A SCHOENFLIES EXTENSION THEOREM FOR A CLASS OF LOCALLY BI-LIPSCHITZ HOMEOMORPHISMS

JASUN GONG

Abstract. In this paper we prove a new version of the Schoenflies extension theorem for collared domains $\Omega$ and $\Omega'$ in $\mathbb{R}^n$: for $p \in [1,n)$, locally bi-Lipschitz homeomorphisms from $\Omega$ to $\Omega'$ with locally $p$-integrable, second-order weak derivatives admit homeomorphic extensions of the same regularity.

Moreover, the theorem is essentially sharp. The existence of exotic 7-spheres shows that such extension theorems cannot hold, for $p > n = 7$.

1. Introduction

1.1. Embeddings of Collars. In point-set topology, the Schoenflies Theorem [Wil79, Thm III.5.9] is a stronger form of the well-known Jordan Curve Theorem: it states that every simple closed curve separates the sphere $S^2$ into two domains, each of which is homeomorphic to $B^2$, the open unit disc. The same statement does not hold in higher dimensions, since the Alexander horned sphere [Ale24] provides a counter-example in $\mathbb{R}^3$. Despite this, Brown [Bro60] proved that for each $n \in \mathbb{N}$, every embedding of $S^{n-1} \times (-\epsilon, \epsilon)$ into $\mathbb{R}^n$ extends to an embedding of $B^n$ into $\mathbb{R}^n$.

Similar extension problems arise by varying the regularity of the embeddings. To this end, we prove a Schoenflies-type theorem for a new class of homeomorphisms. Their regularity is given in terms of Sobolev spaces and Lipschitz continuity.

To begin, recall that a homeomorphism $f : \Omega \rightarrow \Omega'$ is locally bi-Lipschitz if for each $z \in \Omega$, there is a neighborhood $O$ of $z$ and $L \geq 1$ so that the inequality

\begin{equation}
L^{-1} |x - y| \leq |f(x) - f(y)| \leq L |x - y|
\end{equation}

holds for all $x, y \in O$. Recall also that for $p \geq 1$ and $k \in \mathbb{N}$, the Sobolev space $W^{k,p}_{\text{loc}}(\Omega; \Omega')$ consists of maps $f : \Omega \rightarrow \Omega'$, where each component $f_i$ lies in $L^p_{\text{loc}}(\Omega)$ and has weak derivatives of orders up to $k$ in $L^p_{\text{loc}}(\Omega)$.

Definition 1.1. Let $f : \Omega \rightarrow \Omega'$ be a locally bi-Lipschitz homeomorphism. For $p \in [1, \infty)$, we say that $f$ is of class $LW^p_2$ if $f \in W^{2,p}_{\text{loc}}(\Omega; \Omega')$ and $f^{-1} \in W^{2,p}_{\text{loc}}(\Omega'; \Omega)$. If $K$ and $K'$ are closed sets, a homeomorphism $f : K \rightarrow K'$ is of class $LW^p_2$ if the restriction of $f$ to the interior of $K$ is of class $LW^p_2$.

Instead of product sets of the form $S^{n-1} \times (-\epsilon, \epsilon)$, we will consider domains in $\mathbb{R}^n$ of a similar topological type.
Definition 1.2. A bounded domain $D$ in $\mathbb{R}^n$ is Jordan if its boundary $\partial D$ is homeomorphic to $S^{n-1}$. A collared domain (or collar) is a domain in $\mathbb{R}^n$ of the form $D_2 \setminus \overline{D}_1$, where $D_1$ and $D_2$ are Jordan domains with $\overline{D}_1 \subset D_2$.

We now state the extension theorem for homeomorphisms of class $LW^p_2$ between collared domains.

Theorem 1.3. Let $D_1$ and $D_2$ be Jordan domains in $\mathbb{R}^n$ so that $\overline{D}_1 \subset D_2$, let $B_1$ and $B_2$ be balls so that $\overline{B}_1 \subset B_2$, and let $p \in [1, n)$.

If $f : D_2 \setminus \overline{D}_1 \to B_2 \setminus B_1$ is a homeomorphism of class $LW^p_2$ so that $f(\partial D_i) = \partial B_i$ holds, for $i = 1, 2$, then there exists a homeomorphism $F : D_2 \to \overline{B}_2$ of class $LW^p_2$ and a neighborhood $N$ of $\partial D_2$ so that $F|_{(N \cap \overline{D}_2)} = f|_{(N \cap \overline{D}_2)}$.

