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L^2-Boundedness of Gradients of Single Layer Potentials for Elliptic Operators with Coefficients of Dini Mean Oscillation-Type

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Abstract

We consider a uniformly elliptic operator L_A in divergence form associated with an $(n+1)\times(n+1)$-matrix A with real, merely bounded, and possibly non-symmetric coefficients. If

$$\omega_A(r) = \sup_{x\in\mathbb{R}^{n+1}} \fint_{B(x,r)} \left| A(z) - \fint_{B(x,r)} A \right| \mathrm{d}z,$$

then, under suitable Dini-type assumptions on ω_A, we prove the following: if μ is a compactly supported Radon measure in $\mathbb{R}^{n+1}$, $n \geq 2$, and $T_\mu f(x) = \int \nabla_x \Gamma_A(x,y) f(y)\, \mathrm{d}\mu(y)$ denotes the gradient of the single layer potential associated with L_A, then

$$1 + \|T_\mu\|_{L^2(\mu)\to L^2(\mu)} \approx 1 + \|\mathcal{R}_\mu\|_{L^2(\mu)\to L^2(\mu)},$$

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where $\mathcal{R}_\mu$ indicates the n-dimensional Riesz transform. This allows us to provide a direct generalization of some deep geometric results, initially obtained for $\mathcal{R}_\mu$, which were recently extended to T_μ associated with L_A with Hölder continuous coefficients. In particular, we show the following:

(1) If μ is an n-Ahlfors-David-regular measure on $\mathbb{R}^{n+1}$ with compact support, then T_μ is bounded on $L^2(\mu)$ if and only if μ is uniformly n-rectifiable.
(2) Let $E \subset \mathbb{R}^{n+1}$ be compact and $\mathcal{H}^n(E) < \infty$. If $T_{\mathcal{H}^n|_E}$ is bounded on $L^2(\mathcal{H}^n|_E)$, then E is n-rectifiable.
(3) If μ is a non-zero measure on $\mathbb{R}^{n+1}$ such that $\limsup_{r\to 0} \frac{\mu(B(x,r))}{(2r)^n}$ is positive and finite for μ-a.e. $x \in \mathbb{R}^{n+1}$ and $\liminf_{r\to 0} \frac{\mu(B(x,r))}{(2r)^n}$ vanishes for μ-a.e. $x \in \mathbb{R}^{n+1}$, then the operator T_μ is not bounded on $L^2(\mu)$.
(4) Finally, we prove that if μ is a Radon measure on $\mathbb{R}^{n+1}$ with compact support which satisfies a proper set of local conditions at the level of a ball $B = B(x,r) \subset \mathbb{R}^{n+1}$ such that $\mu(B) \approx r^n$ and r is small enough, then a significant portion of the support of $\mu|_B$ can be covered by a uniformly n-rectifiable set. These assumptions include a flatness condition, the $L^2(\mu)$-boundedness of T_μ on a large enough dilation of B, and the smallness of the mean oscillation of T_μ at the level of B.

Key words. Riesz transform · Layer potentials · Second order elliptic equations · Dini mean oscillation · David–Semmes problem · Uniform rectifiability · Rectifiability

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1. Introduction

The aim of this paper is to extend and provide a unified approach to several recent results on the connection of the L^2-boundedness of gradients of single-layer potentials associated with an elliptic operator in divergence form defined on a set E and the geometry of E. The importance of these operators stems from their role in the study of boundary value problems and free boundary problems for harmonic and elliptic measure, as well as the study of analytic capacity (see for instance [2–6,8,18,23,29,30,34,38] and the references therein).

The investigation of geometric properties of singular integrals has produced many important results starting with Calderón's proof in [9] of the boundedness of Cauchy transform on Lipschitz graphs with small Lipschitz constant. A prototypical example of a singular integral operator is the *Riesz transform,* which is the higher dimensional analogue of the Cauchy transform. If μ is a Radon measure on $\mathbb{R}^{n+1}$, $n \geq 1$, its associated (d-dimensional) Riesz transform is defined as

$$\mathcal{R}^d_\mu f(x) = \int \frac{x-y}{|x-y|^{d+1}}\, \mathrm{d}\mu(y), \qquad \text{for } f \in L^1_{\mathrm{loc}}(\mu),$$

whenever the expression above makes sense. For $\delta > 0$, we define the δ-*truncated Riesz transform* as

$$\mathcal{R}^d_{\mu,\delta} f(x) := \int_{|x-y|>\delta} \frac{x-y}{|x-y|^{d+1}} f(y)\, \mathrm{d}\mu(y),$$

and if $f \equiv 1$ on $\mathbb{R}^{n+1}$, we use the notation $\mathcal{R}^d\mu(x) = \mathcal{R}^d_\mu 1(x)$ and $\mathcal{R}^d_\delta\mu(x) = \mathcal{R}^d_{\mu,\delta}1(x)$. We say that $\mathcal{R}^d_\mu$ is *bounded on* $L^2(\mu)$ if $\mathcal{R}^d_{\mu,\delta}$ is bounded on $L^2(\mu)$ uniformly on $\delta > 0$. In this case, we write

$$\|\mathcal{R}^d_\mu\|_{L^2(\mu)\to L^2(\mu)} := \sup_{\delta>0} \|\mathcal{R}^d_{\mu,\delta}\|_{L^2(\mu)\to L^2(\mu)}.$$

Given $x \in \mathbb{R}^{n+1}$ and $r > 0$, we denote by $B(x, r)$ the open ball of center x and radius r. We say that a non-negative Borel measure has *growth of degree d* or, for brevity, d-growth, and we write $\mu \in M^d_+(\mathbb{R}^{n+1})$, if there exists $c_0 > 0$ such that

$$\mu\big(B(x,r)\big) \le c_0 r^d \qquad \text{for all } x \in \mathbb{R}^{n+1},\ r > 0.$$

Any such measure is in fact a Radon measure. Measures with polynomial growth are crucial for the study of singular integrals; for instance, if μ is a non-negative measure on $\mathbb{R}^{n+1}$ without atoms and its associated Riesz transform $\mathcal{R}^d_\mu$ is bounded on $L^2(\mu)$, then $\mu \in M^d_+(\mathbb{R}^{n+1})$ (see [12, p. 56], where this is proved for more general singular integral operators, and also Lemma 3.17).

A Borel measure μ is said *d-Ahlfors-David regular* (also abbreviated by d-AD-regular) if there exists $C > 0$ such that

$$C^{-1} r^d \le \mu\big(B(x,r)\big) \le C r^d \qquad \text{for all } x \in \operatorname{supp}\mu,\ 0 < r < \operatorname{diam}(\operatorname{supp}\mu).$$

If $\mathcal{H}^d$ stands for the d-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$, we say that a set $E \subset \mathbb{R}^{n+1}$ is *d-AD-regular* if $\mathcal{H}^d|_E$ is a d-AD-regular measure.

A set $E \subset \mathbb{R}^{n+1}$ is called *d-rectifiable* if there exists a countable family of Lipschitz maps $f_j : \mathbb{R}^d \to \mathbb{R}^{n+1}$ such that

$$\mathcal{H}^d\Big(E \setminus \bigcup_j f_j(\mathbb{R}^d)\Big) = 0.$$

A measure μ is *d-rectifiable* if it vanishes outside a d-rectifiable set E and it is absolutely continuous with respect to $\mathcal{H}^d|_E$.

We say that a set $E \subset \mathbb{R}^{n+1}$ is *uniformly d-rectifiable* if it is d-AD regular and there exist $\theta, M > 0$ such that for all $x \in E$ and all $r > 0$ there is a Lipschitz mapping g from the ball $B_d(0, r) \subset \mathbb{R}^d$ to $\mathbb{R}^{n+1}$ with $\mathrm{Lip}(g) \le M$ such that

$$\mathcal{H}^d\big(E \cap B(x,r) \cap g(B_d(0,r))\big) \ge \theta r^d.$$

We also say that a measure μ is *uniformly n-rectifiable* if it is d-AD-regular and it vanishes outside of a uniformly d-rectifiable set.

The notion of uniform rectifiability of a set E was introduced by David and Semmes in their seminal works [13,14] as the optimal geometric property that

E should have so that operators in a pretty general subclass of singular integral operators are $L^2(\mathcal{H}^n|_E)$-bounded. They proved in [13] that a d-AD-regular measure μ on $\mathbb{R}^{n+1}$ is uniformly d-rectifiable if and only if *all* the singular integral operators with smooth and anti-symmetric convolution-type kernel are bounded on $L^2(\mu)$. They also raised the question, commonly referred to as *David and Semmes' problem*, if the $L^2(\mu)$-boundedness of the d-Riesz transform $\mathcal{R}_\mu^d$ associated with a d-AD-regular measure μ implies its uniform d-rectifiability.

A positive answer to this question was first provided in the planar case $d = n = 1$ by Mattila, Melnikov, and Verdera in [26], who used the connection of the Cauchy transform with the so-called Menger curvature of a measure. However, their method cannot be generalized to higher dimensions. More recently, Nazarov and Volberg along with the fourth named author proved in [31] the analogous result in the case $d = n$ for any integer $n \geq 1$ using a different set of delicate techniques (we will often refer to it as the 1-codimensional case). We point out that the full David-Semmes' conjecture is still open for d-AD-regular measures of dimension $d = 2, \dots, n-2$.

The n-dimensional Riesz transform in $\mathbb{R}^{n+1}$ has a natural generalization to the context of elliptic PDEs. Let $A(\cdot) = (a_{ij})_{i,j\in\{1,\dots,n+1\}}$ be an $(n+1)\times(n+1)$-matrix whose entries a_{ij} are measurable real-valued functions in $L^\infty(\mathbb{R}^{n+1})$. We say that A is *uniformly elliptic* if there exists $\Lambda > 0$ such that

$$\langle A(x)\xi, \xi\rangle \geq \Lambda^{-1}|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^{n+1} \text{ and a.e. } x \in \mathbb{R}^{n+1}, \tag{1.1}$$

$$\langle A(x)\xi, \eta\rangle \leq \Lambda|\xi||\eta| \qquad \text{for all } \xi, \eta \in \mathbb{R}^{n+1} \text{ and a.e. } x \in \mathbb{R}^{n+1}. \tag{1.2}$$

We consider the second order equation in divergence form

$$L_A u(x) := -\operatorname{div}(A(\cdot)\nabla u(\cdot))(x) = 0, \qquad x \in \mathbb{R}^{n+1}, \tag{1.3}$$

to be understood in the sense of distributions. If A is a uniformly elliptic matrix with bounded measurable coefficients, the operator L_A has a *fundamental solution* $\Gamma_A(x, y)$ which, if δ_y is the Dirac mass at y, satisfies $L_A\Gamma_A(\cdot, y) = \delta_y$ in the sense of distributions. For the construction of the fundamental solution associated with L_A we refer to [21].

For a non-negative Radon measure μ on $\mathbb{R}^{n+1}$ we define the *gradient of the single layer potential*

$$T_\mu f(x) := \int \nabla_1\Gamma_A(x, y) f(y)\, d\mu(y), \qquad \text{for } f \in L^1_{\mathrm{loc}}(\mu), \tag{1.4}$$

to be interpreted in the sense of the truncations

$$T_{\mu,\delta} f(x) := \int_{|x-y|>\delta} \nabla_1\Gamma_A(x, y) f(y)\, d\mu(y), \qquad \text{for } f \in L^1_{\mathrm{loc}}(\mu).$$

For $f \equiv 1$ we use the notations $T_\delta\mu := T_{\mu,\delta}1$ and $T := T_\mu 1$. We also denote that

$$\|T_\mu\|_{L^2(\mu)\to L^2(\mu)} := \sup_{\delta>0} \|T_{\mu,\delta}\|_{L^2(\mu)\to L^2(\mu)}.$$

Furthermore, we observe that in the case $A \equiv Id$, it readily follows by definition that $L_A = -\Delta$ and so $\nabla_1 \Gamma_{Id}$ equals the Riesz kernel up to a dimensional multiplicative constant.

Under the sole assumption that the entries of A are in L^∞, the kernel $\nabla_1 \Gamma_A(\cdot, \cdot)$ does not necessarily satisfy local L^∞ estimates, let alone a modulus of continuity, and so it is not necessarily of Calderón-Zygmund type. We need to impose some additional regularity conditions on A for this to happen. For instance, an adequate framework is provided by matrices with *Hölder continuous entries*. Many important geometric results that are known for the Riesz transform, such as the 1-codimensional version of the David-Semmes' problem, have been successfully generalized by Conde-Alonso, Prat, and the last three named authors (see [10,33,35]). For more details we refer to the discussion of the corollaries of Theorem 1.1.

In the present paper, we are concerned with elliptic operators whose coefficients may have a Lebesgue measure zero set of points of discontinuity. Namely, we will assume that they are of *Dini mean oscillation-type*.

Let $\kappa \geq 1$. We say that a function $\theta \colon [0, \infty] \to [0, \infty]$ is *κ-doubling* if

$$\theta(t) \leq \kappa\, \theta(s) \qquad \text{for } \frac{1}{2}t \leq s \leq t \text{ and } t > 0. \tag{1.5}$$

We denote by $\mathcal{L}^d$ the Lebesgue measure on $\mathbb{R}^d$ and for a set $E \subset \mathbb{R}^{n+1}$, we will also use the notation $\mathcal{L}^{n+1}(E) = |E|$. When we write integrals, we often prefer the more compact and standard notation $\mathrm{d}\mathcal{L}^d(x) = \mathrm{d}x$.

We say that a κ-doubling function θ belongs to the class $\mathrm{DS}(\kappa)$ (*Dini in small scales*), if it is $\mathcal{L}^1$-measurable and

$$\int_0^1 \theta(t)\, \frac{\mathrm{d}t}{t} < \infty. \tag{1.6}$$

Given $d > 0$, we say that θ belongs to the class $\mathrm{DL}_d(\kappa)$ (*d-Dini in large scales*) if it is $\mathcal{L}^1$-measurable and

$$\int_1^\infty \theta(t)\, \frac{\mathrm{d}t}{t^{d+1}} < \infty.$$

We remark that, if $0 < d_1 \leq d_2$, then $\mathrm{DL}_{d_1}(\kappa) \subset \mathrm{DL}_{d_2}(\kappa)$. Moreover, for $\theta \in \mathrm{DS}(\kappa)$ we define

$$\mathfrak{I}_\theta(r) := \int_0^r \theta(t)\, \frac{\mathrm{d}t}{t}, \qquad r > 0 \tag{1.7}$$

and, for $d > 0$ and $\theta \in \mathrm{DL}_d(\kappa)$,

$$\mathfrak{L}_\theta^d(r) := r^d \int_r^\infty \theta(t)\, \frac{\mathrm{d}t}{t^{d+1}}, \qquad r > 0 \tag{1.8}$$

For $x \in \mathbb{R}^{n+1}$, $r > 0$, and an $(n+1) \times (n+1)$-matrix A we denote

$$\bar{A}_{x,r} := \fint_{B(x,r)} A := \frac{1}{|B(x,r)|} \int_{B(x,r)} A(y)\, \mathrm{d}y$$

and, for $p \geq 1$, define its *mean oscillation* function $\omega_A : [0,\infty) \to [0,\infty)$ as

$$\omega_A(r) := \sup_{x\in\mathbb{R}^{n+1}} \fint_{B(x,r)} \left|A(z) - \bar{A}_{x,r}\right| \mathrm{d}z.$$

By [24, p.495][1], there exists a dimensional constant κ such that ω_A satisfies (1.5).

We say that an $(n+1)\times(n+1)$-matrix $A \in \mathrm{DMO}_s$ (resp. $A \in \mathrm{DMO}_\ell$) if $\omega_A \in \mathrm{DS}(\kappa)$ (resp. $\omega_A \in \mathrm{DL}_{n-1}(\kappa)$). We also say that $A \in \mathrm{DDMO}_s$ if $A \in \mathrm{DMO}_s$ and $\mathfrak{I}_{\omega_A}$ satisfies (1.6), i.e.,

$$\int_0^1 \int_0^r \omega_A(t)\, \frac{\mathrm{d}t}{t}\frac{\mathrm{d}r}{r} = -\int_0^1 \omega_A(t)\, \log t\, \frac{\mathrm{d}t}{t} < +\infty. \tag{1.9}$$

Finally, we define

$$\widetilde{\mathrm{DMO}} := \mathrm{DDMO}_s \cap \mathrm{DMO}_\ell.$$

The acronym DMO (resp. DDMO) stands for *Dini mean oscillation* (resp. *double Dini mean oscillation*), and the subscripts in DMO_s and DMO_ℓ indicate that the associated Dini condition is required at small and large scales respectively. Due to (1.9), we may also use the terminology *log-Dini mean oscillation* instead of double Dini mean oscillation.

Furthermore, $\widetilde{\mathrm{DMO}}$ includes the class of matrices with α-Hölder continuous coefficients for $\alpha \in (0,1)$. Indeed, if there exists $C_h > 0$ such that

$$|a_{ij}(x) - a_{ij}(y)| \leq C_h |x-y|^\alpha, \qquad \text{for all } i,j \in \{1,\ldots,n+1\},\ x,y \in \mathbb{R}^{n+1}, \tag{1.10}$$

then $\omega_A(t) \lesssim t^\alpha$ and so $A \in \widetilde{\mathrm{DMO}}$. Our condition even includes matrices that satisfy (1.10) for $\alpha \in (0,1)$ when $|x-y| \lesssim 1$ and $(n-1-\alpha)$ when $|x-y| \gtrsim 1$. In fact, it is clear that if A is uniformly continuous with a Dini modulus of continuity then it is of Dini mean oscillation. In the converse direction, as proved in [22, Appendix A], if A is of Dini mean oscillation, then it agrees (Lebesgue) almost everywhere with a uniformly continuous function with modulus of continuity $\mathfrak{I}_{\omega_A}$. However, as we are mostly interested in sets with Lebesgue measure zero, we highlight that we cannot assume that A is uniformly continuous and thus more delicate arguments are required.

A variant of the example in [15, p. 418] shows that the condition $A \in \widetilde{\mathrm{DMO}}$ is strictly weaker than requiring the matrix A to be Hölder continuous. Indeed, if we define the matrix $a_{ij}(x) = \delta_{ij}$ for $|x| > 1$ and

$$a_{ij}(x) := \delta_{ij}\left(1 + (-\ln|x|)^{-\gamma-1}\right), \qquad \text{for } 0 < |x| \ll 1, 0 < \gamma < 1/2,$$

then, as remarked in [15, p. 418], we have that $\omega_A(r) \approx (-\ln r)^{-\gamma-2}$ for $r \ll 1$. Since ω_A is an increasing function, $A \in \mathrm{DDMO}_s$ but its modulus of continuity does not satisfy the double Dini condition.

[1] The doubling property was proved in [24] for slightly different Dini moduli of oscillation, but a minor variant of that argument works also under a Dini mean oscillation assumption (see also the use of the doubling property in [15, p.424]).

The DMO_s assumption on A guarantees that $\nabla_1\Gamma(\cdot,\cdot)$ is locally of Calderón-Zygmund type, see Lemma 3.9. Indeed, this is possible because of the work of Dong and Kim [15] who proved that, under this hypothesis, weak solutions of $L_A u = 0$ are continuously differentiable providing also local L^∞ and regularity estimates for ∇u. We highlight that one of the crucial technical difficulties in [15] is that the modulus of oscillation ω_A is not monotone as it would be the case if one used $\widetilde{\omega}_A(r) := \sup_{0<\rho\le r}\omega(\rho)$. The proof of the regularity theorem of Dong and Kim is significantly easier for $\widetilde{\omega}_A$. Note that if A is a compactly supported perturbation of the identity matrix Id, then $\omega_A(r) \to 0$ as $r \to \infty$ but $\widetilde{\omega}_A(r)$ does not.[2]

Let us now state the main result of the paper.

Theorem 1.1. *Let A be a uniformly elliptic matrix satisfying $A \in \widetilde{\mathrm{DMO}}$ and let $\mu \in M_+^n(\mathbb{R}^{n+1})$ with compact support, $n \ge 2$. If T_μ is the associated operator given by (1.4), it holds that*

$$1 + \|\mathcal{R}_\mu\|_{L^2(\mu)\to L^2(\mu)} \approx 1 + \|T_\mu\|_{L^2(\mu)\to L^2(\mu)}, \tag{1.11}$$

where the implicit constant depends on n, Λ, c_0, and $\operatorname{diam}(\operatorname{supp}\mu)$.

The role of the $\widetilde{\mathrm{DMO}}$-condition on the matrix A in Theorem 1.1 can be better understood if we relate it to the technical framework of the recent works in the Hölder continuous setting. Indeed, one of the key methods of [10] and the subsequent papers consists in using a proper pointwise estimate of the difference $\nabla_1\Gamma_A(x,y) - \nabla_1\Gamma_{A(x)}(x,y)$, often referred to as *frozen coefficients method* which was proved in [23]. This approach is particularly important because it allows to reduce the study of the operator T_μ to the gradient of the single layer potential associated with a uniformly elliptic equation with *constant coefficients*, which in turn coincides with the Riesz transform modulo a linear change of variables (that depends on x though).

Hence, a crucial difficulty in the proof of Theorem 1.1 is the identification of the right substitute of the frozen coefficients method which adapts to the mean oscillation setting. This issue is resolved in Lemma 3.12, where we estimate the difference between $\nabla_1\Gamma_A(x,y)$ and $\nabla_1\Gamma_{\bar{A}_{x,r}}(x,y)$, for $r := |x-y|/2$. The bound depends on the scale $R > 0$ such that $x, y \in B(0,R)$ and it involves the quantity

$$r^{-n}\,\tau_A(r) := r^{-n}\,\left(\mathfrak{I}_{\omega_A}(r) + \mathfrak{L}^n_{\omega_A}(r)\right) = \frac{1}{r^n}\int_0^r \omega_A(t)\,\frac{\mathrm{d}t}{t} + \int_r^\infty \omega_A(t)\,\frac{\mathrm{d}t}{t^{n+1}} \tag{1.12}$$

and the term

$$R^{-n}\widehat{\tau}_A(R) := R^{-n}\left(\mathfrak{I}_{\omega_A}(R) + \mathfrak{L}^{n-1}_{\omega_A}(R)\right) = \frac{1}{R^n}\int_0^R \omega_A(t)\,\frac{\mathrm{d}t}{t} + \frac{1}{R}\int_R^\infty \omega_A(t)\,\frac{\mathrm{d}t}{t^n}.$$

Observe that the $\widetilde{\mathrm{DMO}}$ assumption implies that $\mathfrak{I}_{\tau_A}(1) < \infty$ and $\widehat{\tau}_A(R) < \infty$ for any $R > 0$. Moreover, if A is α-Hölder continuous, $\tau_A(r) \lesssim r^\alpha$.

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[2] We would like to thank Seick Kim for bringing those facts to our attention motivating us to improve on a previous version of our results where we had used $\widetilde{\omega}_A$.

In the proof of Lemma 3.12 we use a slight variation of the result of Dong and Kim [15] (see Theorem 3.7) and make some delicate PDE estimates obtaining a sharp bound in terms of τ_A and the term $R^{-n}\widehat{\tau}_A(R)$. Since we allow the implicit constant in (1.11) to depend on the diam(supp μ), picking up the term $R^{-n}\widehat{\tau}_A(R)$ not only is it harmless, but it is a term that can become small if the support of the measure has small enough diameter. Its importance will become evident in the proof of Corollary 1.6.

One of the main difficulties is that we do not have scale invariant estimates and in large scales this creates a significant complication. This stands in contrast to the case of Hölder continuous and periodic coefficients, where scale invariant local L^∞ estimates for the gradient of a solution are at our disposal, which makes things work smoothly in scales much larger than R. Let us highlight that, in the present manuscript, we do not require any periodicity assumption on the matrix. In fact, we fill a gap in the use of [23, Lemma 2.2] even for Hölder continuous matrices in the previous works [10,33,35], and [7], where [23, Lemma 2.2] was invoked for non-periodic matrices without any additional justification. To be precise, the bound of (3.56) is the missing component. Another obstacle when working with elliptic operators L_A associated with non-constant matrices is that the kernel $\nabla_1\Gamma(\cdot,\cdot)$ is not anti-symmetric, which, in principle, is rather inconvenient when dealing with its associated single layer potential. If A_0 is a real elliptic measure with constant coefficients, we also write $\Theta(x, y; A_0):=\Gamma_{A_0}(x, y)$.

Our strategy to prove Theorem 1.1 consists in using the frozen coefficients type method in order to bound the L^2-operator norm of the difference of the δ-truncated gradient of the single layer potential and the δ-truncated Riesz transform at the level of a cube in terms of the operator norm of $\mathcal{R}_\mu$. This is a three-step perturbation argument:

(1) The first step is the comparison of $\nabla_1\Gamma_A$ with $\nabla\Theta(\cdot; \bar{A}_{x,|x-y|/2})$ that has already been described above (see Lemma 3.12). The dependence of the second kernel on both x and y requires an additional step.
(2) The second step is to compare $\nabla\Theta(\cdot; \bar{A}_{x,|x-y|/2})$ with $\nabla\Theta(\cdot; \bar{A}_{x,\delta/2})$, where δ is the level of the truncation of the single layer potential (see Lemmas 3.13 and 3.14). This is crucial since it allows us to reduce case to a smooth and odd kernel which is homogeneous of degree $-n$ and independent of the y variable (it is the variable with respect to which we integrate the kernel to construct the integral operator).
(3) The third and final step is the estimate of the difference between $\nabla\Theta(\cdot; \bar{A}_{x,\delta/2})$ and the normalized Riesz kernel. Here we assume that our measure is supported on a cube Q centered at x_Q and $x, y \in Q$. Modulo a change of variables argument, we can assume that the average of A over a ball centered at x_Q with radius comparable to the side-length of the cube is the identity matrix. Contrarily to the previous case, we want to compare $\nabla\Theta(\cdot; \bar{A}_{x,\delta/2})$ with $\nabla\Theta(\cdot; \bar{A}_{x_Q,M\ell(Q)})$ moving up from δ to a higher scale $\ell(Q)$. Pure PDE estimates do not give satisfactory upper bounds and hence, inspired by the approach of [27, Section 1], we study $\nabla\Theta(\cdot; \bar{A}_{x,\delta/2}) - \nabla\Theta(\cdot; Id)$ via the method of *spherical harmonics expansion* (see Lemma 3.16).

More specifically, in the latter step, we prove suitable bounds on the coefficients of the expansion again via PDE estimates. However, we also need proper estimates for the operator norm of singular integrals associated with harmonic polynomials in terms of the norm of the Riesz transform. In order to accomplish this, our argument also relies on some powerful results which have been recently proved by the last named author in [41] (this paper relaxed the assumptions of the main theorem of its companion paper by Dabrowski and the last named author [11], and provided an extension of [39] to higher dimensions). In particular, that work characterizes non-atomic Radon measures in $\mathbb{R}^{n+1}$ whose Riesz transform is L^2-bounded. From that result it was possible to derive the invariance of the L^2-boundedness of the Riesz transform under bilipschitz transformations of the measure. Furthermore, as proved in [41, Corollary 1.4], if μ is a measure in $\mathbb{R}^{n+1}$ with no point masses, and $\mathcal{T}_{K,\mu}$ is the singular integral operator of convolution-type formally defined as

$$\mathcal{T}_{K,\mu} f(x) = \int K(x-y) f(y)\, \mathrm{d}\mu(y) \qquad \text{for } f \in L^1_{\mathrm{loc}}(\mu),$$

where K is antisymetric and satisfies

$$|\nabla^j K(x)| \lesssim \frac{1}{|x|^{n+j}}, \qquad x \in \mathbb{R}^{n+1} \setminus \{0\},\ 0 \le j \le 2,$$

then

$$\|\mathcal{T}_{K,\mu}\|_{L^2(\mu)\to L^2(\mu)} \le C \|\mathcal{R}_\mu\|_{L^2(\mu)\to L^2(\mu)}.$$

Theorem 1.1 has some direct and important applications, which we present below. If μ is a non-zero Borel measure on $\mathbb{R}^{n+1}$ and $s \in (0, n+1]$, we define its *upper s-dimensional density*

$$\Theta^{*,s}(x,\mu) := \limsup_{r\to 0} \frac{\mu(B(x,r))}{(2r)^s} \qquad \text{for } x \in \mathbb{R}^{n+1}$$

and its *lower s-dimensional density*

$$\Theta^{s}_{*}(x,\mu) := \liminf_{r\to 0} \frac{\mu(B(x,r))}{(2r)^s} \qquad \text{for } x \in \mathbb{R}^{n+1}.$$

One of the key tools used in the solution of the codimension-1 David-Semmes' problem in [31] is a variational technique partially inspired by a previous argument of Eiderman, Nazarov and Volberg in [16]. The main result of [16] is that, for $n \le s < n+1$, if a measure μ on $\mathbb{R}^{n+1}$ is such that $0 < \Theta^{*,s}(x,\mu) < \infty$ for μ-almost every x and $\Theta^s_*(x,\mu) = 0$ μ-almost everywhere, then its s-dimensional Riesz transform $\mathcal{R}^s_\mu$ is *not* bounded on $L^2(\mu)$. This was recently generalized in the case $s = n$ by Conde-Alonso together with the second and fourth named authors in [10] for the gradient of the single layer potential T_μ associated with a Hölder continuous matrix A (see also [7] for a version of [10] for Schrödinger operators). Since Hölder continuous matrices belong to $\widetilde{\mathrm{DMO}}$, Theorem 1.1 allows us to obtain an alternative approach to [10, Theorem A], and to extend it to a more general class of elliptic equations.

