonian systems

Dades de l'investigador o beneficiari

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Número d'expedient
10189

Paraules clau: cal que esmenteu cinc conceptes que defineixin el contingut de la vostra memòria.

Sistema Hamiltonià, centre, funció de període, isòcron, polinomi

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Resum del projecte: Cal adjuntar dos resums del document, l'un en anglès i l'altre en la llengua del document, on s'esmenti la durada de l'acció.

Resum en la llengua del projecte (màxim 300 paraules)

La memòria que adjunto és un resum de la investigació que vaig dur a terme durant la meva estada a la School of Mathematics and Statistics de la University of Plymouth des del 16 d'Abril al 15 de Juliol de 2007. Aquesta investigació és encara oberta i la memòria que presento constitueix un informe de la recerca que estem duent a terme actualment.

En aquesta nota estudiem els centres isòcrons dels sistemes Hamiltonians analítics, parant especial atenció en el cas polinomial. Ens centrem en els anomenats *quadratic-like Hamiltonian systems*, i.e., sistemes d'equacions diferencials al pla de la forma

$$\begin{cases} \dot{x} = -H_y(x,y), \\ \dot{y} = H_x(x,y), \end{cases}$$

amb $H(x,y) = A(x) + B(x)y + C(x)y^2$. Diverses propietats dels centres isòcrons d'aquest tipus de sistemes van ser donades a [A. Cima, F. Mañosas and J. Villadelprat, *Isochronicity for several classes of Hamiltonian systems,* J. Differential Equations **157** (1999) 373–413]. Aquell article estava centrat principalment en el cas en que A, B i C fossin funcions analítiques. El nostre objectiu amb l'estudi que estem duent a terme és investigar el cas en el que aquestes funcions són polinomis. En aquesta nota formulem una conjectura concreta sobre les propietats algebraiques que venen forçades per la isocronia del centre i provem alguns resultats parcials.

Resum en anglès (màxim 300 paraules)

The report that I present herewith is a summary of the research carried out during my stay in the School of Mathematics and Statistics of the University of Plymouth from April 16th to July 15th 2007. The research is still open and the report constitutes a review of this work in progress.

In this note we study isochronous centers of analytic Hamiltonian systems giving a special attention to the polynomial case. We focus on the so-called *quadratic-like Hamiltonian systems*, namely the planar differential systems of the form

$$\begin{cases} \dot{x} = -H_y(x,y), \\ \dot{y} = H_x(x,y), \end{cases}$$

where $H(x,y) = A(x) + B(x)y + C(x)y^2$. Several properties of the isochronous centers of this type of systems were given in [A. Cima, F. Mañosas and J. Villadelprat, *Isochronicity for several classes of Hamiltonian systems,* J. Differential Equations **157** (1999) 373–413]. That paper focus mainly on the case that A, B and C are analytic functions. Our aim with the present study is to investigate the case in which these functions are polynomial. In this note we pose a concrete conjecture about some strong algebraic properties that are forced by the isochronicity of the center and some partial results are proved.

Resum en anglès (màxim 300 paraules) – continuació -.

2.- Memòria del treball (informe científic sense limitació de paraules). Pot incloure altres fitxers de qualsevol mena, no més grans de 10 MB cadascun d'ells.

(Vegeu documentació adjunta.)

Algebraic properties of isochronous centers of polynomial quadratic-like Hamiltonian systems*

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July 21, 2008

Abstract. In this note we study isochronous centers of analytic Hamiltonian systems giving a special attention to the polynomial case. We focus on the so-called *quadratic-like Hamiltonian systems*, namely the planar differential systems of the form

$$\begin{cases} \dot{x} = -H_y(x,y), \\ \dot{y} = H_x(x,y), \end{cases}$$

where $H(x,y) = A(x) + B(x)y + C(x)y^2$. Several properties of the isochronous centers of this type of systems were given in [A. Cima, F. Mañosas and J. Villadelprat, *Isochronicity for several classes of Hamiltonian systems,* J. Differential Equations **157** (1999) 373–413]. That paper focus mainly on the case that A, B and C are analytic functions. Our aim with the present study is to investigate the case in which these functions are polynomial. In this note we pose a concrete conjecture about some strong algebraic properties that are forced by the isochronicity of the center and some partial results are proved.