The proof is an adaptation of Gehring’s argument [Geh67, Thm 2’] from the class of quasiconformal homeomorphisms to the class $LW^p_2$. For the locally bi-Lipschitz class, the extension theorem was known to Sullivan [Sul75] and later proved by Tukia and Väisälä [TV81, Thm 5.10]. For more about quasiconformal homeomorphisms, see [Väi71].

As in Gehring’s case, Theorem 1.3 is not quantitative. His extension depends on the distortion (resp. Lipschitz constants) of $g$ as well as the configurations of the collars $D_2 \setminus \overline{D}_1$ and $B_2 \setminus B_1$. In addition, our modification of his extension also depends explicitly on the parameters $p$ and $n$.

1.2. Motivations, Smoothness, and Sharpness. The motivation for Theorem 1.3 comes from the study of Lipschitz manifolds.

Specifically, Heinonen and Keith have recently shown that if an $n$-dimensional Lipschitz manifold, for $n \neq 4$, admits an atlas with coordinate charts in the Sobolev class $W^{2,2}_{loc}(\mathbb{R}^n; \mathbb{R}^n)$, then it admits a smooth structure [HK09].

On the other hand, there are 10-dimensional Lipschitz manifolds without smooth structures [Ker60]. This leads to the following question:

Question 1.4. For $n \neq 4$, does there exist $p \in [1, 2)$ so that every $n$-dimensional Lipschitz manifold admits an atlas of charts in $W^{2,p}_{loc}(\mathbb{R}^n; \mathbb{R}^n)$?

Sullivan has shown that every $n$-dimensional topological manifold, for $n \neq 4$, admits a Lipschitz structure [Sul75]. A key step in the proof is to show that bi-Lipschitz homeomorphisms satisfy a Schoenflies-type extension theorem. One may inquire whether this direction of proof would also lead to the desired Sobolev regularity. Theorem 1.3 would be a first step in this direction. For more about Lipschitz structures on manifolds, see [LV77].

It is worth noting that Theorem 1.3 is not generally true for $p > n$. Recall that for any domain $\Omega$ in $\mathbb{R}^n$, Morrey’s inequality [EG92, Thm 4.5.3.3] gives $W^{2,p}(\Omega) \hookrightarrow C^{1,1-n/p}(\Omega)$, so homeomorphisms of class $LW^p_2$ are necessarily $C^1$-diffeomorphisms.
Indeed, every $C^\infty$-diffeomorphism $\varphi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ admits a radial extension

$$\tilde{\varphi}(x) := |x| \varphi\left(\frac{x}{|x|}\right)$$

that is, a $C^\infty$-diffeomorphism between round annuli. The validity of Theorem 1.3, for $p > n$, would therefore imply that every such $\varphi$ extends to a $C^1$-diffeomorphism of $\mathbb{B}^n$ onto itself. However, for $n = 7$ this conclusion is impossible.

Recall that every such $\varphi$ also determines a $C^\infty$-smooth, $n$-dimensional manifold $M^n_\varphi$ that is homeomorphic to $\mathbb{S}^n$ [Mil56, Construction (C)]. Indeed, $M^n_\varphi$ is the quotient of two copies of $\mathbb{R}^n$ under the relation $x \sim \varphi^*(x)$ on $\mathbb{R}^n \setminus \{0\}$, where

$$\varphi^*(x) := \frac{1}{|x|} \varphi\left(\frac{x}{|x|}\right).$$

If $\varphi$ is the identity map on $\mathbb{S}^{n-1}$, then $\varphi^*$ is the inversion map $x \mapsto |x|^{-2}x$, and $M^n_\varphi$ is precisely $\mathbb{S}^n$. By using invariants from differential topology, Milnor proved the following theorem about such manifolds [Mil56, Thm 3].

**Theorem 1.5** (Milnor, 1956). There exist $C^\infty$-smooth manifolds of the form $M^n_\varphi$ that are homeomorphic, but not $C^\infty$-diffeomorphic, to $\mathbb{S}^7$.

Such manifolds are better known as exotic spheres. The next lemma is an analogue of [Hir94, Thm 8.2.1]; it relates exotic spheres to extension theorems.