Corollary 1.2. *Let A be a uniformly elliptic matrix satisfying $A \in \widetilde{\text{DMO}}$ and let μ be a non-zero measure on $\mathbb{R}^{n+1}$, $n \geq 2$, such that $0 < \Theta^{*,n}(x, \mu) < \infty$ and $\Theta^n_*(x, \mu) = 0$ for μ-a.e. $x \in \mathbb{R}^{n+1}$. Then the associated operator T_μ given by (1.4) is not bounded on $L^2(\mu)$.*

Another important application of our main theorem is the elliptic version of the David and Semmes problem in codimension 1. Via a decomposition in spherical harmonics, it was proved in [10] that, if A is Hölder continuous, T_μ is bounded on $L^2(\mu)$ on uniformly n-rectifiable measures μ with compact support. The converse implication was obtained by Prat, Puliatti, and Tolsa in [33], via a non-trivial adaptation of the scheme of [31]. In particular, we remark that their proof relies on a delicate reflection argument for the matrix across hyperplanes, and it is not clear how to adapt it to the context of uniformly elliptic matrices with Dini mean oscillation. Nevertheless, Theorem 1.1 readily shows the following:

Corollary 1.3. *Let A be a uniformly elliptic matrix satisfying $A \in \widetilde{\text{DMO}}$ and let μ be an n-AD-regular measure on $\mathbb{R}^{n+1}$, $n \geq 2$, with compact support. If T_μ is the associated operator given by (1.4), then T_μ is bounded on $L^2(\mu)$ if and only if μ is uniformly n-rectifiable.*

The combination of [16,31], and a covering argument of Pajot allowed Nazarov, Volberg, and the fourth named author to prove in [32] that if $E \subset \mathbb{R}^{n+1}$ is such that $\mathcal{H}^n(E) < \infty$ and $\mathcal{R}_{\mathcal{H}^n|_E}$ is bounded on $L^2(\mathcal{H}^n|_E)$, then the set E is n-rectifiable. Its elliptic analogue for second order elliptic operators in divergence form associated with Hölder continuous matrices was obtained in [33]. We generalize this result as well.

Corollary 1.4. *Let A be a uniformly elliptic matrix satisfying $A \in \widetilde{\text{DMO}}$ and let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a compact set with $\mathcal{H}^n(E) < \infty$. If T is the associated operator given by (1.4) and $T_{\mathcal{H}^n|_E}$ is bounded on $L^2(\mathcal{H}^n|_E)$, then E is n-rectifiable.*

The main advantage of Theorem 1.1 is that its application gives alternative and more direct proofs of [10,33], and [35] via [16,31,32], and [20], which readily extend to uniformly elliptic matrices in $\widetilde{\text{DMO}}$.

Moreover, if we further assume that the matrix A is Hölder continuous, Corollaries 1.3 and 1.4 are crucial tools in order to prove a rectifiability result for elliptic measure in the context of a non-variational one-phase problem (see [33, Theorem 1.3]) which generalizes [3]. For more details we refer to [33, Section 12].

Finally, we also extend the main result of Girela-Sarrión and the fourth named author [20] as well as its elliptic analogue of the third named author in [35]. Let μ be a Radon measure on $\mathbb{R}^{n+1}$. For a ball $B \subset \mathbb{R}^{n+1}$ of radius $r(B)$ and an integer $N > 0$, we denote that

$$\begin{aligned} \Theta_\mu(B) &= \frac{\mu(B)}{r(B)^n}, \\ \alpha_A(t) &= t + t^\beta + \omega_A(t), \qquad t > 0,\ \beta \in (0, 1] \qquad (1.13) \\ P_{\gamma,\mu}(B) &:= \sum_{j \geq 0} 2^{-\gamma j} \Theta_\mu(2^j B), \qquad \gamma \in (0, 1] \end{aligned}$$

$$\mathcal{P}^N_{\omega,\mu}(B) := \sum_{j \geq N} \alpha_A(2^{-j}) \Theta_\mu(2^j B). \tag{1.14}$$

Given an n-dimensional plane L in $\mathbb{R}^{n+1}$ we denote that

$$\beta^L_{\mu,1}(B) = \frac{1}{r(B)^n} \int_B \frac{\operatorname{dist}(x, L)}{r(B)} \, d\mu(x) \qquad \text{and} \qquad \beta_{\mu,1}(B) = \inf_L \beta^L_{\mu,1}(B),$$

where the infimum is taken over all hyperplanes. Finally, for a set $E \subset \mathbb{R}^{n+1}$ with $\mu(E) > 0$ and $f \in L^1_{\text{loc}}(\mu)$ we write that

$$m_E(f, \mu) = \frac{1}{\mu(E)} \int_E f \, d\mu.$$

Let $M(\mathbb{R}^{n+1})$ be the space of real Borel measures, endowed with the total variation norm $\|\cdot\|$: for $\mu \in M(\mathbb{R}^{n+1})$ we indicate by $|\mu|$ its variation and by $\|\mu\| := |\mu|(\mathbb{R}^{n+1})$.

For our application, we have to determine whether $T_{\mu,\varepsilon} f$ converges pointwise μ-almost everywhere for $\varepsilon \to 0$. In case it does, we denote the limit as

$$\text{pv}T_\mu f(x) = \lim_{\varepsilon \to 0} T_{\mu,\varepsilon} f(x),$$

and we refer to this as the *principal value* of the integral $T_\mu f(x)$. The existence of principal values for gradients of single layer potentials can be proved in our framework via a minor variant of the arguments of [35, Theorem 1.1]: one can study separately the case of rectifiable measures and that of measures with zero density, which can be both analyzed via the frozen coefficients method of Lemma 3.12. Ultimately, this implies

Proposition 1.5. *Let μ be a Radon measure on $\mathbb{R}^{n+1}$, $n \geq 2$, with compact support and with growth of degree n. Let A be a uniformly elliptic matrix satisfying $A \in \widetilde{\text{DMO}}$ and assume that its associated gradient of the single layer potential T_μ is bounded on $L^2(\mu)$. Then the following holds:*

(1) for $1 \leq p < \infty$ and all $f \in L^p(\mu)$, $\text{pv}T_\mu f(x)$ exists for μ-a.e. $x \in \mathbb{R}^{n+1}$.
(2) for all $\nu \in M(\mathbb{R}^{n+1})$, $\text{pv}T\nu(x)$ exists for μ-a.e. $x \in \mathbb{R}^{n+1}$.

Finally, we state the local quantitative rectifiability criterion for Radon measures which generalizes [20] and [35]. We refer to the introductions of the aforementioned articles for a detailed discussion of the result and the role of all the hypotheses. The result is stated in the form of [4, Corollary 3.2].

Corollary 1.6. *Let A be a uniformly elliptic matrix satisfying $A \in \widetilde{\text{DMO}}$ and let μ be a Radon measure with compact support in $\mathbb{R}^{n+1}$, $n \geq 2$. Let $B \subseteq \mathbb{R}^{n+1}$ be an open ball with $\mu(B) > 0$ and let $C_0, C_0' > 0$. Denote by T_μ the gradient of the single layer potential associated with L_A and μ, and let β be as in Lemma 3.9. Suppose that μ and B are such that, for some positive real numbers τ, δ, and λ, and a positive integer N, the following properties hold:*

(1) $r(B) \leq \lambda$.

(2) We have $\mathcal{P}^0_{\omega,\mu}(B) \leq C_0\Theta_\mu(B)$, $\mathcal{P}^N_{\omega,\mu}(B) \leq C_0\mathfrak{I}_{\alpha_A}(2^{-N})\Theta_\mu(2^N B)$, *and it holds* $\Theta_\mu(B(x,r)) \leq C_0\Theta_\mu(2^N B)$ *for all* $x \in B$ *and* $0 < r \leq 2^N r(B)$.

(3) $T_{\mu|_{2^N B}}$ *is bounded on* $L^2(\mu|_{2^N B})$ *and it holds* $\|T_{\mu|_{2^N B}}\|_{L^2(\mu|_{2^N B})\to L^2(\mu|_{2^N B})} \leq C_0'\Theta_\mu(2^N B)$.

(4) There exists some n*-plane* L *passing through the center of* B *such that* $\beta^L_{\mu,1}(B) \leq \delta\Theta_\mu(B)$.

(5) We have that

$$\int_B \left|T_\mu 1(x) - m_B(T_\mu 1, \mu)\right|^2 \mathrm{d}\mu(x) \leq \tau\, \Theta_\mu(2^N B)^2\mu(B). \tag{1.15}$$

Then there exist a choice of δ *and* τ *small enough, possibly depending on* n, Λ, C_0, C_0', *and* $\mathrm{diam}(\mathrm{supp}\,\nu)$, *a choice of* $N = N(\tau, n, \Lambda, \mathrm{diam}(\mathrm{supp}\,\nu), C_0, C_0')$ *large enough, a choice of* $\lambda = \lambda(\tau, N, n, \Lambda, C_0, C_0', \mathrm{diam}(\mathrm{supp}\,\nu))$ *small enough such that* $2^N\lambda$ *is also sufficiently small, for which the following holds: if* μ *satisfies (1)–(6), there exist a uniformly* n*-rectifiable set* Γ *and* $\theta \in (0,1)$ *such that*

$$\mu(B \cap \Gamma) \geq \theta\mu(B).$$

The UR constants of Γ *depend on all the constants above.*

In the condition (1.15) we identify $T_\mu 1$ with $\mathrm{pv}T_\mu 1$. Moreover, the well-posedness of the expression on the left hand side of (1.15) can be justified via the existence of principal values and the fact the measure μ has compact support. For the details we refer to [20, Section 2.4] (see also [35, Section 3]).

The proof of Corollary 1.6 consists in showing that, for $\lambda \ll 1$, the smallness in the mean oscillation assumption (1.15) implies smallness for the analogous quantity associated to the Riesz transform. As it is more coherent with the notation of the rest of the paper, we will equivalently prove this for cubes in $\mathbb{R}^{n+1}$; we also remark that Girela-Sarrión and Tolsa reduced the proof of their theorem to [20, Main Lemma 3.1], which is formulated for cubes itself.

Finally, we remark that for $0 < \beta < 1$, $N \geq 1$, $\mu \in M^n_+(\mathbb{R}^{n+1})$, and a ball $B \subset \mathbb{R}^{n+1}$ such that $2^N B$ satisfies the $P_{\beta,\mu}$-doubling condition

$$P_{\beta,\mu}(2^N B) \lesssim \Theta_\mu(2^N B), \tag{1.16}$$

elementary calculations show that

$$\begin{aligned}\sum_{j\geq N} 2^{-j\beta}\Theta_\mu(2^j B) &= 2^{-N\beta}\sum_{j\geq N} 2^{-(j-N)\beta}\Theta_\mu\big(2^{j-N}(2^N B)\big)\\ &= 2^{-N\beta}\sum_{j\geq 0} 2^{-j\beta}\Theta_\mu\big(2^j(2^N B)\big) \lesssim 2^{-N\beta}\Theta_\mu(2^N B).\end{aligned}$$

Thus, if $\omega_A(t) \lesssim t^\beta$, the assumption (2) in Corollary 1.6 is satisfied if we can guarantee that both B and $2^N B$ are $P_{\beta,\mu}$-doubling in the sense of (1.16).

Previous versions of Corollary 1.6 had been applied to solve the non-variational two-phase problem for harmonic measure and elliptic measure associated with an elliptic operator with Hölder continuous coefficients. In particular, those were needed to analyze the measures at the level of points of zero density. One of the main ingredients was [4, Lemma 6.1], which shows that if μ is an n-dimensional measure, for any $\beta \in (0, 1)$, there exists $M > 0$ depending on β and the dimension, such that for μ-a.e. $x \in \mathbb{R}^{n+1}$, there exists a sequence of M-$P_{\beta,\mu}$-doubling balls $B(x, r_i)$ with $r_i \to 0$ as $i \to \infty$. In fact, if A is a β-Hölder continuous matrix, Corollary 1.6 is equivalent to [35, Theorem 1.2] for the study of the two-phase problem for the elliptic measure as in [35]. Indeed, if x_0 is a point for which we may find a sequence of M-$P_{\beta,\mu}$-doubling balls, we fix $B_0 = B(x_0, r_0)$ to be a ball with radius $r_0 \leq \lambda$. Then, we pick the largest M-$P_{\beta,\mu}$-doubling ball $B = B(x_0, \tilde{r})$ such that $0 < \tilde{r} \leq 2^{-N} r_0$ for which, if $r_0/2 < 2^{N_0}\tilde{r} \leq r_0$ and $N \leq N_0$, it holds that

$$P_{\beta,\mu}(2^{N_0} B) \leq M 2^{-N_0\beta}\Theta_\mu(2^{N_0} B).$$

Hence, under these circumstances, Corollary 1.6 gives the desired property of big pieces of uniformly n-rectifiable measure in B, equivalently to the main results of [20] and [35].

Finally, let us mention that generalizing the one-phase and two-phase problems for elliptic measure as in [3,33] and [6,35] to elliptic measures associated with L_A, $A \in \widetilde{\text{DMO}}$, presents significant difficulties; for instance, the lack of a proper $T(1)$-theorem for suppressed kernels with such general modulus of continuity. Therefore, those problems should be treated separately.

Structure of the paper

In Section 2 we present the general notation which we adopt in the paper and recall some properties of Dini functions along with their relation to integral operators with proper reproducing kernels (see Lemma 2.5).

Section 3 contains the PDE bulk of the paper. In the first part we describe how the elliptic operator L_A and the gradient of the single layer potential transform under a bilipschitz change of variables. Furthermore, in Lemma 3.5 we introduce a specific linear map S so that, given a ball $B \subset \mathbb{R}^{n+1}$, the average of the symmetric part of the transformed matrix $\hat{A}$ equals the identity matrix on $S^{-1}(B)$. This turns out to be crucial for the proof of Main Lemma I, as it allows to compare $\nabla_1\Gamma_A$ at the level of a given cube with the Riesz kernel effectively. In the second part of Section 3 we adapt the arguments of [15] in order to prove that $\nabla_1\Gamma_A$ can be interpreted locally as a Calderón-Zygmund kernel (see Lemma 3.9), and we gather other auxiliary PDE lemmas.

Section 3.3 contains the previously described three-step perturbation argument: we deal with the frozen coefficients type estimate in Lemma 3.12, in Lemma 3.13 and Lemma 3.14 we prove pointwise bounds which allow us to compare $\nabla_1\Gamma_{\bar{A}_{x,|x-y|/2}}$ and $\nabla_1\Gamma_{\bar{A}_{x,\delta/2}}$ for $\delta > 0$, and finally in Lemma 3.16 we implement the techniques based on spherical harmonic decomposition in order to estimate the difference of $\nabla\Theta(\cdot; \bar{A}_{x,\delta/2})$ and $\nabla\Theta(\cdot; \bar{A}_{\Omega_Q})$, where $\bar{A}_{\Omega_Q}$ denotes the integral average of A on a set at the level of al cube Q.

Section 4 covers the proof of the two main lemmas. In Main Lemma I we gather all the results of the previous section and estimate the $L^2(\mu)$-norm of the difference of $T_{\mu,\delta}$ and a proper normalization of $\mathcal{R}_{\mu,\delta}$. Main Lemma II is the main tool for the proof of Corollary 1.6: under suitable hypotheses on the measure μ, a duality argument and the local Calderón-Zygmund character of $\nabla_1\Gamma_A(x, y)$ allow us to transfer the smallness of the $L^2(\mu)$-mean oscillation of T_μ to the Riesz transform, which is the crucial step in order to invoke [20].

In Section 5 we introduce and discuss the properties of an auxiliary measure ν_ε, which we obtain as the convolution of the measure ν supported on a cube with a proper cut-off function. This guarantees that the measure ν_ε is absolutely continuous with respect to $\mathcal{L}^{n+1}$, which entails that the $L^2(\nu_\varepsilon)$-norm of the Riesz transform associated with ν_ε applied to a Lipschitz function with compact support is (qualitatively) finite. This is needed in the last section, as it allows to absorb the norm of the Riesz transform in (6.3) in the left hand side of that expression.

The final Section 6 contains the proof of Theorem 1.1 and its corollaries. In particular, we show how to prove those results combining the lemmas of Section 4 and Section 5 via the change of variables introduced in Lemma 3.5.

2. Preliminaries and Notation

General notation

- For $\lambda > 0$ and an open ball $B = B(x, r)$, we define its dilation $\lambda B := B(x, \lambda r)$. Analogously, given a Euclidean cube Q in $\mathbb{R}^{n+1}$ with center x_Q and side-length $\ell(Q)$, we denote by λQ the cube with center x_Q and side-length $\lambda\ell(Q)$.
- For $0 < r \leq R < \infty$, we indicate
$$A(x, r, R) := B(x, R) \setminus \overline{B(x, r)} = \{y \in \mathbb{R}^{n+1} : r < |x - y| < R\}.$$
- We denote by $\mathbb{S}^n = \partial B(0, 1)$ the unit sphere in $\mathbb{R}^{n+1}$, by σ its surface measure, and we define $\omega_n := \sigma(\mathbb{S}^n)$.
- Given $A \subset \mathbb{R}^d$, we denote by χ_A its characteristic function.
- We endow the space of matrices $\mathbb{R}^{n_1\times n_2}$ with the norm $|A| := \max_{i,j} |a_{ij}|$, for $A = (a_{ij})_{i,j} \in \mathbb{R}^{n_1\times n_2}$.
- We write $a \lesssim b$ if there is $C > 0$ so that $a \leq Cb$, and $a \lesssim_t b$ to specify that the constant C depends on the parameter t. We write $a \approx b$ to mean $a \lesssim b \lesssim a$, and define $a \approx_t b$ similarly.

Dini functions and integral operators

Let θ be a κ-doubling function in the sense of (1.5) for $\kappa > 0$. For $\eta \in \left(0, \frac{1}{2}\right)$ denote by N_η the positive integer such that $2^{-N_\eta-1} \leq \eta < 2^{-N_\eta}$. Hence, if $r > 0$ and $\eta r \leq t \leq r$ then

$$\begin{aligned}\int_{\eta r}^{r} \theta(t)\,\frac{\mathrm{d}t}{t} &\leq \sum_{\ell=0}^{N_\eta} \int_{2^{-\ell-1}r}^{2^{-\ell}r} \theta(t)\,\frac{\mathrm{d}t}{t} \leq \kappa \sum_{\ell=0}^{N_\eta} \theta(2^{-\ell-1}r) \leq \kappa \sum_{\ell=0}^{N_\eta} \kappa^{N_\eta-\ell-1}\theta(2^{-N_\eta}r)\\ &= \frac{\kappa^{N_\eta+1}-1}{\kappa-1}\theta(2^{-N_\eta}r) \leq \kappa\frac{\kappa^{N_\eta+1}-1}{\kappa-1}\theta(\eta r) =: C(\kappa,\eta)\,\theta(\eta r),\end{aligned}$$

where we used that θ is κ-doubling. Therefore,

$$\int_0^R \theta(t)\,\frac{\mathrm{d}t}{t} = \sum_{j=0}^{\infty} \int_{\eta^{j+1}R}^{\eta^j R} \theta(t)\,\frac{\mathrm{d}t}{t} \le C(\kappa,\eta) \sum_{j=0}^{\infty} \theta(\eta^{j+1}R).$$

Moreover, if $r > 0$ and $\eta r \le t \le r$ it holds that

$$\begin{aligned}\theta(r) = \theta(r) \fint_{2^{-N_\eta}r}^{r} \mathrm{d}t &\le \kappa \frac{\theta(r)}{(1-2^{-N_\eta})r} \sum_{\ell=0}^{N_\eta-1} \frac{2^{-\ell}r}{\theta(2^{-\ell}r)} \int_{2^{-\ell-1}r}^{2^{-\ell}r} \theta(t)\frac{\mathrm{d}t}{t} \\ &\le \frac{\kappa}{(1-2^{-N_\eta})} \sum_{\ell=0}^{N_\eta-1} \kappa^\ell 2^{-\ell} \int_{2^{-\ell-1}r}^{2^{-\ell}r} \theta(t)\frac{\mathrm{d}t}{t} \lesssim \max(1,(\kappa/2)^{N_\eta}) \int_{\eta r}^{r} \theta(t)\,\frac{\mathrm{d}t}{t}.\end{aligned} \tag{2.1}$$

Thus, for $R > 0$,

$$\begin{aligned}\sum_{j=0}^{\infty} \theta(\eta^j R) &\lesssim \sum_{j=0}^{\infty} \max(1,(\kappa/2)^{N_\eta}) \int_{\eta^{j+1}R}^{\eta^j R} \theta(t)\,\frac{\mathrm{d}t}{t} \\ &= \max(1,(\kappa/2)^{N_\eta}) \int_0^R \theta(t)\,\frac{\mathrm{d}t}{t}.\end{aligned} \tag{2.2}$$

In particular, θ belongs to $\mathrm{DS}(\kappa)$ if and only if the doubling property (1.5) holds and $\sum_{j=0}^{\infty} \theta(2^{-j}) < +\infty$. One can analogously show that, if θ verifies (1.5), and $0 < d \le n$, we have

$$\sum_{k=1}^{\infty} \frac{\theta(2^k R)}{(2^k R)^d} \lesssim \int_R^\infty \theta(t)\,\frac{\mathrm{d}t}{t^{d+1}}, \qquad R > 0. \tag{2.3}$$

Moreover, by the doubling property of θ,

$$\theta(r) \le \kappa \int_{r/2}^{r} \theta(t)\,\frac{\mathrm{d}t}{t} \le \kappa\, \mathfrak{I}_\theta(r), \qquad r > 0. \tag{2.4}$$

Lemma 2.1. *Assuming that for fixed $d > 0$*

$$\mathfrak{L}_\theta^d(t) = t^d \int_t^\infty \theta(s)\,\frac{ds}{s^{d+1}} < +\infty \quad \text{for any } t > 0,$$

then $\mathfrak{L}_\theta^d$ is a 2^d-doubling function. Moreover, it holds that

(1) If $\mathfrak{I}_\theta(1) < \infty$ and $\mathfrak{L}_\theta^d(1) < \infty$, then $\mathfrak{L}_\theta^d \in \mathrm{DS}\left(2^d\right)$.
(2) If $\mathfrak{I}_{\mathfrak{I}_\theta}(1) < \infty$ and $\mathfrak{I}_{\mathfrak{L}_\theta^d}(1) < \infty$, then $\mathfrak{L}_{\mathfrak{I}_\theta}^d \in \mathrm{DS}\left(2^d\right)$.

Proof. If $t/2 \le s \le t$, then

$$\mathfrak{L}^d_\theta(t) = t^d \int_t^\infty \theta(r)\,\frac{\mathrm{d}r}{r^{d+1}} \le t^d \int_s^\infty \theta(r)\,\frac{\mathrm{d}r}{r^{d+1}} \le 2^d s^d \int_s^\infty \theta(r)\,\frac{\mathrm{d}r}{r^{d+1}} \le 2^d \mathfrak{L}^d_\theta(s).$$

which proves that $\mathfrak{L}^d_\theta$ is 2^d-doubling. Moreover, to show (1), we use Fubini's theorem to get

$$\begin{aligned}\int_0^1 \mathfrak{L}^d_\theta(t)\,\frac{\mathrm{d}t}{t} &= \int_0^1 t^d \int_t^1 \theta(s)\,\frac{\mathrm{d}s}{s^{d+1}}\,\frac{\mathrm{d}t}{t} + \int_0^1 t^d \int_1^\infty \theta(s)\,\frac{\mathrm{d}s}{s^{d+1}}\,\frac{\mathrm{d}t}{t}\\ &= \frac{1}{d}\int_0^1 \theta(s)\,\frac{\mathrm{d}s}{s} + \frac{1}{d}\int_1^\infty \theta(s)\,\frac{\mathrm{d}s}{s^{d+1}} < \infty.\end{aligned}$$

To prove (2), we apply Fubini's theorem and for $r > 0$, it holds

$$\begin{aligned}\mathfrak{L}^d_{\mathfrak{I}_\theta}(r) = r^d \int_r^\infty \mathfrak{I}_\theta(t)\,\frac{\mathrm{d}t}{t^{d+1}} &= \frac{1}{d}\int_0^r \theta(t)\,\frac{\mathrm{d}t}{t} + \frac{r^d}{d}\int_r^\infty \theta(t)\,\frac{\mathrm{d}t}{t^{d+1}}\\ &= \frac{1}{d}\big(\mathfrak{I}_\theta(r) + \mathfrak{L}^d_\theta(r)\big) < \infty.\end{aligned} \tag{2.5}$$

□

Remark 2.2. If we assume that $\theta \in \mathrm{DS}(\kappa)$, the condition (2.2) implies that $\theta(\eta^j R) \to 0$ as $j \to \infty$ for all $R > 0$. In particular, the previous lemma implies that if $\theta \in \mathrm{DS}(\kappa) \cap \mathrm{DL}_d(\kappa)$, then $\mathfrak{L}^d_\theta(\eta^j R) \to 0$ as $j \to \infty$ for all $R > 0$.

Definition 2.3. Let θ be a κ-doubling function and $0 < d \le n+1$. We say that a function $K : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \{(0,0)\} \to \mathbb{R}$ is a (θ, d)-*kernel* if it is continuous and there exists $C > 0$ such that

$$|K(x,y)| \le C\frac{\theta(|x-y|)}{|x-y|^d} \quad \text{for } x \ne y.$$

The latter estimate for (θ, d)-kernels is directly connected with the Dini integral.

Lemma 2.4. *Let $\theta \in \mathrm{DS}(\kappa)$. Let $0 < d \le n+1$, and assume that $\mu \in M^d_+(\mathbb{R}^{n+1})$ with d-growth constant $c_0 > 0$. For $\rho > 0$ we have that*

$$\int_{B(x,\rho)} \frac{\theta(|x-z|)}{|x-z|^d}\,\mathrm{d}\mu(z) \lesssim_{c_0,\kappa} \int_0^\rho \theta(t)\,\frac{\mathrm{d}t}{t}, \tag{2.6}$$

and the right hand side of (2.6) *tends to* 0 *as* $\rho \to 0$.

Proof. The proof of (2.6) follows from a standard estimate of the integral on dyadic annuli, the d-growth of μ, and (2.2). Indeed,

$$\begin{aligned}\int_{B(x,\rho)} \frac{\theta(|x-z|)}{|x-z|^d}\,\mathrm{d}\mu(z) &= \sum_{j=0}^\infty \int_{A(x,2^{-j-1}\rho,2^{-j}\rho)} \frac{\theta(|x-z|)}{|x-z|^d}\,\mathrm{d}\mu(z)\\ &\lesssim \sum_{j=0}^\infty \frac{\theta(2^{-j}\rho)}{2^{-jd}\rho^d}\,\mu\big(B(x,2^{-j}\rho)\big) \lesssim \sum_{j=0}^\infty \theta(2^{-j}\rho) \approx \int_0^\rho \theta(t)\,\frac{\mathrm{d}t}{t}.\end{aligned}$$

□

Lemma 2.5. *Let* $0 < d \leq n+1$ *and let* μ *be a measure with compact support in* $\mathbb{R}^{n+1}$ *and* d*-growth with constant* $c_0 > 0$*. If* K *is a* (θ, d)*-kernel for* $\theta \in \mathrm{DS}(\kappa)$*, then its associated integral operator*

$$Tf(x) := \int K(x, y) f(y) \,\mathrm{d}\mu(y), \qquad x \in \mathbb{R}^{n+1}$$

is bounded on $L^2(\mu)$*. More specifically, if* $C > 0$ *is as in Definition 2.3 and* $R := \operatorname{diam}(\operatorname{supp}\mu)$*, we have that*

$$\|T\|_{L^2(\mu)\to L^2(\mu)} \lesssim_{\kappa, C} c_0 \int_0^R \theta(t) \,\frac{\mathrm{d}t}{t}. \tag{2.7}$$

Proof. The lemma is a direct consequence of Schur's Test (see for instance [17, Theorem 6.18]) and Lemma 2.4. Indeed, for all $x \in \mathbb{R}^{n+1}$, we have that

$$\int |K(x, y)| \,\mathrm{d}\mu(y) \lesssim \int \frac{\theta(|x - y|)}{|x - y|^d} \,\mathrm{d}\mu(y) \lesssim \int_0^R \theta(t) \,\frac{\mathrm{d}t}{t} < +\infty$$

and, analogously,

$$\int |K(x, y)| \,\mathrm{d}\mu(x) \lesssim \int_0^R \theta(t) \,\frac{\mathrm{d}t}{t} \qquad \text{for all } y \in \mathbb{R}^{n+1}.$$

□

3. Change of Variables and Pointwise Estimates for the Gradient of the Fundamental Solution

3.1. Change of variables

Our arguments involve a change of variables with respect to a particular bilipschitz map, which will be specified later on. For this reason, we state some result concerning how the elliptic operator, its fundamental solution, and its associated gradient of the single layer potential change under such a transformation.