1 Introduction

In this paper we study isochronous centers of analytic Hamiltonian systems giving a special attention to the polynomial case. The problem of characterizing isochronous centers has attracted the attention of several authors. However there are very few families of polynomial differential systems in which a complete classification of the isochronous centers has been found. Quadratic systems were classified by Loud [6] and cubic systems with homogeneous nonlinearities by Pleshkan [9]. Kukles' systems were classified by Christopher and Devlin [2]. Concerning Hamiltonian systems, i.e., planar differential systems of the form

$$\begin{cases} \dot{x} = -H_y(x,y), \\ \dot{y} = H_x(x,y), \end{cases} \tag{1}$$

there are also very few results. It is well known, see for instance [1], that the only polynomial potential system having an isochronous center is the linear one. Several authors proved independently (see [2, 4, 10])

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*This note is a report of the second author about some of the research carried out during his stay in the School of Mathematics and Statistics of the University of Plymouth working with the first author. This research is still open and the present note constitutes a review of this work in progress.

that there are no Hamiltonian systems with homogeneous nonlinearities having an isochronous center at the origin. The cubic Hamiltonian systems, i.e., differential systems of the form (1) with $H(x, y)$ being a fourth degree polynomial, were classified in [3]. Apart from few other special cases, the knowledge of polynomial systems with isochronous centers is slight. For some other results on isochronicity we refer the reader to [2, 5, 7, 8] and references therein.

For any center p of a planar differential system, the largest punctured neighbourhood of p which is entirely covered by periodic orbits is called the *period annulus* of p and we will denote it by $\mathcal{P}$. A center is said to be *global* when its period annulus is $\mathbb{R}^2 \setminus \{p\}$. The function which associates to each periodic orbit in $\mathcal{P}$ its period is called the *period function.* The center is called a *isochronous* when the period function is constant. By a result that goes back to Poincaré, it is well known that a center of an analytic differential system is isochronous if and only if it is *linearizable*, i.e., there exists an analytic local change of coordinates transforming the initial system to the linear center

$$\begin{cases} \dot{x} = -k\,y, \\ \dot{y} = k\,x. \end{cases}$$

In particular, only non-degenerate centers can be isochronous. The boundary of the period annulus has two connected components; the center itself and a polycycle. It can be shown that this polycycle can only have critical points at infinity in case that the center is isochronous. When the differential system is analytic this implies that the period annulus of an isochronous center is unbounded.

Our aim with the present study is to go further in the knowledge of the isochronicity in the so-called *quadratic-like* Hamiltonian systems, namely the planar differential systems (1) associated to a Hamiltonian of the form

$$H(x, y) = A(x) + B(x)y + C(x)y^2.$$

This kind of Hamiltonian system were previously studied in [3], where the authors obtained several results for the case in which A, B and C are real analytic functions or polynomials. In these notes we shall focus on the latter case. Clearly there is no loss of generality in assuming that the center is at the origin and, since it must be non-degenerate, we can perform a rotation and a constant rescalling of time so that $A(x) = \frac{1}{2}x^2 + \mathrm{o}(x^2)$, $B(x) = \mathrm{o}(x)$ and $C(0) = \frac{1}{2}$. Let us define $G = 4AC - B^2$ and note that then $G(x) = x^2 + \mathrm{o}(x^2)$. Accordingly

$$g(x) := \operatorname{sgn}(x)\sqrt{\frac{G}{C}(x)}$$

is a local diffeomorphism at $x = 0$. With this notation we can now state Theorem C in [3] as follows:

Theorem 1.1. *Consider the Hamiltonian system* (1) *taking* $H(x, y) = A(x) + B(x)y + C(x)y^2$ *with* A, B *and* C *analytic functions on* $\mathbb{R}$. *Then the center is isochronous if and only if*

$$\int_{g^{-1}(-x)}^{g^{-1}(x)} \frac{ds}{\sqrt{C(s)}} = 2x \ \textit{ for all } x \approx 0. \tag{2}$$

As we already mentioned, we are interested on isochronous centers of polynomial systems and, on account of this, we shall first study the above equality in case that G and C are polynomial. With this aim in view we introduce the following definition:

Definition 1.2 We say that (G, C) is a *polynomial isochronous pair* if G and C are polynomials with $C(0) = \frac{1}{2}$ and $G(x) = x^2 + \mathrm{o}(x^2)$ such that (2) holds. $\square$

Since $x \longmapsto \frac{G}{C}(x)$ has a local minimum at $x = 0$, there exists a unique analytic involution σ, defined in neighbourhood of $x = 0$, such that

$$\frac{G}{C}(x) = \frac{G}{C}\big(\sigma(x)\big) \ \text{ for all } x \approx 0.$$