**Lemma 1.6.** Let $\varphi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be a $C^\infty$-diffeomorphism and let $\tilde{\varphi} : \mathbb{B}^n \setminus \{0\} \to \mathbb{B}^n \setminus \{0\}$ be its radial (diffeomorphic) extension. If there exists a $C^1$-diffeomorphism $\Phi : \mathbb{B}^n \to \mathbb{B}^n$ that agrees with $\tilde{\varphi}$ on a neighborhood of $\mathbb{S}^{n-1}$ in $\mathbb{B}^n$, then $M^n_\varphi$ is $C^1$-diffeomorphic to $\mathbb{S}^n$.

**Proof of Lemma 1.6.** Let $\varphi^*$ be the diffeomorphism defined in Equation (1.2). By construction, there is an atlas of charts $\{M_i\}_{i=1}^2$ for $M^n_\varphi$ with homeomorphisms $\psi_i : M_i \to \mathbb{R}^n$ that satisfy $\psi_1 \circ \psi_1^{-1} = \varphi^*$.

Let $\pi_1, \pi_2 : \mathbb{R}^n \to \mathbb{S}^n$ be stereographic projections relative to the “north” and “south” poles on $\mathbb{S}^n$, respectively, so $\pi_1^{-1} \circ \pi_1 = \text{id}^* = (\text{id}^*)^{-1}$.

Observe that

$$((\text{id}^*)^{-1} \circ \varphi^*)(x) = \frac{\varphi^*(x)}{|\varphi^*(x)|^2} = |x| \varphi\left(\frac{x}{|x|}\right) = \tilde{\varphi}(x)$$

holds for all $x \in \mathbb{R}^n \setminus \{0\}$. It follows that

$$x \mapsto \begin{cases} (\pi_1^{-1} \circ \psi_1)(x), & \text{if } x \in M_1 \\ (\pi_2^{-1} \circ \Phi \circ \psi_2)(x), & \text{if } x \in M_2 \end{cases}$$

is a $C^1$-diffeomorphism of $M^n_\varphi$ onto $\mathbb{S}^n$. \qed
By [Hir94, Thm 2.2.10], if two $C^\infty$-smooth manifolds are $C^1$-diffeomorphic, then they are $C^\infty$-diffeomorphic. It follows that there exist $C^1$-diffeomorphisms of collars in $\mathbb{R}^7$ that do not admit diffeomorphic extensions of class $LW^p_2$, for any $p > 7$.

The next result follows from the inclusion $W^{2,p}_{\text{loc}}(\Omega; \Omega') \subseteq W^{2,q}_{\text{loc}}(\Omega; \Omega')$, for $q \leq p$.

**Corollary 1.7.** Let $n = 7$. For $p > n$, there exist collars $\Omega, \Omega'$ in $\mathbb{R}^n$ and homeomorphisms $\varphi : \Omega \to \Omega'$ of class $LW^p_2$ that admit homeomorphic extensions of class $LW^q_2$, for every $1 \leq q < n$, but not of class $LW^p_2$.

Since the above discussion relies crucially on Sobolev embedding theorems, it leaves open the borderline case $p = n$.

**Question 1.8.** Is Theorem 1.3 true for the case $p = n$?

For $p > n$, the main obstruction to an extension theorem is the existence of exotic $n$-spheres. It is known that no exotic spheres exist for $n = 1, 2, 3, 5, 6$ [KM63], and the case $n = 1$ can be done by hand.

It would be interesting to determine whether other geometric obstructions arise.

**Question 1.9.** For $n = 2, 3, 5, 6$, is Theorem 1.3 true for all $p \geq 1$?

The outline of the paper is as follows. In Section 2 we review basic facts about Lipschitz mappings, Sobolev spaces, and the class $LW^p_2$. In Section 3 we prove extension theorems in the setting of doubly-punctured domains. Section 4 addresses the case of homeomorphisms between collars, by employing suitable generalizations of inversion maps and reducing to previous cases.

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2. **Notation and Basic Facts**

For $A \subset \mathbb{R}^n$, we write $A^c$ for the complement of $A$ in $\mathbb{R}^n$. The open unit ball in $\mathbb{R}^n$ is denoted $B^n$; if the dimension is understood, we will write $B$ for $B^n$.

