On locally finite transitive two-ended digraphs *

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1 Introduction

During these four months visiting the Montanuniversit¨at Leoben, Austria (from May 14-June 15, Sept 26-Dec 20 2006), I have joined the group of Prof. Norbert Seifter in the Department of Applied Mathematics. Our goal was to investigate in detail locally finite transitive (connected) two-ended digraphs, and in particular, highly arc transitive two-ended digraphs.

Since decades transitive graphs are a topic of great interest. The study of s-edge transitive (undirected) graphs goes back to Tutte [13], who showed that finite cubic graphs cannot be s-edge transitive for s> 5. Weiss [14] proved several years later that the only finite connected s-edge transitive graphs with s = 8 are the cycles. Considering directed graphs the situation is much more involved. Praeger [9] gave infinite families of finite s-arc transitive
digraphs for each positive integer s and each degree. And Mansilla and Serra

[6] showed that given an arbitrary regular finite digraph and an arbitrary positive integer s, there are infinitely many s-arc transitive finite digraphs which cover the original digraph.
Knowing the above mentioned results about transitivity in finite graphs and digraphs the following question immediately arises: Do there exist infinite digraphs which are k-arc transitive for some large finite k but not highly arc transitive? In this paper we cannot give a complete answer to this question, but we give a partial answer following previous work of Seifter in [10] and disproving a conjecture of the mentioned work in Section 3.

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2 Basic definitions

A digraph or directed graph D =(V, A) consists of a set V of vertices and a subset A of ordered pairs from V, called arcs. If(x, y) . A is an arc from x to y we say that x is adjacent to y and also that y is adjacent from x. The in

neighbourhood of a vertex x, denoted by N

-D

(x)= {y . V |(y, x) . A}. Furthermore, the in-degree of
(x), is the cardinality of its in-neighbourhood.
-D

(x), is the set of vertices adjacent

to x, that is, N

x, denoted by d

the out-neighbourhood, N+(x), and the out-degree, d+ (x), of a vertex x are

D D

defined. A digraph is regular of degree r or r-regular when all vertices have in-degree and out-degree r.

If not stated otherwise, digraphs considered in this paper are infinite, locally finite and connected, that is, all vertices have both finite in-and outdegrees and there is a walk (not necessarily directed) from each given vertex

to any other one in the digraph.

By Aut D we denote the automorphism group of D. If Aut D acts transi

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Similarly,

tively on V then all vertices have the same in-degree, denoted by d

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D, and the
same out-degree, d+

D. For a positive integer k,a k-arc of a digraph D =(V, A)
is a sequence (x0,...,xk) of k + 1 vertices of D such that, for each 0 = i<k,
(xi,xi+1) is an arc of the digraph and xi = xi+1. A digraph D is k-arc tran
sitive if it has an automorphism group G< Aut D which acts transitively
on k-arcs. A digraph is said to be highly arc transitive if Aut D is k-arc
transitive for all finite k = 0. Usually a digraph is called transitive if Aut D

acts transitively on V.

Let P and Q be two one-way infinite paths (not necessarily directed) in

D. P and Q are equivalent in D if there are infinitely many disjoint paths
in the underlying undirected graph, connecting vertices in P to vertices in
Q. The equivalence classes of all infinite paths with respect to this relation
are called the (undirected) ends of D. The concepts of ends can be defined in
several different ways; this definition is due to Halin [2].
3 An infinite family of sharply 2-arc transitive two-ended digraphs

We disclaim in this section Conjecture 1 posed by Seifter in [10], that stated
that a connected locally finite digraph with more than one end is either 0-,
1- or highly arc transitive.

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In [10] the author gave a partial solution to the conjecture, for regular
digraphs of prime degree with a connected D-cut. In the following, we describe
an infinite family of 2-arc transitive two-ended digraphs, not 3-arc transitive,

of degree 2. Thus, we disprove Seifter’s Conjecture in the general case, even
for prime degree. Nonetheless, the partial solution given by Seifter in [10] is
in some sense best possible and hence the existence of a connected D-cut in
the digraph essential.

Theorem 3.1 Let Dn be the digraph defined by the following vertex-set and
arc-set:

V (Dn)= Zn × Zn × Z

A(Dn)= {((i, j, k), (j, i, k + 1)), ((i, j, k), (j, i +1,k + 1))|(i, j, k) . V
(D)}

for n = 3.

Then, Dn is a connected 2-regular 2-arc transitive digraph, homomorphic
onto Z, which is not 3-arc transitive.

4 Bounds of k in reachability relations of twoended digraphs

Malni.c, Maru.si.c et al. in [5] introduced a new kind of equivalence relations of vertices in a digraph, called reachability relations, that generalises and originates from a problem posed in [1], where the class of the equivalence relation were edges contained in some alternating walk of the digraph . Malni.c et al. in the above paper established close connections between properties of reachability relations and the end structure and growth properties of locally finite transitive digraphs. In our work, we completely characterised reachability relations for locally finite transitive two-ended digraphs in terms of the
decomposition into prime numbers of the number of lines that spanned the two-ended digraph.

Let us introduce some more notation before stating the characterizations of two-ended digraphs of this section.

Given x, y . V, a walk W (x, y) is a sequence (v0,e
1
1 ,v1,...,e
n
n ,vn),n 1,i . {-1, 1}, 1 = i = n, of n + 1 vertices x = v0,...,vn = y and n arcs
e1,...,en where ei =(vi-1,vi) if i = 1 and ei =(vi,vi-1) if i = -1. If the vertices v0,...,vn are pairwise different we call W a path.

