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An equivariant Reeb–Beltrami correspondence and the Kepler–Euler flow

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We prove that the correspondence between Reeb and Beltrami vector fields presented in Etnyre & Ghrist (Etnyre, Ghrist 2000 *Nonlinearity* **13**, 441–458 (doi:10.1088/0951-7715/13/2/306)) can be made equivariant whenever additional symmetries of the underlying geometric structures are considered. As a corollary of this correspondence, we show that energy levels above the maximum of the potential energy of mechanical Hamiltonian systems can be viewed as stationary fluid flows, though the metric is not prescribed. In particular, we showcase the emblematic example of the n-body problem and focus on the Kepler problem. We explicitly construct a compatible Riemannian metric that makes the Kepler problem of celestial mechanics a stationary fluid flow (of Beltrami type) on a suitable manifold, the *Kepler–Euler flow*.

1. Introduction

Reeb and Beltrami vector fields are two important classes of vector fields that appear naturally in classical mechanics and hydrodynamics, respectively. In mechanical systems, Reeb vector fields are often obtained by restriction of Hamiltonian vector fields to level-sets of their Hamiltonians. Furthermore, the classical Weinstein conjecture asserts that Reeb vector fields exhibit at least one periodic orbit on closed manifolds.[1] On the other hand, Beltrami vector fields are in a sense (see §2b) the most interesting stationary solutions of the incompressible Euler equations from a dynamical point of view. A more comprehensive introduction to Reeb and Beltrami vector fields is given in §2a and §2b below.

Sullivan envisaged that these vector fields, as well as their associated geometric structures (contact forms for Reeb vector fields and Riemannian metrics for Beltrami vector fields), are closely interconnected [2]. Etnyre & Ghrist formalized this idea in [3], showing that they are in fact related through a simple correspondence.

Theorem 1.1 ([3]). *Let M be a three-dimensional manifold. For any contact form on M and any non-zero rescaling of the associated Reeb vector field, there is a Riemannian metric for which the rescaled Reeb vector field is Beltrami. Conversely, for any Riemannian metric on M and any non-vanishing Beltrami vector field, there is a contact form for which the Beltrami vector field is a rescaling of the Reeb vector field.*

This correspondence has been used in various settings to translate results and techniques between contact geometry and hydrodynamics. On one hand, tools from contact geometry are adequate to construct Reeb vector fields with arbitrary topological complexity, so through the correspondence one obtains results about topological complexity of Beltrami fields for adapted Riemannian metrics. When the metric of the ambient space is fixed, analogous results are extremely hard to prove (e.g. [4]). For instance, Etnyre & Ghrist used this technique to prove that there is a non-vanishing C^ω Beltrami field on $\mathbb{S}^3$ for some adapted Riemannian metric which exhibits periodic flowlines of all possible knot and link types [5]. Following a similar philosophy, the second and the third author of this article used the correspondence to construct a Beltrami field on $\mathbb{S}^3$ for some Riemannian metric, which is Turing complete (i.e. one can perform arbitrary computations by following streamlines of the flow) [6]. On the other hand, analytical techniques on the hydrodynamics side were used to give lower bounds on the number of escape orbits of b-singular Reeb vector fields using a singular version of the correspondence [7,8].

However, in all these references the study of the symmetries of the systems is missing. In this article, we fill this gap and consider the effect of symmetries on both sides of the correspondence. In §3, we extend theorem 1.1 by showing that we can also carry over any symmetries of the underlying geometric structures, that is, the correspondence is equivariant.

Theorem 1.2. *Let M be a* $(2n+1)$*-dimensional manifold and* $\rho : G \times M \to M$ *a compact Lie group action on M. For any non-vanishing* ρ*-invariant Beltrami vector field X with* ρ*-invariant Riemannian metric g, there is a* ρ*-invariant contact form for which X is a rescaling of the corresponding Reeb vector field. Conversely, for each* ρ*-invariant rescaling of a Reeb vector field X with contact form* α*, there is a* ρ*-invariant Riemannian metric for which X is Beltrami.*

This result is particularly relevant when studying physical systems, whose symmetries are key in their treatment. Indeed, at the end of §3, we use theorem 1.2 to obtain the following corollary.

Corollary 1.3. *The flow of any mechanical Hamiltonian system (i.e. a system where the Hamiltonian is the sum of kinetic and potential energies) on an energy level above the maximum value of the potential is a stationary Beltrami solution to the Euler equations on the energy level with some adapted Riemannian metric. Furthermore, symmetries of the Hamiltonian system translate to isometries of the metric.*

This provides countless examples of Beltrami fields with nice symmetries. Indeed, any integrable mechanical Hamiltonian system will yield highly symmetric Beltrami fields. We remark, however, that the metrics that allow the aforementioned Hamiltonian systems to be seen as Beltrami flows are not prescribed *a priori*, and it is generally very hard to understand their curvature.

In §4, we give an explicit example of this equivariant correspondence, showing how it can be used to view the classical Kepler problem of celestial mechanics—the motion of a satellite orbiting a planet—along with its symmetries, as a stationary Beltrami solution to the Euler equations on

[1]This has been proved in dimension 3 [1] and in multiple other scenarios

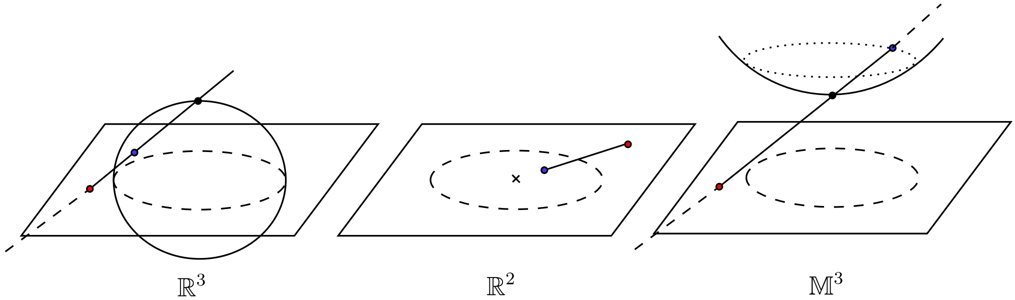

Figure 1. The maps which induce the Riemannian metrics whose geodesic flow is the regularized Kepler flow.

an adapted Riemannian manifold. We call this fluid flow the Kepler–Euler flow. The example is based on the well-known Moser–Osipov–Belbruno regularization of the Kepler problem (see theorem 4.3), which provides a way to rescale time to regularize collisions (i.e. when the satellite collides with the planet at the origin) without otherwise affecting the dynamics of the system. After mapping the flow on energy levels of the Kepler problem to adequate manifolds (shown in figure 1), we can state the result about the Kepler–Euler flow as follows.