We write $A \lesssim B$ for inequalities of the form $A \leq kB$, where $k$ is a fixed dimensional constant and does not depend on $A$ or $B$.

For domains $\Omega$ and $\Omega'$ in $\mathbb{R}^n$, recall that a map $f : \Omega \to \Omega'$ is Lipschitz whenever

$$L(f) := \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in \Omega, x \neq y \right\} < \infty.$$
The map $f$ is \textit{locally Lipschitz} if every point in $\Omega$ has a neighborhood on which $f$ is Lipschitz. A homeomorphism $f : \Omega \rightarrow \Omega'$ is \textit{bi-Lipschitz} (resp. \textit{locally bi-Lipschitz}) if $f$ and $f^{-1}$ are both Lipschitz (resp. locally Lipschitz); compare Equation (1.1).

The following lemmas about bi-Lipschitz maps are used in Section 2. The first is a special case of [TV81, Lemma 2.17]; the second one is elementary, so we omit the proof.

\textbf{Lemma 2.1} (Tukia-Väisälä). Let $O$ and $O'$ be open, connected sets in $\mathbb{R}^n$ and let $K$ be a compact subset of $O$. If $f : O \rightarrow O'$ is locally bi-Lipschitz, then $f|K$ is bi-Lipschitz, where $L((f|K)^{-1})$ depends only on $O$, $K$, and $L(f)$.

\textbf{Lemma 2.2.} For $i = 1, 2$, let $h_i : \Omega_i \rightarrow \mathbb{R}^n$ be locally bi-Lipschitz embeddings so that $h_1(\Omega_1 \setminus \Omega_2) \cap h_2(\Omega_2 \setminus \Omega_1) = \emptyset$. If $h_1 = h_2$ holds on all of $\Omega_1 \cap \Omega_2$, then

$$h(x) = \begin{cases} h_1(x), & \text{if } x \in \Omega_1 \\ h_2(x), & \text{if } x \in \Omega_2 \setminus \Omega_1 \end{cases}$$

is also a locally bi-Lipschitz embedding.

For $f \in W^{2,p}(\Omega; \Omega')$, we will use the Hilbert-Schmidt norm for the weak derivatives $Df(x) := [\partial_j f_i(x)]_{i=1}^n$ and $D^2 f(x) := [\partial_k \partial_j f_i(x)]_{i,j,k=1}^n$. That is,

$$|Df(x)| := \left[ \sum_{i,j=1}^n |\partial_j f_i(x)|^2 \right]^{1/2}, \quad |D^2 f(x)| := \left[ \sum_{i,j,k=1}^n |\partial_k \partial_j f_i(x)|^2 \right]^{1/2}.$$

In what follows, we will use basic facts about Sobolev spaces, such as the change of variables formula [Zie89, Thm 2.2.2] and that Lipschitz functions on $\Omega$ are characterized by the class $W^{1,\infty}(\Omega)$ [EG92, Thm 4.2.3.5]. The lemma below gives a gluing procedure for Sobolev functions.

\textbf{Lemma 2.3.} For $i = 1, 2$, let $O_i$ be a domain in $\mathbb{R}^n$ and let $f_i \in W^{1,p}_{loc}(O_i)$. If $f_1 = f_2$ holds a.e. on $O_1 \cap O_2$, then $\chi_{O_1} f_1 + \chi_{O_2 \setminus O_1} f_2 \in W^{1,p}_{loc}(O_1 \cup O_2)$.

\textbf{Proof.} Let $O$ be a bounded domain in $\mathbb{R}^n$ so that $\bar{O} \subset O_1 \cup O_2$. For each $x \in O$, there exists $r > 0$ so that $B(x, r)$ lies entirely in $O_1$ or in $O_2$. Since $\bar{O}$ is compact, there exists $N \in \mathbb{N}$ and a collection of balls $\{B(x_i, r_i)\}_{i=1}^N$ whose union covers $\bar{O}$.