Lemma 3.1. (see [2], Lemma 2.4) *Let* A *be a uniformly elliptic matrix with real entries and let* $\phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *be a bilipschitz map. If we denote*

$$A_\phi := |\det D(\phi)| D(\phi^{-1})(A \circ \phi) D(\phi^{-1})^T,$$

where $D(\cdot)$ *denotes the differential matrix, then* A_ϕ *is a uniformly elliptic matrix in* $\mathbb{R}^{n+1}$*. Furthermore,* $u\colon \mathbb{R}^{n+1} \to \mathbb{R}$ *is a weak solution of* $L_A u = 0$ *if and only if* $\tilde{u} := u \circ \phi$ *is a weak solution of* $L_{A_\phi} \tilde{u} = 0$ *in* $\mathbb{R}^{n+1}$*.*

Lemma 3.2. (see [33], Lemma 5.3) *Let $\phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a bilipschitz map and let $A(\cdot)$ be a uniformly elliptic matrix with coefficients in $L^\infty(\mathbb{R}^{n+1})$. Let Γ_A be the fundamental solution of $L_A = -\mathrm{div}(A\nabla\cdot)$. Set $A_\phi := |\det\phi| D(\phi^{-1})(A \circ \phi)D(\phi^{-1})^T$. Then*

$$\Gamma_{A_\phi}(x, y) = \Gamma_A(\phi(x), \phi(y)) \qquad \textit{for } x, y \in \mathbb{R}^{n+1}, \tag{3.1}$$

and

$$\nabla_1\Gamma_{A_\phi}(x, y) = D(\phi)^T(x)\nabla_1\Gamma_A(\phi(x), \phi(y)) \qquad \textit{for } x, y \in \mathbb{R}^{n+1}. \tag{3.2}$$

Lemma 3.3. (see [33], Lemma 5.4) *Let $\phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a bilipschitz map. Let μ be a Radon measure on $\mathbb{R}^{n+1}$ and $\phi\sharp\mu$ be its image measure. For every $x \in \mathbb{R}^{n+1}$ we have that*

$$T_\phi\mu(x) = D(\phi)^T(x)T\phi\sharp\mu(\phi(x)). \tag{3.3}$$

If ϕ and μ are as in the previous lemma, $\nu := \phi^{-1}\sharp\mu$, and for $\delta > 0$ we define

$$T_{\phi,\delta}\nu(x) := \int_{|x-y|>\delta} \nabla_1\Gamma_{A_\phi}(x, y)\,\mathrm{d}\nu(y) \tag{3.4}$$

and its variant

$$\widetilde{T}_{\phi,\delta}\nu(x) := \int_{|\phi(x)-\phi(y)|>\delta} \nabla_1\Gamma_{A_\phi}(x, y)\,\mathrm{d}\nu(y),$$

then the arguments of [33, Lemma 5.4] show that

$$\widetilde{T}_{\phi,\delta}\nu(x) = D(\phi)^T(x)T_\delta\mu(\phi(x)),$$

For $f \in L^1_{\mathrm{loc}}(\nu)$ we also denote $T_{\phi,\nu,\delta}f(x) := T_{\phi,\delta}(f\nu)(x)$ and $\widetilde{T}_{\phi,\nu,\delta}f(x) := \widetilde{T}_{\phi,\delta}(f\nu)(x)$. In particular, by [33, Section 6, p. 740], if T_μ is bounded on $L^2(\mu)$ then the operators $\widetilde{T}_{\phi,\nu,\delta}$ are bounded on $L^2(\nu)$ uniformly on $\delta > 0$ and

$$\|\widetilde{T}_{\phi,\nu,\delta}\|_{L^2(\nu)\to L^2(\nu)} \approx \|T_{\mu,\delta}\|_{L^2(\mu)\to L^2(\mu)}, \tag{3.5}$$

where the implicit constant depends on the bilipschitz constant of ϕ.

Moreover we can prove the following lemma:

Lemma 3.4. *Let A, T, and μ be as in Theorem 1.1, and $L\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be an invertible linear map. Then, for any $\delta > 0$,*

$$1 + \|\widetilde{T}_{L,\mu,\delta}\|_{L^2(\mu)\to L^2(\mu)} \approx 1 + \|T_{\mu,\delta}\|_{L^2(\mu)\to L^2(\mu)}, \tag{3.6}$$

where the implicit constant depends on $\|L\|_{\mathrm{op}}$, n, Λ, c_0, and $\mathrm{diam}(\mathrm{supp}\,\mu)$.

Proof. Let $f \in L^2(\mu)$. Let $\psi \in C_c^\infty(\mathbb{R}^{n+1}\setminus B(0,1))$ be a function such that $0 \le \psi \le 1$ in $\mathbb{R}^{n+1}$, $\psi = 1$ in $\mathbb{R}^{n+1}\setminus B(0,2)$, and $\|\nabla\psi\|_\infty \lesssim 1$, and we define $\psi_\delta(\cdot) := \psi(\frac{\cdot}{\delta})$. By standard estimates, we can prove that

$$\begin{aligned}&\left|\widetilde{T}^{\psi}_{L,\mu,\delta} f(x) - \widetilde{T}_{L,\mu,\delta} f(x)\right| \\ &:= \left|\int \psi_\delta\left(L(x) - L(y)\right) \nabla_1 \Gamma_{A_L}(x,y) f(y)\, \mathrm{d}\mu(y) - \widetilde{T}_{L,\mu,\delta} f(x)\right| \lesssim \mathcal{M}_\mu f(x),\end{aligned}$$

where $\mathcal{M}_\mu$ is the centered Hardy-Littlewood maximal function with respect to μ. Apparently the same estimate holds for $L = \mathrm{Id}$. Moreover, by the mean value theorem, (3.2), and Lemma 3.9-(2) (which we will prove later), if $M_1 := \min\{\|L\|_{\mathrm{op}}, 1\}$, $M_2 := 2\max\{\|L\|_{\mathrm{op}}^{-1}, 1\}$, it holds that

$$\begin{aligned}&\left|\widetilde{T}^{\psi}_{L,\mu,\delta} f(x) - \widetilde{T}^{\psi}_{\mathrm{Id},\mu,\delta} f(x)\right| \\ &\lesssim \|\nabla\psi\|_{L^\infty}\|L\|_{\mathrm{op}}\|L - \mathrm{Id}\|_{\mathrm{op}} \int_{A(x,M_1\delta,M_2\delta)} \frac{|x-y|}{\delta} \frac{|f(y)|}{|L(x)-L(y)|^n}\, \mathrm{d}\mu(y) \\ &\lesssim_{\|L\|_{\mathrm{op}}} \mathcal{M}_\mu f(x),\end{aligned}$$

and our result readily follows by triangle inequality, the $L^2(\mu)$-boundedness of $\mathcal{M}_\mu$, and the fact that $\widetilde{T}_{\mathrm{Id},\mu,\delta} f = T_{\mu,\delta} f$. □

As a direct application we have that, for ϕ, A_ϕ, and ν as in (3.4), and if we further assume that ϕ is an invertible linear map, Lemma 3.4 reads as

$$1 + \|T_{\phi,\nu,\delta}\|_{L^2(\nu)\to L^2(\nu)} \approx 1 + \left\|\widetilde{T}_{\phi,\nu,\delta}\right\|_{L^2(\nu)\to L^2(\nu)},$$

so

$$\|T_{\phi,\nu,\delta}\|_{L^2(\nu)\to L^2(\nu)} \lesssim 1 + \left\|\widetilde{T}_{\phi,\nu,\delta}\right\|_{L^2(\nu)\to L^2(\nu)} \overset{(3.5)}{\lesssim} 1 + \|T_{\mu,\delta}\|_{L^2(\mu)\to L^2(\mu)}. \tag{3.7}$$

We remark that, if A is a uniformly elliptic matrix with real entries and $A_s := (A + A^T)/2$ is its symmetric part, for every $x \in \mathbb{R}^{n+1}$ and $r > 0$ the matrix $(\bar{A}_s)_{x,r}$ is a symmetric and uniformly elliptic matrix with real entries. In particular, it admits a unique square root $\sqrt{(\bar{A}_s)_{x,r}}$, which is symmetric and uniformly elliptic, too.

A particularly useful change of variables is the one that transforms the symmetric part of the matrix at a given point into the identity (see [2, Lemma 2.5]). A standard application of Lemma 3.1 and change of variables allows us to state the following adaptation to the context of the present paper.

Lemma 3.5. *Let A be a uniformly elliptic matrix with real entries and let A_s be its symmetric part. For a fixed point $x \in \mathbb{R}^{n+1}$ and $r > 0$, define $S = \sqrt{(\bar{A}_s)_{x,r}}$. If*

$$\hat{A}(\cdot) := \frac{|\det S||S^{-1}(B(x,r))|}{|B(x,r)|} S^{-1}(A \circ S)(\cdot) S^{-1}, \tag{3.8}$$

then $\hat{A}$ is uniformly elliptic, $⨍_{S^{-1}(B(x,r))} \hat{A}_s = Id$ and u is a weak solution of $L_A u = 0$ in $\mathbb{R}^{n+1}$ if and only if $\tilde{u} = u \circ S$ is a weak solution of $L_{\hat{A}}\tilde{u} = 0$ in $\mathbb{R}^{n+1}$.

Proof. In light of Lemma 3.1, we only have to verify that $\fint_{S^{-1}(B(x,r))} \hat{A}_s = Id$. This follows from a change of variables and the definition of S. In particular, we have

$$\begin{aligned}\fint_{S^{-1}(B(x,r))} \hat{A}_s(y)\,\mathrm{d}y &= \frac{|\det S||S^{-1}(B(x,r))|}{|B(x,r)|} S^{-1}\Big(\fint_{S^{-1}(B(x,r))} A_s(Sy)\,\mathrm{d}y\Big)S^{-1} \\ &= \frac{|\det S||S^{-1}(B(x,r))|}{|B(x,r)|} S^{-1}\Big(\frac{1}{|S^{-1}(B(x,r))|}\int_{B(x,r)} A_s(z)|\det S^{-1}|\,\mathrm{d}z\Big)S^{-1} \\ &= S^{-1}\Big(\frac{1}{|B(x,r)|}\int_{B(x,r)} A_s(z)\,\mathrm{d}z\Big)S^{-1} = Id.\end{aligned}$$

This concludes the proof of the lemma. □

The change of variables S defined in the previous lemma is a bilipschitz function with bi-Lipschitz constant $\Lambda^{1/2}$, and it maps balls to ellipsoids. In particular, we have that $\Lambda^{-1/2} \leq |S| \leq \Lambda^{1/2}$. For more details we refer to [2, Section 2]. We also remark that in the setting of Lemma 3.5 we have that, for $\zeta \in \mathbb{R}^{n+1}$, $\rho > 0$, and denoting $\tilde{\zeta} := S\zeta$, it holds that

$$B\big(\tilde{\zeta}, \Lambda^{-1/2}\rho/2\big) \subset S\big(B(\zeta,\rho)\big) \subset B\big(\tilde{\zeta}, 2\Lambda^{1/2}\rho\big). \tag{3.9}$$

Furthermore, we equivalently have that

$$B(\zeta,\rho) \subset S^{-1}\big(B(S\zeta, 2\Lambda^{1/2}\rho)\big) \subset B(\zeta, 4\Lambda\rho). \tag{3.10}$$

The mean oscillation of the transformed matrix under the change of variables in (3.8) can be controlled according to the following lemma.

Lemma 3.6. *Let A be a uniformly elliptic matrix with real entries, S and $\hat{A}$ be as in Lemma 3.5, and define*

$$\hat{\omega}_{\hat{A}}(\rho) := \sup_{z\in\mathbb{R}^{n+1}} \fint_{S^{-1}(B(\zeta,\rho))} \Big|\hat{A}(y) - \fint_{S^{-1}(B(\zeta,\rho))} \hat{A}\Big|\,\mathrm{d}y \qquad \text{for } \rho > 0.$$

Then we have that

$$\hat{\omega}_{\hat{A}}(\rho) \approx_\Lambda \omega_A(\rho) \approx_{\Lambda,\kappa} \omega_{\hat{A}}(\rho) \qquad \text{for all} \qquad \rho > 0, \tag{3.11}$$

where Λ is the uniform ellipticity constant of A. In particular, if $A \in \mathrm{DMO}_s$ then $\hat{A} \in \mathrm{DMO}_s$ as well.

Proof. The upper bound in the first equality of (3.11) is an easy consequence of the definition of $\hat{A}$, the change of variables formula, the uniform ellipticity of A and the fact that $|S^{-1}| \approx_\Lambda 1 \approx_\Lambda |\det S|$. In particular, for any ball $B \subset \mathbb{R}^{n+1}$ of radius $r(B)$ we have that

$$\begin{aligned}&\fint_{S^{-1}(B)} \Big|\hat{A}(y) - \fint_{S^{-1}(B)} \hat{A}\Big|\,\mathrm{d}y \\ &\quad\approx_\Lambda \frac{1}{|S^{-1}(B)|^2}\int_{S^{-1}(B)}\Big|\int_{S^{-1}(B)} \big(S^{-1}A(Sz)S^{-1} - S^{-1}A(Sw)S^{-1}\big)\,\mathrm{d}w\Big|\,\mathrm{d}z \\ &\quad\lesssim_\Lambda \frac{1}{|B|^2}\int_B |S^{-1}|\Big|\int_B \big(A(z)-A(w)\big)\,\mathrm{d}w\Big||S^{-1}|\,\mathrm{d}z \lesssim_\Lambda \frac{1}{|B|}\int_B\Big|A(z) - \fint_B A\Big|\,\mathrm{d}z.\end{aligned}$$

The proof of the lower bound is analogous.

In order to prove the second estimate in (3.11), we observe that, given a ball $B{:=}B(\zeta,\rho)$ and denoting $\tilde{B}{:=}B(S\zeta, 2\Lambda^{1/2}\rho)$, the inclusion (3.10), the first estimate in (3.11) and the doubling assumption yield

$$\begin{aligned}\fint_B \Big|\hat{A}(y) - \fint_B \hat{A}\Big|\, \mathrm{d}y &\le \fint_B \Big|\hat{A}(y) - \fint_{S^{-1}(\tilde{B})} \hat{A}\Big|\, \mathrm{d}y + \Big|\fint_{S^{-1}(\tilde{B})} \hat{A} - \fint_B \hat{A}\Big| \\ &\lesssim_\Lambda \fint_{S^{-1}(\tilde{B})} \Big|\hat{A}(y) - \fint_{S^{-1}(\tilde{B})} \hat{A}\Big|\, \mathrm{d}y \le \hat{\omega}_{\hat{A}}\big(2\Lambda^{1/2}\rho\big) \lesssim_{\Lambda,\kappa} \omega_A(\rho).\end{aligned}$$

The converse inequality can be proved analogously. □

We recall that, if A_0 is a uniformly elliptic matrix with constant coefficients, then

$$\Gamma_{A_0}(x, y) = \Gamma_{(A_0)_s}(x, y) \qquad \text{for all } x, y \in \mathbb{R}^{n+1}, x \neq y. \tag{3.12}$$

Let $A \in \mathrm{DMO}_s$. The quantity $\mathfrak{I}_{\omega_A}(r)$ defined in (1.7) satisfies $\mathfrak{I}_{\omega_A}(2^{-j}r) \to 0$ for every $0 < r < 1$ by Remark 2.2. We remark that it is not necessary to assume that $\omega_A(r)$ vanishes as $r \to 0^+$ for this property to hold.

3.2. Estimates for the gradient of the fundamental solutions

The following theorem is an easy adaptation of one of the main results of [15]:

Theorem 3.7. *Let A be a uniformly elliptic matrix satisfying $A \in \mathrm{DMO}_s$. Let $0 < \eta < 1/2$, and set $N{:=}3(\frac{4}{3})^{N_\eta}$ for N_η such that $2^{-N_\eta - 1} \le \eta < 2^{-N_\eta}$. Assume that $g\colon B(0, N+1) \to \mathbb{R}^m$ is a function that satisfies the Dini mean oscillation condition*

$$\int_0^1 \mathring{\omega}_g^{0,N}(t)\,\frac{\mathrm{d}t}{t} < +\infty, \quad \textit{where} \quad \mathring{\omega}_g^{0,k}(t){:=} \sup_{w\in B(0,k)} \fint_{B(w,t)} |g(x) - \bar{g}_{w,t}|\,\mathrm{d}x, \qquad k \ge 1. \tag{3.13}$$

Let u be a weak solution of

$$\operatorname{div}(A(x)\nabla u) = \operatorname{div} g, \qquad \textit{in } B(0, N+1). \tag{3.14}$$

There exists an absolute value of η such that, if g and u satisfy (3.13) and (3.14), then $u \in C^1\big(\overline{B(0,1)}; \mathbb{R}^m\big)$. Furthermore, it holds that

$$\|\nabla u\|_{L^\infty(B(0,2))} \lesssim \|\nabla u\|_{L^1(B(0,4))} + \int_0^1 \mathring{\omega}_g^{0,N}(t)\,\frac{\mathrm{d}t}{t}, \tag{3.15}$$

and, for $x, y \in B(0,1)$ such that $|x-y| < 1/2$,

$$\begin{aligned}|\nabla u(x) - \nabla u(y)| &\lesssim \|\nabla u\|_{L^1(B(0,4))}|x-y|^\beta \\ &+ \left(\|\nabla u\|_{L^1(B(0,4))} + \int_0^1 \mathring{\omega}_g^{0,N}(t)\,\frac{\mathrm{d}t}{t}\right)\int_0^{|x-y|} \mathring{\omega}_A^{0,N}(t)\,\frac{\mathrm{d}t}{t} + \int_0^{|x-y|} \mathring{\omega}_g^{0,N}(t)\,\frac{\mathrm{d}t}{t},\end{aligned} \tag{3.16}$$

where $\beta > 0$ and the implicit constants depend on n.

Proof. For the proof of the fact that u belongs to $C^1\big(\overline{B(0,1)}\big)$, we refer to [15, Theorem 1.5]. The inequality (3.15) is a variant of [15, (2.17)], which is formulated in terms of a slightly different modulus of oscillation (see [15, (2.15)]). In order to prove (3.15), we fix an exponent $0 < p < 1$ and define

$$\phi(\bar{x}, r) := \inf_{c\in\mathbb{R}^{n+1}} \left(\fint_{B(\bar{x},r)} |\nabla u(z) - c|^p \, \mathrm{d}z \right)^{1/p}, \qquad \bar{x} \in \mathbb{R}^{n+1},\ 0 < r < 1/3.$$

It was proved in [15, p. 424] that, for $0 < \eta < 1/2$ sufficiently small (depending on p and an absolute constant, see the bottom of [15, p. 423]) and $j = 1, 2, \ldots$,

$$\phi\big(\bar{x}, \eta^j r\big) \lesssim 2^{-j}\phi(\bar{x}, r) + \|\nabla u\|_{L^\infty(B(x,r))} \sum_{i=1}^{j} 2^{1-i} \mathring{\omega}_A^{0,3}\big(\eta^{j-i} r\big) + \sum_{i=1}^{j} 2^{1-i} \mathring{\omega}_g^{0,3}\big(\eta^{j-i} r\big). \tag{3.17}$$

Now notice that

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j} 2^{1-i} \mathring{\omega}_g^{0,3}\big(\eta^{j-i} r\big) = \sum_{i=1}^{\infty}\sum_{j=i}^{\infty} 2^{1-i} \mathring{\omega}_g^{0,3}\big(\eta^{j-i} r\big) = 2\sum_{j=0}^{\infty} \mathring{\omega}_g^{0,3}\big(\eta^{j} r\big). \tag{3.18}$$

By [24, p. 495], there exists a constant $\kappa > 0$ depending only on n such that

$$\mathring{\omega}_g^{0,3}(s) \le \kappa \mathring{\omega}_g^{0,4}(t), \qquad \text{for } \frac{s}{2} \le t < s. \tag{3.19}$$

Moreover, by definition of N and a minor variant of (2.1) and (2.2), we have that

$$\sum_{j=0}^{\infty} \mathring{\omega}_g^{0,3}(\eta^j r) \lesssim \int_0^r \mathring{\omega}_g^{0,N}(t)\, \frac{\mathrm{d}t}{t}. \tag{3.20}$$

Analogous properties hold for $\mathring{\omega}_A^{0,3}$. Thus,

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j} 2^{1-i} \mathring{\omega}_g^{0,3}\big(\eta^{j-i} r\big) \overset{(3.18)}{\le} 2\sum_{j=0}^{\infty} \mathring{\omega}_g^{0,3}\big(\eta^{j} r\big) \overset{(3.20)}{\lesssim} \int_0^r \mathring{\omega}_g^{0,N}(t)\, \frac{\mathrm{d}t}{t}.$$

The same computations can be repeated to handle the second summand on the right hand side of (3.17). Finally, the inequalities (3.15) and (3.16) follow as in [15, (2.17) and (2.19)]. □

Remark 3.8. Let $0 < r < R_0$, $x_0 \in \mathbb{R}^{n+1}$, and $g\colon B(x_0, (N+1)R_0) \to \mathbb{R}$, where $N := 3(\frac{4}{3})^{N_\eta}$ for η as in (3.17) and N_η such that $2^{-N_\eta - 1} \le \eta < 2^{-N_\eta}$. For $0 < t < r$ and $1 \le k \le N$, we define

$$\mathring{\omega}_g^{x_0,kr}(t) = \sup_{w\in B(x_0,kr)} \fint_{B(w,tr)} \big|g(x) - \bar{g}_{w,tr}\big| \, \mathrm{d}x.$$

Let A be a uniformly elliptic such that $A \in \mathrm{DMO}_s$ and assume that, for N as above,

$$\int_0^1 \mathring{\omega}_g^{x_0,NR_0}(t)\, \frac{\mathrm{d}t}{t} < \infty.$$

Then by the proof of Theorem 3.7, it holds that if u is a weak solution to $-\operatorname{div}(A(x)\nabla u) = -\operatorname{div} g$ in $B(x_0, (N+1)R_0)$, we obtain that $u \in C^1(\overline{B(x_0, r)})$ and satisfies the estimates

$$\|\nabla u\|_{L^\infty(B(x_0,2r))} \lesssim_{R_0} \fint_{B(x_0,4r)} |\nabla u(x)|\,\mathrm{d}x + \int_0^1 \mathring{\omega}_g^{x_0,Nr}(t)\,\frac{\mathrm{d}t}{t}, \tag{3.21}$$

and, for $x, y \in B(x_0, r)$ such that $|x-y| < r/2$,

$$\begin{aligned} |\nabla u(x) - \nabla u(y)| &\lesssim_{R_0} \frac{|x-y|^\beta}{r^\beta} \fint_{B(x_0,4r)} |\nabla u(z)|\,\mathrm{d}z \\ &+ \left(\fint_{B(x_0,4r)} |\nabla u(z)|\,\mathrm{d}z + \int_0^1 \mathring{\omega}_g^{x_0,Nr}(t)\,\frac{\mathrm{d}t}{t} \right) \int_0^{\frac{|x-y|}{r}} \mathring{\omega}_A^{x_0,Nr}(t)\,\frac{\mathrm{d}t}{t} \\ &+ \int_0^{\frac{|x-y|}{r}} \mathring{\omega}_g^{x_0,Nr}(t)\,\frac{\mathrm{d}t}{t}. \end{aligned} \tag{3.22}$$

Furthermore, we have that the implicit constants blow up logarithmically as $R_0 \to \infty$. If $0 < R_0 < 1$, they only depend on ellipticity, dimension, and the Dini Mean Oscillation condition.

An important consequence of the pointwise bounds of Theorem 3.7, are the following estimates for the fundamental solution and its derivatives:

Lemma 3.9. *Let $A(\cdot) = (a_{ij})$ be a uniformly elliptic matrix in $\mathbb{R}^{n+1}$, $n \geq 2$, satisfying $A \in \mathrm{DMO}_s$. Let $R > 0$, and let $\beta > 0$ be as in Theorem 3.7. Then there exists $C = C(n, \Lambda, R) > 0$ such that the fundamental solution Γ_A satisfies the following pointwise bounds:*

(1) $|\Gamma_A(x,y)| \leq C|x-y|^{-(n-1)}$ *for* $x, y \in \mathbb{R}^{n+1}$, $0 < |x-y| < R$.
(2) $|\nabla_1\Gamma_A(x,y)| + |\nabla_2\Gamma_A(x,y)| \leq C|x-y|^{-n}$ *for* $x, y \in \mathbb{R}^{n+1}$, $0 < |x-y| < R$.
(3) $|\nabla_1\nabla_2\Gamma_A(x,y)| \leq C|x-y|^{-(n+1)}$ *for* $x, y \in \mathbb{R}^{n+1}$, $0 < |x-y| < R$.
(4) We have

$$\begin{aligned} &|\nabla_1\Gamma_A(x,y) - \nabla_1\Gamma_A(x,z)| + |\nabla_1\Gamma_A(y,x) - \nabla_1\Gamma_A(z,x)| \\ &\quad \leq C\left(\frac{|y-z|^\beta}{|x-y|^\beta} + \int_0^{\frac{|y-z|}{|x-y|}} \omega_A(t)\,\frac{\mathrm{d}t}{t} \right) |x-y|^{-n}, \end{aligned} \tag{3.23}$$

for $2|y-z| \leq |x-y| < R$.

Proof. For the proof of (1) we refer to [21, Section 5]. The bounds (2) and (4) follow directly from (1), the fact that the function $u_y(\cdot) := \Gamma_A(\cdot, y)$ satisfies $L_A u_y = 0$ in $B(x, |x-y|/8)$, and Remark 3.8. The bound for $\nabla_1\nabla_2\Gamma_A$ can be proved analogously, observing that $w_y(\cdot) := \nabla_2\Gamma_A(\cdot, y)$ satisfies $L_A w_y = 0$ in $B(x, |x-y|/8)$. □

If $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is an open set and $2^* := \frac{2(n+1)}{n-1}$, we define $Y^{1,2}(\Omega)$ as the space of all weakly differentiable functions $u \in L^{2^*}(\Omega)$, whose weak derivatives belong to $L^2(\Omega)$. We endow $Y^{1,2}(\Omega)$ with the norm

$$\|u\|_{Y^{1,2}(\Omega)} := \|u\|_{L^{2^*}(\Omega)} + \|\nabla u\|_{L^2(\Omega)}, \qquad u \in Y^{1,2}(\Omega).$$

We denote by $Y_0^{1,2}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $Y^{1,2}(\Omega)$ and remark that $Y^{1,2}(\mathbb{R}^{n+1}) = Y_0^{1,2}(\mathbb{R}^{n+1})$ (see for instance [25, p. 46]).

Let A be a uniformly elliptic matrix in $\mathbb{R}^{n+1}$, $n \geq 2$, with coefficients in $L^\infty(\mathbb{R}^{n+1})$. For $g \in L^2(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$, we denote by $L_A^{-1}\nabla \cdot g$ the unique solution u of the variational Dirichlet problem $L_A u = -\mathrm{div} g$ and $u \in Y_0^{1,2}(\mathbb{R}^{n+1})$. By a modification of the argument that proves [21, (3.47)] (for the details see, for instance, the proof of [28, (6.3)]), for $g \in L^2(\mathbb{R}^{n+1})$, we have that

$$L_A^{-1}\nabla \cdot g(x) = \int \nabla_2 \Gamma_A(x, y) \cdot g(y)\, \mathrm{d}y, \tag{3.24}$$

and as in [21, (3.10)-(3.11)], one can prove that

$$\|\nabla L_A^{-1}\nabla \cdot g\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|g\|_{L^2(\mathbb{R}^{n+1})}. \tag{3.25}$$

The proof of the next lemma is a standard adaptation of the one of [21, Corollary 3.5], which we present below for the reader's convenience.

Lemma 3.10. *If A and $\tilde{A}$ are uniformly elliptic matrices in $\mathbb{R}^{n+1}$, $n \geq 2$, so that $A, \tilde{A} \in \mathrm{DMO}_s$, then for $R > 0$ and all $x, y \in \mathbb{R}^{n+1}$ such tha $0 < |x - y| < R$, it holds that*

$$\Gamma_{\tilde{A}}(x, y) - \Gamma_A(x, y) = \int \nabla_2 \Gamma_A(x, z) \cdot \big(A(z) - \tilde{A}(z)\big) \nabla_1 \Gamma_{\tilde{A}}(z, y)\, \mathrm{d}z. \tag{3.26}$$

Proof. Set $\Gamma{:=}\Gamma_A$, $\tilde{\Gamma}{:=}\Gamma_{\tilde{A}}$, $\Gamma_*{:=}\Gamma_{A^T}$, $\tilde{\Gamma}_*{:=}\Gamma_{\tilde{A}^T}$, and $r{:=}|x-y|/4$. For $0 < \rho < r$ we denote as Γ^ρ the *averaged fundamental solution* (see [21, Section 3.1]) which can be defined, via Lax-Milgram theorem, as the unique function in $Y_0^{1,2}(\mathbb{R}^{n+1})$ such that

$$\int A \nabla_1 \Gamma^\rho(\cdot, y) \cdot \nabla u = \fint_{B(y,\rho)} u = \int f_{\rho,y} u \qquad \text{for all } u \in Y_0^{1,2}(\mathbb{R}^{n+1}), \tag{3.27}$$

for $f_{\rho,y}(z){:=}|B(y,\rho)|^{-1}\chi_{B(y,\rho)}(z)$. Note that, if B is a ball such that $B \cap B(y, \rho) = \varnothing$ and $u \in C_c^\infty(B)$, then the right hand side of (3.27) vanishes, i.e., $L_A \Gamma^\rho(\cdot, y) = 0$ in B. Thus, the De Giorgi-Nash-Moser theorem implies that $\Gamma^\rho(\cdot, y)$ is locally Hölder continuous in $\mathbb{R}^{n+1} \setminus \overline{B(y, \rho)}$. Moreover, by [21, (3.45)], Γ^ρ admits the representation

$$\Gamma^\rho(x, y){:=}\fint_{B(y,\rho)} \Gamma(x, z)\, \mathrm{d}z. \tag{3.28}$$

We define $\tilde{\Gamma}^\rho, \Gamma_*^\rho, \tilde{\Gamma}_*^\rho \in Y_0^{1,2}(\mathbb{R}^{n+1})$ analogously and it holds that

$$\int \nabla_1 \tilde{\Gamma}_*^\rho(\cdot, x) \cdot \tilde{A} \nabla u = \fint_{B(x,\rho)} u \qquad \text{for all } u \in Y_0^{1,2}(\mathbb{R}^{n+1}). \tag{3.29}$$

We claim that

$$\Gamma^\rho(x, y) = \int \nabla_2 \tilde{\Gamma}(x, \cdot) \cdot \tilde{A}(\cdot) \nabla_1 \Gamma^\rho(\cdot, y). \tag{3.30}$$

Let $0 < \rho' < \rho < r$. If we apply (3.29) for $\tilde{\Gamma}_*^{\rho'}$ and $u = \Gamma^\rho(\cdot, y) \in Y_0^{1,2}(\mathbb{R}^{n+1})$, then it holds that

$$\int \nabla_1 \tilde{\Gamma}_*^{\rho'}(z, x) \cdot \tilde{A} \nabla_1 \Gamma^\rho(z, y)\, \mathrm{d}z = \fint_{B(x,\rho')} \Gamma^\rho(z, y)\, \mathrm{d}z. \tag{3.31}$$

Since $\Gamma^\rho(\cdot, y)$ is continuous away from y, by Lebesgue's differentiation theorem,

$$\lim_{\rho' \to 0} \fint_{B(x,\rho')} \Gamma^\rho(z, y)\, \mathrm{d}z = \Gamma^\rho(x, y). \tag{3.32}$$

Moreover, by [21, (3.22) and (3.24)], there exist $C_1, C_2 > 0$ such that

$$\int_{\mathbb{R}^{n+1} \setminus B(x,r)} \left| \nabla_1 \Gamma_{\tilde{A}^T}^{\rho'}(z, x) \right|^2 \mathrm{d}z \leq C_1 r^{1-n} \tag{3.33}$$

and

$$\left\| \nabla_1 \Gamma_{\tilde{A}^T}^{\rho'}(\cdot, x) \right\|_{L^p(B(x,r))} \leq C_2 r^{\frac{n+1}{p} - n} \qquad \text{for } p \in [1, (n+1)/n), \tag{3.34}$$

where C_1 and C_2 do not depend on ρ'.