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1 n

If W (v0,vn)=(v0,e 1P,v1,...,en ,vn) is a walk in D, we define the height (W) of W as (W)= ni=1 i. If W is a trivial walk consisting of only one

i+1 j

vertex, the we set (W)=0. The subsequence iWj =(vi,e i+1 ,...,e j ,vj), 0 i = j = n, of W is called a subwalk of W.

We say that two vertices x, y . V are k-plus related for some positive k = 1, in symbols xRk
+ y, if there exists a walk W (x, y) of height (W)=0
and (0Wj) . [0,k] for every subwalk 0Wj . W. Analogously we say that two vertices x, y are k-minus related for some positive integer k = 1, in symbols xRk-y, if there exists a walk W (x, y) of height (W) = 0 and (0Wj) . [-k, 0] for every subwalk 0Wj . W. Note that Rk
+ . R+ and Rk-. R-for all k,

k+1 k+1

which means that these relations form two ascending sequences. We set

R+ = [k2Z+ R+
k ,R- = [k2Z+ Rk-.

In this work, we characterize locally finite transitive two-ended graphs in terms of the ascending sequence Rk
+ and furthermore, we are able to completely determine the quotient graphs with respect to the imprimitivity systems induced by Rk
+ for this particular case.

Theorem 4.1 Let D be a locally finite transitive (connected) digraph with two ends, homomorphic to the two-way directed infinite line, spanned by m

lines. Let m = in
=1 pi be the decomposition of m into prime numbers (not necessarily pairwise distinct).

Then, R+ = R+ for k = n +1.

k

Theorem 4.2 Let D be a locally finite transitive (connected) digraph with two ends, homomorphic to the directed infinite path, spanned by m lines. Let

m = in
=1 pi be the decomposition of m into prime numbers (not necessarily pairwise distinct).

Then, R+ = R+ for k = n.

k

Theorem 4.3 Let D be a locally finite transitive (connected) digraph with two ends, homomorphic to the directed infinite path with loops, spanned by

m lines. Let m = ni=1 pi be the decomposition of m into prime numbers (not necessarily pairwise distinct).

Then, R+
k for k = n +1.

= R+

Theorem 4.4 Let D be a locally finite transitive (connected) digraph with two ends, homomorphic to the two-way directed infinite line with loops, spanned

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□
by m lines. Let m = ni=1 pi be the decomposition of m into prime numbers (not necessarily pairwise distinct).
Then, R+ = R+ for k = n +1.

k

5 Open Problems

We enumerate next some open problems we worked on during the stay. Some of the problems are classical problems in the area and some of them turned up during the investigations. Our goal is to make some progress in the problems of the list in the future and to have an updated list of open problems in the area, that might be useful for other researches as well.

1. Let D be connected highly arc-transitive digraph with property Z and f : D Za digraph epimorphism such that the inverse image of □-1(0)
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(and hence of any integer) is finite. In [[1], Conjecture 3.3.] Cameron et al. suggested that the equivalence classes of alternating walks will span subdigraphs that are all isomorphic to a complete bipartite digraph.

We note that R¨ognvaldur M¨oller published a counterexample to this in
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[7], but later Primo.s Sparl found out that the counterexample was not even 1-arc transitive [8], so the Conjecture is still a conjecture.

2. Cameron et al. [[1], Question 1.2.] asked about the existence of locally finite, highly arc transitive digraphs for which the reachability relation is universal.
For a partial solution see [4].

3. Let D be a locally finite highly arc transitive digraph with linear growth. Is D/R1+ highly arc transitive?
4. Let D be a locally finite highly arc transitive digraph with linear

growth. Assume that D/R+
1 is highly arc transitive and that R2+ = R+ .
Do the equivalence classes of alternating walks span complete bipartite subdigraphs?
5. Let D be a locally finite one-ended transitive digraph with exponential growth. Is one of the sequences (Rk
+)
k2Z+ and (Rk-)
k2Z+ infinite? (if
so, then by [[12], Proposition 2.4.] is homomorphic to Z.)
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We remark that in [[5], Theorem 3.2.] a similar result is proven, which states that D with more than two ends and property Z implies at least of the sequences infinite.

6. Let D be a locally finite transitive digraph with at least one of the sequences (R+
k2Z+ and (R-
sition 2.4.] is hom to Z and by [[5], Theorem 3.2.] exponential growth.)

k

k

)

)infinite (and hence by [[12], Propo

k2Z+

Is its out-spread an integer?
In [[7], Theorem 2.] M¨oller proves that the out-spread of a locally finite highly arc-transitive digraph D is an integer. Cameron et al. in [[1], Theorem 3.6.] prove that if the out-spread of a highly arc transitive digraph is 1 then the digraph has property Z (and many other results about the out-spread.)

7. Let D be a digraph with polynomial growth and take a covering map t
k

of a central element without arcs in the orbits. Does R+
imply R+ = R+ in D?

k

= R+ in D/t

8. Let D be a locally finite highly arc transitive digraph with more than two ends with a connected D-cut. Does it always cover a locally finite highly arc transitive digraph with two ends having an induced isomorphic D-cut?
9. Do all locally finite 2-arc transitive digraphs with linear growth have reachability relation universal?
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