Let $\{\varphi_i\}_{i=1}^N$ be a smooth partition of unity that is subordinate to the cover $\{B(x_i, r_i)\}_{i=1}^N$. For each $i = 1, 2, \ldots, N$, one of $f_1 \varphi_i$ or $f_2 \varphi_i$ is well-defined and lies in $W^{1,p}(O)$; call it $\psi_i$. We now observe that $\psi := \sum_{i=1}^N \psi_i$ also lies in $W^{1,p}(O)$ and by construction, it agrees with $\chi_{O_1} f_1 + \chi_{O_2 \setminus O_1} f_2$. \hfill $\square$

It is a fact that the class $LW^p_2$ is preserved under composition. This is stated as a lemma below, and it follows directly from the product rule [EG92, Thm 4.2.2.4] and the change of variables formula [Zie89, Thm 2.2.2].
Lemma 2.4. Let $p \geq 1$. If $f : \Omega \to \Omega'$ and $g : \Omega' \to \Omega''$ are homeomorphisms of class $LW^p$, then so is $h := g \circ f$.

In addition, for a.e. $x \in \Omega$ and for all $i, j, k \in \{1, \cdots, n\}$, the weak derivatives satisfy

\[
\begin{aligned}
\partial_j h_i(x) &= \sum_{l=1}^n \partial_l g_l(f(x)) \partial_j f_l(x) \\
\partial_{kj} h_i(x) &= \sum_{l=1}^n \left[ \partial_l g_l(f(x)) \partial_{kj} f_l(x) + \sum_{m=1}^n \partial_{ml} g_l(f(x)) \partial_k f_m(x) \partial_j f_l(x) \right].
\end{aligned}
\]  

(2.1)

Remark 2.5. Linear maps (homeomorphisms) such as dilation and translation, are clearly of class $LW^2$. So if $g : \Omega \to \Omega'$ is any homeomorphism of class $LW^2$, then by Lemma 2.4, its composition with such linear maps is also of class $LW^2$. In what follows, we will implicitly use this fact to obtain convenient geometrical configurations.

3. Extensions for Homeomorphisms of Class $LW^p$ between Doubly-Punctured Domains

First we formulate the extension theorem in a different geometric configuration.

Theorem 3.1. Let $p \geq 1$, let $E_1$ and $E_2$ be Jordan domains so that $\overline{E_1} \cap \overline{E_2} = \emptyset$, and let $B_1$ and $B_2$ be balls so that $\overline{B_1} \cap \overline{B_2} = \emptyset$.

If $g : (E_2 \cup E_1)^c \to (B_1 \cup B_2)^c$ is a homeomorphism of class $LW^p$ so that $g(\partial E_i) = \partial B_i$ holds, for $i = 1, 2$, then there exists a homeomorphism $G : E_2^c \to B_2^c$ of class $LW^p$ and a neighborhood $N$ of $\partial E_2$ so that $g(N \cap E_2^c) = G(N \cap E_2^c)$.

Following the outline of [Geh67, Sect 3], we begin with a special case.

Lemma 3.2. Theorem 3.1 holds under the additional assumption that

\[
g|B^c = \text{id}|B^c
\]  

(3.1)

where $B$ is an open ball that contains $\overline{E_1}$ and $\overline{E_2}$.

Proof. Step 1. By composing with linear maps, we may assume that $B = \mathbb{R}$, and that there exist $a, b \in \mathbb{R}$ so that $a < b$ and $B_1 \subset \{x_n < a\}$ and $B_2 \subset \{x_n > b\}$.

Put $c = (b - a)/2$. Define an odd, $C^{1,1}$-smooth function $s_0 : \mathbb{R} \to [-1, 1]$ by

\[
s_0(t) := \begin{cases} 
1 - (t - c)^2/c^2, & \text{if } 0 \leq t \leq c \\
1, & \text{if } t > c
\end{cases}
\]

and using the auxiliary function $s : \mathbb{R} \to [0, 3]$, given by

\[
s(t) := \frac{3}{2} \left(s_0 \left(t - \frac{a + b}{2}\right) + 1\right)
\]

we define a bi-Lipschitz homeomorphism $S : \mathbb{R}^n \to \mathbb{R}^n$ by

\[
S(x) = x - s(x_n) e_1.
\]  

(3.2)
It is clear that $S$ is of class $LW^p$ and satisfies the a.e. estimate
\begin{equation}
|D^2S| \leq 2c^{-2}.
\end{equation}