Theorem 1.4 (The Kepler–Euler flow). *The regularized Kepler flow on the c-energy level is a stationary Beltrami solution to the Euler equations on*

— $S^*\mathbb{S}^2_c$ *if* $c < 0$,
— $S^*\mathbb{R}^2$ *if* $c = 0$ *and*
— $S^*\mathbb{H}^2_c$ *if* $c > 0$,

where S^* *denotes the cosphere bundle (see definition 4.2). The Riemannian metrics are the lifts to the cosphere bundles of the natural constant* $(-2c)$*-curvature metrics, and the symmetries of the Kepler problem correspond to isometries of the Riemannian metrics.*

Remark 1.5. In the theorem above, the flow lines are lifted geodesics. The Kepler flow on the plane is recovered from the natural stereographic projections of the respective surfaces, or from the involution $x \mapsto 2x/|x|^2$ when $c = 0$, as shown in figure 1. We note that the fact that symmetries of the Kepler problem correspond to symmetries of the respective metrics was already well known since Moser's regularization.

This new perspective on the Kepler problem evokes the metaphoric image of a fluid moving planets and stars through space, but it also raises the more *down to Earth* question, *what systems of classical mechanics are solutions to fluid equations?* We take the example further, noting that we can also reinterpret the n-body problem on positive energies as a stationary fluid flow (by corollary 1.3), though we do not compute the metric explicitly.

Corollary 1.6. *The flow of the n-body problem on a positive energy level is a stationary Beltrami solution to the Euler equations for some adapted Riemannian metric.*

In the next section, we briefly introduce Reeb and Beltrami vector fields and proceed with the proofs of theorems 1.2 and 1.4 in §§3 and 4 below.

2. Basic results

(a) Contact geometry and Reeb vector fields

Recall that a symplectic form on an even-dimensional manifold is a closed, non-degenerate differential 2-form ω, and that a Hamiltonian system on a symplectic manifold (M, ω) with

Hamiltonian function $H \in C^\infty(M, \mathbb{R})$ is given by the vector field X_H such that

$$\iota_{X_H}\omega = \omega(X_H, \cdot) = -\mathrm{d}H.$$

For example, the Kepler problem is a Hamiltonian system on $(T^*(\mathbb{R}^2 \setminus \{0\}), \omega)$, where $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ is the standard symplectic form, and the Hamiltonian function is given by

$$H(q,p) = \frac{|p|^2}{2} - \frac{1}{|q|}. \tag{2.1}$$

One of the fundamental problems when studying a Hamiltonian system is to describe its periodic orbits. For example, for the 3-body problem of celestial mechanics (the motion of three massive bodies interacting through gravitational force), only in very particular cases are periodic orbits known to exist (for a thorough exposition of work on the three-body problem since Poincaré we recommend [9]). Since energy is conserved along the flow of a Hamiltonian system, we can immediately restrict our search for periodic orbits to energy levels of the Hamiltonian to reduce the system by one dimension.

Reeb vector fields come into play when energy levels carry a contact form which is a primitive of the symplectic form.

Definition 2.1 (Contact form). A *contact form* on a $(2n+1)$-dimensional manifold Σ is a differential 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$, or equivalently,

$$T\Sigma = \ker \alpha \oplus \ker d\alpha.$$

The *Reeb vector field* associated with α is the unique vector field R on Σ such that

— $\langle R \rangle = \ker d\alpha$ and
— $\alpha(R) = 1$.

We say that a vector field X is *Reeb-like* if $X = fR$ for some smooth factor $f > 0$.

Remark 2.2. If a contact form on a regular energy level $\Sigma = H^{-1}(c)$ is a primitive of the symplectic form, that is, $d\alpha = \omega_{|\Sigma}$, then the Reeb vector field is parallel to the Hamiltonian vector field X_H on the energy level,

$$\langle R \rangle = \ker d\alpha = \ker \omega_{|\Sigma} = \langle X_H \rangle.$$

Therefore, the Hamiltonian vector field has periodic orbits if and only if the Reeb vector field does. This is relevant because Weinstein's conjecture states that every Reeb vector field on a closed manifold has a closed orbit, so contact geometry and Reeb vector fields can be useful to prove the existence of periodic orbits of Hamiltonian systems. This is precisely what is done in [10] to prove that the regularized circular, planar, restricted three-body problem has periodic orbits on compactified energy levels below the first Lagrange point L_1.

Remark 2.3. When energy levels are not compact, saying something interesting about the dynamics becomes more complicated. One can either resort to results such as Berg–Pasquotto–Vandervorst's [11], or compactify the energy level, somehow extending the dynamics in a physically meaningful way. This last option, especially when compactifying unbounded energy levels with contact forms, often gives rise to singularities in the extended contact form along the added boundary. This is the case, for example, when positive energy levels of the restricted 3-body problem are compactified using a McGehee change of coordinates [8]. Studying singular contact forms and finding an equivalent to Weinstein's conjecture in this setting thus becomes an important question. In this line of research, we have shown that for a generic set within a class of so-called b-contact forms (contact forms having a logarithmic singularity along a hypersurface), one can give lower bounds on the number of escape orbits of the system [7,8].