We now split

$$\begin{aligned} &\int \nabla_1 \tilde{\Gamma}_*^{\rho'}(z, x) \cdot \tilde{A} \nabla_1 \Gamma^\rho(z, y)\, \mathrm{d}z \\ &\quad = \int_{B(x,r)} + \int_{\mathbb{R}^{n+1} \setminus B(x,r)} \nabla_1 \tilde{\Gamma}_*^{\rho'}(z, x) \cdot \tilde{A} \nabla_1 \Gamma^\rho(z, y)\, \mathrm{d}z =: I_{\rho'} + II_{\rho''}. \end{aligned} \tag{3.35}$$

Since $A \in \mathrm{DMO}_s$ and $L_A \Gamma^\rho(\cdot, y) = 0$ in $B(x, r)$, by Lemma 3.9 and (3.28), for any $z \in B(x, r)$, it holds that

$$|\nabla_1 \Gamma_A^\rho(z, y)| \leq \fint_{B(y,\rho)} |\nabla_1 \Gamma(z, w)|\, \mathrm{d}w \lesssim \fint_{B(y,\rho)} \frac{1}{|w - z|^n}\, \mathrm{d}w \approx r^{-n} \approx |x - y|^{-n},$$

where in the penultimate inequality we used that $|w - z| \approx r$ for $w \in B(y, \rho)$, and so $\nabla_1 \Gamma^\rho(\cdot, y) \in L^\infty(B(x, r))$.

Fix $p \in \left(1, \frac{n+1}{n}\right)$ and consider a sequence $\rho'_j \to 0$ such that $\rho'_j < \rho < r$ for all j. As in the proof of [21, Theorem 3.1], by (3.34) and weak compactness of $W^{1,p}(B(x, r))$, we may pass to a subsequence, which we still denote by ρ'_j, such that $\nabla_1 \tilde{\Gamma}_*^{\rho'_j}(\cdot, x) \rightharpoonup \nabla_1 \tilde{\Gamma}_*(\cdot, x)$ in $L^p(B(x, r))$. Hence

$$\lim_{j \to \infty} I_{\rho'_j} = \int_{B(x,r)} \nabla_1 \tilde{\Gamma}_*(z, x) \cdot \tilde{A} \nabla_1 \Gamma^\rho(z, y)\, \mathrm{d}z. \tag{3.36}$$

Furthermore, once again as in the proof of [21, Theorem 3.1], the bound (3.33) implies that, by passing to another subsequence if necessary, $\nabla_1 \tilde{\Gamma}_*^{\rho'_j}(\cdot, x) \rightharpoonup \nabla_1 \tilde{\Gamma}_*(\cdot, x)$ in $L^2(\mathbb{R}^{n+1} \setminus B(x, r))$. Thus, since $\Gamma^\rho(\cdot, y) \in Y_0^{1,2}(\mathbb{R}^{n+1})$,

$$\lim_{j \to \infty} II_{\rho'_j} = \int_{\mathbb{R}^{n+1} \setminus B(x,r)} \nabla_1 \tilde{\Gamma}_*(z, x) \cdot \tilde{A} \nabla_1 \Gamma^\rho(z, y)\, \mathrm{d}z. \tag{3.37}$$

Therefore, (3.36), (3.37), and (3.32), imply

$$\Gamma^\rho(x,y) = \int \nabla_1 \tilde{\Gamma}_*(z,x) \cdot \tilde{A} \nabla_1 \Gamma^\rho(z,y)\,\mathrm{d}z = \int \nabla_2 \tilde{\Gamma}(x,z) \cdot \tilde{A} \nabla_1 \Gamma^\rho(z,y)\,\mathrm{d}z, \tag{3.38}$$

where in the last equality we used that $\Gamma_{\tilde{A}^T}(w,z) = \Gamma_{\tilde{A}}(z,w)$ for all $z, w \in \mathbb{R}^{n+1}$, $z \neq w$ (see [21, (3.43)]). This concludes the proof of (3.30).

Let us now split

$$\begin{aligned}\int \nabla_2 \tilde{\Gamma}(x,\cdot) \cdot A(\cdot) \nabla_1 \Gamma^{\rho_j}(\cdot,y) &= \int_{B(x,r)} \\ &+ \int_{B(y,r)} + \int_{\mathbb{R}^{n+1}\setminus(B(x,r)\cup B(y,r))} =: I^1_{\rho_j} + I^2_{\rho_j} + I^3_{\rho_j}.\end{aligned}$$

Since $\tilde{\Gamma}(x,\cdot) \in Y^{1,2}(\mathbb{R}^{n+1}\setminus B(x,r))$, by the weak convergence $\nabla_1 \Gamma^{\rho_j}(\cdot,y) \rightharpoonup \nabla_1 \Gamma(\cdot,y)$ in $L^2(\mathbb{R}^{n+1}\setminus B(y,r))$, we have that

$$\lim_{j\to\infty} I^3_{\rho_j} = \int_{\mathbb{R}^{n+1}\setminus(B(x,r)\cup B(y,r))} \nabla_2 \tilde{\Gamma}(x,z) \cdot A(z) \nabla_1 \Gamma(z,y)\,\mathrm{d}z.$$

By Lemma 3.9, it holds that $|\nabla_2 \tilde{\Gamma}(x,z)| \lesssim |x-y|^{-n}$, for all $z \in B(y,r)$, and since $\nabla_1 \Gamma^{\rho_j}(\cdot,y) \rightharpoonup \nabla_1 \Gamma(\cdot,y)$ in $L^p(B(y,r))$, we get that

$$\lim_{j\to\infty} I^2_{\rho_j} = \int_{B(y,r)} \nabla_2 \tilde{\Gamma}(x,z) \cdot A(z) \nabla_1 \Gamma(z,y)\,\mathrm{d}z.$$

Lastly, by Lemma 3.9, $\nabla_1 \Gamma^{\rho_j}(\cdot,y)$ is a uniformly bounded equicontinuous family of functions in $B(x,r)$, and so, after passing to a subsequence, we get that $\nabla_1 \Gamma^{\rho_j}(\cdot,y) \to \nabla_1 \Gamma(\cdot,y)$ uniformly in $B(x,r)$. By [21, (3.52)], it holds that

$$\|\nabla_2 \tilde{\Gamma}(x,\cdot)\|_{L^1(B(x,r))} \lesssim r,$$

and thus, we deduce that

$$\lim_{j\to\infty} I^1_{\rho_j} = \int_{B(x,r)} \nabla_2 \tilde{\Gamma}(x,z) \cdot A(z) \nabla_1 \Gamma(z,y)\,\mathrm{d}z.$$

Hence, as $\lim_{j\to\infty} \Gamma^{\rho_j}(x,y) = \Gamma(x,y)$, by (3.38), we conclude that

$$\Gamma(x,y) = \int \nabla_2 \tilde{\Gamma}(x,\cdot) \cdot A \nabla_1 \Gamma(\cdot,y). \tag{3.39}$$

Similarly, we can be prove that

$$\tilde{\Gamma}(x,y) = \tilde{\Gamma}_*(y,x) = \int \nabla_2 \Gamma_*(y,\cdot) \cdot \tilde{A}^T \nabla_1 \tilde{\Gamma}_*(\cdot,x) = \int \nabla_2 \tilde{\Gamma}(x,\cdot) \cdot \widetilde{A} \nabla_1 \Gamma(\cdot,y),$$

which, together with (3.39), concludes the proof of the lemma. □

For a real elliptic $(n+1)\times(n+1)$-matrix A_0 with constant coefficients, we use the notation $\Theta(x, y; A_0) = \Gamma_{A_0}(x, y)$ to denote the fundamental solution of L_{A_0}. In particular, we recall that $\Theta(x, y; A_0) = \Theta(x-y, 0; A_0)$ and

$$\Theta(z, 0; A_0) = \Theta(z, 0; A_{0,s}) = \begin{cases} \dfrac{-1}{(n-1)\omega_n\sqrt{\det A_{0,s}}}\dfrac{1}{\langle A_{0,s}^{-1}z, z\rangle^{(n-1)/2}} & \text{for } n \geq 2, \\[2ex] \dfrac{1}{4\pi\sqrt{\det A_{0,s}}}\log\left(\langle A_{0,s}^{-1}z, z\rangle\right) & \text{for } n = 1, \end{cases} \tag{3.40}$$

where $A_{0,s}:=\frac{1}{2}(A_0 + A_0^T)$ is the symmetric part of A_0 and ω_n is the surface measure of the unit sphere $\mathbb{S}^n$ (see also Section 2). Moreover it holds

$$\nabla_1\Theta(z, 0; A_0) = \frac{\omega_n^{-1}}{\sqrt{\det A_{0,s}}}\frac{A_{0,s}^{-1}z}{\langle A_{0,s}^{-1}z, z\rangle^{(n+1)/2}}, \qquad \text{for } z \in \mathbb{R}^{n+1}\setminus\{0\}. \tag{3.41}$$

Finally we observe that, for any integer $k \geq 0$, we have

$$\left|\nabla_1^k\Theta(z, 0; A_0)\right| \lesssim \frac{1}{|z|^{n+k-1}} \qquad \text{for } z \in \mathbb{R}^{n+1}\setminus\{0\}, \tag{3.42}$$

where the implicit constant depends on dimension, the ellipticity constants of $A_{0,s}$, and the order of differentiation k.

In the next lemma we show that $\mathfrak{I}_{\omega_A}(r)$ controls the mean oscillation of a matrix A at *all* scales below $r > 0$. This is used in the proof of Lemma 3.12.

Lemma 3.11. *Let $A = (a_{ij})_{1\leq i,j\leq n+1}$ be a matrix such that $a_{ij} \in L^\infty(\mathbb{R}^{n+1})$ for $1 \leq i, j \leq n+1$ and $A \in \mathrm{DMO}_s$. Then, for $r > 0$ and $p \geq 1$,*

$$\sup_{0<\rho\leq r}\sup_{x\in\mathbb{R}^{n+1}}\left(⨍_{B(x,\rho)}|A(z) - \bar{A}_{x,\rho}|^p\,dz\right)^{1/p} \lesssim p\,\mathfrak{I}_{\omega_A}(r). \tag{3.43}$$

Moreover, if $\mathcal{C}_j:=A(x, 2^j r, 2^{j+1}r)$, $j \in \mathbb{N}$, we have

$$\left(⨍_{\mathcal{C}_j}|A(z) - \bar{A}_{x,r}|^p\,dz\right)^{1/p} \lesssim p\,\mathfrak{I}_{\omega_A}(2^j r). \tag{3.44}$$

Proof. By the John-Nirenberg inequality (see for instance [37, p. 144]), it holds

$$\begin{aligned} &\sup_{0<\rho\leq r}\sup_{x\in\mathbb{R}^{n+1}}\left(⨍_{B(x,\rho)}|A(z) - \bar{A}_{x,\rho}|^p\,dz\right)^{1/p} \\ &\lesssim p\sup_{0<\rho\leq r}\sup_{x\in\mathbb{R}^{n+1}}⨍_{B(x,\rho)}|A(z) - \bar{A}_{x,\rho}|\,dz. \end{aligned}$$

Let $x \in \mathbb{R}^{n+1}$, $r > 0$, and $\rho \in [0, r]$. We denote by N the positive integer such that $2^N \rho \leq r < 2^{N+1}\rho$. Then

$$\begin{aligned}
&\fint_{B(x,\rho)} |A(z) - \bar{A}_{x,\rho}| \,\mathrm{d}z \leq 2 \fint_{B(x,\rho)} |A(z) - \bar{A}_{x,r}| \,\mathrm{d}z \\
&\quad \lesssim 2 \fint_{B(x,\rho)} |A(z) - \bar{A}_{x,2\rho}| \,\mathrm{d}z + \sum_{j=2}^{N+1} |\bar{A}_{x,2^j\rho} - \bar{A}_{x,2^{j-1}\rho}| + |\bar{A}_{x,2^{N+1}\rho} - \bar{A}_{x,r}| \\
&\quad \lesssim \sum_{j=1}^{N+1} \omega_A(2^j\rho) \lesssim \int_0^{2^{N+1}\rho} \omega_A(t) \frac{\mathrm{d}t}{t} \leq \int_0^{2r} \omega_A(t) \frac{\mathrm{d}t}{t} \lesssim \mathfrak{I}_{\omega_A}(r),
\end{aligned}$$

where the last bound is a consequence of the doubling property of ω_A. Thus, taking the supremum for $\rho \in [0, r)$, we obtain (3.43).

Let us now prove (3.44). Let B_k^j, $k = 1, \ldots, M_n$, be a collection of balls of radius $\frac{5}{4}2^j r$ which covers $\mathcal{C}_j$, for a dimensional constant $M_n > 1$. Fix a ball $\widetilde{B}$ in this family, denote by $\tilde{x}$ its center, and define $L := \{tx + (1-t)\tilde{x} : t \in [0, 1]\}$. There exists a sequence of balls $B_0, \ldots, B_j$ centered at L such that $B_0 = B(x, r)$, $B_j = \widetilde{B}$, $r(B_k) \approx 2^k r$, and $B_{k+1} \cap B_k \neq \varnothing$. Moreover, for every $k = 1, \ldots, M_n$, there exists a ball B_k' centered at $L \cap B_k \cap B_{k+1}$ such that $r(B_k') \approx 2^k r$ and $B_k \cup B_{k+1} \subset B_k'$. Hence, if we denote $\bar{A}_{B_k} := \fint_{B_k} A$, it holds that

$$\begin{aligned}
\left|\bar{A}_{B_k} - \bar{A}_{B_{k+1}}\right| &\leq \left|\bar{A}_{B_k} - \bar{A}_{B_k'}\right| + \left|\bar{A}_{B_{k+1}} - \bar{A}_{B_k'}\right| \\
&\leq \fint_{B_k} |A(z) - \bar{A}_{B_k'}| \,\mathrm{d}z + \fint_{B_{k+1}} |A(z) - \bar{A}_{B_k'}| \,\mathrm{d}z \\
&\lesssim \fint_{B_k'} |A(z) - \bar{A}_{B_k'}| \,\mathrm{d}z \lesssim \omega_A(r(B_k')) \lesssim \omega_A(2^k r),
\end{aligned} \tag{3.45}$$

where the last bound follows from the doubling property of ω_A. Thus,

$$\left|\bar{A}_{B(x,r)} - \bar{A}_{\widetilde{B}}\right| \leq \sum_{k=0}^{j} \left|\bar{A}_{B_{k+1}} - \bar{A}_{B_k}\right| \overset{(3.45)}{\lesssim} \sum_{k=0}^{j} \omega_A(2^k r) \overset{(2.2)}{\lesssim} \mathfrak{I}_{\omega_A}(2^j r), \tag{3.46}$$

and so

$$\begin{aligned}
&\left(\fint_{\mathcal{C}_j} |A(z) - \bar{A}_{x,r}|^p \,\mathrm{d}z\right)^{1/p} \\
&\quad \leq \sum_{k=0}^{M_n} \left(\fint_{B_k^j} |A(z) - \bar{A}_{B_k^j}|^p \,\mathrm{d}z\right)^{1/p} + \sum_{k=0}^{M_n} \left|\bar{A}_{B(x,r)} - \bar{A}_{B_k^j}\right| \\
&\quad \overset{(3.46)}{\lesssim_n} \sum_{k=0}^{M_n} \left(\fint_{B_k^j} |A(z) - \bar{A}_{B_k^j}|^p \,\mathrm{d}z\right)^{1/p} + \mathfrak{I}_{\omega_A}(2^j r).
\end{aligned} \tag{3.47}$$

Observe that, for $k = 1, \ldots, M_n$, if we apply (3.43) we have

$$\left(\fint_{B_k^j} \left|A(z) - \bar{A}_{B_k^j}\right|^p \,\mathrm{d}z\right)^{1/p} \lesssim p\, \mathfrak{I}_{\omega_A}\big(r(B_k^j)\big) \lesssim p\, \mathfrak{I}_{\omega_A}(2^j r), \tag{3.48}$$

where in the last inequality we used the doubling property of $\mathfrak{I}_{\omega_A}$. Thus, by (3.47) and (3.48) we obtain (3.44) and conclude the proof of the lemma. □

3.3. The three-step perturbations

An important component of our method is the comparison of $\nabla_1 \Gamma_A$ to the gradient of the fundamental solution associated with the averaged matrix. This is what we prove in the next lemma.

Lemma 3.12. *Let A be a uniformly elliptic matrix in $\mathbb{R}^{n+1}$, $n \geq 2$, satisfying $A \in \mathrm{DMO}_s \cap \mathrm{DMO}_\ell$. For $R_0 > 0$, there exists $C = C(n, \Lambda, R_0) > 0$ such that, for $x, y \in \mathbb{R}^{n+1}$ such that $0 < |x-y| < R < R_0$, and*

$$\mathcal{K}^1_\Theta(x,y) := \nabla_1 \Gamma_A(x,y) - \nabla_1 \Theta\big(x, y; \bar{A}_{x,|x-y|/2}\big),$$

we have

$$|\mathcal{K}^1_\Theta(x,y)| \leq C\,\frac{\tau_A(r)}{r^n} + C\,\frac{\widehat{\tau}_A(R)}{R^n} \qquad \text{for } r := |x-y|/2, \tag{3.49}$$

where

$$\tau_A(r) := \mathfrak{I}_{\omega_A}(r) + \mathfrak{L}^n_{\omega_A}(r) = \int_0^r \omega_A(t)\,\frac{\mathrm{d}t}{t} + r^n \int_r^\infty \omega_A(t)\,\frac{\mathrm{d}t}{t^{n+1}}$$

and

$$\widehat{\tau}_A(R) = \mathfrak{I}_{\omega_A}(R) + \mathfrak{L}^{n-1}_{\omega_A}(R) = \int_0^R \omega_A(t)\,\frac{\mathrm{d}t}{t} + R^{n-1} \int_R^\infty \omega_A(t)\,\frac{\mathrm{d}t}{t^n}.$$

In particular, if $R_0 = 1$, the constant C only depends on ellipticity, dimension, and the Dini Mean Oscillation condition.

Proof. Let $x, y \in B(0, R)$. Let us denote by $\bar{a}_{ij}$ the coefficients of $\bar{A}_{x,r}$ and, for brevity, write $\Theta(z, y) := \Theta(z, y; \bar{A}_{x,r})$. By (3.26), it holds that

$$\begin{aligned}
\nabla_1 \Gamma(\cdot, y) - \nabla_1 \Theta(\cdot, y) &= \int \nabla_1 \nabla_2 \Gamma(\cdot, z)\big(\bar{A}_{x,r} - A(z)\big)\nabla_1 \Theta(z, y)\,\mathrm{d}z \\
&=: \int \Phi(\cdot, y, z)\,\mathrm{d}z \\
&= \int_{B(x,r)} \Phi(\cdot, y, z)\,\mathrm{d}z + \int_{B(y,r)} \Phi(\cdot, y, z)\,\mathrm{d}z \\
&\quad + \int_{\mathbb{R}^{n+1}\setminus(B(x,r)\cup B(y,r))} \Phi(\cdot, y, z)\,\mathrm{d}z =: I_r(\cdot) + II_r(\cdot) + III_r(\cdot).
\end{aligned} \tag{3.50}$$

First, let us estimate I_r. Setting $\varepsilon_{x,r}(z) := (\bar{A}_{x,r} - A(z))\,\chi_{B(x,r)}(z)$, we have that

$$\begin{aligned}
I_r(w) &= \int \nabla_1 \nabla_2 \Gamma(w, z) \cdot \varepsilon_{x,r}(z) \nabla_1 \Theta(z, y)\,\mathrm{d}z \\
&= \int \nabla_1 \nabla_2 \Gamma(w, z) \cdot \varepsilon_{x,r}(z)\big(\nabla_1 \Theta(z, y) - \nabla_1 \Theta(x, y)\big)\big)\,\mathrm{d}z \\
&\quad + \int \nabla_1 \nabla_2 \Gamma(w, z) \cdot \varepsilon_{x,r}(z) \nabla_1 \Theta(x, y)\,\mathrm{d}z \\
&=: I_{r,1}(w) + I_{r,2}(w).
\end{aligned}$$

For $w = x$, in order to estimate $I_{r,1}(x)$, we use Lemma 3.9 and write

$$|I_{r,1}(x)| \lesssim \int |\varepsilon_{x,r}(z)| \frac{\left|\nabla_1\Theta(z,y) - \nabla_1\Theta(x,y)\right|}{|x-z|^{n+1}}\,\mathrm{d}z$$
$$\lesssim \int \frac{|\varepsilon_{x,r}(z)|}{|x-z|^n|x-y|^{n+1}}\,\mathrm{d}z \approx \frac{1}{r^{n+1}}\int_{B(x,r)} \frac{|A(z)-\bar{A}_{x,r}|}{|x-z|^n}\,\mathrm{d}z.$$

Then, we estimate the last integral by splitting the domain of integration into dyadic annuli $A(x, 2^{-j-1}r, 2^{-j}r)$:

$$\begin{aligned}
|I_{r,1}(x)| &\lesssim \frac{1}{r^{n+1}}\sum_{j=0}^{\infty}\int_{A(x,2^{-j-1}r,2^{-j}r)} \frac{\left|A(z)-\bar{A}_{x,r}\right|}{|x-z|^n}\,\mathrm{d}z \\
&\lesssim \frac{1}{r^{n+1}}\sum_{j=0}^{\infty}\frac{r}{2^j}\fint_{B(x,2^{-j}r)}\left|A(z)-\bar{A}_{x,r}\right|\mathrm{d}z \\
&\leq \frac{1}{r^{n}}\sum_{j=0}^{\infty}\frac{1}{2^j}\fint_{B(x,2^{-j}r)}\Big(\left|A(z)-\bar{A}_{x,2^{-j}r}\right| + |\bar{A}_{x,2^{-j}r}-\bar{A}_{x,r}|\Big)\,\mathrm{d}z \\
&\lesssim \frac{1}{r^{n}}\sum_{j=0}^{\infty}\frac{1}{2^j}\bigg(\omega_A(2^{-j}r) + \sum_{k=0}^{j-1}\fint_{B(x,2^{-k}r)}\left|A(z)-\bar{A}_{x,2^{-k}r}\right|\mathrm{d}z\bigg) \\
&\lesssim \frac{1}{r^{n}}\sum_{j=0}^{\infty}\frac{1}{2^j}\bigg(\omega_A(2^{-j}r) + \sum_{k=0}^{j-1}\omega_A(2^{-k}r)\bigg) \\
&\lesssim \frac{1}{r^{n}}\sum_{j=0}^{\infty}\frac{j+1}{2^j}\omega_A(2^{-j}r) \lesssim \frac{1}{r^n}\int_0^r \omega_A(t)\frac{\mathrm{d}t}{t},
\end{aligned} \tag{3.51}$$

where we use that $j+1 \leq 2^j$ for any $j \geq 0$ integer.

The estimate of $I_{r,2}(x)$ is slightly more delicate. Let us denote $g{:=}\varepsilon_{x,r}(\cdot)\nabla_1\Theta(x,y)$ and observe that $L_A^{-1}\nabla\cdot g$ is a weak solution to $L_A\big(L_A^{-1}\nabla\cdot g\big) = \mathrm{div} g$ in $B(x, r/3)$. We claim that, for N as in Remark 3.8 and $\rho{:=}r/(3N+3)$,

$$\mathring{\omega}_g^{x,N\rho}(t) \lesssim \frac{\omega_A(t\rho)}{r^n}, \qquad 0 < t < 1, \tag{3.52}$$

with the implicit constant depending on n and the doubling parameter C_1. In order to prove (3.52) we first observe that, for $0 < t < 1$ and $\varepsilon(\cdot){:=}\bar{A}_{x,r} - A(\cdot)$, we have that

$$\begin{aligned}
\mathring{\omega}_g^{x,N\rho}(t) &= \sup_{w\in B(x,N\rho)}\fint_{B(w,t\rho)}\left|g(z) - \fint_{B(w,t\rho)} g(u)\,\mathrm{d}u\right|\mathrm{d}z \\
&= \sup_{w\in B(x,N\rho)}\fint_{B(w,t\rho)}\left|\bigg(\chi_{B(x,r)}(z)\varepsilon(z) - \fint_{B(w,t\rho)}\varepsilon(u)\chi_{B(x,r)}(u)\,\mathrm{d}u\bigg)\cdot\nabla_1\Theta(x,y)\right|\mathrm{d}z \\
&\overset{(3.42)}{\lesssim} r^{-n}\sup_{w\in B(x,N\rho)}\fint_{B(w,t\rho)}\left|\chi_{B(x,r)}(z)\varepsilon(z) - \fint_{B(w,t\rho)}\varepsilon(u)\chi_{B(x,r)}(u)\,\mathrm{d}u\right|\mathrm{d}z \\
&{=:}r^{-n}\sup_{w\in B(x,N\rho)}\mathcal{I}(w,t\rho).
\end{aligned}$$

For $w \in B(x, N\rho)$ and $t < 1$ by triangle inequality we have $B(w, t\rho) \subset B(x, r)$. Thus, for such w and t we can write

$$\mathcal{I}(w, t\rho) = \fint_{B(w,t\rho)} \Big|\varepsilon(z) - \fint_{B(w,t\rho)} \varepsilon\Big| \,\mathrm{d}z = \fint_{B(w,t\rho)} \big|A(z) - \bar{A}_{w,t\rho}\big| \,\mathrm{d}z \le \omega_A(t\rho),$$

which implies (3.52). Hence, by Theorem 3.7 and (3.21) (see the end of Remark 3.8),

$$\begin{aligned} |I_{r,2}(x)| &\le \sup_{w\in B(x,2\rho)} \Big|\int \nabla_1\nabla_2\Gamma(w, z)\varepsilon_{x,r}(z)\nabla_1\Theta(x, y)\,\mathrm{d}z\Big| \\ &\lesssim \left(\frac{1}{\rho^{n+1}}\int_{B(x,4\rho)} \big|\nabla L_A^{-1}\nabla\cdot g(w)\big|^2 \,\mathrm{d}w\right)^{1/2} + \int_0^1 \mathring{\omega}_g^{x,N\rho}(t)\,\frac{\mathrm{d}t}{t} \\ &\overset{(3.52)}{\lesssim}_N \left(\frac{1}{\rho^{n+1}}\int_{B(x,4\rho)} \big|\nabla L_A^{-1}\nabla\cdot g(w)\big|^2 \,\mathrm{d}w\right)^{1/2} + \frac{1}{r^n}\int_0^r \omega_A(t)\,\frac{\mathrm{d}t}{t}. \end{aligned} \tag{3.53}$$

The operator $(\nabla L_A^{-1}\nabla\cdot)$ is bounded from $L^2(\mathcal{L}^{n+1})$ to $L^2(\mathcal{L}^{n+1})$ and satisfies

$$\big\|\nabla L_A^{-1}\nabla\cdot g\big\|_{L^2(\mathcal{L}^{n+1})} \overset{(3.25)}{\lesssim} \|g\|_{L^2(\mathcal{L}^{n+1})}. \tag{3.54}$$

By Lemma 3.9 and the fact that $|x - y| = 2r$ we have that

$$\begin{aligned} \|g\|_{L^2(\mathcal{L}^{n+1})} &\le \left(\int_{B(x,r)} |\varepsilon(z)|^2 |\nabla_1\Theta(x, y)|^2 \,\mathrm{d}z\right)^{1/2} \\ &= |\nabla_1\Theta(x, y)| \left(\int_{B(x,r)} \big|A(z) - \bar{A}_{x,r}\big|^2 \,\mathrm{d}z\right)^{1/2} \overset{(3.43)}{\lesssim} r^{-n} r^{\frac{n+1}{2}}\,\mathfrak{I}_{\omega_A}(r). \end{aligned} \tag{3.55}$$

Hence, combining (3.51), (3.53), (3.54), and (3.55), we obtain

$$|I_r(x)| \lesssim r^{-n}\mathfrak{I}_{\omega_A}(r).$$

Let us bound $II_r(x)$. For $z \in B(y, r)$, we have $|x - z| \ge |x - y| - |z - y| > r$. Hence, by Lemma 3.9, triangle inequality and analogous calculations to those that proved (3.51), we have that

$$\begin{aligned} |II_r(x)| &\lesssim \int_{B(y,r)} \frac{|A(z) - \bar{A}_{x,r}|}{|x - z|^{n+1}|y - z|^n}\,\mathrm{d}z \le \frac{1}{r^{n+1}}\int_{B(y,r)} \frac{|A(z) - \bar{A}_{x,r}|}{|y - z|^n}\,\mathrm{d}z \\ &\le \frac{1}{r^{n+1}}\int_{B(y,r)} \frac{|\bar{A}_{x,r} - \bar{A}_{y,4r}|}{|y - z|^n}\,\mathrm{d}z + \frac{1}{r^{n+1}}\int_{B(y,4r)} \frac{|A(z) - \bar{A}_{y,4r}|}{|y - z|^n}\,\mathrm{d}z \\ &\overset{(3.51)}{\lesssim} \frac{1}{r^{n+1}}\left(\int_{B(y,r)} \frac{1}{|z - y|^n}\fint_{B(x,r)} |A(w) - \bar{A}_{y,4r}|\,\mathrm{d}w\,\mathrm{d}z\right) + \frac{1}{r^n}\int_0^{4r} \omega_A(t)\,\frac{\mathrm{d}t}{t}, \\ &\lesssim \frac{1}{r^n}\int_0^{4r} \omega_A(t)\,\frac{\mathrm{d}t}{t} + \frac{\omega_A(4r)}{r^n} \lesssim \frac{1}{r^n}\int_0^r \omega_A(t)\,\frac{\mathrm{d}t}{t} + \frac{\omega_A(r)}{r^n} \overset{(2.4)}{\lesssim} r^{-n}\,\mathfrak{I}_{\omega_A}(r), \end{aligned}$$

where in the penultimate inequality we used the doubling property of ω_A and $\mathfrak{I}_{\omega_A}$.