**Step 2.**
For $k \in \mathbb{Z}$, put $\tau_k(x) = x + 3ke_1$ and consider the sets
\begin{equation}
\Omega := \left( \bigcup_{k=0}^{\infty} \tau_k(E_1) \cup \tau_k(E_2) \right)^c \quad \text{and} \quad \Omega' := \left( \bigcup_{k=0}^{\infty} \tau_k(B_1) \cup \tau_k(B_2) \right)^c.
\end{equation}
We now modify $g$ into a new homeomorphism $g_* : \Omega \to \Omega'$, as follows:
\begin{equation}
g_*(x) := \begin{cases} (\tau_k \circ g \circ \tau_{-k})(x), & \text{if } x \in \Omega \cap \tau_k(\mathbb{B}), \text{ for some } k \geq 0 \\ x, & \text{if } x \in \Omega \setminus \bigcup_{k=0}^{\infty} \tau_k(\mathbb{B}). \end{cases}
\end{equation}
By our hypotheses, there exists $r \in (0, 1)$ so that $E_1 \cup E_2 \subset B(0, r)$ and so that $g|\mathbb{B} \setminus B(0, r) = \text{id}$. Putting $\Omega_1 := \tau_k(\mathbb{B}) \cap \Omega$ and $\Omega_2 := \Omega \setminus \bigcup_{k=0}^{\infty} \tau_k(B(0, r))$ for each $k \in \mathbb{N}$, Lemma 2.2 implies that $g_*$ is locally bi-Lipschitz.

Similarly, for any bounded domain $O$ in $\Omega$ that meets $\tau_k(\partial \mathbb{B})$, put $O_1 := O \cap \Omega$ and $O_2 := O \setminus \tau_k(\overline{B(0, r)})$. For $f_1 := D(\tau_k \circ g \circ \tau_{-k})$ and $f_2 := D(\text{id})$, Lemma 2.3 implies that $g_* \in W^{2,p}(O)$ and therefore $g_* \in W^{2,p}_{loc}(\Omega; \Omega')$. By symmetry, the same is true of $g_*^{-1}$, so $g_*$ is of class $LW^p_2$.

**Step 3.** Consider the bi-Lipschitz homeomorphism given by
\begin{equation}
G_* := \tau_1 \circ g_*^{-1} \circ S \circ g_*.
\end{equation}
By Lemma 2.4, it is also of class $LW^p_2$. We now define $G : E_2^\infty \to B_2^\infty$ as
\begin{equation}
G(x) := \begin{cases} G_*(x), & \text{if } x \in \Omega \\ \tau_1(x), & \text{if } x \in \bigcup_{k=0}^{\infty} \tau_k(E_1) \\ x, & \text{if } x \in \bigcup_{k=1}^{\infty} \tau_k(E_2). \end{cases}
\end{equation}
By the same argument as [Geh67, pp. 153-4], the map $G$ is a homeomorphism. We also note that $G$ is “periodic” in the sense that, for each $k \in \mathbb{N}$,
\begin{equation}
(\tau_k \circ G \circ \tau_{-k})|\tau_k(\mathbb{B} \setminus E_2) = G|\tau_k(\mathbb{B} \setminus E_2).
\end{equation}
To see that $G$ extends $g$, consider the set $\sigma_{ab} := g_*^{-1}\{a \leq x_n \leq b\}$. Its complement $\mathbb{R}^n \setminus \sigma_{ab}$ consists of two (connected) components. Let $\sigma_b$ be the component containing the vector $e_n$, let $\sigma_a$ be the component containing $-e_n$, and consider
the open set \( N := \mathbb{B} \cap \sigma_b \). By assumption, \( \bar{B}_2 \) lies in \( \mathbb{B} \cap \{ x_n > b \} \), so \( \bar{E}_2 \) lies in \( N \). From before, we have \( g_* = g \) on \( \mathbb{B} \) and \( S = \tau_{-1} \) on \( \{ x_n > b \} \), which imply that

\[
(S \circ g_*)(N) = (\tau_{-1} \circ g)(N) = \tau_{-1}(\mathbb{B} \cap \{ x_n > b \}) \subset \tau_{-1}(\mathbb{B}).
\]

By hypothesis we have \( g_*^{-1} = \text{id} \) on \( \tau_{-1}(\mathbb{B}) \) and hence on \( (S \circ g)(N) \). It follows that

\[
G|N = G_*|N = (\tau_1 \circ g_*^{-1} \circ S \circ g_*)|N = (\tau_1 \circ \text{id} \circ \tau_{-1} \circ g)|N = g|N.
\]

As a result, \( G \) agrees with \( g \) on \( N \cap E_2 \).