(b) Hydrodynamics and Beltrami vector fields

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We now move on to a subject at first glance completely unrelated to celestial mechanics: hydrodynamics. The mathematical model of a fluid is that it is composed of *fluid particles*, each with a certain velocity which can be jointly written as a vector field X on a smooth manifold M in which the fluid is contained. We call this vector field X the *velocity field*, and it can depend on time and position on M, so $X = X(t, x)$. Fluid equations are equations that such a velocity field must satisfy, and the simplest such equations are the *Euler equations*. To formulate them, a Riemannian metric g and a distinguished volume form μ are required. For most physical applications, μ is the Riemannian volume form. We assume that the density of the fluid is constant and equal to one. Applying Newton's second law to the trajectory of a fluid particle $x(t)$, we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) = F \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}X(t, x(t)) = \frac{\partial X}{\partial t} + \nabla_X X = F,$$

where ∇_X is the Levi–Civita connection given by g and F is the force acting on the particle. The incompressible Euler equations are obtained assuming that the only force acting on the fluid particles comes from the internal pressures of the fluid, so that if P is the pressure at each point in the fluid, $F = -\nabla P$. Furthermore, it is assumed that the fluid is incompressible, which translates to $\operatorname{div}(X)\mu = \mathcal{L}_X\mu = 0$. In this article, we are concerned with the special case in which the velocity field X is stationary, so $\partial X/\partial t = 0$. The stationary incompressible Euler equations are

$$\left.\begin{aligned} &\nabla_X X = -\nabla P\\ \text{and}\qquad &\mathcal{L}_X\mu = 0. \end{aligned}\right\}$$

It is convenient to reformulate the stationary Euler equations in the language of differential forms, see [12]. To do so, we dualize the equations by the Riemannian metric, thus obtaining

$$\left.\begin{aligned} &\iota_X d\iota_X g = -\mathrm{d}B\\ \text{and}\qquad &\mathcal{L}_X\mu = 0 \end{aligned}\right\}, \tag{2.2}$$

where $B = P + (1/2)||X||^2$ is known as the *Bernoulli function*. The last remaining object to be introduced in this section is the *curl* of a vector field, which in Euclidean $\mathbb{R}^3$ is the usual curl operator, $\operatorname{curl} X = \nabla \times X$. On general Riemannian manifolds and in the language of differential forms, the curl operator is generalized to

Definition 2.4 (Curl operator). Given a $(2n + 1)$-dimensional Riemannian manifold with distinguished volume form (M, g, μ), the *curl* of a vector field X with respect to g and μ is the unique vector field $\operatorname{curl} X$ that satisfies

$$\iota_{\operatorname{curl} X}\mu = (d\iota_X g)^n.$$

When X is a velocity field, $\operatorname{curl} X$ is commonly known as the *vorticity field* of the flow. A particularly relevant class of stationary solutions to the Euler equations are velocity fields that (in addition to being divergence-free) are parallel to their vorticity field: $\operatorname{curl} X = fX$. In this case, the Bernoulli function is constant $B = c$.

Definition 2.5 (Beltrami vector field). A divergence-free (with respect to the volume form μ) vector field X on (M, g, μ) is a *Beltrami vector field* if $\operatorname{curl} X = fX$. We call a Beltrami field *non-singular* if neither it nor f vanish at any point.

The following proposition is standard and shows that there is a natural contact form associated with a Beltrami field. We expand on this observation in the next section.

Proposition 2.6. *A divergence-free non-singular Beltrami vector field X on (M, g, μ) is a solution to the stationary Euler equations. Furthermore, $\iota_X g$ is a contact form on M.*

Proof. Since X is Beltrami, noticing that $\mu = h\mu_g$ for some factor $h > 0$ (μ_g is the Riemannian volume), we have

$$\iota_X g \wedge (d\iota_X g)^n = f\iota_X g \wedge \iota_X \mu = fh\iota_X g \wedge \iota_X \mu_g = fh||X||^2 \mu_g \neq 0.$$

This means that $\iota_X g$ is a contact form. Moreover,

$$\iota_X (d\iota_X g)^n = \iota_X \iota_{\mathrm{curl}\, X} \mu = \iota_X \iota_{fX} \mu = 0, \tag{2.3}$$

and since $\iota_X g$ is a contact form, it is obvious that X is in the kernel of $d\iota_X g$, i.e. $\iota_X d\iota_X g = 0$. This shows that X is a solution to the stationary Euler equations (2.2) on (M, g, μ) with constant Bernoulli function. ■

Remark 2.7. Assume that $\mathrm{curl}\, X = fX$ for some non-vanishing factor f and volume form μ, and we do not assume that X preserves the volume form μ. Then

$$\mathcal{L}_X \mu = \mathrm{d}\iota_X \mu = \mathrm{d}\left(\frac{1}{f}\iota_{\mathrm{curl}\, X}\mu\right) = \mathrm{d}\left(\frac{1}{f}(\mathrm{d}\iota_X g)^n\right) = \mathrm{d}\frac{1}{f} \wedge (\mathrm{d}\iota_X g)^n, \tag{2.4}$$

which is not necessarily zero. However, since the proportionality function f is non-vanishing, it is obvious that X preserves the volume form $\tilde{\mu} = f\mu$. Additionally, if we denote by $\widetilde{\mathrm{curl}}$ the curl operator computed with the volume form $\tilde{\mu}$, it is elementary to check that

$$\widetilde{\mathrm{curl}} X = X.$$

Accordingly, X is a Beltrami field with constant proportionality factor for the volume form $\tilde{\mu}$.

For a much more comprehensive introduction to stationary Euler flows, we recommend [4,12,13].

3. The equivariant correspondence

In this section, we show that Etnyre and Ghrist's correspondence between Reeb and Beltrami vector fields of theorem 1.1 can be made to preserve symmetries of the metrics or contact forms.

Theorem 3.1 (The equivariant correspondence). *Let M be a $(2n+1)$-dimensional smooth manifold and $\rho : G \times M \to M$ a compact Lie group action on M. For each ρ-invariant non-singular Beltrami field (X, g), there is a ρ-invariant contact form for which X is Reeb-like. Conversely, for each ρ-invariant Reeb-like field (X, α), there is a ρ-invariant Riemannian metric for which X is non-singular Beltrami.*

When we refer to Reeb and Beltrami vector fields, we refer to the vector field *and the subjacent geometric structure*, so that when we say that (X, g) is ρ-invariant, for example, we mean that $\rho_{\sigma *} X = X$ and $\rho_\sigma^* g = g$ for every $\sigma \in G$.

Before continuing with the proof, we recall the definitions of a couple of objects that will be used therein. The *Haar measure* on a compact Lie group G is the unique measure η on G such that η is left invariant and $\eta(G) = 1$.

Remark 3.2. Given a compact Lie group action $\rho : G \times M \to M$, it is standard that the Haar measure provides a way to *average* tensor fields on M to make them invariant by the action. Indeed, if T is a tensor field on M, the tensor field defined pointwise as

$$\tilde{T}_p = \int_{\sigma \in G} (\rho_\sigma^* T)_p \, \mathrm{d}\eta,$$

is a ρ-invariant tensor field. The integral is taken over all $\sigma \in G$ and with respect to the Haar measure.