We are left with the estimate of III_r.[3] Let us observe that $B(x,r)\cup B(y,r)\subset B(x,4r)$. Given $j\geq 0$, we denote $\mathcal{C}_j:=A(x,2^j r,2^{j+1}r)$ and we split

$$III_r(x)=\int_{B(x,8r)\setminus\big(B(x,r)\cup B(y,r)\big)}\Phi(x,y,z)\,\mathrm{d}z+\sum_{j\geq 3}\int_{\mathcal{C}_j}\Phi(x,y,z)\,\mathrm{d}z=:\mathcal{I}_r^0+\mathcal{I}_r^1, \tag{3.56}$$

where Φ is the function defined in (3.50). The term $\mathcal{I}_r^0$ can be readily estimated using Lemma 3.9 and the fact that, for $z\in B(x,4r)\setminus\big(B(x,r)\cup B(y,r)\big)$, $|x-z|>r$ and $|y-z|>r$. In particular,

$$\begin{aligned}|\mathcal{I}_r^0|&\lesssim\int_{B(x,8r)\setminus\big(B(x,r)\cup B(y,r)\big)}\frac{\big|A(z)-\bar{A}_{x,r}\big|}{|x-z|^{n+1}|y-z|^n}\,\mathrm{d}z\\&\leq r^{-2n-1}\int_{B(x,8r)}\big|A(z)-\bar{A}_{x,r}\big|\,\mathrm{d}z\\&\lesssim\frac{1}{r^n}\fint_{B(x,8r)}\big|A(z)-\bar{A}_{x,8r}\big|\,\mathrm{d}z+\frac{1}{r^n}\big|\bar{A}_{x,8r}-\bar{A}_{x,r}\big|\,\mathrm{d}z\\&\lesssim\frac{\omega_A(8r)}{r^n}\lesssim\frac{\omega_A(r)}{r^n}\overset{(2.4)}{\lesssim}r^{-n}\,\mathfrak{I}_{\omega_A}(r).\end{aligned} \tag{3.57}$$

For $w\in\mathbb{R}^{n+1}$ we denote

$$v_j(w):=\int_{\mathcal{C}_j}\nabla_2\Gamma(w,z)\big(\bar{A}_{x,r}-A(z)\big)\nabla_1\Theta(z,y)\,\mathrm{d}z$$

so that

$$\mathcal{I}_r^1=\sum_{j\geq 3}\nabla v_j(x)=\sum_{j=3}^{j_0}\nabla v_j+\sum_{j\geq j_0+1}\nabla v_j=:\mathcal{I}_r^{1,1}+\mathcal{I}_r^{1,2},$$

where j_0 is such that $2^{j_0-3}r\leq R<2^{j_0-2}r$. Remark that v_j is a weak solution to $L_A v_j=0$ in $B(x,2^j r)$ for all $j\geq 3$.

Let us estimate $\mathcal{I}_r^{1,1}$. For $j\in\{3,\dots,j_0\}$[4], by (3.21) we have that

$$\begin{aligned}|\nabla v_j(x)|&\leq\sup_{w\in B(x,2^{j-4}r)}|\nabla v_j(w)|\lesssim_{R_0}\Big(\fint_{B(x,2^{j-3}r)}|\nabla v_j|^2\Big)^{1/2}\\&\lesssim\frac{1}{2^j r}\Big(\fint_{B(x,2^{j-2}r)}|v_j|^2\Big)^{1/2},\end{aligned} \tag{3.58}$$

[3] This is exactly the part we mentioned in the introduction that was missing in the justification of [23, Lemma 2.2] when the coefficients are not periodic (even in the Hölder continuous case).

[4] The method for $j\in\{3,\dots,j_0\}$ could be significantly simplified using the pointwise estimates for $\nabla_1\nabla_2\Gamma_A$, but as we will repeat this argument to handle the case $j>j_0$, we decided to use it for both. Here we only use the DMO_s condition to bound $|\nabla v_j(x)|$ by its L^2-average on the ball, while the last inequality of (3.58) holds for solutions of elliptic equations with L^∞ coefficients without any additional regularity assumption.

where the last bound follows from Caccioppoli inequality, and the second inequality does not depend on R_0 if $R_0 = 1$.

For $z \in \mathcal{C}_j$, we have $|x - z| \approx |y - z| \approx 2^j r$ and so (3.42) implies that $|\nabla_1 \Theta(z, y)| \lesssim |z - y|^{-n} \approx (2^j r)^{-n}$. If $w \in B(x, 2^{j-2} r)$, by Cauchy-Schwarz inequality and the latter estimate, we have that

$$
\begin{aligned}
|v_j(w)| &= \Big| \int_{\mathcal{C}_j} \nabla_2 \Gamma(w, z) \big(\bar{A}_{x,r} - A(z)\big) \nabla_1 \Theta(z, y) \, \mathrm{d}z \Big| \\
&\lesssim \frac{1}{(2^j r)^n} \Big(\int_{\mathcal{C}_j} |\nabla_2 \Gamma(w, z)|^2 \, \mathrm{d}z \Big)^{1/2} \Big(\int_{\mathcal{C}_j} \big| \bar{A}(z) - A_{x,r} \big|^2 \, \mathrm{d}z \Big)^{1/2}.
\end{aligned}
\tag{3.59}
$$

Thus, since $|z - w| < 2^{j+2} r$ for $z \in \mathcal{C}_j$, by Caccioppoli inequality (see [19, Theorem 4.4, p. 63]) and Lemma 3.9 we have that

$$
\begin{aligned}
\Big(\int_{\mathcal{C}_j} |\nabla_2 \Gamma(w, z)|^2 \, \mathrm{d}z \Big)^{1/2} &\leq \Big(\int_{B(w, 2^{j+2} r)} |\nabla_2 \Gamma(w, z)|^2 \, \mathrm{d}z \Big)^{1/2} \\
&\lesssim \frac{1}{2^j r} \Big(\int_{A(w, 2^{j+2} r, 2^{j+3} r)} |\Gamma(w, z)|^2 \, \mathrm{d}z \Big)^{1/2} \\
&\lesssim \frac{|B(w, 2^{j+3} r)|^{1/2}}{(2^j r)^n} \lesssim (2^j r)^{\frac{(1-n)}{2}}.
\end{aligned}
\tag{3.60}
$$

Inequality (3.44) yields

$$
\begin{aligned}
\Big(\int_{\mathcal{C}_j} \big| A(z) - \bar{A}_{x,r} \big|^2 \, \mathrm{d}z \Big)^{1/2} &= |\mathcal{C}_j|^{1/2} \Big(⨍_{\mathcal{C}_j} \big| A(z) - \bar{A}_{x,r} \big|^2 \, \mathrm{d}z \Big)^{1/2} \\
&\lesssim (2^{j+1} r)^{\frac{(n+1)}{2}} \mathfrak{I}_{\omega_A}(2^j r).
\end{aligned}
\tag{3.61}
$$

In view of (3.59), (3.60), and (3.61) we obtain

$$
|v_j(x)| \lesssim_n (2^j r)^{1-n} \mathfrak{I}_{\omega_A}(2^j r),
\tag{3.62}
$$

which, by (3.58), implies

$$
|\nabla v_j(x)| \lesssim_{R_0, n} (2^j r)^{-n} \mathfrak{I}_{\omega_A}(2^j r),
\tag{3.63}
$$

and hence

$$
\begin{aligned}
|\mathcal{I}_r^{1,1}| &\leq \sum_{j=3}^{j_0} |\nabla v_j(x)| \lesssim_{R_0, n} \sum_{j=3}^{j_0} (2^j r)^{-n} \mathfrak{I}_{\omega_A}(2^j r) \leq \sum_{j=1}^{\infty} (2^j r)^{-n} \mathfrak{I}_{\omega_A}(2^j r) \\
&\overset{(2.3)}{\lesssim} \int_r^\infty \mathfrak{I}_{\omega_A}(t) \, \frac{\mathrm{d}t}{t^{n+1}} \overset{(2.5)}{=} \frac{1}{n r^n} \int_0^r \omega_A(t) \, \frac{\mathrm{d}t}{t} + \frac{1}{n} \int_r^\infty \omega_A(t) \, \frac{\mathrm{d}t}{t^{n+1}} \\
&= \frac{1}{n r^n} \left(\mathfrak{I}_{\omega_A}(r) + \mathfrak{L}^n_{\omega_A}(r) \right).
\end{aligned}
\tag{3.64}
$$

We are left with the estimate of $\mathcal{I}_r^{1,2}$. For $j \geq j_0 + 1$, the rescaled version of (3.15) and Cacciopoli inequality give

$$|\nabla v_j(x)| \lesssim_{R_0} \Big(\fint_{B(x,R)} |\nabla v_j|^2 \Big)^{1/2} \lesssim \frac{1}{R} \Big(\fint_{B(x,2R)} |v_j(w)|^2 \, \mathrm{d}w \Big)^{1/2}. \tag{3.65}$$

We remark that the first inequality in (3.65) does not depend on R_0 if $R_0 = 1$.

If $z \in \mathcal{C}_j$ and $w \in B(x, 2R)$, since $j \geq j_0 + 1$ and $R \leq 2^{j_0 - 2} r$, it holds that $|z - w| \approx 2^j r$, and, arguing as above, we can prove (3.62). Therefore,

$$|\nabla v_j(x)| \overset{(3.62)}{\lesssim_{R_0}} \frac{(2^j r)^{1-n}}{R} \mathfrak{I}_{\omega_A}(2^j r),$$

which infers that

$$\begin{aligned} \sum_{j \geq j_0+1} |\nabla v_j(x)| &\lesssim_{R_0} \sum_{j \geq j_0+1} \frac{(2^j r)^{1-n}}{R} \mathfrak{I}_{\omega_A}(2^j r) \overset{(2.3)}{\lesssim} \frac{1}{R} \int_R^\infty \mathfrak{I}_{\omega_A}(t) \frac{\mathrm{d}t}{t^n} \\ &\overset{(2.5)}{\approx_n} \frac{1}{R^n} \int_0^R \omega_A(t) \frac{\mathrm{d}t}{t} + \frac{1}{R} \int_R^\infty \omega_A(t) \frac{\mathrm{d}t}{t^n} \\ &\approx_n R^{-n} \big(\mathfrak{I}_{\omega_A}(R) + \mathfrak{L}_{\omega_A}^{n-1}(R) \big). \end{aligned} \tag{3.66}$$

Therefore, combining (3.57), (3.64), and (3.66), we obtain

$$|III_r(x)| \leq |\mathcal{I}_r^0| + |\mathcal{I}_r^1| \lesssim r^{-n} \left(\mathfrak{I}_{\omega_A}(r) + \mathfrak{L}_{\omega_A}^n(r) \right) + R^{-n} \left(\mathfrak{I}_{\omega_A}(R) + \mathfrak{L}_{\omega_A}^{n-1}(R) \right).$$

Gathering the bounds for $I_r(x)$, $II_r(x)$, and $III_r(x)$, we conclude (3.49). □

In the next lemma we bound the difference of the averages of a matrix at two distinct scales, and compare the gradients of the respective fundamental solutions. This is crucial for the second step of our perturbation argument.

Lemma 3.13. *Let A be a uniformly elliptic matrix in $\mathbb{R}^{n+1}$, $n \geq 2$, satisfying $A \in \mathrm{DMO}_s$. Let $0 < \delta < r < 1$ and $x \in \mathbb{R}^{n+1}$, and assume that $\Omega_{x,\delta} \subset \mathbb{R}^{n+1}$ is a Borel set such that for some constant $M \geq 1$,*

$$B(x, \delta) \subset \Omega_{x,\delta} \subset B(x, M\delta).$$

If we denote $\bar{A}_{\Omega_{x,\delta}} := \fint_{\Omega_{x,\delta}} A$, then

$$\left| \bar{A}_{x,r/2} - \bar{A}_{\Omega_{x,\delta}} \right| \lesssim_M \int_\delta^r \omega_A(t) \frac{\mathrm{d}t}{t} \leq \tau_A(r) \tag{3.67}$$

and

$$\left| (\bar{A}_{x,r/2})_s - (\bar{A}_{\Omega_{x,\delta}})_s \right| \lesssim_M \int_\delta^r \omega_A(t) \frac{\mathrm{d}t}{t} \leq \tau_A(r). \tag{3.68}$$

Moreover, for

$$\mathcal{K}_\Theta^2(x, y) := \nabla_1 \Theta(x - y, 0; \bar{A}_{x,r/2}) - \nabla_1 \Theta(x - y, 0; \bar{A}_{x,\delta/2})$$

and all $z \in \mathbb{R}^{n+1} \setminus \{0\}$, it holds

$$\left| \mathcal{K}_\Theta^2(z, 0) \right| \lesssim_{n,\Lambda,M} \frac{1}{|z|^n} \int_\delta^r \omega_A(t) \frac{\mathrm{d}t}{t}. \tag{3.69}$$

Proof. Let $N_0 \geq 1$ be such that $2^{-N_0}r \leq \delta < 2^{-N_0+1}r$ and $2^{N_1-1} \leq M < 2^{N_1}$. Therefore, if $N = N_0 - N_1$, we have that

$$
\begin{aligned}
&\left|\bar{A}_{x,r/2} - \bar{A}_{\Omega_{x,\delta}}\right| \leq \left|\bar{A}_{x,r} - \bar{A}_{x,2^{-N+1}r}\right| + \left|\bar{A}_{\Omega_{x,\delta}} - \bar{A}_{x,2^{-N+1}r}\right| \\
&\quad \lesssim_M \sum_{j=1}^{N-2}\left|\bar{A}_{x,2^{-j}r} - \bar{A}_{x,2^{-j-1}r}\right| + \omega_A(2^{-N+1}r) \lesssim \sum_{j=1}^{N-2}\omega_A(2^{-j}r) + \omega_A(2^{-N+1}r) \\
&\quad = \sum_{j=1}^{N-1}\omega_A(2^{-j}r) \lesssim \int_\delta^r \omega_A(t)\,\frac{\mathrm{d}t}{t}.
\end{aligned}
$$

The bound (3.68) follows directly from (3.67).

Since A is uniformly elliptic with constant Λ, it is invertible and its inverse is uniformly elliptic as well with the same constant. Moreover, as all its eigenvalues are bounded from above by Λ and below by Λ^{-1}, and so is its determinant $\det(A)$ (as the product of its $n+1$ eigenvalues). The same considerations apply to $\mathcal{A}_r := (\bar{A}_{x,r/2})_s$ and $\mathcal{A}_\delta := (\bar{A}_{\Omega_{x,\delta}})_s$. By standard calculations we can write

$$
\begin{aligned}
&\left|\nabla_1\Theta(z,0;\bar{A}_{x,r/2}) - \nabla_1\Theta(z,0;\bar{A}_{\Omega_{x,\delta}})\right| \overset{(3.40)}{=} \left|\nabla_1\Theta(z,0;\mathcal{A}_r) - \nabla_1\Theta(z,0;\mathcal{A}_\delta)\right| \\
&\quad = \frac{1}{\omega_n}\left|\frac{\mathcal{A}_r^{-1}z}{\sqrt{\det\mathcal{A}_r}\langle\mathcal{A}_r^{-1}z,z\rangle^{(n+1)/2}} - \frac{\mathcal{A}_\delta^{-1}z}{\sqrt{\det\mathcal{A}_\delta}\langle\mathcal{A}_\delta^{-1}z,z\rangle^{(n+1)/2}}\right| \\
&\quad \lesssim_{\Lambda,n} \frac{1}{|z|^{2n+2}}\left|\sqrt{\det\mathcal{A}_\delta}\langle\mathcal{A}_\delta^{-1}z,z\rangle^{(n+1)/2}\mathcal{A}_r^{-1}z - \sqrt{\det\mathcal{A}_r}\langle\mathcal{A}_r^{-1}z,z\rangle^{(n+1)/2}\mathcal{A}_\delta^{-1}z\right|.
\end{aligned}
\tag{3.70}
$$

We remark that by elementary calculations and using the ellipticity of $\mathcal{A}_\delta$ and $\mathcal{A}_r$ we have

$$
\begin{aligned}
|\mathcal{A}_\delta^{-1}z - \mathcal{A}_r^{-1}z| &= |\mathcal{A}_\delta^{-1}\mathcal{A}_r\mathcal{A}_r^{-1}z - \mathcal{A}_\delta^{-1}\mathcal{A}_\delta\mathcal{A}_r^{-1}z| \\
&= \left|\mathcal{A}_\delta^{-1}(\mathcal{A}_r - \mathcal{A}_\delta)\mathcal{A}_r^{-1}z\right| \lesssim_\Lambda |\mathcal{A}_r - \mathcal{A}_\delta||z|.
\end{aligned}
\tag{3.71}
$$

The mean value theorem implies that, for $0 < a < b$ we have

$$
|a^{(n+1)/2} - b^{(n+1)/2}| \leq \frac{n+1}{2}\max_{t\in[a,b]} t^{(n-1)/2}(b-a) = \frac{n+1}{2}b^{(n-1)/2}(b-a).
$$

The symmetric inequality holds for $0 < b \leq a$ and, for the choices $a = \langle\mathcal{A}_r^{-1}z,z\rangle^{(n+1)/2}$ and $b = \langle\mathcal{A}_\delta^{-1}z,z\rangle^{(n+1)/2}$, gives

$$
\begin{aligned}
&\left|\langle\mathcal{A}_r^{-1}z,z\rangle^{(n+1)/2} - \langle\mathcal{A}_\delta^{-1}z,z\rangle^{(n+1)/2}\right| \\
&\quad \lesssim_n \left|\langle\mathcal{A}_r^{-1}z,z\rangle - \langle\mathcal{A}_\delta^{-1}z,z\rangle\right|\left|\langle\mathcal{A}_r^{-1}z,z\rangle^{(n-1)/2} + \langle\mathcal{A}_\delta^{-1}z,z\rangle^{(n-1)/2}\right| \\
&\quad \overset{(3.71)}{\lesssim} |\mathcal{A}_r - \mathcal{A}_\delta|\,|z|^2\,|z|^{n-1} = |z|^{n+1}|\mathcal{A}_r - \mathcal{A}_\delta|,
\end{aligned}
\tag{3.72}
$$

by the ellipticity of $\mathcal{A}_r^{-1}$ and $\mathcal{A}_\delta^{-1}$.

Let us recall that the map $\det : \mathbb{R}^{(n+1)\times(n+1)} \to \mathbb{R}$ is a polynomial in the entries of the matrix and, more specifically, that Jacobi's formula gives

$$
\frac{\partial}{\partial\tilde{a}_{ij}}\det(\tilde{\mathcal{A}}) = (\mathrm{adj}\tilde{\mathcal{A}})_{ij} \qquad \text{for} \quad \tilde{\mathcal{A}} = (\tilde{a}_{ij})_{i,j} \in \mathbb{R}^{(n+1)\times(n+1)},
$$

where $\mathrm{adj}\tilde{\mathcal{A}} = \det(\tilde{\mathcal{A}})\tilde{\mathcal{A}}^{-1}$ is the adjugate matrix of $\tilde{\mathcal{A}}$. In particular, the map $\det(\cdot)$ is locally Lipschitz continuous and, for $|\tilde{\mathcal{A}}| \leq \Lambda$, its Lipschitz constant depends only on Λ and n. Moreover,

$$|\sqrt{a} - \sqrt{b}| = \big(a^{1/2} + b^{1/2}\big)^{-1}|a - b| \qquad \text{for } a, b > 0,$$

which implies

$$\begin{aligned} |\sqrt{\det \mathcal{A}_r} - \sqrt{\det \mathcal{A}_\delta}| &= \big((\det \mathcal{A}_r)^{1/2} + (\det \mathcal{A}_\delta)^{1/2}\big)^{-1}|\det \mathcal{A}_r - \det \mathcal{A}_\delta| \\ &\lesssim_{\Lambda,n} |\det \mathcal{A}_r - \det \mathcal{A}_\delta| \lesssim_{\Lambda,n} |\mathcal{A}_r - \mathcal{A}_\delta|. \end{aligned} \tag{3.73}$$

Finally, (3.70), triangle inequality, the bounds (3.72), (3.73), (3.71), and the uniform ellipticity of $\mathcal{A}_r$ and $\mathcal{A}_\delta$ yield

$$\begin{aligned} &\big|\nabla_1\Theta(z, 0; \mathcal{A}_r) - \nabla_1\Theta(z, 0; \mathcal{A}_\delta)\big| \\ &\lesssim_{\Lambda,n} \frac{1}{|z|^{2n+2}}\Big(|\sqrt{\det \mathcal{A}_\delta} - \sqrt{\det \mathcal{A}_r}|\big|\langle \mathcal{A}_\delta^{-1}z, z\rangle^{(n+1)/2}\mathcal{A}_r^{-1}z\big| \\ &\quad + |\sqrt{\det \mathcal{A}_r}|\big|\langle \mathcal{A}_\delta^{-1}z, z\rangle^{(n+1)/2}(\mathcal{A}_r^{-1}z - \mathcal{A}_\delta^{-1}z)\big| \\ &\quad + |\sqrt{\det \mathcal{A}_r}|\big|\langle \mathcal{A}_\delta^{-1}z, z\rangle^{(n+1)/2} - \langle \mathcal{A}_r^{-1}z, z\rangle^{(n+1)/2}\big||\mathcal{A}_\delta^{-1}z|\Big) \\ &\lesssim_{n,\Lambda} \frac{|\mathcal{A}_r - \mathcal{A}_\delta|}{|z|^n} \overset{(3.68)}{\lesssim} \frac{1}{|z|^n}\int_\delta^r \omega_A(t)\,\frac{\mathrm{d}t}{t}. \end{aligned}$$

This concludes the proof of the lemma. □

We can also demonstrate the following lemma:

Lemma 3.14. *Let A be a uniformly elliptic matrix in $\mathbb{R}^{n+1}$, $n \geq 2$, satisfying $A \in \mathrm{DMO}_s$. Assume that $Q \subset \mathbb{R}^{n+1}$ is a dyadic cube and $\Omega_Q \subset \mathbb{R}^{n+1}$ is a Borel set such that, for some constant $M \geq 1$,*

$$B(x_Q, \ell(Q)) \subset \Omega_Q \subset B(x_Q, M\ell(Q)).$$

If we denote $\bar{A}_{\Omega_Q} := ⨍_{\Omega_Q} A$, then if $x \in Q$,

$$\big|\bar{A}_{\Omega_Q} - \bar{A}_{x,\delta/2}\big| \lesssim_M \int_0^{\ell(Q)} \omega_A(t)\,\frac{\mathrm{d}t}{t} \leq \tau_A(\ell(Q)), \quad \textit{for } \delta < \sqrt{n+1}\ell(Q), \tag{3.74}$$

and, for all $z \in \mathbb{R}^{n+1} \setminus \{0\}$ and $\delta < \sqrt{n+1}\ell(Q)$,

$$\big|\nabla_1\Theta(z, 0; \bar{A}_{\Omega_Q}) - \nabla_1\Theta(z, 0; \bar{A}_{x,\delta/2})\big| \lesssim_{n,\Lambda,M} \frac{1}{|z|^n}\int_0^{\ell(Q)} \omega_A(t)\,\frac{\mathrm{d}t}{t}. \tag{3.75}$$

Proof. The proof is a routine adaptation of the one of Lemma 3.13 and is left as an exercise to the interested reader. □

Let $\delta > 0$ and, for an affine function L, define the truncated integral operator

$$\widehat{T}^j_{L,\mu,\delta} f(x) := \int_{|Lx-Ly|>\delta} \mathcal{K}^j_\Theta(x,y) f(y)\,\mathrm{d}\mu(y), \quad j = 1, 2,$$

where we set $r = |x-y|$ in the definition of $\mathcal{K}^2_\Theta$ given in Lemma 3.13.

Lemma 3.15. *Let A be a uniformly elliptic matrix in $\mathbb{R}^{n+1}$, $n \geq 2$, satisfying $A \in \widetilde{\mathrm{DMO}}$. Let $R_0 > 0$ and $\mu \in M^n_+(\mathbb{R}^{n+1})$ with compact support so that* $\mathrm{diam}(\mathrm{supp}\,\mu) = R \leq R_0$. *Then $\widehat{T}^j_{L,\mu,\delta}: L^2(\mu) \to L^2(\mu)$, $j = 1, 2$, satisfying*

$$\sup_{\delta>0} \|\widehat{T}^1_{L,\mu,\delta} f\|_{L^2(\mu)\to L^2(\mu)} \lesssim \mathfrak{I}_{\tau_A}(R) + \widehat{\tau}_A(R)$$
$$\sup_{\delta>0} \|\widehat{T}^2_{L,\mu,\delta} f\|_{L^2(\mu)\to L^2(\mu)} \lesssim \mathfrak{I}_{\tau_A}(R),$$

where the implicit constants depend on $\|L\|_{\mathrm{op}}$, Λ, n, and R_0. In particular, if $R_0 = 1$, the implicit constants only depend on ellipticity, dimension, and the Dini Mean Oscillation condition.

Proof. In view of Lemmas 3.12 and 3.13, we can apply Lemma 2.5 to the integral operators with kernel $\mathcal{K}^j_\Theta(x,y)\chi_{B(Lx,\delta/2)^c}(Ly)$ and deduce the result. □

Lemma 3.16. *Let A be a uniformly elliptic matrix in $\mathbb{R}^{n+1}$, $n \geq 2$, satisfying $A \in \widetilde{\mathrm{DMO}}$. Let Q be a cube in $\mathbb{R}^{n+1}$ with center x_Q and side-length $\ell(Q) \lesssim 1$, and let $\nu \in M^n_+(\mathbb{R}^{n+1})$ be supported on Q and have n-growth constant $c_0 > 0$. Assume also that there exist a Borel set Ω_Q and a constant $M \geq 1$ such that*

$$B(x_Q, \ell(Q)) \subset \Omega_Q \subset B(x_Q, M\ell(Q)).$$

For $\delta > 0$ let us define

$$\mathcal{K}^3_\Theta(x,z) := \nabla_1\Theta(z,0;\bar{A}_{x,\delta/2}) - \nabla_1\Theta\big(z,0;\bar{A}_{\Omega_Q}\big) \tag{3.76}$$

and

$$T_{\mathcal{K}^3_\Theta,\nu,\delta} f(x) := \int_{|x-y|>\delta} \mathcal{K}^3_\Theta(x,x-y) f(y)\,\mathrm{d}\nu(y), \qquad f \in L^1_{\mathrm{loc}}(\nu).$$

Then, there exist a positive constant $C'' = C''(n,\Lambda,M,c_0)$ such that we have

$$\|T_{\mathcal{K}^3_\Theta,\nu,\delta}\|_{L^2(\nu)\to L^2(\nu)} \leq C''\mathfrak{I}_{\omega_A}(\ell(Q))^{1/2}\|\mathcal{R}_{\nu,\delta}\|_{L^2(\nu)\to L^2(\nu)}. \tag{3.77}$$

Proof. For brevity, let us denote $\mathcal{K} := \mathcal{K}^3_\Theta$. Observe that the function $\mathcal{K}(x,\cdot)$ is homogeneous of degree $-n$ for any $x \in \mathbb{R}^{n+1}$, namely,

$$\mathcal{K}(x,z) = \frac{1}{|z|^n}\mathcal{K}\Big(x, \frac{z}{|z|}\Big) \quad \text{for all } z \in \mathbb{R}^{n+1},$$

and satisfies $\|\mathcal{K}(x,\cdot)\|_{L^2(\mathbb{S}^n)} \lesssim_{n,\Lambda} 1$. Indeed, by (3.41) and ellipticity of $(\bar{A}_{x,\delta/2})_s$ we have that

$$\|\mathcal{K}(x,\cdot)\|^2_{L^2(\mathbb{S}^n)} \lesssim \int_{\mathbb{S}^n} \left| \frac{\omega_n^{-1}}{\sqrt{\det(\bar{A}_{x,\delta/2})_s}} \frac{(\bar{A}_{x,\delta/2})_s^{-1}\zeta}{\langle(\bar{A}_{x,\delta/2})_s^{-1}\zeta,\zeta\rangle^{(n+1)/2}} \right|^2 \mathrm{d}\sigma(\zeta) \\ + \int_{\mathbb{S}^n} \left|\omega_n^{-1}\frac{\zeta}{|\zeta|^{n+1}}\right|^2 \mathrm{d}\sigma(\zeta) \lesssim_{n,\Lambda} \int_{\mathbb{S}^n} |\zeta|^{-2n}\,\mathrm{d}\sigma(\zeta) = \omega_n.$$

Let $\{\varphi_{j,\ell}\}_{j\geq 1, 1\leq \ell\leq N_j}$ be an orthonormal basis of $L^2(\mathbb{S}^n)$ of spherical harmonics of degree j. In particular N_j satisfies the asymptotic estimate

$$N_j = O(j^{n-1}) \qquad \text{for } j \gg 1, \tag{3.78}$$

for which we refer, for instance, to [1, display (2.12)]. Hence, we decompose $\mathcal{K}$ into spherical harmonics in the L^2-sense and write

$$\mathcal{K}(x,z) = \frac{1}{|z|^n}\mathcal{K}\Big(x,\frac{z}{|z|}\Big) = \frac{1}{|z|^n}\sum_{j\geq 1}\sum_{\ell=1}^{N_j}\big\langle\mathcal{K}(x,\cdot),\varphi_{j,\ell}\big\rangle_{L^2(\mathbb{S}^n)}\varphi_{j,\ell}\Big(\frac{z}{|z|}\Big) \\ =: \frac{1}{|z|^n}\sum_{j\geq 1}\sum_{\ell=1}^{N_j} k_{j,\ell}(x)\varphi_{j,\ell}\Big(\frac{z}{|z|}\Big).$$

We observe that, as $\mathcal{K}(x,\cdot)$ is an odd function, $k_{j,\ell}(x) = 0$ if j is even. Furthermore, since $\mathcal{K}(x,\cdot)$ is smooth on $\mathbb{S}^n$, then by [1, Thorem 2.36], the series $\sum_{j\geq 1}\sum_{\ell=1}^{N_j} k_{j,\ell}(x)\varphi_{j,\ell}(\cdot)$ converges uniformly on $\mathbb{S}^n$.