Lastly, \( G = \text{id} \) holds on \( \sigma_b \setminus E_2 \) and \( G = \tau_1 \) holds on \( \sigma_a \). Using these domains for \( \Omega_1 \) and \( \mathbb{R}^n \setminus \bigcup_{k=0}^{\infty} \tau_k(\mathbb{B}) \) for \( \Omega_2 \), Lemma 2.2 implies that \( G \) is locally bi-Lipschitz. With the same choice of domains, Lemma 2.3 further implies that \( G \in W^{2,p}_{lo}(E_2; B_2) \).

For the case of \( G^{-1} \), note that the inverse is given by

\[
G^{-1}(x) = \begin{cases} 
G_*^{-1}(x), & \text{if } x \in \Omega \setminus \tau_{-1}(B_2) \\
\tau_{-1}(x), & \text{if } x \in \bigcup_{k=0}^{\infty} \tau_k(E_1) \\
x, & \text{if } x \in \bigcup_{k=1}^{\infty} \tau_k(E_2).
\end{cases}
\]
Arguing similarly with $g_*(N)$ for $N$, it follows that $G^{-1} \in W_{loc}^{2p}(B_2^c; E_2^c)$, which proves the lemma.

We now observe that Lemma 3.2 holds true even when $B_1$ and $B_2$ are not balls. In the preceding proof it is enough that, up to rotation, there is a slab $\{c_1 < x_n < c_2\}$ that separates $B_1$ from $B_2$. This result, stated below, is used in Section 4.

**Lemma 3.3.** Let $p \geq 1$ and let $E_1$, $E_2$, $C_1$, and $C_2$ be Jordan domains so that $\overline{E_1} \cap \overline{E_2} = \emptyset$ and $\overline{C_1} \cap \overline{C_2} = \emptyset$. If $g : (E_1 \cup E_2)^c \to (C_1 \cup C_2)^c$ is a homeomorphism of class $LW_1^2$ so that

1. $g(\partial E_i) = \partial B_i$ holds, for $i = 1, 2$,
2. there exists a ball $B$ containing $E_1$ and $E_2$ so that $g|B = id|B^c$,
3. there exist a rotation $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ and numbers $c_1, c_2 \in \mathbb{R}$, with $c_1 < c_2$, so that $\Theta(C_1) \subset \{x_n < c_1\}$ and $\Theta(C_2) \subset \{x_n > c_2\}$,

then there is a homeomorphism $G : E_2^c \to C_2^c$ of class $LW_1^2$ and a neighborhood $N$ of $\partial E_2$ so that $g|(N \cap E_2^c) = G|(N \cap E_2^c)$.

Though the regularity of the extension $G$ is local in nature, it nonetheless enjoys certain uniform properties. We summarize them in the next lemma.

**Lemma 3.4.** Let $E_1$, $E_2$, $C_1$, $C_2$, $B$, and $g$ be as in Lemma 3.3. If $G$ is the extension of $g$ as defined in Equation (3.6), then

1. $DG \in L^\infty(E_2^c)$ and $DG^{-1} \in L^\infty(C_2^c)$;
2. the restriction $G|B^c$ is a bi-Lipschitz homeomorphism.

**Proof.** From Lemma 3.3, the map $G$ is already locally bi-Lipschitz. To prove item (1), we will give a uniform bound for $L(G|K)$ over all compact subsets $K$ of $B^c$. Let $B = \mathbb{B}$ and let $S$ and $g_*$ be as defined in the proof of Lemma 3.2.

Again, let $\sigma_{ab} := g_*^{-1}\{a \leq x_n \leq b\}$ and let $\sigma_b$ and $\sigma_a$ be the (connected) components of $\mathbb{R}^n \setminus \sigma_{ab}$ containing the vectors $e_n$ and $-e_n$, respectively.

By Equation (3.2), we have $S\{x_n < a\} = id$ and $S\{x_n > b\} = \tau_{-1}$, which imply, respectively, the bounds $L(G|B^c \cap \sigma_a) \leq 1$ and $L(G|B^c \cap \sigma_b) \leq 1$.