In particular, we will use this averaging procedure for Riemannian metrics. Since the space of positive definite symmetric bilinear forms is convex, it is well known that averaging a Riemannian metric by the procedure above yields a Riemannian metric that is invariant by the group action.

Recall, also, that an almost complex structure on a vector bundle $E \to M$ over a smooth manifold M is a section $J : M \to End(E)$ such that $J^2 = -Id$. Almost complex structures establish a deep connection between complex and symplectic geometry. However, the only property we will use is that they provide a way to construct Riemannian metrics from non-degenerate 2-forms: for every non-degenerate differential 2-form ω on a vector bundle $E \to M$, there is an almost complex structure J such that $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is a positive definite, symmetric bilinear form on E (see [14]).

Proof. Let (X, g) be a ρ-invariant non-singular Beltrami field. The construction of a ρ-invariant contact form is the one in the proof of proposition 2.6. Invariance of X and α are automatic because X was already taken to be invariant and α is a contraction of the invariant metric g with an invariant vector field. Let us now prove the equivariance of the converse direction.

Let (X, α) be a ρ-invariant Reeb-like field, with $X = hR$, where R is the Reeb field with respect to α and $h > 0$. Since X is ρ-invariant, h must also be so, as R is invariant by the invariance of α. Using the splitting of TM into $\ker \alpha \oplus \ker \mathrm{d}\alpha$, we can define a Riemannian metric on each component and impose mutual orthogonality to naturally construct a Riemannian metric following [15]. Consider

$$\tilde{g} = \frac{1}{h}\alpha \otimes \alpha + \mathrm{d}\alpha(\cdot, J\cdot),$$

where J is an almost complex structure on the vector bundle $\ker \alpha \to M$ (the complex structure) making $\mathrm{d}\alpha(\cdot, J\cdot)$ a Riemannian metric on this bundle. Note that in general $\tilde{g}$ is not ρ-invariant because J is not. Nevertheless, we can average this metric using the Haar measure on G to obtain a ρ-invariant metric

$$g = \int_G \rho^*(\tilde{g})\,\mathrm{d}\eta = \int_G \rho^*\left(\frac{1}{h}\alpha \otimes \alpha\right)\mathrm{d}\eta + \int_G \rho^*(\mathrm{d}\alpha(\cdot, J\cdot))\,\mathrm{d}\eta = \frac{1}{h}\alpha \otimes \alpha + \int_G \rho^*(\mathrm{d}\alpha(\cdot, J\cdot))\,\mathrm{d}\eta,$$

where we have used that the first summand is ρ-invariant.

We claim that X is Beltrami with respect to g. Indeed, the contraction of X with the second summand of the metric vanishes, since $X \in \ker d\alpha$, and therefore

$$\iota_X g = \frac{1}{h}\alpha(hR)\alpha = \frac{h}{h}\alpha(R)\alpha = \alpha.$$

Taking the volume form $\mu = \frac{1}{h}\alpha \wedge (\mathrm{d}\alpha)^n \neq 0$, we finally arrive at

$$(\mathrm{d}\iota_X g)^n = (\mathrm{d}\alpha)^n = \iota_X \mu,$$

and therefore $\operatorname{curl} X = X$. Also note that for the chosen μ,

$$\mathcal{L}_X \mu = \mathrm{d}\iota_X \mu = \mathrm{d}(\mathrm{d}\alpha)^n = 0,$$

so X is also divergence-free and therefore X is Beltrami with respect to (g, μ). This completes the proof of the theorem. ■

An interesting consequence of the equivariant Reeb–Beltrami correspondence is that it allows us to view many physical systems as Beltrami fluid flows with all the symmetries of the original system. In particular, the study of integrable systems—Hamiltonian systems with a high degree of symmetry—is central to mechanics, and countless interesting examples have been considered in the literature. In the rest of this section, we highlight a couple of families of examples which, though not known for their symmetries, we consider to be particularly nice.

(a) Mechanical Hamiltonian systems as fluid flows and the *n*-body problem

The n-body problem of celestial mechanics is the problem of determining the dynamics of a system of n bodies in $\mathbb{R}^d$ moving according to classical laws of gravitation. This system can be expressed in the Hamiltonian formalism as follows. Take canonical coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ on $T^*\mathbb{R}^{dn}$ with the canonical symplectic form $\omega = dp \wedge dq$, where q_i

represents the position of the ith body and p_i its momentum. Denoting by m_i the mass of the ith body, the Hamiltonian describing the system is

$$H(q,p) = K(p) + U(q), \tag{3.1}$$

with

$$K(p) = \sum_{1\leq i\leq n} \frac{|p_i|^2}{2m_i},$$

the kinetic energy and

$$U(q) = -\sum_{1\leq i<j\leq n} \frac{m_i m_j}{|q_i - q_j|},$$

the potential energy. The Hamiltonian is an example of a *mechanical Hamiltonian*. Mechanical Hamiltonians (also known as *natural Hamiltonians*) are Hamiltonians consisting of a sum of kinetic and potential energies depending only on momenta and positions, respectively.

Lemma 3.3 (Lemma 2.6.3 and remark 2.6.5 of [16]). *Let M be a smooth manifold and $H = K(p) + U(q)$ a mechanical Hamiltonian on T^*M, where $K(p) = |p|_g^2/2$ for some Riemannian metric g on M. Then for $c > \max U$, the Hamiltonian vector field X_H restricted to the energy level $\Sigma = H^{-1}(c)$ is Reeb-like with respect to the canonical Liouville form $\alpha = pdq$ restricted to Σ.*

Proof. As in the discussion of §2a, on one hand, we have

$$\omega|_\Sigma = \mathrm{d}\alpha|_\Sigma \Rightarrow \langle X_H\rangle = \ker\,\mathrm{d}\alpha|_\Sigma.$$

Furthermore, α is contact on Σ because $\alpha(X_H) = K(p) > 0$ on Σ, so

$$\ker d\alpha|_\Sigma \cap \ker\alpha|_\Sigma = 0,$$

which yields the contact condition. ■

An immediate consequence of the above lemma and theorem 3.1 is the following reinterpretation of the n-body problem as a stationary incompressible fluid flow on some Riemannian manifold.