We claim that

$$|k_{j,\ell}(x)| \lesssim_{n,\Lambda} \mathfrak{I}_{\omega_A}(\ell(Q)) \qquad \text{for all } x \in \mathbb{R}^{n+1}. \tag{3.79}$$

Indeed, the bound (3.75) readily implies

$$|k_{j,\ell}(x)| \leq \int_{\mathbb{S}^n} |\mathcal{K}(x,\zeta)||\varphi_{j,\ell}(\zeta)|\,\mathrm{d}\sigma(\zeta) \lesssim_{n,\Lambda} \int_0^{\ell(Q)} \omega_A(t)\,\frac{\mathrm{d}t}{t}\int_{\mathbb{S}^n}\frac{|\varphi_{j,\ell}(\zeta)|}{|\zeta|^n}\,\mathrm{d}\sigma(\zeta) \\ \lesssim_n \int_0^{\ell(Q)} \omega_A(t)\,\frac{\mathrm{d}t}{t},$$

where the last inequality holds because of the normalization $\|\varphi_{j,\ell}\|_{L^2(\mathbb{S}^n)} = 1$ and Cauchy-Schwarz inequality. On the other hand we observe that, for $m \geq 1$, the bound (3.42) and the definition of $\mathcal{K}$ yield

$$\big|\nabla_z^{2m}\mathcal{K}\big(x,z\big)\big| \lesssim_{n,\Lambda,m} \frac{1}{|z|^{n+2m}}, \qquad \text{for } z \in \mathbb{R}^{n+1}\setminus\{0\}.$$

Thus, denoting by $\Delta^m_{\mathbb{S}^n}$ the m-th iteration of the Laplace-Beltrami operator on $\mathbb{S}^n$, we have

$$\big|\Delta^m_{\mathbb{S}^n,\zeta}\mathcal{K}\big(x,\zeta\big)\big| \lesssim_{n,\Lambda,m} 1, \tag{3.80}$$

where the subscript in $\Delta^m_{\mathbb{S}^n,\zeta}$ denotes that the operator is applied with respect to the ζ-variable. Moreover, as $\mathcal{K}(x,\cdot)$ is infinitely differentiable on $\mathbb{S}^n$, [36, III 3.1.5] gives that

$$\int_{\mathbb{S}^n} \Delta^m_{\mathbb{S}^n} \mathcal{K}(x,\zeta)\, \varphi_{j,\ell}(\zeta)\, \mathrm{d}\sigma(\zeta) = k_{j,\ell}(x)[-j(j+n-1)]^m \qquad \text{for all } m \geq 1.$$

Hence, Cauchy-Schwarz inequality implies that

$$\begin{aligned}[j(j+n-1)]^m \left|k_{j,\ell}(x)\right| &\lesssim \left(\int_{\mathbb{S}^n} |\Delta^m_{\mathbb{S}^n}\mathcal{K}(x,\cdot)|^2\,\mathrm{d}\sigma\right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^n} |\varphi_{j,\ell}|^2\,\mathrm{d}\sigma\right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{S}^n} |\Delta^m_{\mathbb{S}^n}\mathcal{K}(x,\cdot)|^2\,\mathrm{d}\sigma\right)^{\frac{1}{2}} \overset{(3.80)}{\lesssim}_{n,\Lambda,m} 1,\end{aligned} \tag{3.81}$$

where we remark that the bound is uniform on x and y. Hence, for every $m > 0$ we have that

$$\left|k_{j,\ell}(x)\right| \lesssim_{n,\Lambda,m} j^{-2m}.$$

In particular, the choice $m = (n+5)(n-1)/2$ yields

$$\left|k_{j,\ell}(x)\right| \lesssim_{n,\Lambda} j^{-(n+5)(n-1)}. \tag{3.82}$$

Taking the geometric mean of (3.79) and (3.82), we obtain

$$\left|k_{j,\ell}(x)\right| \lesssim_{n,\Lambda} \frac{\left(\mathfrak{I}_{\omega_A}(\ell(Q))\right)^{1/2}}{j^{(n+5)(n-1)/2}}. \tag{3.83}$$

Now, let us define the kernel

$$K_{j,\ell}(z) := \frac{1}{|z|^n}\varphi_{j,\ell}\left(\frac{z}{|z|}\right), \qquad \text{for } z \in \mathbb{R}^{n+1}\setminus\{0\}.$$

By [36, p. 276] we have that

$$\sup_{|x|=1}\left|\frac{\partial^\alpha \varphi_{j,\ell}(x)}{\partial x^\alpha}\right| \lesssim_\alpha j^{\frac{n+1}{2}+|\alpha|}, \quad \text{for } \alpha \in \mathbb{N}^{n+1}.$$

so $K_{j,\ell}$ satisfies the estimate

$$|\partial^\alpha_z K_{j,\ell}(z)| \lesssim_\alpha \frac{j^{\frac{n+1}{2}+|\alpha|}}{|z|^{n+|\alpha|}} \quad \text{for } \alpha \in \mathbb{N}^{n+1}, z \in \mathbb{R}^{n+1}. \tag{3.84}$$

Thus, for j odd and by [41, Corollary 1.4], its associated singular integral operator

$$T_{K_{j,\ell},\nu} f(x) := \int K_{j,\ell}(x-y) f(y)\,\mathrm{d}\nu(y)$$

is bounded on $L^2(\nu)$ with norm

$$\|T_{K_{j,\ell},\nu}\|_{L^2(\nu)\to L^2(\nu)} \lesssim j^{\frac{n+5}{2}} \|\mathcal{R}_\nu\|_{L^2(\nu)\to L^2(\nu)}. \tag{3.85}$$

On the other hand we recall that, by the discussion before display (3.79), we have that $k_{j,\ell}(x) = 0$ if j is even.

Let $T_{K_{j,\ell},\nu,\delta}$ be the associated δ-truncated operator. For $\tilde{N} > 1$ and $\psi \in L^\infty(\nu)$, we have that

$$\begin{aligned}
&\int \left| \int_{|x-y|>\delta} \sum_{j\geq 1}^{\tilde{N}} \sum_{\ell=1}^{N_j} k_{j,\ell}(x) K_{j,\ell}(x-y)\psi(y)\, \mathrm{d}\nu(y) \right|^2 \mathrm{d}\nu(x) \\
&\quad = \int \left| \sum_{j\geq 1}^{\tilde{N}} \sum_{\ell=1}^{N_j} \int_{|x-y|>\delta} k_{j,\ell}(x) K_{j,\ell}(x-y)\psi(y)\, \mathrm{d}\nu(y) \right|^2 \mathrm{d}\nu(x) \qquad (3.86) \\
&\quad = \int \left| \sum_{j\geq 1}^{\tilde{N}} \sum_{\ell=1}^{N_j} k_{j,\ell}(x) T_{K_{j,\ell},\nu,\delta}\psi(x) \right|^2 \mathrm{d}\nu(x).
\end{aligned}$$

Hence by (3.86), (3.83) and (3.78) we get that

$$\begin{aligned}
&\left(\int \left| \sum_{j\geq 1}^{\tilde{N}} \sum_{\ell=1}^{N_j} k_{j,\ell}(x) T_{K_{j,\ell},\nu,\delta}\psi(x) \right|^2 \mathrm{d}\nu(x) \right)^{1/2} \\
&\quad \leq \sum_{j\geq 1}^{\tilde{N}} \sum_{\ell=1}^{N_j} \|k_{j,\ell}\|_{L^\infty(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})} \|T_{K_{j,\ell},\nu,\delta}\psi\|_{L^2(\nu)} \\
&\quad \lesssim \left(\int_0^{\ell(Q)} \omega_A(t)\, \frac{\mathrm{d}t}{t} \right)^{1/2} \|\mathcal{R}_\nu\|_{L^2(\nu)\to L^2(\nu)} \sum_{j\geq 1} \frac{N_j}{j^{n+1}} \|\psi\|_{L^2(\nu)} \\
&\quad \lesssim \left(\int_0^{\ell(Q)} \omega_A(t)\, \frac{\mathrm{d}t}{t} \right)^{1/2} \|\mathcal{R}_\nu\|_{L^2(\nu)\to L^2(\nu)} \|\psi\|_{L^2(\nu)}.
\end{aligned}$$

By the uniform convergence of the decomposition in spherical harmonics and dominated convergence theorem, the estimate above implies that

$$\|T_{\mathcal{K},\nu,\delta}\psi\|_{L^2(\nu)} \lesssim \left(\int_0^{\ell(Q)} \omega_A(t)\, \frac{\mathrm{d}t}{t} \right)^{1/2} \|\mathcal{R}_\nu\|_{L^2(\nu)\to L^2(\nu)} \|\psi\|_{L^2(\nu)}.$$

Arguing by density, this proves the lemma. □

Lemma 3.17. *If A is a uniformly elliptic matrix in $\mathbb{R}^{n+1}$, $n \geq 2$, satisfying $A \in \widetilde{\mathrm{DMO}}$ and if $T_\mu \colon L^2(\mu) \to L^2(\mu)$ for some $\mu \in M_+(\mathbb{R}^{n+1})$ without atoms is bounded, then there exist $r_0 = r_0(n, \Lambda, \mathrm{diam}(\operatorname{supp}\mu)) \in (0, \mathrm{diam}(\operatorname{supp}\mu))$ small enough and $c_0 > 0$ so that $\mu(B(x,r)) \leq c_0 r^n$, for all $x \in \mathbb{R}^{n+1}$ and $r < r_0$.*

Proof. Let $K(x,y) := \nabla_1 \Gamma_A(x,y)$ for $x, y \in \mathbb{R}^{n+1}$, $x \neq y$. The lemma follows from [12, Proposition 1.4, p. 56] once we show that there exists $r_0 \in (0, \mathrm{diam}(\operatorname{supp}\mu))$ such that for any fixed cube Q of side length $\ell(Q) < r_0$, it holds $|K(x,y)| \gtrsim$

$|x-y|^{-n}$ for all $x \neq y \in Q$. To this end, fix a cube Q, and for $x \neq y \in Q$ we have that the matrix $\mathcal{A} := \bar{A}_{x,|x-y|/2}$ is elliptic with constant Λ, which yields

$$|\nabla_1 \Theta(x-y, 0; \mathcal{A})| \overset{(3.40)}{=} \frac{\omega_n^{-1}}{\sqrt{\det \mathcal{A}_s}} \frac{|\mathcal{A}_s^{-1}(x-y)|}{\langle \mathcal{A}_s^{-1}(x-y), x-y \rangle^{(n+1)/2}} \geq \frac{C(n,\Lambda)}{|x-y|^n} \tag{3.87}$$

for some $C(n,\Lambda) > 0$. Let $C > 0$ be as Lemma 3.12 for $R = \operatorname{diam}(\operatorname{supp} \mu)$ and $R_0 = 2R$. Then, for $|x-y| < R$,

$$\begin{aligned} |K(x,y) - \nabla_1 \Theta(x-y, 0; \mathcal{A})| &\leq C \frac{\tau_A(r)}{r^n} + C \frac{\widehat{\tau}_A(R)}{R^n} \\ &\leq \frac{C\tau_A(r) + C\widehat{\tau}_A(R) R^{-n} r^n}{r^n}, \qquad r = \frac{|x-y|}{2}. \end{aligned} \tag{3.88}$$

Thus, (3.87), (3.88), and the triangle inequality imply

$$\begin{aligned} |K(x,y)| &\geq |\nabla_1 \Theta(x-y,0;\mathcal{A})| - |K(x,y) - \nabla_1 \Theta(x-y,0;\mathcal{A})| \\ &\geq \frac{C(n,\Lambda) - C\tau_A(r) - C\widehat{\tau}_A(R) R^{-n} r^n}{r^n}. \end{aligned}$$

Since $\tau_A(2^{-j}) \to 0$ as $j \to \infty$ and also τ_A is c_{db}-doubling for some constant $c_{db} > 1$, there exists $j_0 \in \mathbb{N}$ such that, for every $j > j_0$,

$$C\tau_A(2^{-j}) + C\widehat{\tau}_A(R) R^{-n} 2^{-jn} < \frac{C(n,\Lambda)}{2\max(c_{db}, 2^n)}.$$

Therefore, if $2^{-N} \leq |x-y| < 2^{-N+1}$ for some $N \in \mathbb{N}$ so that $N > j_0$, it holds that

$$\begin{aligned} &C\tau_A(|x-y|/2) + C\widehat{\tau}_A(R) R^{-n} \frac{|x-y|^n}{2^n} \\ &< \max(c_{db}, 2^n) \frac{C(n,\Lambda)}{2\max(c_{db}, 2^n)} = C(n,\Lambda)/2. \end{aligned}$$

Therefore, for $|x-y|/2 < 2^{-j_0} =: r_0$,

$$|K(x,y)| \gtrsim |x-y|^{-n},$$

which proves the lemma. □

4. The Main Lemmas

Let A be a uniformly elliptic matrix as in Lemma 3.12. In particular, we recall that we introduced the function τ_A in (1.12) and ω_n in Section 2.

Lemma 4.1. (Main Lemma I) *Let A be a uniformly elliptic matrix in $\mathbb{R}^{n+1}$, $n \geq 2$, satisfying $A \in \widetilde{\mathrm{DMO}}$. Let Q be a cube in $\mathbb{R}^{n+1}$ with center x_Q and side-length $\ell(Q) \lesssim 1$, and let $\nu \in M_+^n(\mathbb{R}^{n+1})$ be supported on Q and have n-growth constant $c_0 > 0$. Assume also that there exist a Borel set Ω_Q and a constant $M \geq 1$ such that*

$$B(x_Q, \ell(Q)) \subset \Omega_Q \subset B(x_Q, M\ell(Q))$$

and $(\bar{A}_{\Omega_Q})_s := \fint_{\Omega_Q} A_s = Id$. Let T_ν denote the gradient of the single layer potential associated with the matrix A. Then, there exist positive constants $C' = C'(n, c_0)$ and $C'' = C''(n, \Lambda, M, c_0)$ such that for $\delta > 0$ we have that

$$\|T_{\nu,\delta} - \omega_n^{-1}\mathcal{R}_{\nu,\delta}\|_{L^2(\nu)\to L^2(\nu)} \leq C'\mathfrak{I}_{\tau_A}(\ell(Q)) + C'\widehat{\tau}_A(\ell(Q)) + C''\mathfrak{I}_{\omega_A}(\ell(Q))^{1/2}\|\mathcal{R}_{\nu,\delta}\|_{L^2(\nu)\to L^2(\nu)}. \tag{4.1}$$

Proof. Let $\delta > 0$ and

$$\bar{T}_{\nu,\delta} f(x) := \int_{|x-y|>\delta} \nabla_1\Theta(x-y, 0; \bar{A}_{x,\delta/2}) f(y)\, d\nu(y).$$

By Lemma 3.15 we have that

$$\|T_{\nu,\delta} - \bar{T}_{\nu,\delta}\|_{L^2(\nu)\to L^2(\nu)} \lesssim_{n,\Lambda} \mathfrak{I}_{\tau_A}(\ell(Q)) + \widehat{\tau}_A(\ell(Q)). \tag{4.2}$$

Moreover,

$$\begin{aligned}\mathcal{K}^3_\Theta(x,z) &\overset{(3.76)}{=} \nabla_1\Theta(z, 0; \bar{A}_{x,\delta/2}) - \nabla_1\Theta\big(z, 0; \bar{A}_{\Omega_Q}\big)\\ &= \nabla_1\Theta(z, 0; \bar{A}_{x,\delta/2}) - \omega_n^{-1}\frac{z}{|z|^{n+1}},\end{aligned}$$

where the second equality holds because of the assumption $(\bar{A}_{\Omega_Q})_s = Id$. Hence $\bar{T}_{\nu,\delta} - \omega_n^{-1}\mathcal{R}_{\nu,\delta} = T_{\mathcal{K}^3_\Theta,\nu,\delta}$, so Lemma 3.16 concludes the proof of (4.1). □

In the next lemma we denote by $\mathcal{R}_\mu$ and T_μ the principal values of the corresponding singular integral operators.

Lemma 4.2. (Main Lemma II) *Let A be a uniformly elliptic matrix in $\mathbb{R}^{n+1}$, $n \geq 2$, satisfying $A \in \widetilde{\mathrm{DMO}}$, and let Q be a cube in $\mathbb{R}^{n+1}$ with center x_Q and side-length $\ell(Q) \lesssim 1$. Let also Ω_Q be a Borel set and $M \geq 1$ a constant such that*

$$B(x_Q, \ell(Q)) \subseteq \Omega_Q \subseteq B(x_Q, M\ell(Q))$$

and $\bar{A}_{\Omega_Q} := \fint_{\Omega_Q} A_s = Id$. Let μ be a non-negative Radon measure on $\mathbb{R}^{n+1}$ with compact support. Assume that for some integer $N > 0$ we have $2^N\ell(Q) \leq$ diam(supp μ) and for a constant $C_0 > 0$, the measure μ is such that

$$\Theta_\mu\big(B(x,r)\cap 2^N Q\big) \leq C_0\Theta_\mu(2^N Q), \qquad \text{for all } x \in 2^N Q,\ 0 < r \leq 2^N\ell(Q), \tag{4.3}$$

$$\mathcal{P}^N_{\omega,\mu}(Q) \leq C_0\mathfrak{I}_{\alpha_A}\big(2^{-N}\big)\Theta_\mu(2^N Q), \tag{4.4}$$

where Θ_μ *and* $\mathcal{P}^N_{\omega,\mu}$ *were defined in* (1.13) *and* (1.14) *respectively. If* T_μ *denotes the gradient of the single layer potential associated with the matrix* A *and there exists* $\tau \in (0,1)$ *such that*

$$\Big(⨍_Q \Big|T_\mu 1(x) - ⨍_Q T_\mu 1\Big|^2 \, d\mu(x)\Big)^{1/2} \le \tau^{1/2}\Theta_\mu(2^N Q), \tag{4.5}$$

then it holds that

$$\Big(⨍_Q \Big|\mathcal{R}_\mu 1(x) - ⨍_Q \mathcal{R}_\mu 1\Big|^2 \, d\mu(x)\Big)^{1/2} \le C_1 \Theta_\mu(2^N Q)\Big(\tau^{1/2} + \mathfrak{I}_{\alpha_A}\big(2^{-N}\big)\Big)$$
$$+ C_1 \Theta_\mu(2^N Q)\Big(\vartheta\big(2^N \ell(Q)\big) + \mathfrak{I}_{\omega_A}(2^N \ell(Q))^{1/2} \|\mathcal{R}_\mu\|_{L^2(\mu|_{2^N Q}) \to L^2(\mu|_{2^N Q})}\Big),$$

where $\vartheta(8\ell(Q)) := \mathfrak{I}_{\tau_A}(8\ell(Q)) + \widehat{\tau}_A(8\ell(Q))$*, and* C_1 *depends on* n, Λ, c_0, C_0, M*, and* $\mathrm{diam}(\mathrm{supp}\,\mu)$.

Proof. Note that (4.1) still holds if we replace the truncated singular integrals on its left hand-side by their principal values. This is an easy application of Fatou's lemma and the existence of principal values given by Proposition 1.5.

For brevity, we write

$$m_Q(f,\mu) = ⨍_Q f \, d\mu.$$

We define

$$L^2_0(\mu, Q) := \big\{ f \in L^2(\mu) : \mathrm{supp}\, f \subset Q, \quad m_Q(f,\mu) = 0 \big\}$$

and by $L^2_0(\mu, Q; \mathbb{R}^{n+1})$ its vector-valued analogue. The space $L^2_0(\mu, Q)$ endowed with the norm $\|\cdot\|_{L^2(\mu)}$ is a Hilbert space whose Banach dual is the space of functions in $L^2(\mu)$ modulo an additive constant and equipped with the norm $\|\cdot\|_{L^2(\mu)}$ (see e.g. [37, 1.2.2, p. 143]). Moreover, for $f \in L^2(\mu, Q)$ it holds that

$$\begin{aligned} \|f - m_Q(f,\mu)\|_{L^2(\mu,Q)} &\approx \sup_{\substack{g \in L^2_0(\mu,Q), \\ \|g\|_{L^2(\mu)}=1}} \int (f - m_Q(f,\mu)) g \, d\mu \\ &= \sup_{\substack{g \in L^2_0(\mu,Q), \\ \|g\|_{L^2(\mu)}=1}} \int f g \, d\mu, \end{aligned} \tag{4.6}$$

where the second identity follows from $m_Q(g,\mu) = 0$.

Then we have that

$$\begin{aligned} \Big(\int_Q \big|\mathcal{R}_\mu 1(x) - m_Q(\mathcal{R}_\mu 1, \mu(x))\big|^2 \, d\mu\Big)^{1/2} &\approx \sup_{\substack{\vec{g} \in L^2_0(\mu,Q;\mathbb{R}^{n+1}), \\ \|\vec{g}\|_{L^2(\mu;\mathbb{R}^{n+1})}=1}} \Big|\int \mathcal{R}_\mu 1 \cdot \vec{g} \, d\mu\Big| \\ &= \sup_{\substack{\vec{g} \in L^2_0(\mu,Q;\mathbb{R}^{n+1}), \\ \|\vec{g}\|_{L^2(\mu;\mathbb{R}^{n+1})}=1}} \Big|\int \mathcal{R}^*_\mu \vec{g} \, d\mu\Big|, \end{aligned}$$

where

$$\mathcal{R}_\mu^* \vec{g}(x) := \int \tfrac{y-x}{|y-x|^{n+1}} \cdot \vec{g}(y)\, \mathrm{d}\mu(y).$$

We also denote that

$$\mathcal{R}_\mu \cdot \vec{g}(x) := \int \tfrac{x-y}{|x-y|^{n+1}} \cdot \vec{g}(y)\, \mathrm{d}\mu(y),$$

so that $\mathcal{R}_\mu^* g(x) = -\mathcal{R}_\mu \cdot \vec{g}(x)$ for all $x \in \mathbb{R}^{n+1}$.

Then, for $\vec{g} \in L_0^2(\mu, Q; \mathbb{R}^{n+1})$ with $\|\vec{g}\|_{L^2(\mu;\mathbb{R}^{n+1})} = 1$ and $\mathcal{S}_\mu \cdot \vec{g} := \omega_n T_\mu \cdot \vec{g} - \mathcal{R}_\mu \cdot \vec{g}$, triangle inequality yields

$$\begin{aligned} \left| \int \mathcal{R}_\mu^* \vec{g}\, \mathrm{d}\mu \right| &= \left| \int \mathcal{R}_\mu \cdot \vec{g}\, \mathrm{d}\mu \right| \\ &\lesssim \left| \int T_\mu \cdot \vec{g}\, \mathrm{d}\mu \right| + \left| \int \mathcal{S}_\mu \cdot \vec{g}\, \mathrm{d}\mu \right| =: I + II. \end{aligned}$$

By (4.6) and the hypothesis (4.5) we have that

$$I \lesssim \|T_\mu 1 - m_Q(T_\mu 1, \mu)\|_{L^2(\mu)} \le \tau^{1/2} \Theta_\mu(2^N Q) \mu(Q)^{1/2}. \tag{4.7}$$

We denote

$$\mathfrak{K}(x, y) := \omega_n \nabla_1 \Gamma_A(x, y) - \frac{x-y}{|x-y|^{n+1}}, \qquad x, y \in \mathbb{R}^{n+1}, x \neq y.$$

In order to estimate I, we first observe that Lemma 3.9 and the standard Calderón-Zygmund properties of the Riesz kernel imply that

$$|\mathfrak{K}(x, y) - \mathfrak{K}(x, z)| \lesssim_{n,\Lambda,R} \alpha_A\Big(\frac{|y-z|}{|x-y|}\Big) |x-y|^{-n} \tag{4.8}$$

for $2|y-z| \le |x-y| \le R$, where

$$\alpha_A(t) := t^\beta + t + \omega_A(t), \qquad t > 0. \tag{4.9}$$

Moreover, $A \in \widetilde{\mathrm{DMO}}$ implies $\alpha_A \in \mathrm{DS}(\kappa)$.

Now, we write

$$II \le \left| \int_{2^N Q} \mathcal{S}_\mu \cdot \vec{g}\, \mathrm{d}\mu \right| + \left| \int_{\mathbb{R}^{n+1} \setminus 2^N Q} \mathcal{S}_\mu \cdot \vec{g}\, \mathrm{d}\mu \right| =: II_1 + II_2.$$

In order to estimate II_1 we apply Lemma 4.1, the Cauchy-Schwarz inequality and the assumption $\|\vec{g}\|_{L^2(\mu;\mathbb{R}^{n+1})} = 1$, which give that

$$\begin{aligned} II_1 &\le C' \Theta_\mu(2^N Q) \vartheta\big(2^N \ell(Q)\big) \mu(Q)^{1/2} \\ &\quad + C'' \Theta_\mu(2^N Q) \mathfrak{I}_{\omega_A}\big(2^N \ell(Q)\big)^{1/2} \|\mathcal{R}_\mu\|_{L^2(\mu|_{2^N Q}) \to L^2(\mu|_{2^N Q})} \mu(Q)^{1/2}, \end{aligned}$$

where the multiplicative factor $\Theta_\mu(2^N Q)$ on the right hand side is a consequence of (4.3).

Denote by x_Q the center of the cube Q. To estimate II_2, observe that $\vec{g} \in L^2_0(\mu, Q; \mathbb{R}^{n+1})$ and (4.8) imply that

$$\begin{aligned} II_2 &\leq \int_{\mathbb{R}^{n+1}\setminus 2^N Q} \int_Q |\mathfrak{K}(x,y) - \mathfrak{K}(x,x_Q)||\vec{g}(y)| \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) \\ &\lesssim \int_{\mathbb{R}^{n+1}\setminus 2^N Q} \int_Q \alpha_A\Big(\frac{|y-x_Q|}{|x-y|}\Big)\frac{1}{|x-y|^n}|\vec{g}(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) \\ &\leq \sum_{j\geq N}\int_{2^{j+1}Q\setminus 2^j Q}\int_Q \alpha_A\Big(\frac{|y-x_Q|}{|x-y|}\Big)\frac{1}{|x-y|^n}|\vec{g}(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) \\ &\lesssim \mathcal{P}^N_{\omega,\mu}(Q)\mu(Q)^{1/2}, \end{aligned} \tag{4.10}$$

where the last inequality follows from the definition of $\mathcal{P}^N_{\omega,\mu}(Q)$, the doubling property of α_A, and the assumption $\|\vec{g}\|_{L^2(\mu;\mathbb{R}^{n+1})} = 1$.

The bounds (4.7), (4.10), (4.10), and the assumption (4.4) conclude the proof of the lemma. □

5. The Approximating Measures

Let Q be a cube in $\mathbb{R}^{n+1}$ and let $\nu \in M^n_+(Q)$. We fix a function $\varphi \in C^\infty_c(\mathbb{R}^{n+1})$ such that $\operatorname{supp}\varphi \subset B(0,1)$, $0 \leq \varphi \leq 2$, and $\|\varphi\|_{L^1(\mathbb{R}^{n+1})} = 1$. Given $\varepsilon > 0$, we denote

$$\varphi_\varepsilon(z) := \frac{1}{\varepsilon^{n+1}}\varphi\Big(\frac{z}{\varepsilon}\Big) \qquad \text{for } z \in \mathbb{R}^{n+1},$$

and we define

$$\nu_\varepsilon := \nu * \varphi_\varepsilon. \tag{5.1}$$

The measure ν_ε here introduced is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^{n+1}$, its support is contained in the set $\{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, \operatorname{supp}\nu) \leq \varepsilon\}$ and it satisfies $\|\nu_\varepsilon\| = \|\nu\|$ for all $\varepsilon > 0$. The following lemma shows that, under our hypotheses on the matrix A, the $L^2(\nu)$-boundedness of T_ν controls the $L^2(\nu_\varepsilon)$-boundedness of T_{ν_ε}.