(Recall that an involution is a function $\sigma \neq \mathrm{Id}$ such that $\sigma^2 = \mathrm{Id}$.) In this case it is easy to show that the above involution can be written as $\sigma(x) = g^{-1}\big(-g(x)\big)$. This function can be extremely complicated, even in case that G and C are polynomial. We want to show that in order to study the polynomial isochronous pairs one can reduce, essentially, to the simplest involution, namely, the Möbius transformation

$$\sigma(x) = \frac{-x}{1+\kappa x} \text{ with } \kappa \in \mathbb{R}.$$

More precisely:

Conjecture 1. *If (G, C) is a polynomial isochronous pair then G is a squared polynomial, i.e., $G = G_0^2$ with $G_0 \in \mathbb{R}[x]$, and there exist $h, \ell \in \mathbb{R}[x]$ such that:*

(a) $G_0 = \frac{h(t)}{t'}$ and $C = \frac{\ell(t)}{(t')^2}$ for some $t \in \mathbb{R}[x]$ with $t(x) = x + o(x)$,

(b) (h^2, ℓ) is an isochronous pair and the involution defined by $x \longmapsto \frac{h^2}{\ell}(x)$ is $\sigma(x) = \frac{-x}{1+\kappa x}$ with $\kappa \in \mathbb{R}$.

Remark 1.3 In fact the reverse implication is clear. If $(\bar{G}, \bar{C})$ is an analytic isochronous pair and t is a local diffeomorphism at $x = 0$, then $G = \frac{\bar{G}\circ t}{(t')^2}$ and $C = \frac{\bar{C}\circ t}{(t')^2}$ provides an analytic isochronous pair. Indeed, following the obvious notation, we have that $g = \bar{g} \circ t$ and then it turns out that

$$\int_{g^{-1}(-x)}^{g^{-1}(x)} \frac{ds}{\sqrt{C(s)}} = \int_{(\bar{g}\circ t)^{-1}(-x)}^{(\bar{g}\circ t)^{-1}(x)} \frac{t'(s)ds}{\bar{C}\big(t(s)\big)^{1/2}} = \int_{\bar{g}^{-1}(-x)}^{\bar{g}^{-1}(x)} \frac{du}{\bar{C}(u)^{1/2}} = 2x,$$

where in the second equality we performed the change $u = t(s)$ and in the last one we took the assumption that $(\bar{G}, \bar{C})$ is an isochronous pair into account. □

2 Around the proof of Conjecture 1

In this section we gather some results and ideas to prove Conjecture 1. The first one is the following lemma, that will be used to show that G must be a squared polynomial.

Lemma 2.1. *If F is an algebraic function such that $F'(x) = \sqrt{\chi}(x)$ for some $\chi \in \mathbb{C}(x)$, then there exists $q \in \mathbb{C}(x)$ and $\eta \in \mathbb{C}$ such that $F = q\sqrt{\chi} + \eta$.*

The following result, namely Lüroth's Theorem, will be used to study the generating function of the involution.

Theorem 2.2 (Lüroth)**.** *Let $k(t)$ be a transcendental extension of a field k. Any subfield $K \subset k(t)$, such that $k \subsetneq K$, is of the form $K = k(r)$ for some $r \in k(t)$.*

Idea of the proof of Conjecture 1. Since (G, C) is an isochronous pair, from the derivation of the equality in (2) it follows easily that

$$\frac{(g^{-1})'(x)}{\sqrt{C\big(g^{-1}(x)\big)}} + \frac{(g^{-1})'(-x)}{\sqrt{C\big(g^{-1}(-x)\big)}} = 2 \text{ for all } x \approx 0.$$

For the sake of shortness from now on we shall use the notation $M := \frac{G}{C}$. By definition $g^2 = M$ and thus, using implicit derivation, it follows that $(g^{-1})'(x) = \frac{2\sqrt{M}}{M'}$. Hence the above equality yields to

$$\frac{\sqrt{G}}{CM'}\big(g^{-1}(x)\big) + \frac{\sqrt{G}}{CM'}\big(g^{-1}(-x)\big) = 1 \text{ for all } x \approx 0.$$

Accordingly, taking $\sigma(x) = g^{-1}\big(-g(x)\big)$ into account,

$$\frac{\sqrt{G}}{CM'}(x) + \frac{\sqrt{G}}{CM'}\left(\sigma(x)\right) = 1 \text{ for all } x \approx 0.$$