Corollary 3.4. *The flow of the n-body problem on a positive energy level is a stationary Beltrami solution to the Euler equations on a hypersurface of $T^*\mathbb{R}^{dn}$ with some Riemannian metric. Any symmetries of the Hamiltonian field correspond to isometries of the corresponding metric.*

In §4, we describe in detail the particular case of the 2-body problem, which can be reduced to the *Kepler problem*. This system is integrable and we interpret the system's symmetries explicitly on the Beltrami side of the correspondence.

(b) Magnetic Hamiltonians and magnetic geodesic flows

Magnetic Hamiltonian functions are a more general class than mechanical Hamiltonians, and are of the form

$$H(q,p) = K(p) + L(q,p) + U(q), \tag{3.2}$$

where K and U are defined as above, and L (often referred to as the *magnetic terms*) is linear in momenta. As the name suggests, they are used to model the motion of charged particles in a magnetic field, or otherwise under the influence of velocity-dependent forces.

An interesting example of a magnetic Hamiltonian, again from celestial mechanics, is given by the *circular restricted three-body problem*. This is the problem of determining the motion of a (virtually) massless object (say a satellite) near two massive bodies (like the Earth and the Moon). Indeed, let $q = (q_1, q_2)$ and $p = (p_1, p_2)$ denote the positions and momenta of the satellite.

Normalizing the masses of the massive bodies to μ and $1-\mu$ with $\mu \in (0,1)$, under a non-inertial change of coordinates the Hamiltonian of this system is given by

$$H(q,p) = \frac{1}{2}|p|^2 + q_1p_2 - q_2p_1 - \frac{\mu}{|q-(1-\mu,0)|} - \frac{1-\mu}{|q-(-\mu,0)|}.$$

The magnetic terms $L(q,p) = q_1p_2 - q_2p_1$ are what gives rise to the *Coriolis force* in this non-inertial reference frame (see [10], §4). In this case, however, we cannot apply lemma 3.3 to easily conclude that the dynamics of the system on given energy levels are generated by a contact form. In fact, it is a highly non-trivial matter to establish when this is true for magnetic Hamiltonian systems. Nevertheless, in [10], the authors show that certain energy levels of the circular restricted three-body problem do carry contact forms that generate the dynamics, giving rise to an interpretation of this system as a Beltrami flow through theorem 3.1.

Other examples of magnetic systems that have garnered a lot of attention are magnetic flows on *homogeneous spaces* (see [17] and references therein). Magnetic flows on these spaces are more naturally formulated by incorporating the magnetic terms of equation (3.2) directly into the symplectic form on $T^*(G/H)$ by 'twisting' it with a closed two-form. More specifically, given a closed two-form Ω on G/H, the symplectic form on $T^*(G/H)$ twisted by Ω is

$$\omega = \omega_0 + \pi^*\Omega,$$

where ω_0 is the natural symplectic form on $T^*(G/H)$, and $\pi : T^*(G/H) \to G/H$ is the natural projection. With this twisted symplectic form, mechanical Hamiltonian functions exhibit the same dynamical properties as magnetic Hamiltonian functions with the natural symplectic form. These systems are particularly interesting because they naturally exhibit many symmetries. In their work [17], the authors prove integrability of the magnetic geodesic flow of the normal metrics for specific classes of homogeneous spaces including the coadjoint orbits. Integrable systems in the sense of Liouville have associated a semilocal torus action in the neighbourhood of their compact fibres (Liouville tori). In the non-commutative integrable case, the associated Lie groups can be more complex. Whenever the energy levels are contact, these symmetries will naturally be transferred to symmetries of the associated fluid flows models by considering the corresponding Lie group. An object of further study is to look at what the highly symmetric metrics look like after applying theorem 3.1. Nevertheless, determining when these systems are governed by contact dynamics is generally challenging.

More generally, a natural question in view of these examples is if the metric on an energy level which makes the Hamiltonian flow Beltrami can be characterized in any way. A natural guess for such a metric may be the *canonical lift* of the base metric to the cotangent bundle (described in the next section), restricted to the energy level. In the next section, we show that this is the case when the Hamiltonian is purely kinetic energy, but it is not true in general. As mentioned above, in the next section, we also present an explicit example of corollary 3.4, namely we obtain an explicit formulation of the Kepler problem as an Euler flow.

4. The Kepler–Euler flow

The 2-body problem can be reduced to the Kepler problem, which is that of describing the dynamics of a system comprised of a large body, which we call the *star* and assume centred at the origin of the plane $\mathbb{R}^2$, and a body of comparatively negligible mass orbiting around the first, which we call the *planet*. In particular, since the planet exerts negligible force on the star, the star remains stationary and therefore we need only to describe the motion of the planet. Taking canonical coordinates $(q,p) = (q_1,q_2,p_1,p_2)$ on $(T^*\mathbb{R}^2, \omega = p_1 \wedge q_1 + p_2 \wedge q_2)$, the Hamiltonian describing the system (ignoring masses and other constants) is

$$H : T^*(\mathbb{R}^2 \setminus \{0\}) \to \mathbb{R}, \quad (q,p) \mapsto \frac{|p|^2}{2} - \frac{1}{|q|}.$$

By corollary 3.4, the Kepler problem on a positive energy level is a solution to the Euler equations for some Riemannian metric. Our goal is to find such a metric. We recall that a metric for which a Reeb-like vector field is Beltrami is called an *adapted metric* to the Reeb-like field. Adapted metrics were introduced by Chern & Hamilton in [18], though, as mentioned in the introduction, it was Sullivan who first understood the connection with Beltrami fields in hydrodynamics.

(a) Regularizing the problem

When the planet is moving directly towards the star, a *collision* occurs and the planet's velocity blows up in finite time. The flow is therefore not complete and it is advantageous to reparametrize time to fix this issue. Since we are mainly interested in the qualitative behaviour of trajectories, reparametrizing time is not a problem and we will consider two orbits to be identical if they are so up to reparametrization. This process is known as *regularization* of the system and it is standard in dynamical systems. To regularize the Kepler problem, we follow Arnold in [19]. We include a proof for the sake of completeness.