Lemma 5.1. *Let A be a uniformly elliptic matrix in $\mathbb{R}^{n+1}$, $n \geq 2$, satisfying $A \in \widetilde{\mathrm{DMO}}$. Let $\nu \in M^n_+(Q)$ with growth constant $c_0 > 0$, and let ν_ε be as in (5.1) for $\varepsilon > 0$. Then*

$$\|T_{\nu_\varepsilon}\|_{L^2(\nu_\varepsilon)\to L^2(\nu_\varepsilon)} \lesssim 1 + \|T_\nu\|_{L^2(\nu)\to L^2(\nu)}, \tag{5.2}$$

where the implicit constant depends on n, c_0, Λ and $\operatorname{diam}(\operatorname{supp}\nu)$.

Proof. Given $f \in L^2(\nu_\varepsilon)$ and $\varepsilon > 0$, we define $\sigma_\varepsilon := f\nu_\varepsilon$. Let $N \in \mathbb{N}$ be such that $2^{-N-1}\ell(Q) \leq \varepsilon < 2^{-N}\ell(Q)$. Let $\{Q_i\}_i$ be a family of $\widetilde{N} := 3^{n+1}2^{N(n+1)}$ cubes with disjoint interior and side-length $2^{-N}\ell(Q)$ that cover $3Q$. We denote by x_{Q_i} the center of the cube Q_i. For any $i = 1, \ldots, \widetilde{N}$ we define the set $v(Q_i) := 3Q_i \cap 3Q$, which consists of the union of at most 3^{n+1} cubes of side-length $2^{-N}\ell(Q)$. We also define

$$\tilde{\sigma}_{\varepsilon,i} := \frac{\sigma_\varepsilon(Q_i)}{\nu(v(Q_i))}\nu|_{v(Q_i)} = \Big(\frac{\sigma_\varepsilon(Q_i)}{\nu(v(Q_i))}\chi|_{v(Q_i)}\Big)\nu =: \tilde{f}_{\varepsilon,i}\,\nu, \tag{5.3}$$

and also

$$\tilde{\sigma}_\varepsilon := \sum_{i=1}^{\tilde{N}} \tilde{\sigma}_{\varepsilon,i} = \sum_{i=1}^{\tilde{N}} \tilde{f}_{\varepsilon,i}\,\nu =: \tilde{f}_\varepsilon\,\nu. \tag{5.4}$$

We claim that $\tilde{f}_\varepsilon \in L^2(\nu)$ satisfies

$$\|\tilde{f}_\varepsilon\|_{L^2(\nu)} \lesssim_n \|f\|_{L^2(\nu_\varepsilon)}. \tag{5.5}$$

Observe that the choices of ε and N yield

$$\{z \in \mathbb{R}^{n+1} : \operatorname{dist}(z, Q_i) < \varepsilon\} \subset v(Q_i).$$

Thus, an application of Fubini's theorem and the choice of the cut-off function φ gives that

$$\begin{aligned} \nu_\varepsilon(Q_i) &= \int_{Q_i}\int \varphi_\varepsilon(x-y)\,\mathrm{d}\nu(y)\,\mathrm{d}x \\ &= \int_{\{z\in\mathbb{R}^{n+1}:\operatorname{dist}(z,Q_i)<\varepsilon\}}\int_{Q_i} \varphi_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}\nu(y) \\ &\leq \nu\big(\{z \in \mathbb{R}^{n+1} : \operatorname{dist}(z, Q_i) < \varepsilon\}\big) \leq \nu(v(Q_i)). \end{aligned} \tag{5.6}$$

Note that there exists a dimensional constant $c_n > 0$ such that, for $i = 1, \dots, \tilde{N}$, the set $v(Q_i)$ has non-empty intersection with at most c_n different sets of the form $v(Q_j)$ for $j = 1, \dots, \tilde{N}$. Analogously, every cube Q_j can be contained in at most c_n different sets of the form $v(Q_i)$. For $i = 1, \dots, \tilde{N}$ we also define the cube Q_i^* as a cube such that $v(Q_i^*) \cap v(Q_i) \neq \varnothing$ and, for all j such that $v(Q_j) \cap v(Q_i) \neq \varnothing$, it holds that

$$\frac{|\sigma_\varepsilon(Q_j)|}{\nu(v(Q_j))} \leq \frac{|\sigma_\varepsilon(Q_i^*)|}{\nu(v(Q_i^*))}. \tag{5.7}$$

Hence, by elementary inequalities and the definitions above we obtain

$$\begin{aligned} \|\tilde{f}_\varepsilon\|^2_{L^2(\nu)} &= \int \Big|\sum_{i=1}^{\tilde{N}} \frac{\sigma_\varepsilon(Q_i)}{\nu(v(Q_i))}\chi_{v(Q_i)}(x)\Big|^2 \mathrm{d}\nu(x) \\ &= \sum_{i=1}^{\tilde{N}} \frac{\sigma_\varepsilon(Q_i)^2}{\nu(v(Q_i))} + \sum_{i,j=1,i\neq j}^{\tilde{N}} \int \frac{\sigma_\varepsilon(Q_i)}{\nu(v(Q_i))}\frac{\sigma_\varepsilon(Q_j)}{\nu(v(Q_j))}\chi_{v(Q_i)\cap v(Q_j)}(x)\,\mathrm{d}\nu(x) \\ &\overset{(5.7)}{\leq} \sum_{i=1}^{\tilde{N}} \frac{\sigma_\varepsilon(Q_i)^2}{\nu(v(Q_i))} + c_n \sum_{i=1}^{\tilde{N}} \int \frac{\sigma_\varepsilon(Q_i^*)^2}{\nu(v(Q_i^*))^2}\chi_{v(Q_i)}(x)\,\mathrm{d}\nu(x). \end{aligned} \tag{5.8}$$

The first sum on the right hand side of (5.8) satisfies

$$\sum_{i=1}^{\widetilde{N}} \frac{(\sigma_\varepsilon(Q_i))^2}{\nu(v(Q_i))} = \sum_{i=1}^{\widetilde{N}} \frac{1}{\nu(v(Q_i))}\Big(\int_{Q_i} f\, \mathrm{d}\nu_\varepsilon\Big)^2 \\ \leq \sum_{i=1}^{\widetilde{N}} \frac{\nu_\varepsilon(Q_i)}{\nu(v(Q_i))}\|f\chi_{Q_i}\|^2_{L^2(\nu_\varepsilon)} \overset{(5.6)}{\leq} \|f\|^2_{L^2(\nu_\varepsilon)} \tag{5.9}$$

and, analogously, we have that

$$\sum_{i=1}^{\widetilde{N}} \int \frac{\sigma_\varepsilon(Q_i^*)^2}{\nu(v(Q_i^*))^2}\chi_{v(Q_i)}(x)\,\mathrm{d}\nu(x) \lesssim_n \|f\|^2_{L^2(\nu_\varepsilon)}. \tag{5.10}$$

Hence, by (5.9) and (5.10), we get (5.5).

Let $K(x, y):=\nabla_1\Gamma(x, y; A)$, for $x, y \in \mathbb{R}^{n+1}$ with $x \neq y$ and, for $\delta > 0$, we define $K_\delta(x, y):=K(x, y)\chi_{B(0,\delta)^c}(x - y)$. For $\delta \in (0, \varepsilon/2)$ we write

$$T_\delta\sigma_\varepsilon(x) = \int_{|x-y|<\varepsilon} K_\delta(x, y)\,\mathrm{d}\sigma_\varepsilon(y) + \int_{|x-y|\geq\varepsilon} K_\delta(x, y)\, d\big(\sigma_\varepsilon - \tilde{\sigma}_\varepsilon\big)(y) \\ + \int_{|x-y|\geq\varepsilon} K_\delta(x, y)\, d\tilde{\sigma}_\varepsilon(y)=:I_{\delta,\varepsilon}(x) + II_{\delta,\varepsilon}(x) + III_{\delta,\varepsilon}(x). \tag{5.11}$$

In order to estimate the first term, we observe that, by the definition of φ and the growth of ν, we have

$$\nu_\varepsilon\big(B(x, 2^{-k}\varepsilon)\big) = \frac{1}{\varepsilon^{n+1}}\int_{B(x,2^{-k}\varepsilon)}\int \varphi\Big(\frac{y-z}{\varepsilon}\Big)\,\mathrm{d}\nu(z)\,\mathrm{d}y \\ \leq \frac{2}{\varepsilon^{n+1}}\int_{B(x,2^{-k}\varepsilon)} \nu(B(y, \varepsilon))\,\mathrm{d}y \leq \frac{2}{\varepsilon}\big|B(x, 2^{-k}\varepsilon)\big| \lesssim_n \frac{\varepsilon^n}{2^{k(n+1)}}. \tag{5.12}$$

Hence, if $\mathcal{M}_{\nu_\varepsilon}$ stands for the centered Hardy-Littlewood maximal function

$$\mathcal{M}_{\nu_\varepsilon} g(x):=\sup_{r>0}\frac{1}{\nu_\varepsilon(B(x, r))}\int_{B(x,r)}|g(y)|\,\mathrm{d}\nu_\varepsilon(y), \qquad \text{for } g \in L^1_{\mathrm{loc}}(\nu_\varepsilon),$$

the decay of K, a standard integration over dyadic annuli, and the definition of σ_ε give that

$$\big|I_{\delta,\varepsilon}(x)\big| \lesssim \sum_{k=0}^{\infty}\int_{A(x,2^{-k-1}\varepsilon,2^{-k}\varepsilon)}\frac{1}{|x-y|^n}\, d|\sigma_\varepsilon|(y) \\ \lesssim \sum_{k=0}^{\infty}\frac{2^{nk}}{\varepsilon^n}|\sigma_\varepsilon|\big(B(x, 2^{-k-1}\varepsilon)\big) \\ \lesssim \sum_{k=0}^{\infty}\frac{2^{nk}}{\varepsilon^n}\nu_\varepsilon\big(B(x, 2^{-k}\varepsilon)\big)\mathcal{M}_{\nu_\varepsilon} f(x) \overset{(5.12)}{\lesssim} \mathcal{M}_{\nu_\varepsilon} f(x). \tag{5.13}$$

Let us estimate $II_{\delta_\varepsilon}(x)$. Let $\beta > 0$ be as in Lemma 3.9. For $i \in \{1, \dots, \tilde{N}\}$, the fact that $\sigma_{\varepsilon,i} := \sigma_\varepsilon|_{Q_i}$ and $\tilde{\sigma}_{\varepsilon,i}$ have equal total mass, Lemma 3.9, triangle inequality, and the choice of N in the construction yield

$$\begin{aligned}
&\left| \int_{|x-y|>\varepsilon} K_\delta(x, y)\, d\big(\sigma_{\varepsilon,i} - \tilde{\sigma}_{\varepsilon,i}\big)(y) \right| \\
&= \left| \int_{|x-y|>\varepsilon} \big(K_\delta(x, y) - K_\delta(x, x_{Q_i})\big)\, d\big(\sigma_{\varepsilon,i} - \tilde{\sigma}_{\varepsilon,i}\big)(y) \right| \\
&\leq \int_{|x-y|>\varepsilon} |K(x, y) - K(x, x_{Q_i})|\, d\big(|\sigma_{\varepsilon,i}| + |\tilde{\sigma}_{\varepsilon,i}|\big)(y) \\
&\lesssim \int_{|x-y|>\varepsilon} \frac{|y - x_{Q_i}|^\beta}{|x - y|^{n+\beta}}\, d\big(|\sigma_{\varepsilon,i}| + |\tilde{\sigma}_{\varepsilon,i}|\big)(y) \\
&\quad + \int_{|x-y|>\varepsilon} \frac{1}{|x - y|^n} \int_0^{\frac{|y-x_{Q_i}|}{|x-y|}} \omega_A(t)\, \frac{\mathrm{d}t}{t}\, d\big(|\sigma_{\varepsilon,i}| + |\tilde{\sigma}_{\varepsilon,i}|\big)(y) \\
&=: II'_{\delta,\varepsilon} + II''_{\delta,\varepsilon}.
\end{aligned} \tag{5.14}$$

Thus

$$\begin{aligned}
II'_{\delta,\varepsilon} &\lesssim \varepsilon^\beta \int_{|x-y|>\varepsilon} \frac{1}{|x - y|^{n+\beta}}\, d\big(|\sigma_{\varepsilon,i}| + |\tilde{\sigma}_{\varepsilon,i}|\big)(y) \\
&\lesssim \sum_{k=0}^{\infty} \frac{\varepsilon^\beta}{(2^k\varepsilon)^{n+\beta}} \Big(|\sigma_\varepsilon|\big(B(x, 2^{k+1})\big) + |\tilde{\sigma}_\varepsilon|\big(B(x, 2^{k+1})\big) \Big),
\end{aligned} \tag{5.15}$$

where the last inequality follows from a standard integration on dyadic annuli. Similarly, the second term can be bounded using the monotonicity of the function $\mathfrak{I}_{\omega_A}$ and integration on dyadic annuli. More precisely, we have that

$$\begin{aligned}
II''_{\delta,\varepsilon} &\lesssim \int_{|x-y|>\varepsilon} \frac{1}{|x - y|^n} \mathfrak{I}_{\omega_A}\Big(\frac{\varepsilon}{|x - y|}\Big)\, d\big(|\sigma_{\varepsilon,i}| + |\tilde{\sigma}_{\varepsilon,i}|\big)(y) \\
&\lesssim \sum_{k=0}^{\infty} \frac{\tau_A(2^{-k})}{(2^k\varepsilon)^n} \Big(|\sigma_\varepsilon|\big(B(x, 2^{k+1})\big) + |\tilde{\sigma}_\varepsilon|\big(B(x, 2^{k+1})\big) \Big),
\end{aligned} \tag{5.16}$$

where we also used the fact that $\mathfrak{I}_{\omega_A}(\cdot) \leq \tau_A(\cdot)$.

Now, for $\mu \in M(\mathbb{R}^{n+1})$, we define the n-dimensional truncated radial maximal operator

$$\mathcal{M}_\varepsilon \mu(x) := \sup_{r \geq 2\varepsilon} \frac{|\mu|\big(B(x, r)\big)}{r^n},$$

and the truncated centered maximal function

$$\mathcal{M}_{\mu,\varepsilon} g(x) := \sup_{r > 2\varepsilon} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |g(y)|\, \mathrm{d}\mu(y), \qquad \text{for } g \in L^1_{\mathrm{loc}}(\mu).$$

So if we gather (5.14), (5.15), and (5.16), and sum over $i = 1, \dots, \tilde{N}$, in view of the fact that $\tau_A \in \mathrm{DS}(\kappa)$ for some $\kappa = \kappa(n)$ and (2.2), we deduce that

$$\begin{aligned} |II_{\delta,\varepsilon}(x)| &\lesssim \sum_{k=0}^{\infty}\big(2^{-k\beta} + \tau_A(2^{-k})\big)\big(\mathcal{M}_\varepsilon\sigma_\varepsilon(x) + \mathcal{M}_\varepsilon\tilde{\sigma}_\varepsilon(x)\big) \\ &\lesssim \Big(\sum_{k=0}^{\infty} 2^{-k\beta} + \int_0^1 \tau_A(t)\,\frac{\mathrm{d}t}{t}\Big)\big(\mathcal{M}_\varepsilon\sigma_\varepsilon(x) + \mathcal{M}_\varepsilon\tilde{\sigma}_\varepsilon(x)\big) \\ &\lesssim \mathcal{M}_\varepsilon\sigma_\varepsilon(x) + \mathcal{M}_\varepsilon\tilde{\sigma}_\varepsilon(x). \end{aligned} \tag{5.17}$$

We claim that the operator $\mathcal{M}_\varepsilon$ is bounded from $M(\mathbb{R}^{n+1})$ to $L^{1,\infty}(\nu_\varepsilon)$, with operator norm independent on ε. Indeed let $\mu \in M(\mathbb{R}^{n+1})$, consider $\lambda, m > 0$, and let us denote

$$A_\lambda := \big\{x \in \mathbb{R}^{n+1} : \mathcal{M}_\varepsilon\mu(x) > \lambda\big\} \qquad \text{and} \qquad A_{\lambda,m} := A_\lambda \cap B(0,m).$$

Thus, for every $x \in A_{\lambda,m}$ there exists $r_x > 2\varepsilon$ such that

$$|\mu|(B(x,r_x)) > \lambda r_x^n.$$

Fubini's theorem, the normalization $\|\varphi\|_{L^1(\mathbb{R}^{n+1})} = 1$, and the choice of r_x imply that, for all $x \in A_{\lambda,m}$, we have that

$$\nu_\varepsilon(B(x,r_x)) = \int_{B(x,r_x)}\int \varphi_\varepsilon(y-z)\,\mathrm{d}\nu(z)\mathrm{d}y \le \nu(B(x, r_x+\varepsilon)) \lesssim \Big(\frac{3}{2}r_x\Big)^n. \tag{5.18}$$

Besicovitch covering Lemma implies that there exists a countable collection of balls $\{B_i\}_i \subset \big\{B(x,r_x)\big\}_{x\in A_{\lambda,m}}$ with bounded overlaps that covers $A_{\lambda,m}$. Hence

$$\nu_\varepsilon(A_{\lambda,m}) \le \nu_\varepsilon\Big(\bigcup_i B_i\Big) \le \sum_i \nu_\varepsilon(B_i) \overset{(5.18)}{\lesssim} \frac{3^n}{2^n}\sum_i \frac{|\mu|(B_i)}{\lambda} \lesssim \frac{3^n}{2^n\lambda}\|\mu\|.$$

Since the latter estimate holds for every m, our claim follows.

Moreover, the fact that ν and ν_ε have n-polynomial growth implies that $\mathcal{M}_{\nu_\varepsilon,\varepsilon}$ and $\mathcal{M}_{\nu,\varepsilon}$ are bounded from $L^\infty(\nu_\varepsilon)$ to $L^\infty(\nu_\varepsilon)$ and from $L^\infty(\nu)$ to $L^\infty(\nu_\varepsilon)$, respectively. Thus, Marcinkiewicz interpolation theorem implies that

$$\|\mathcal{M}_{\nu_\varepsilon,\varepsilon}\, g\|_{L^2(\nu_\varepsilon)} \lesssim \|g\|_{L^2(\nu_\varepsilon)} \quad \text{and} \quad \|\mathcal{M}_{\nu,\varepsilon}\, g\|_{L^2(\nu_\varepsilon)} \lesssim \|g\|_{L^2(\nu)}. \tag{5.19}$$

Analogously, we can prove that the operator $\mathcal{M}_{\nu,\varepsilon}$ is bounded from $L^2(\nu)$ to $L^2(\nu)$, namely

$$\|\mathcal{M}_{\nu,\varepsilon}\, g\|_{L^2(\nu)} \lesssim \|g\|_{L^2(\nu)}. \tag{5.20}$$

Now we turn our attention to $III_{\delta,\varepsilon}(x)$. Since we assumed $\delta < \varepsilon/2$, we have that

$$|III_{\delta,\varepsilon}(x)| \le \big|T_\varepsilon\tilde{\sigma}_\varepsilon(x)\big| + \mathcal{M}_\varepsilon\tilde{\sigma}_\varepsilon(x).$$

Therefore, by (5.11), (5.13), (5.17), and the inequality above, we infer that

$$|T_{\nu_\varepsilon,\delta} f(x)| = |T_\delta\sigma_\varepsilon(x)| \lesssim \mathcal{M}_\varepsilon\sigma_\varepsilon(x) + \mathcal{M}_\varepsilon\tilde{\sigma}_\varepsilon(x) + \mathcal{M}_{\nu_\varepsilon} f(x) + \big|T_\varepsilon\tilde{\sigma}_\varepsilon(x)\big|$$

$$= \mathcal{M}_{\nu,\varepsilon} f_\varepsilon(x) + \mathcal{M}_{\nu,\varepsilon} \tilde{f}_\varepsilon(x) + \mathcal{M}_{\nu_\varepsilon} f(x) + \left|T_\varepsilon \tilde{\sigma}_\varepsilon(x)\right|. \tag{5.21}$$

We claim that, for $i = 1, \ldots, \tilde{N}$, it holds that

$$|T_\varepsilon \tilde{\sigma}_\varepsilon(x) - T_\varepsilon \tilde{\sigma}_\varepsilon(x')| \lesssim \mathcal{M}_\varepsilon \tilde{\sigma}_\varepsilon(z) \qquad \text{for all } x, x', z \in 3Q_i. \tag{5.22}$$

Indeed, for $x, x' \in 3Q_i$, observe that the choice of ε implies $|x-x'| \leq \operatorname{diam}(3Q_i) = 3\sqrt{n+1}\ell(Q_i) \leq 6n\varepsilon$. Furthermore, by triangle inequality we have that

$$B(x', \varepsilon) \subset B(x, 10n\varepsilon) \subset B(x', 20n\varepsilon).$$

Thus, we can write

$$\begin{aligned}
|T_\varepsilon \tilde{\sigma}_\varepsilon(x) - T_\varepsilon \tilde{\sigma}_\varepsilon(x')| &= \Big| \int_{|x-y|>\varepsilon} K(x,y)\, d\tilde{\sigma}_\varepsilon(y) - \int_{|x'-y|>\varepsilon} K(x',y)\, d\tilde{\sigma}_\varepsilon(y) \Big| \\
&\leq \int_{|x-y|>10n\varepsilon} |K(x,y) - K(x',y)|\, d|\tilde{\sigma}_\varepsilon|(y) + \int_{\varepsilon<|x-y|\leq 10n\varepsilon} |K(x,y)|\, d|\tilde{\sigma}_\varepsilon|(y) \\
&\quad + \int_{\varepsilon<|x'-y|\leq 20n\varepsilon} |K(x',y)|\, d|\tilde{\sigma}_\varepsilon|(y) =: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\end{aligned}$$

The Calderón-Zygmund properties of K in Lemma 3.9, a standard integration over dyadic annuli analogous to (5.14), the inequality $\mathfrak{I}_{\omega_A}(\cdot) \leq \tau_A(\cdot)$, and the assumption $\tau_A \in \mathrm{DS}(\kappa)$ imply

$$\begin{aligned}
\mathcal{J}_1 &\lesssim \int_{|x-y|>\varepsilon} \Big(\frac{|x-x'|^\beta}{|x-y|^{n+\beta}} + \frac{\int_0^{\frac{|x'-y|}{|x-y|}} \omega_A(t) t^{-1}\, \mathrm{d}}{|x-y|^n} \Big) d|\tilde{\sigma}_\varepsilon|(y) \\
&\lesssim \int_{|x-y|>\varepsilon} \Big(\frac{\varepsilon^\beta}{|x-y|^{n+\beta}} + \frac{\mathfrak{I}_{\omega_A}\big(\varepsilon/|x-y|\big)}{|x-y|^n} \Big) d|\tilde{\sigma}_\varepsilon|(y) \\
&\lesssim \sum_{k=0}^{\infty} \Big(\frac{\varepsilon^\beta}{(2^k\varepsilon)^{n+\beta}} + \frac{\tau_A(2^{-k})}{(2^k\varepsilon)^n} \Big) |\tilde{\sigma}_\varepsilon|\big(B(x, 2^k\varepsilon)\big) \\
&\lesssim \sum_{k=0}^{\infty} \big(2^{-k\beta} + \tau_A(2^{-k})\big) \mathcal{M}_\varepsilon \tilde{\sigma}_\varepsilon(z) \\
&\overset{(2.2)}{\lesssim} \Big(1 + \int_0^1 \tau_A(t)\, \frac{\mathrm{d}t}{t}\Big) \mathcal{M}_\varepsilon \tilde{\sigma}_\varepsilon(z) \lesssim \mathcal{M}_\varepsilon \tilde{\sigma}_\varepsilon(z).
\end{aligned} \tag{5.23}$$

Analogously, we can prove that

$$\begin{aligned}
\mathcal{J}_2 + \mathcal{J}_3 &\lesssim \int_{\varepsilon<|x-y|<10n\varepsilon} \frac{1}{|x-y|^n}\, d|\tilde{\sigma}_\varepsilon|(y) + \int_{\varepsilon<|x'-y|<20n\varepsilon} \frac{1}{|x'-y|^n}\, d|\tilde{\sigma}_\varepsilon|(y) \\
&\lesssim \mathcal{M}_\varepsilon \tilde{\sigma}_\varepsilon(z).
\end{aligned} \tag{5.24}$$

Combining (5.23) and (5.24), we get (5.22).

Finally, in light of $\nu_\varepsilon(Q_i) \le \nu(3Q_i)$, the definition of ν_ε, and (5.22), we have that

$$\begin{aligned}\int |T_\varepsilon\tilde{\sigma}_\varepsilon(x)|^2\,\mathrm{d}\nu_\varepsilon(x) &= \sum_{i=1}^{\tilde{N}} \int_{Q_i} |T_\varepsilon\tilde{\sigma}_\varepsilon(x)|^2\,\mathrm{d}\nu_\varepsilon(x) \le \sum_{i=1}^{\tilde{N}} \Big(\nu_\varepsilon(Q_i) \sup_{x\in 3Q_i} |T_\varepsilon\tilde{\sigma}_\varepsilon(x)|^2\Big)\\ &\le \sum_{i=1}^{\tilde{N}} \Big[\nu_\varepsilon(Q_i)\Big(\inf_{z\in 3Q_i} |T_\varepsilon\tilde{\sigma}_\varepsilon(z)|^2 + \inf_{z\in 3Q_i} \big(\mathcal{M}_\varepsilon\tilde{\sigma}_\varepsilon(z)\big)^2\Big)\Big]\\ &\le \sum_{i=1}^{\tilde{N}} \int_{3Q_i} |T_\varepsilon\tilde{\sigma}_\varepsilon(x)|^2\,\mathrm{d}\nu(x) + \sum_{i=1}^{\tilde{N}} \int_{3Q_i} \big(\mathcal{M}_\varepsilon\tilde{\sigma}_\varepsilon(x)\big)^2\,\mathrm{d}\nu(x)\\ &\le \|T_\varepsilon\tilde{\sigma}_\varepsilon\|^2_{L^2(\nu)} + \|\mathcal{M}_\varepsilon\tilde{\sigma}_\varepsilon\|^2_{L^2(\nu)} = \|T_{\nu,\varepsilon}\tilde{f}_\varepsilon\|^2_{L^2(\nu)} + \|\mathcal{M}_{\nu,\varepsilon}\tilde{f}_\varepsilon\|^2_{L^2(\nu)}.\end{aligned} \tag{5.25}$$

Hence, by (5.20), (5.21), (5.25), and (5.19), we infer that

$$\begin{aligned}\|T_{\nu_\varepsilon,\delta} f\|_{L^2(\nu_\varepsilon)} &\lesssim \|\mathcal{M}_{\nu_\varepsilon} f\|_{L^2(\nu_\varepsilon)} + \|\mathcal{M}_{\nu_\varepsilon,\varepsilon} f\|_{L^2(\nu_\varepsilon)} + \|\mathcal{M}_{\nu,\varepsilon}\tilde{f}_\varepsilon\|_{L^2(\nu_\varepsilon)}\\ &\quad + \|T_\varepsilon\tilde{\sigma}_\varepsilon\|_{L^2(\nu_\varepsilon)} + \|\mathcal{M}_{\nu,\varepsilon}\tilde{f}_\varepsilon\|_{L^2(\nu)}\\ &\lesssim \|f\|_{L^2(\nu_\varepsilon)} + \|\tilde{f}_\varepsilon\|_{L^2(\nu)} + \|T_\nu\|_{L^2(\nu)\to L^2(\nu)}\|\tilde{f}_\varepsilon\|_{L^2(\nu)}\\ &\overset{(5.9)}{\lesssim} \big(1 + \|T_\nu\|_{L^2(\nu)\to L^2(\nu)}\big)\|f\|_{L^2(\nu_\varepsilon)},\end{aligned}$$

which concludes the proof of the lemma. □

Conversely to the previous lemma, we prove that the uniform $L^2(\nu_\varepsilon)$-boundedness of T_{ν_ε} with respect to ε controls the $L^2(\nu)$-boundedness of T_ν at small scales.

Lemma 5.2. *Let ν, ν_ε, A and Q be as in Lemma 5.1. Let us also assume that there exists $C > 0$ such that $\|T_{\nu_\varepsilon}\|_{L^2(\nu_\varepsilon)\to L^2(\nu_\varepsilon)} \le C$ for all $\varepsilon > 0$. Then for any fixed $\delta > 0$,*

$$\lim_{\varepsilon\to 0}\big\|T_{\nu_\varepsilon,\delta}\big\|_{L^2(\nu_\varepsilon)} = \|T_{\nu,\delta}\|_{L^2(\nu)} \quad \textit{for any fixed } \delta > 0. \tag{5.26}$$

In particular $\|T_\nu\|_{L^2(\nu)\to L^2(\nu)} \le C$.