Recall at this point that $z = \sigma(x)$ is precisely the involution defined by means of $M(x) = M(z)$. Therefore, setting $N := \frac{\sqrt{G}}{CM'}$, we have shown that if (G, C) is an isochronous pair, then there exists an analytic function $z = -x + \mathrm{o}(x)$ such that

$$M(x) = M(z) \text{ and } N(x) + N(z) = 1. \tag{3}$$

Let $\mathscr{F}$ denote the field of all rational functions $r \in \mathbb{C}(x)$ such that $r(x) = r(z)$. Clearly $\mathbb{C} \subset \mathscr{F}$ but, for instance, x does not lie in $\mathscr{F}$ because $z = -x + \mathrm{o}(x)$. Consequently, since M is a non-trivial element of $\mathscr{F}$, we have

$$\mathbb{C} \subsetneq \mathscr{F} \subsetneq \mathbb{C}(x).$$

By Lüroth's Theorem there exists a non-trivial rational function R so that $\mathscr{F} = \mathbb{C}(R)$. Taking $z = -x + \mathrm{o}(x)$ into account once again, it is easy to show that there is no loss of generality in assuming that the generator verifies $R(x) = x^2 + \mathrm{o}(x^2)$. From (3) we have that $R'(x)dx = R'(z)dz$ and $N'(x)dx = -N'(z)dz$, so that

$$\frac{R'}{N'}(x) = -\frac{R'}{N'}(z).$$

On account of $(N')^2 \in \mathbb{C}(x)$, the above equality shows that $\frac{(R')^2}{(N')^2} \in \mathscr{F}$. We can thus assert that there exists $\psi_1 \in \mathbb{C}(x)$ such that $N' = \sqrt{\psi_1(R)}R'$. Hence

$$N(x) - N(0) = \int_0^{R(x)} \sqrt{\psi_1(s)}ds,$$

and this implies that the primitive of $\sqrt{\psi_1}$ is algebraic. Then, by applying Lemma 2.1, there exists $q \in \mathbb{C}(x)$ and $\eta \in \mathbb{C}$ such that, setting $\psi_2 := q^2\psi_1$, we can write

$$N(x) = \sqrt{\psi_2\big(R(x)\big)} + \eta.$$

On the other hand, since $N(x)+N(z) = 1$ with $z = -x+\mathrm{o}(x)$, it turns out that $N(x) = \frac{1}{2}+\alpha x^{2k-1}+\mathrm{o}(x^{2k-1})$ for some $k \in \mathbb{N}$ and $\alpha \neq 0$. This, on account of the above equality, implies that $\psi_2(x) = \Delta x^{2k-1} + \mathrm{o}(x^{2k-1})$ with $\Delta \neq 0$ and $\eta = \frac{1}{2}$. In particular $\psi_2 \circ R = (N - \frac{1}{2})^2 = N^2 + N + \frac{1}{4}$. This shows that N is a rational function because so they are $\psi_2 \circ R$ and N^2. Consequently, taking $N = \frac{\sqrt{G}}{CM'}$ into account, it follows that G is a squared polynomial, i.e., $G = G_0^2$ with $G_0 \in \mathbb{C}[x]$.

Let us fix that the generator is $R = r/s$, where $r, s \in \mathbb{C}[x]$ with $r(x) = x^2 + \mathrm{o}(x^2)$ and $s(0) = 1$. Then, if the degree of the rational function ψ_2 is m, we can write

$$\psi_2\big(R(x)\big) = \prod_{i=1}^{m} \frac{\alpha_i r(x) + \beta_i s(x)}{\alpha_{m+i} r(x) + \beta_{m+i} s(x)}$$

for some $\alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, 2, \ldots, 2m$. Due to $\sqrt{\psi_2 \circ R} \in \mathbb{C}(x)$, it follows that any factor in the above decomposition repeated an odd number of times must be a square. Note that r must be one of those since ψ_2 has order $2k-1$ at $x = 0$. So $r = t^2$ with $t \in \mathbb{C}[x]$ and $t(x) = x + \mathrm{o}(x)$. This implies that at least another factor in the numerator or the denominator should appear an odd number of times. Hence there exists $j \in \{1, 2, \ldots, 2m\}$ such that $\alpha_j r + \beta_j s = u^2$ with $u \in \mathbb{C}[x]$. Note at this point that if R is a generator of $\mathscr{F}$, then so it is $\frac{\alpha R+\beta}{\gamma R+\delta}$ for any constants such that $\alpha\delta-\beta\gamma\neq 0$. Accordingly, choosing $\frac{R}{\alpha_j R+\beta_j}$ as alternative generator of $\mathscr{F}$, and making an abuse of notation, we have that

$$R(x)=\frac{t(x)^2}{u(x)^2}$$

with $u,t\in\mathbb{C}[x]$. Therefore, since ψ_2 has odd order at $x=0$ we have shown that

$$N=\frac{G_0}{CM'}=\frac{1}{2}+\frac{t}{u}\sqrt{\psi(R)}\ \text{ where } \psi\in\mathbb{C}(x). \tag{4}$$