Lemma 4.1. *Let $\gamma(t)$ be an integral curve of energy c of a Hamiltonian H. If we reparametrize time by $\tau \mapsto t(\tau)$ where $dt/d\tau = G(x)$, then $\gamma(\tau) = \gamma(t(\tau))$ is an integral curve of energy 0 for the Hamiltonian $\bar{H} = G(H - c)$. If we take $G = (H + k)$ for a constant k, we can take $\bar{H} = (1/2)(H + k)^2$ and $\gamma(\tau)$ will have energy $(c + k)^2/2$ instead.*

Proof. We must check that $(\mathrm{d}\gamma/\mathrm{d}\tau)(\tau) = X_{\bar{H}}(\gamma(\tau))$. Applying the chain rule, we have

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}(\tau) = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = X_H G.$$

Now,

$$\omega\left(\frac{\mathrm{d}\gamma}{\mathrm{d}\tau}(\tau), \cdot\right) = G\omega(X_H, \cdot) = -G\mathrm{d}H = -(G\mathrm{d}H + (H - c)\mathrm{d}G) = -\mathrm{d}(G(H - c)) = -\mathrm{d}\bar{H},$$

where we have used that $(H - c) = 0$ along γ. From this, we see that $(\mathrm{d}\gamma/\mathrm{d}\tau)(\tau) = X_{\bar{H}}(\gamma(\tau))$. If $G = (H + k)$, we have

$$-G\mathrm{d}H = -(H + k)\,\mathrm{d}H = -\mathrm{d}\left(\frac{1}{2}(H + k)^2\right) = -\mathrm{d}\bar{H},$$

as desired. ■

With the help of this lemma we reparametrize time, slowing it down as the planet approaches the star taking $G = |q|$. On an energy level c, the regularized Kepler–Hamiltonian becomes

$$\bar{H}(q, p) = G(H - c) = |q|\left(\frac{|p|^2}{2} - c\right) - 1.$$

For reasons that will shortly become clear, it is convenient to apply the lemma again with $G = (\bar{H} + 1)$ to obtain another equivalent Hamiltonian

$$K_c(q, p) = \frac{|q|^2}{2}\left(\frac{|p|^2 - 2c}{2}\right)^2,$$

with integral curves on the energy level $K_c = 1/2$ identical to integral curves of the original Hamiltonian on the c-energy level up to reparametrization. We henceforth refer to the flow of K as the *regularized Kepler flow* on the c-energy level, omitting c when it does not lead to confusion. This Hamiltonian is particularly interesting because after a symplectic coordinate change called

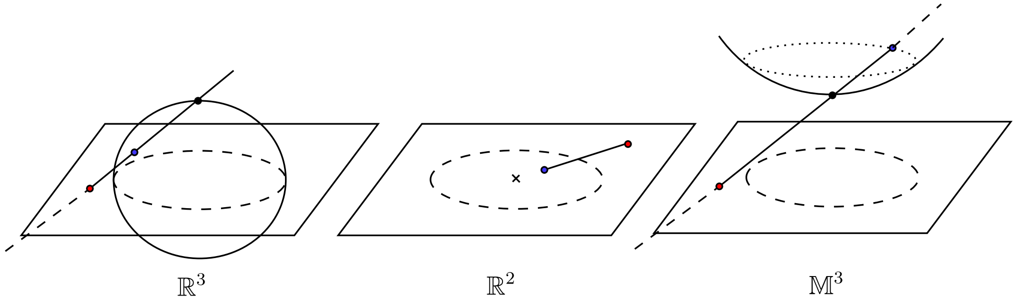

Figure 2. The maps that induce the metrics whose lifted geodesic flows are the regularized Kepler flows on the respective energy levels. The sphere corresponds to negative energy levels and has radius $1/\sqrt{2|c|}$. The plane corresponds to the null energy level and the map is given by $x \mapsto 2x/|x|^2$. The hyperboloid in Minkowski space $\mathbb{M}^3$ with signature $++-$ corresponds to positive energy levels and is defined by the equation $x^2+y^2-z^2=-2c$.

symplectic switch given by $(x, y) \mapsto (q, p) = (y, -x)$, the Hamiltonian becomes

$$K(x, y) = \frac{|y|^2}{2}\left(\frac{|x|^2-2c}{2}\right)^2 = \frac{|y|^2_{g^*}}{2}, \tag{4.1}$$

where g^* is the inverse of the conformally flat metric $g = (2/(|x|^2-2c))^2\langle\cdot,\cdot\rangle_{euc}$. The physical interpretation is that K consists only of kinetic energy. Such Hamiltonians are called *kinetic* and they are of particular interest because their trajectories are lifted geodesics of the metric [16]. Physically, since there is no potential, there is no external force acting on the system. It is easy to check that g is a constant $-2c$ curvature metric, and that the *stereographic projections* shown in the following figure are the maps that induce these metrics on $\mathbb{R}^2$ (this is shown, for example, in propositions 8.30 and 8.38 of [20]). When $c<0$ the variable x takes values on the whole $\mathbb{R}^2$, when $c=0$ then $x \in \mathbb{R}^2\backslash\{0\}$, and for $c>0$ the configuration space is $\{x \in \mathbb{R}^2 : |x|^2 > 2c\}$.

The energy levels $K = 1/2$ are more precisely the *cosphere bundles* of these surfaces.

Definition 4.2. The *cosphere bundle* of a Riemannian manifold (M, g) is defined as

$$S^*M = \{\alpha \in T^*M \,|\, |\alpha|_{g^*} = 1\} \subseteq T^*M.$$

Putting these last remarks together, we reach a formulation of the well-known *Moser–Osipov–Belbruno regularization*. It was Moser who first gave a regularization of this sort for negative energy levels in [21]. Osipov and Belbruno gave the corresponding positive energy regularization in [22,23], respectively. The regularizations can be condensed into one theorem covering all energy levels.

Theorem 4.3 (Moser–Osipov–Belbruno). *The dynamics of the Kepler problem on the energy level $H=c$ are equivalent to the lifted geodesic flow on*

— $S^*\mathbb{S}^2_c$ *for* $c<0$,
— $S^*\mathbb{R}^2$ *for* $c=0$ *and*
— $S^*\mathbb{H}^2_c$ *for* $c>0$

after applying the stereographic maps described in figure 2.

For an elementary geometric proof of this result, we recommend Geiges' exposition [20].