Proof. It is clear that $\|T_\nu\|_{L^2(\nu)\to L^2(\nu)} \le C$ follows form (5.26) and $\|T_{\nu_\varepsilon}\|_{L^2(\nu_\varepsilon)\to L^2(\nu_\varepsilon)} \le C$ and so it suffices to prove (5.26). To this end, if $f \in C_c^\infty(Q)$ and $\delta > 0$ is fixed, we have that

$$\begin{aligned}&\left|\int \big|T_{\nu_\varepsilon,\delta} f(x)\big|^2\,\mathrm{d}\nu_\varepsilon(x) - \int \big|T_{\nu,\delta} f(x)\big|^2\,\mathrm{d}\nu(x)\right|\\ &\quad\le \int \big|T_{\nu,\delta} f(x)\big|^2\, d\big(\nu_\varepsilon - \nu\big)(x) + \int \Big|\big|T_{\nu_\varepsilon,\delta} f(x)\big|^2 - \big|T_{\nu,\delta} f(x)\big|^2\Big|\,\mathrm{d}\nu_\varepsilon(x)\\ &\quad =: I_{\delta,\varepsilon} + II_{\delta,\varepsilon}.\end{aligned}$$

The first summand $I_{\delta,\varepsilon}$ vanishes as $\varepsilon \to 0$ because $|T_{\nu,\delta} f(x)|^2$ is a bounded and continuous function, ν is compactly supported, and ν_ε converges weakly to ν.

If $\psi \in C^\infty$ is a non-negative smooth function such that $\chi_{B(0,2)^c} \leq \psi \leq \chi_{B(0,1)^c}$ with $\|\nabla\psi\|_\infty \lesssim 1$. We set $\psi_\delta(\cdot) = \psi(\cdot/\delta)$ and

$$K(x,y):=\nabla_1\Gamma_A(x,y),\ \ x,y \in \mathbb{R}^{n+1}\setminus\{0\} \quad \text{and} \quad K_\delta(x,y):=K(x,y)\psi_\delta(x-y).$$

As Lemma 3.9 entails $|K(x,y)| \lesssim |x-y|^{-n}$ for all $x, y \in B(0,R)$, it follows that

$$|K_\delta(x,y)| \lesssim \delta^{-n} \qquad \text{for all } x,y \in B(0,R), x \neq y. \tag{5.27}$$

Moreover, if $x, y_1, y_2 \in B(0,R)$ such that $y_1 \neq y_2$ and $2|y_1 - y_2| < \min\{|x - y_1|, |x - y_2|\}$, by the mean value theorem and (3.23), it holds that

$$\begin{aligned}
&|K_\delta(x,y_1) - K_\delta(x,y_2)| \\
&\quad \leq |K_\delta(x,y_1)|\,|\psi_\delta(x-y_1) - \psi_\delta(x-y_2)| + |K_\delta(x,y_1) - K_\delta(x,y_2)| \\
&\quad \lesssim \frac{|y_1-y_2|}{\delta^{n+1}} + \left(\frac{|y_1-y_2|^\beta}{\delta^\beta} + \int_0^{\frac{|y_1-y_2|}{\delta}} \omega_A(t)\,\frac{\mathrm{d}t}{t}\right)\frac{1}{\delta^n} \\
&\quad \overset{(1.13)}{\lesssim_\delta} \alpha_A\Big(\frac{|y_1-y_2|}{\delta}\Big).
\end{aligned}$$

If $|y_1 - y_2| < \varepsilon \ll \delta$, then, since $\delta < \min\{|x-y_1|, |x-y_2|\}$, we have that

$$|K_\delta(x,y_1) - K_\delta(x,y_2)| \lesssim_{\Lambda,n,\delta} \alpha_A(\varepsilon/\delta) \to 0, \ \ \text{as } \varepsilon \to 0. \tag{5.28}$$

In order to estimate $II_{\delta,\varepsilon}$ we first observe that (5.27) and the fact that $\nu_\varepsilon(\mathbb{R}^{n+1}) = \nu(\mathbb{R}^{n+1})$ readily imply

$$\left|T_{\nu_\varepsilon,\delta} f(x) + T_{\nu,\delta} f(x)\right| \lesssim \delta^{-n}\|f\|_{L^\infty(\mathbb{R}^{n+1})}\nu(\mathbb{R}^{n+1}), \tag{5.29}$$

where the implicit constant is independent of ε. Also, the definition of ν_ε and Fubini's theorem yield

$$\begin{aligned}
&\left|T_{\nu_\varepsilon,\delta} f(x) - T_{\nu,\delta} f(x)\right| = \left|\int K_\delta(x,y) f(y)\,\mathrm{d}\nu_\varepsilon(y) - \int K_\delta(x,z) f(z)\,\mathrm{d}\nu(z)\right| \\
&\quad = \left|\int K_\delta(x,y) f(y) \int \varphi_\varepsilon(y-z)\,\mathrm{d}\nu(z)\,\mathrm{d}y\right. \\
&\qquad \left. - \int K_\delta(x,z) f(z) \int \varphi_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}\nu(z)\right| \\
&\quad = \left|\int \left[K_\delta(x,y) f(y)\varphi_\varepsilon(y-z)\,\mathrm{d}y - \int K_\delta(x,z) f(z)\varphi_\varepsilon(y)\,\mathrm{d}y\right]\mathrm{d}\nu(z)\right| \\
&\quad \leq \int \int_{B(0,\varepsilon)} \left|K_\delta(x,y+z) f(y+z) - K_\delta(x,z) f(z)\right| \varphi_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}\nu(z).
\end{aligned}$$

Furthermore, the estimate (5.27), the mean value theorem, and (5.28) imply

$$\begin{aligned}
&\left|T_{\nu_\varepsilon,\delta} f(x) - T_{\nu,\delta} f(x)\right| \\
&\quad \leq \int \int_{B(0,\varepsilon)} \left|K_\delta(x,y+z)\right| |f(y+z) - f(z)| \varphi_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}\nu(z) \\
&\qquad + \int \int_{B(0,\varepsilon)} |f(z)| \left|K_\delta(x,y+z) - K_\delta(x,z)\right| \varphi_\varepsilon(y)\,\mathrm{d}y\,\mathrm{d}\nu(z) \\
&\quad \lesssim \frac{\varepsilon}{\delta^n}\|\nabla f\|_{L^\infty(\mathbb{R}^{n+1})}\nu(\mathbb{R}^{n+1}) + \alpha_A(\varepsilon/\delta)\|f\|_{L^\infty(\mathbb{R}^{n+1})}\nu(\mathbb{R}^{n+1}).
\end{aligned}$$

Hence, by(5.29) and the latter inequality we infer that

$$II_{\delta,\varepsilon} = \int \left|T_{\nu_\varepsilon,\delta} f(x) + T_{\nu,\delta} f(x)\right|\left|T_{\nu_\varepsilon,\delta} f(x) - T_{\nu,\delta} f(x)\right| \mathrm{d}\nu_\varepsilon(x) \lesssim_{\delta,f,\nu} \varepsilon + \alpha_A(\varepsilon/\delta)$$

and so $II_{\delta,\varepsilon} \to 0$ as $\varepsilon \to 0$. This proves (5.26), which concludes the proof of the lemma. □

6. The Proof of Theorem 1.1

Proof of Theorem 1.1. Let $R := \operatorname{diam}(\operatorname{supp}\mu)$ and assume that $\operatorname{supp}\mu \subset Q_0$ for some cube Q_0 such that $\ell(Q_0) = R$. First, we prove that

$$\|\mathcal{R}_\mu\|_{L^2(\mu)\to L^2(\mu)} \lesssim 1 + \|T_\mu\|_{L^2(\mu)\to L^2(\mu)}. \tag{6.1}$$

For $N \in \mathbb{N}$ we consider a collection of cubes $\{Q_i\}_{1\le i\le N^{n+1}}$ with disjoint interior such that $\ell(Q_i) = R/N$ for all i and $Q_0 = \bigcup_i Q_i$.

We also denote $\mu_i := \mu|_{Q_i}$ and observe that T_{μ_i} is bounded on $L^2(\mu_i)$ and satisfies

$$\|T_{\mu_i}\|_{L^2(\mu_i)\to L^2(\mu_i)} \le \|T_\mu\|_{L^2(\mu)\to L^2(\mu)}. \tag{6.2}$$

We recall that $Q_i \subset B\big(x_{Q_i}, \sqrt{n+1}\,\ell(Q_i)\big) \subset B(x_{Q_i}, 2n\ell(Q_i))$. Moreover, we denote $S_i := \sqrt{(\bar{A}_s)_{x_{Q_i},4n\Lambda\ell(Q_i)}}$ and we define the change of variables $\psi_i(x) := S_i x$ for all $x \in \mathbb{R}^{n+1}$. Finally, we consider the measure $\nu_i := (\psi_i^{-1})_\sharp\mu_i$, and we denote by $\hat{A}^i$ the matrix defined in (3.8), namely

$$\hat{A}^i(x) := \frac{|\det S_i|\left|S_i^{-1}\big(B(x_{Q_i}, 4n\Lambda\ell(Q_i))\big)\right|}{|B(x_{Q_i}, 4n\Lambda\ell(Q_i))|} S_i^{-1}(A\circ S_i)(x)S_i^{-1} \qquad \text{for all } x \in \mathbb{R}^{n+1}.$$

By Lemma 3.5 we have that $⨍_{S_i^{-1}\left(B(x_{Q_i},4n\Lambda\ell(Q_i))\right)} \hat{A}^i_s = Id$ and, by Lemma 3.6, the moduli of oscillation $\omega_{\hat{A}^i}$ and $\tau_{\hat{A}^i}$ belong to $\widetilde{\mathrm{DMO}}$. Moreover, by (3.9) it holds that

$$\operatorname{supp}\nu_i \subset S_i^{-1}(Q_i) \subset S_i^{-1}\big(B(x_{Q_i}, 2n\ell(Q_i))\big) \subseteq B\big(S_i^{-1}x_{Q_i}, 4n\Lambda^{1/2}\ell(Q_i)\big)$$

and the ball $B\big(S_i^{-1}x_{Q_i}, 4n\Lambda^{1/2}\ell(Q_i)\big)$ is contained in a cube $\tilde{Q}_i$ with center $x_{\tilde{Q}_i} = S_i^{-1}x_{Q_i}$ and side-length $\ell(\tilde{Q}_i) = 4n\Lambda^{1/2}\ell(Q_i)$.

Given $M := 4n\Lambda^{3/2}$, the inclusions (3.9) imply

$$B\big(x_{\tilde{Q}_i}, \ell(\tilde{Q}_i)\big) \subset S_i^{-1}\big(B(x_{Q_i}, 4n\Lambda\ell(Q_i))\big) \subset B\big(x_{\tilde{Q}_i}, M\ell(\tilde{Q}_i)\big).$$

The definition of ν_i and the bilipschitz character of ψ_i yield

$$\nu_i(B(x,r)) \le c_0 M r^n \qquad \text{for all } x \in \mathbb{R}^{n+1}, r > 0.$$

Hence, the measure ν_i belongs to $M^n_+(\mathbb{R}^{n+1})$ and is supported on $\tilde{Q}_i$. For $\varepsilon > 0$, let $\nu_{i,\varepsilon}$ be the auxiliary measure defined as in (5.1). Let f be a compactly supported Lipschitz function in $L^2(\nu_{i,\varepsilon})$ (this class is clearly dense in $L^2(\nu_{i,\varepsilon})$), and observe that $T_{\psi_i,\nu_{i,\varepsilon}}$ defined according to the notation introduced in (3.4) satisfies the assumptions of Main Lemma 4.1. By triangle inequality, the Main Lemma 4.1, and Lemma 3.6 we have

$$\begin{aligned}\|\mathcal{R}_{\nu_{i,\varepsilon}} f\|_{L^2(\nu_{i,\varepsilon})} &\leq \omega_n \|\omega_n^{-1}\mathcal{R}_{\nu_{i,\varepsilon}} f - T_{\psi_i,\nu_{i,\varepsilon}} f\|_{L^2(\nu_{i,\varepsilon})} + \omega_n \|T_{\psi_i,\nu_{i,\varepsilon}} f\|_{L^2(\nu_{i,\varepsilon})} \\ &\leq C'\left(\mathfrak{I}_{\tau_A}(\ell(\tilde{Q}_i)) + \widehat{\tau}_A(\ell(\tilde{Q}_i))\right)\|f\|_{L^2(\nu_{i,\varepsilon})} \\ &\quad + \omega_n \|T_{\psi_i,\nu_{i,\varepsilon}} f\|_{L^2(\nu_{i,\varepsilon})} + C''\left(\int_0^{\ell(\tilde{Q}_i)} \omega_A(t)\,\frac{\mathrm{d}t}{t}\right)^{1/2} \|\mathcal{R}_{\nu_{i,\varepsilon}} f\|_{L^2(\nu_{i,\varepsilon})} \\ &\leq \left(\|f\|_{L^2(\nu_{i,\varepsilon})} + \omega_n \|T_{\psi_i,\nu_{i,\varepsilon}} f\|_{L^2(\nu_{i,\varepsilon})}\right) + \frac{1}{2}\|\mathcal{R}_{\nu_{i,\varepsilon}} f\|_{L^2(\nu_{i,\varepsilon})},\end{aligned} \tag{6.3}$$

where in the last inequality we used Lemma 5.1 and chose N big enough (see also Remark 2.2) so that

$$C''\left(\int_0^{\ell(\tilde{Q}_i)} \omega_A(t)\,\frac{\mathrm{d}t}{t}\right)^{1/2} \leq 1/2$$
$$C'\left(\mathfrak{I}_{\tau_A}\left(\ell(\tilde{Q}_i)\right) + \widehat{\tau}_A\left(\ell(\tilde{Q}_i)\right)\right) \leq 1.$$

Hence, since $\|\mathcal{R}_{\nu_{i,\varepsilon}} f\|_{L^2(\nu_{i,\varepsilon})} < \infty$ by construction of the approximating measures $\nu_{i,\varepsilon}$ and the fact that f is Lipschitz with compact support, the estimate (6.3) implies that

$$\begin{aligned}\|\mathcal{R}_{\nu_{i,\varepsilon}} f\|_{L^2(\nu_{i,\varepsilon})} &\leq 2\,\|f\|_{L^2(\nu_{i,\varepsilon})} + 2\omega_n \|T_{\psi_i,\nu_{i,\varepsilon}} f\|_{L^2(\nu_{i,\varepsilon})} \\ &\overset{(5.2)}{\leq} \left(2 + 2\omega_n \|T_{\psi_i,\nu_i}\|_{L^2(\nu_i)\to L^2(\nu_i)}\right)\|f\|_{L^2(\nu_{i,\varepsilon})}.\end{aligned}$$

So, if we take the supremum over all compactly supported Lipschitz functions in $L^2(\nu_{i,\varepsilon})$, by density, we have that for any $\delta > 0$,

$$\begin{aligned}\|\mathcal{R}_{\nu_{i,\varepsilon},\delta}\|_{L^2(\nu_{i,\varepsilon})\to L^2(\nu_{i,\varepsilon})} &\leq 2 + 2\omega_n \|T_{\psi_i,\nu_i}\|_{L^2(\nu_i)\to L^2(\nu_i)} \\ &\overset{(3.7)}{\leq} \widetilde{C}_1 + \widetilde{C}_1 \|T_{\mu_i}\|_{L^2(\mu_i)\to L^2(\mu_i)}.\end{aligned}$$

Taking limits as $\varepsilon \to 0$, by Lemma 5.2, we infer

$$\|\mathcal{R}_{\nu_i,\delta}\|_{L^2(\nu_i)\to L^2(\nu_i)} \leq \widetilde{C}_1 + \widetilde{C}_1 \|T_{\mu_i}\|_{L^2(\mu_i)\to L^2(\mu_i)}, \tag{6.4}$$

uniformly in $\delta > 0$, and, applying [41, Corollary 1.3], there exists $\widetilde{C}_2 > 0$ depending on dimension and Λ, such that

$$\begin{aligned}\|\mathcal{R}_{\mu_i}\|_{L^2(\mu_i)\to L^2(\mu_i)} &\leq \widetilde{C}_2 \|\mathcal{R}_{\nu_i}\|_{L^2(\nu_i)\to L^2(\nu_i)} \leq \widetilde{C}_1\widetilde{C}_2 + \widetilde{C}_1\widetilde{C}_2 \|T_{\mu_i}\|_{L^2(\mu_i)\to L^2(\mu_i)} \\ &\overset{(6.2)}{\leq} \widetilde{C}_1\widetilde{C}_2 + \widetilde{C}_1\widetilde{C}_2 \|T_\mu\|_{L^2(\mu)\to L^2(\mu)}.\end{aligned} \tag{6.5}$$

Thus [40, Proposition 2.25] and [40, Theorem 2.21] imply that

$$\|\mathcal{R}_\mu\|_{L^2(\mu)\to L^2(\mu)} \lesssim \|\mathcal{R}_*\|_{M(\mathbb{R}^{n+1})\to L^{1,\infty}(\mu)} \lesssim_{n,N} \sum_{i=1}^{N^{n+1}} \|\mathcal{R}_*\|_{M(\mathbb{R}^{n+1})\to L^{1,\infty}(\mu_i)}$$

$$\lesssim_{n,N} \sum_{i=1}^{N^{n+1}} (1 + \|\mathcal{R}_{\mu_i}\|_{L^2(\mu_i)\to L^2(\mu_i)})$$
$$\overset{(6.5)}{\lesssim}_{n,N,\Lambda,R} 1 + \|T_\mu\|_{L^2(\mu)\to L^2(\mu)},$$

which concludes the proof of (6.1) since N depends on Λ, n, and R.

In order to prove the converse inequality it is enough to observe that, for N big enough, (6.1) yields

$$\begin{aligned}\|T_{\psi_i,\nu_i}\|_{L^2(\nu_{i,\varepsilon})\to L^2(\nu_i)} &\le \|T_{\psi_i,\nu_i} - \omega_n^{-1}\mathcal{R}_{\nu_i}\|_{L^2(\nu_{i,\varepsilon})\to L^2(\nu_i)} + \omega_n^{-1}\|\mathcal{R}_{\nu_i}\|_{L^2(\nu_{i,\varepsilon})\to L^2(\nu_i)}\\ &\lesssim 1 + \|\mathcal{R}_{\nu_i}\|_{L^2(\nu_i)\to L^2(\nu_i)} \lesssim 1 + \|\mathcal{R}_{\mu_i}\|_{L^2(\mu_i)\to L^2(\mu_i)},\end{aligned}$$

where the last inequality follows from [41, Corollary 1.3]. Then we apply (3.5) and (3.6) and deduce that $\|T_{\mu_i}\|_{L^2(\mu)\to L^2(\mu)} \lesssim 1 + \|\mathcal{R}_{\mu_i}\|_{L^2(\mu_i)\to L^2(\mu_i)}$. Finally, we can repeat the argument above using [40, Theorem 2.21] and [40, Proposition 2.25], which are still true for the operator T_μ (the hypothesis that τ_A is a Dini function makes possible to argue via estimates in terms of the centered maximal function which closely resemble (5.23)), and conclude the proof of the theorem. □

Proof of Corollary 1.3. Let μ be an n-AD-regular measure on $\mathbb{R}^{n+1}$ with compact support such that the gradient of the single layer potential T_μ is bounded on $L^2(\mu)$. Then, in particular $\mu \in M_+^n(\operatorname{supp}\mu)$ so Proposition 1.1 implies that $\mathcal{R}_\mu$ is bounded on $L^2(\mu)$. Hence, the main result of [31] yields that μ is uniformly n-rectifiable. Conversely, it is immediate that Theorem 1.1 and the boundedness of the Riesz transform on uniformly n-rectifiable sets imply that the boundedness of T_μ on uniformly n-rectifiable sets. □

Proof of Corollary 1.2. Let μ be a non-trivial totally irregular measure on $\mathbb{R}^{n+1}$, i.e., it satisfies $0 < \Theta^{*,n}(x,\mu) < \infty$ and $\Theta_*^n(x,\mu) = 0$ for μ-a.e. $x \in \mathbb{R}^{n+1}$. Arguing by contradiction, we assume that $\|\mathcal{T}_\mu\|_{L^2(\mu)\to L^2(\mu)} < \infty$ and so, by Lemma 3.17, μ has n-polynomial growth. Thus, by Theorem 1.1, we have that $\|\mathcal{R}_\mu\|_{L^2(\mu)\to L^2(\mu)} < \infty$, which contradicts the main result of [16]. □

Proof of Corollary 1.4. This is a direct consequence of Theorem 1.1 and the main result of [32]. □

Proof of Corollary 1.6. We first apply Lemma 4.2 for N large enough (depending on n, τ, C_1, and $\operatorname{diam}(\operatorname{supp}\nu)$) so that

$$\mathfrak{I}_{\alpha_A}(2^{-N}) \le \tau^{1/2},$$

where we recall that $\alpha_A(t) = t + t^\beta + \omega_A(t)$, $t > 0$.

Then, in view of (6.5), we choose $\ell(Q)$ small enough (depending on N and τ) so that $2^N\ell(Q)$ is as small as required in the proof of Theorem 1.1 in order for the following estimate to hold:

$$\|\mathcal{R}_\mu\|_{L^2(\mu|_{2^NQ})\to L^2(\mu|_{2^NQ})} \le 1 + C_2\|T_\mu\|_{L^2(\mu|_{2^NQ})\to L^2(\mu|_{2^NQ})} \le 1 + C_0' C_2.$$

Here C_2 depends on n, Λ, M, and C_0. Finally, we may choose $\ell(Q)$ even smaller (depending on n, N, τ, C_0', C_1, C_2, and $\operatorname{diam}(\operatorname{supp}\nu)$) so that

$$\mathfrak{I}_{\tau_A}\big(2^N\ell(Q)\big) + \widehat{\tau}_A\big(2^N\ell(Q)\big) \leq \tau^{1/2}$$

and also

$$\mathfrak{I}_{\omega_A}(2^N\ell(Q))^{1/2}\|\mathcal{R}_\mu\|_{L^2(\mu|_{2^N Q})\to L^2(\mu|_{2^N Q})} \leq (1+C_0'\,C_2)\mathfrak{I}_{\omega_A}(2^N\ell(Q))^{1/2} \leq \tau^{1/2}.$$

Collecting all the above estimates and combining them with Lemma 4.2, we deduce that

$$\left(⨍_Q \left|\mathcal{R}_\mu 1(x) - ⨍_Q \mathcal{R}_\mu 1\right|^2 \mathrm{d}\mu(x)\right)^{1/2} \leq 4C_1\,\tau^{1/2}\Theta_\mu(2^N Q).$$

If $\tilde{\tau} = 4C_1\tau^{1/2}$ is small enough depending on C_0, C_0', n, Λ, M, and $\operatorname{diam}(\operatorname{supp}\mu)$, in light of Theorem 1.1 applied to the measure $\mu|_{2^N Q}$, we can implement [20, Theorem 1.1] and conclude the proof of the corollary. □

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References

1. Atkinson, K., Han, W.: Spherical Harmonics and Approximations on the Unit Sphere. An Introduction. Lecture Notes in Mathematics 2044. Springer, Berlin (2012)
2. Azzam, J., Garnett, J., Mourgoglou, M., Tolsa, X.: Uniform rectifiability, elliptic measure, square functions, and ε-approximability via an ACF monotonicity formula. *Int. Math. Res. Not. (IMRN)* (2021). https://doi.org/10.1093/imrn/rnab095
3. Azzam, J., Hofmann, S., Martell, J.M., Mayboroda, S., Mourgoglou, M., Tolsa, X., Volberg, A.: Rectifiability of harmonic measure. *Geom. Funct. Anal.* **26**(3), 703–728, 2016

4. Azzam, J., Mourgoglou, M., Tolsa, X.: Mutual absolute continuity of interior and exterior harmonic measure implies rectifiability. *Commun. Pure Appl. Math.* **70**(11), 2121–2163, 2017
5. Azzam, J., Mourgoglou, M., Tolsa, X.: A two-phase free boundary problem for harmonic measure and uniform rectifiability. *Trans. Am. Math. Soc.* **373**(6), 4359–4388, 2020
6. Azzam, J., Mourgoglou, M., Tolsa, X., Volberg, A.: On a two-phase problem for harmonic measure in general domains. *Am. J. Math.* **141**(5), 1259–1279, 2019
7. Bailey, J., Morris, A., Reguera, M.C.: Unboundedness of potential dependent Riesz transforms for totally irregular measures. *J. Math. Anal. Appl.* **494**(1), 124570, 2021
8. Bortz, S., Hofmann, S.: A singular integral approach to a two phase free boundary problem. *Proc. Am. Math. Soc.* **144**(9), 3959–3973, 2016
9. Calderón, A.: Cauchy integrals on Lipschitz curves and related operators. *Proc. Natl. Acad. Sci. USA* **74**, 1324–1327, 1977
10. Conde-Alonso, J.M., Mourgoglou, M., Tolsa, X.: Failure of L^2 boundedness of gradients of single layer potentials for measures with zero low density. *Math. Ann.* **373**(1–2), 253–285, 2019
11. Dabrowski, D., Tolsa, X.: The measures with L^2-bounded Riesz transform satisfying a subcritical Wolff-type energy condition. (2021). arXiv:2106.00303
12. David, G.: *Wavelets and singular integrals on curves and surfaces*. Lecture Notes in Mathematics. 1465. Springer-Verlag, Berlin etc. x, pp. 109 (1992)
13. David, G., Semmes, S.: *Singular integrals and rectifiable sets in Rn. Au-delà des graphes lipschitziens*. Astérisque, 193. Montrouge: Société Mathématique de France, p. 145 (1991)
14. David, G., Semmes, S.: Analysis of and On Uniformly Rectifiable Sets. Mathematical Surveys and Monographs, vol. 38. American Mathematical Society, Providence (1993)
15. Dong, H., Kim, S.: On C^1, C^2, and weak type-(1,1) estimates for linear elliptic operators. *Commun. Partial Differ. Equ.* **43**(3), 417–435, 2017
16. Eiderman, V., Nazarov, F., Volberg, A.: The s-Riesz transform of an s-dimensional measure in $\mathbb{R}^2$ is unbounded for $1 \geq s \geq 2$. *J. Anal. Math.* **122**, 1–23 (2014)
17. Folland, G.: Real Analysis. Modern Techniques and Their Applications. 2nd ed. Pure and Applied Mathematics. A Wiley-Interscience Series of Texts, Monographs, and Tracts. Wiley, New York (1999)
18. Garnett, J., Mourgoglou, M., Tolsa, X.: Uniform rectifiability from Carleson measure estimates and ε-approximability of bounded harmonic functions. *Duke Math. J.* **167**(8), 1473–1524, 2018
19. Giaquinta, M., Martinazzi, L.: An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs, 2nd edn. Edizioni della Normale, Pisa (2012)
20. Girela-Sarrión, D., Tolsa, X.: The Riesz transform and quantitative rectifiability for general Radon measures. *Calc. Var. Partial. Differ. Equ.* **57**(1), 16, 2018
21. Hofmann, S., Kim, S.: The Green function estimates for strongly elliptic systems of second order. *Manuscr. Math.* **124**(2), 139–172, 2007
22. Hwang, S., Kim, S.: Green's function for second order elliptic equations in non-divergence form. *Potential Anal.* **52**, 27–39, 2020
23. Kenig, C., Shen, Z.: Layer potential methods for elliptic homogenization problems. *Commun. Pure Appl. Math.* **64**(1), 1–44, 2011
24. Li, Y.: On the C^1 regularity of solutions to divergence form elliptic systems with Dini-continuous coefficients. *Chin. Ann. Math. Ser. B* **38**(2), 489–496, 2017
25. Malý, J., Ziemer, W.P.: Fine Regularity of Solutions of Elliptic Partial Differential Equations. American Mathematical Society, Providence (1997)
26. Mattila, P., Melnikov, M., Verdera, J.: The Cauchy integral, analytic capacity, and uniform rectifiability. *Ann. of Math. (2)* **144**(1), 127–136, 1996
27. Mitrea, M., Taylor, M.: Boundary layer methods for Lipschitz domains in Riemannian manifolds. *J. Funct. Anal.* **163**, 181–251, 1999

28. Mourgoglou, M.: Regularity theory and Green's function for elliptic equations with lower order terms in unbounded domains. (2019). arXiv:1904.04722
29. Mourgoglou, M., Tolsa, X.: Harmonic measure and Riesz transform in uniform and general domains. *J. Reine Angew. Math.* **758**, 183–221, 2020
30. Mourgoglou, M., Tolsa, X.: The regularity problem for the Laplace equation in rough domains. (2021). arXiv:2110.02205
31. Nazarov, F., Tolsa, X., Volberg, A.: On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1. *Acta Math.* **2**(213), 237–321, 2014
32. Nazarov, F., Tolsa, X., Volberg, A.: The Riesz transform, rectifiability, and removability for Lipschitz harmonic functions. *Publ. Mat.* **58**(2), 517–532, 2014
33. Prat, L., Puliatti, C., Tolsa, X.: L^2-boundedness of gradients of single-layer potentials and uniform rectifiability. *Anal. PDE* **14**(3), 717–791, 2021
34. Prats, M., Tolsa, X.: The two-phase problem for harmonic measure in VMO. *Calc. Var. Partial. Differ. Equ.* **59**(3), 102, 58, 2020
35. Puliatti, C.: Gradient of the single layer potential and quantitative rectifiability for general Radon measures. *J. Funct. Anal.* **282**(6), 109376, 2022
36. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton Mathematical Series 30, vol. XIV, p. 287. Princeton University Press, Princeton (1970)
37. Stein, E.M.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton Mathematical Series. 43, vol. xiii, p. 695. Princeton University Press, Princeton (1993)
38. Tolsa, X.: Painlevé's problem and the semiadditivity of analytic capacity. *Acta Math.* **190**(1), 105–149, 2003
39. Tolsa, X.: Bilipschitz maps, analytic capacity, and the Cauchy integral. *Ann. Math.* **162**(3), 1241–1302, 2005
40. Tolsa, X.: Analytic Capacity, the Cauchy Transform, and Non-homogeneous Calderón–Zygmund Theory. Progress in Mathematics 307, vol. xiii, p. 396. Birkhäuser/Springer, Basel (2014)
41. Tolsa, X.: The measures with L^2-bounded Riesz transform and the Painlevé problem for Lipschitz harmonic functions. (2021). arXiv:2106.00680

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