On the other hand, M belongs to $\mathscr{F}$ as well, so it turns out that

$$M=\frac{G_0^2}{C}=\phi(R)\ \text{ where } \phi\in\mathbb{C}(x) \text{ with } \phi(x)=x+\mathrm{o}(x). \tag{5}$$

Following the same ideas as in the previous paragraph one can show that it is possible to choose the generator in such a way that $u=1+\kappa t$ with $\kappa\in\mathbb{C}$. On account of this, the combination of (4) and (5) gives

$$N=\frac{G_0}{CM'}=\frac{M}{G_0M'}=\frac{\phi(R)}{G_0\phi'(R)R'}=\frac{\phi(R)}{G_0\phi'(R)}\frac{ut}{2Rt'},$$

where in the last equality we used that $R=(t/u)^2$ with $u=1+\kappa t$. Taking $N=\frac{1}{2}+\frac{t}{u}\sqrt{\psi(R)}$ into account, from the above equality we obtain

$$G_0=\frac{\phi(R)}{\phi'(R)}\frac{ut}{Rt'}\frac{1}{1+2\frac{t}{u}\sqrt{\psi(R)}}. \tag{6}$$

We claim now that t divides G_0. To show this assume that $t(z_0)=0$ with $z_0\in\mathbb{C}$. Then

$$\frac{\phi\big(R(z_0)\big)}{R(z_0)}=1\ \text{ and }\ \phi'\big(R(z_0)\big)=1$$

because $R(z_0)=t(z_0)^2=0$ and $\phi(x)=x+\mathrm{o}(x)$. Thus, taking $\frac{t}{t'}(z_0)=0$ and $u(z_0)=1+\kappa t(z_0)=1$ also into account and the fact that $\psi(x)$ is well defined at $x=0$, from (6) it follows that $G_0(z_0)=0$. This proves the claim. Now, on account of the claim and using (6) once again, we can assert that

$$G_0=\frac{th(t)}{t'}\ \text{ with } h\in\mathbb{C}[x].$$

Thus, since $G_0^2=C\phi(R)$, we also have that $C=\frac{\ell(t)}{(t')^2}$ with $\ell\in\mathbb{R}[x]$.

Finally let us turn to the assertions in (b). Since $R=\left(\frac{t}{1+\kappa t}\right)^2$ belongs to $\mathscr{F}$, we note that the involution associated to $\frac{G_0^2}{C}(x)=\frac{h^2}{\ell}\big(t(x)\big)$ is given by

$$\frac{t(x)}{1+\kappa t(x)}=\frac{-t(z)}{1+\kappa t(z)}.$$

Thus, the one associated to $\frac{h^2}{\ell}(x)$ is given by

$$\frac{x}{1+\kappa x}=\frac{-z}{1+\kappa z},$$

that is, $z=\frac{-x}{1+2\kappa x}$. ■

References

[1] C. Chicone and M. Jacobs, *Bifurcation of critical periods,* Trans. Amer. Math. Soc. **312** (1989) 433-486.

[2] C.J. Christopher and C.J. Devlin, *Isochronous centres in planar polynomial systems,* SIAM Jour. Math. Anal. **28** (1997) 162-177.

[3] A. Cima, F. Mañosas and J. Villadelprat, *Isochronicity for several classes of Hamiltonian systems,* J. Differential Equations **157** (1999) 373–413.

[4] A. Gasull, A. Guillamon, V. Mañosa and F. Mañosas, *The period function for Hamiltonian systems with homogeneous nonlinearities,* J. Differential Equations **139** (1997) 237-260.

[5] L. Gavrilov, *Isochronicity of plane polynomial Hamiltonian systems,* Nonlinearity **10** (1997) 433-448.

[6] W.S. Loud, *Behaviour of the period of solutions of certain plane autonomous systems near centers,* Contrib. Differential Equations **3** (1964) 21-36.

[7] P. Mardesic, L. Moser-Jauslin and C. Rousseau, *Darboux linearization and isochronous centers with a rational first integral,* J. Differential Equations **134** (1997) 216-268.

[8] P. Mardesic, C. Rousseau and B. Toni, *Linearization of isochronous centers,* J. Differential Equations **121** (1995) 67-108.

[9] I. Pleshkan, *A new method of investigating the isochronicity of a system of two differential equations,* Differential Equations **5** (1969) 796-802.

[10] B. Schuman, *Sur la forme normale de Birkhoff et les centres isochrones,* C.R. Acad. Sci. Paris **322** (1996) 21-24.