(b) The contact form and adapted metric

We have already seen in lemma 3.3 that the contact form for which the regularized Kepler flow is Reeb-like on an energy level is the Liouville form $\alpha = y dx$ restricted to the energy level. If we want the contact form on the original phase space, we simply pull back the Liouville form by the maps that induce the metrics and then by the symplectic switch. However, we continue the exposition in the context of constant curvature surfaces, as this results in more elegant and concise formulations of the statements.

The following proposition is well known and gives an equivalent characterization of adapted metrics to Reeb vector fields. We will use it to obtain the adapted metric to the Kepler–Reeb vector field explicitly.

Proposition 4.4. *Let (α, R) be a Reeb field. A Riemannian metric g such that R is g-orthogonal to* $\ker\alpha$ *and $|R|_g^2$ is constant is an adapted metric to (α, R).*

Proof. Since $\ker \iota_R g = \ker\alpha$, necessarily $\iota_R g = h\alpha$ for some $h \neq 0$. Furthermore, $|R|_g^2 = h\alpha(R) = h$ is constant by hypothesis, say $h = 1$. Taking $\mu = \alpha \wedge (d\alpha)^n$, which is a volume form preserved by R, we obtain

$$\iota_R\mu = (\mathrm{d}\alpha)^n = (\mathrm{d}\iota_R g)^n,$$

so that curl $R = R$, as we wanted to prove. ∎

Theorem 4.5. *Let (M, g) be a Riemannian manifold. The Reeb field on S^*M with respect to the canonical Liouville form α is orthogonal to* $\ker\alpha$ *and of constant magnitude with respect to the canonical lift S^*g of g to the cosphere bundle.*

Before the proof of this theorem, we recall how to construct S^*g. We begin by emphasizing that g is a tensor field that takes vectors of TM, while S^*g is a tensor field that takes vectors of $T(S^*M)$. We construct S^*g by first constructing the cotangent lift T^*g on T^*M and then restricting it to S^*M. The cotangent lift was introduced by Mok in [24] following Sasaki's construction of the tangent lift metric in [25].

The tangent bundle of the cotangent bundle $T^*M \xrightarrow{\pi} M$ splits on each $\xi \in T^*M$ into the *vertical tangent space* $V_\xi = \ker T_\xi\pi$ and a *horizontal tangent space* H_ξ, which without additional structure cannot be chosen canonically. It is only required that $T_\xi(T^*M) = V_\xi \oplus H_\xi$. In fact, a smooth choice of horizontal tangent spaces is equivalent to a choice of a connection on M. Precisely for this reason, the metric g provides a canonical choice of H through the Levi–Civita connection,

$$H_\xi = \{(\dot\gamma, \dot\theta) \in T_\xi(T^*M) | (\gamma(t), \theta(t)) \subseteq T^*M \text{ and } \nabla_{\dot\gamma}\theta = 0\}.$$

It is easy to check that indeed $T_\xi(T^*M) = V_\xi \oplus H_\xi$ and that H_ξ is naturally isomorphic to $T_{\pi(\xi)}M$, since H_ξ is identified with the space of geodesics on M passing through $\pi(\xi)$. On the other hand, $V_\xi \cong T^*_{\pi(\xi)}M$, because if we take paths of the form $(\pi(\xi), t\theta)$, their tangent fields are in V_ξ and they are identified with $\theta \in T^*_{\pi(\xi)}M$. Figure 3 illustrates this splitting.

With these natural identifications, we finally obtain

$$T_\xi(T^*M) \cong T^*_{\pi(\xi)}M \oplus T_{\pi(\xi)}M,$$

and since there are natural inner products on each component, given by g^* and g respectively, we can define an inner product on $T_\xi(T^*M)$ by imposing that the components are orthogonal to each other.

A coordinate expression for the resulting metric is the following. If $V = (\dot\alpha, \dot\theta)$ and $W = (\dot\beta, \dot\omega)$,

$$T^*g(V, W) = g(T\pi(V), T\pi(W)) + g^*(\nabla_{\dot\alpha}\theta, \nabla_{\dot\beta}\omega).$$

The first component projects onto $H_\xi \cong T_{\pi(\xi)}M$ and the second onto $V_\xi \cong T^*_{\pi(\xi)}M$.

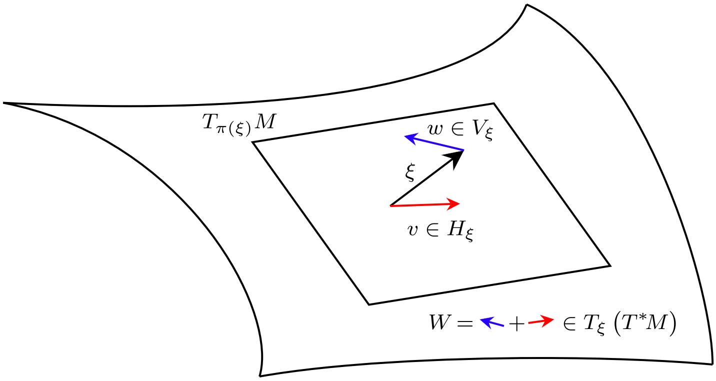

Figure 3. Splitting of $W = (v, w) \in T_\xi(T^*M)$ in components v of a horizontal tangent space and w of the vertical tangent space. The horizontal component v can be thought of as moving the base point $\pi(\xi)$ while the vertical component w moves the covector ξ within the cotangent space at $\pi(\xi)$.

Proof. We first check that R is orthogonal to $\ker\alpha$ with respect to S^*g, which, again, is T^*g restricted to $T(S^*M)$. Taking natural coordinates (x, y), we have

$$K(x,y) = \frac{1}{2}g^{ij}(x)y_iy_j \Rightarrow \mathrm{d}K(x,y) = \frac{1}{2}g^{ij}_{,k}y_iy_j\,\mathrm{d}x^k + g^{ij}y_i\,\mathrm{d}y_j,$$

where we are using Einstein notation to avoid notational clutter. In this case, the Reeb vector field is the Hamiltonian vector field, so it has the following expression,

$$R = X_K = \frac{1}{2}g^{ij}(x)_{,k}y_iy_j\frac{\partial}{\partial y_k} - g^{ij}y_i\frac{\partial}{\partial x^j}.$$

On the other hand, a vector field $Y \in \ker\alpha$ is necessarily of the form $Y = \alpha^i(\partial/\partial x^i) + \beta_j(\partial/\partial y_j)$ with

$$\alpha(Y) = y_k\,\mathrm{d}x^k(\alpha^i\frac{\partial}{\partial x^i} + \beta_j\frac{\partial}{\partial y_j}) = y_i\alpha^i = 0.$$

Thus, since the projection of R onto the horizontal tangent spaces vanishes for being a geodesic vector field, the inner product of R and Y by S^*g is

$$S^*g(R, Y) = g(T\pi(R), T\pi(Y)) = g\left(-g^{ij}y_i\frac{\partial}{\partial x^j}, \alpha^i\frac{\partial}{\partial x^i}\right) = -g_{kl}g^{ik}y_i\alpha^l = -y_i\alpha^i = 0.$$

Therefore, R is orthogonal to $\ker\alpha$. Lastly, for the constant magnitude condition, we see that

$$S^*g(R,R) = g\left(-g^{ij}y_i\frac{\partial}{\partial x^j}, -g^{ij}y_i\frac{\partial}{\partial x^j}\right) = g_{kl}g^{ik}y_ig^{jl}y_j = g^{ij}y_iy_j = 1$$

for being always on S^*M. ■

Combining proposition 4.4 and theorem 4.5, we finally get an interpretation of the Kepler problem as a stationary Beltrami solution to the Euler equations.

Corollary 4.6 (The Kepler–Euler flow). *The regularized Kepler flow on the c-energy level is a stationary Beltrami solution to the Euler equations on*

— $S^*\mathbb{S}^2_c$ *if* $c < 0$,
— $S^*\mathbb{R}^2$ *if* $c = 0$ *and*
— $S^*\mathbb{H}^2_c$ *if* $c > 0$

with the liftings to the cosphere bundles of the respective constant $(-2c)$-curvature metrics. The flow lines are lifted geodesics. The Kepler flow on the plane is recovered from the natural stereographic projections of the respective surfaces, or from the involution $x \mapsto 2x/|x|^2$ when $c=0$.

5. Further remarks on the Kepler–Euler flow

We conclude with a more detailed description of the lifted metrics and dynamics of the Kepler–Euler flow, relating them to other known Beltrami fields when possible.

(a) The metrics

The coordinate expressions for the metrics lifted to the cosphere bundles are rather long and messy, so we do not give them here. However, the spherical tangent lift of the round sphere metric was studied by Klingenberg & Sasaki [26], and the lifts of general metrics on surfaces were studied by Nagy [27]. Since the tangent and cotangent bundles (with the lifted metrics) are isometric [24], the results therein also apply to our case. In the latter article, Nagy showed that the spherical tangent bundle of a surface is of constant curvature only when the base surface is of constant curvature 0 or 1. In these cases, the curvature of the spherical tangent bundles are 0 and 1/4, respectively. In particular, the Kepler–Euler flow on energy level c takes place on a constant curvature 3-manifold exactly when $c=0$ or $c=-(1/2)$.

Sasaki further classified geodesics on the spherical tangent bundles of space forms into *horizontal*, *vertical* and *oblique* geodesics [28]. Horizontal geodesics are geodesics in which the unit tangent vector is parallel transported along the projection of the geodesic to the base manifold; vertical geodesics are rotations along the fibres of the sphere bundle, the base point remaining stationary; oblique geodesics have some combination of motion through the base and fibres. Since the Kepler–Euler flow is the lift to the cosphere bundle of base geodesics, all of the trajectories are horizontal geodesics according to Sasaki's classification.

(b) The Beltrami vector fields

First, we give the Hamiltonian vector field for the regularized Kepler Hamiltonian given in equation (4.1). In those coordinates, which are the stereographic coordinates (x_1, x_2, α) on $S_c^*\mathbb{R}^2$, where the cosphere bundle is taken with respect to the metric corresponding to the c-energy level, the vector field is expressed as

$$X_c = \frac{|x|^2-2c}{2}(\cos\alpha\,\partial_{x_1} + \sin\alpha\,\partial_{x_2}) + (x_1\sin\alpha - x_2\sin\alpha)\partial_\alpha.$$

A straightforward computation shows that for all c, these Beltrami fields have eigenvalue 1, that is, $X_c = \mathrm{curl}_c\, X_c$, where curl_c is the curl with respect to the lift of the corresponding constant curvature metric. We now give a few more details on the dynamics of these Beltrami fields according to the sign of the energy level.

When $c<0$, in spherical coordinates (ϕ, θ, α) for $S^*\mathbb{S}_c^2 \cong \mathbb{RP}^3$, the vector field is expressed as

$$X_c = \rho\sqrt{-2c}\left(\frac{\cos\alpha}{\sin\theta}\partial_\phi + \sin\theta\sin\alpha\,\partial_\theta + \frac{\cos\theta}{\sin^2\theta}\cos^3\alpha\,\partial_\alpha\right),$$

where $\rho = (\cos^2\alpha + \sin^2\theta\sin^2\alpha)^{-(1/2)}$. While this expression may be somewhat messy, the dynamics are clear: For all $c<0$, the flow is simply a constant rescaling of the flow on $c=-(1/2)$, and it is well known that the lift of the geodesic flow to $S^*\mathbb{S}^2 \cong \mathbb{RP}^3$ is the quotient of the Hopf flow on $\mathbb{S}^3$ by the antipodal action (see, for example, ch 4 in [16]). Thus, the orbits are all the Hopf orbits under the antipodal map quotient.

When $c = 0$, applying the involution $x \mapsto 2x/|x|^2$, we can reexpress X_0 as

$$X_0 = \cos\alpha\,\partial_{x_1} + \sin\alpha\,\partial_{x_2}.$$

This flow on $S^*\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{S}^1$ is a *shear* type Beltrami flow (meaning that the flow consists of parallel stream lines that twist as the angle α changes).

Finally, when $c > 0$, in half-plane coordinates (x, y, α) for $S^*\mathbb{H}^2_c \cong \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^1$ the vector field is expressed as

$$X_c = y\sqrt{2c}(\cos\alpha\,\partial_x + \sin\alpha\,\partial_y) + \sqrt{2c}\cos\alpha\,\partial_\alpha.$$

As in the case $c < 0$, this Beltrami field is the same for all values of c up to a constant rescaling, so the dynamics are the same. Naturally, all streamlines are open and tend to $y = 0$ or $y = +\infty$.

Data accessibility. This article has no additional data.

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

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