It remains to estimate $L(G|B^c \cap \sigma_{ab})$. For each $k \in \mathbb{N}$, the set $\sigma_{ab}^k := \sigma_{ab} \cap \tau_k(\mathbb{B})$ is compact, so by Lemma 2.1, the restriction $G|\sigma_{ab}^k$ is bi-Lipschitz. Equation (3.7) then implies that

$L(G|\sigma_{ab}^k) = L(G|\sigma_{ab})$ holds for each $k \in \mathbb{N}$.

The remaining set $\sigma_{ab} \setminus \bigcup_{k=0}^{\infty} \tau_k(\mathbb{B})$ consists of infinitely many components, one of which is an unbounded subset $U$ of $\{x_1 < 0\}$ and the others are translates of a compact subset $K_0$ of $\sigma_{ab} \cap B(0, 3)$. Since $g|U = id$, it follows that

$G|\sigma = (\tau_1 \circ g_1^{-1} \circ S \circ g_*)|\sigma = (\tau_1 \circ S)|\sigma$

from which $L(G|\sigma) \leq L(S)$ follows. By the ‘periodicity’ of $G$ (Equation (3.7)), for all $k \in \mathbb{N}$ we also have $L(G|\tau_k(K_0)) = L(G|K_0)$. Item (1) of the lemma follows
from [EG92, Thm 4.2.3.5] and from the above estimates, where
\[ \|DG\|_{L^\infty(B^c)} \leq \max \left \{ 1, L(G|K_0), L(G|G_{ab}), L(S) \right \}. \]

Using the explicit formula in Equation (3.8), the case of \( G^{-1} \) follows similarly.

To prove item (2), let \( \ell \) be any line segment that does not intersect \( \mathbb{B} \). The restriction \( G|\ell \) is bi-Lipschitz with \( L(G|\ell) \leq C \). Since \( \partial \mathbb{B} \) is compact, it follows from Lemma 2.1 that the restriction \( G|\partial \mathbb{B} \) is bi-Lipschitz.

Let \( x_1 \) and \( x_2 \) be arbitrary points in \( \mathbb{B}^c \) and let \( \ell \) be the line segment in \( \mathbb{R}^n \) which joins \( x_1 \) to \( x_2 \). If \( \ell \) crosses through \( \mathbb{B} \), then let \( y_1 \) and \( y_2 \) be points on \( \ell \cap \partial \mathbb{B} \), where \( |x_1 - y_1| < |x_1 - y_2| \). Since \( \ell \) is a geodesic, we have the identity
\[ |x_1 - x_2| = |x_1 - y_1| + |y_1 - y_2| + |y_2 - x_2|. \]

The Triangle inequality then implies that
\[
|G(x_1) - G(x_2)| \leq |G(x_1) - G(y_1)| + |G(y_1) - G(y_2)| + |G(y_2) - G(x_2)| \\
\leq C \left( |x_1 - y_1| + |y_2 - x_2| \right) + L(G|\partial \mathbb{B}) |y_1 - y_2| \\
\leq (C + L(G|\partial \mathbb{B})) (|x_1 - y_1| + |y_1 - y_2| + |y_2 - x_2|) \\
= (C + L(G|\partial \mathbb{B})) |x_1 - x_2|.
\]

Again, the argument is symmetric for \( G^{-1} \), so this proves the lemma. \( \square \)

Theorem 3.1 now follows easily from Lemma 3.2, and a more general version of the theorem follows from Lemma 3.3. As in [Geh67, Lemma 2], one takes compositions with the extension, its inverse, and a radial stretch map.

Proof of Theorem 3.1. By composing \( g \) with linear maps, we may assume that \( E_1, E_2, B_1 \) and \( B_2 \) are subsets of \( \mathbb{B} \), that \( 0 \in E_2 \), and that \( \mathbb{B}^c \subset g(\mathbb{B}^c) \).

Choose \( r_1, r_2 \in (0,1) \) so that \( B(0,r_1) \subset E_2 \) and that \( E_1 \cup E_2 \subset B(0,r_2) \).

Let \( \rho : [0, \infty) \to [0, \infty) \) be a smooth increasing function so that \( \rho([0,r_1]) = [0,r_2] \) and \( \rho([1, \infty)) = [1, \infty) \). Define a homeomorphism \( R : \mathbb{R}^n \to \mathbb{R}^n \) by
\[
R(x) := \begin{cases} 
\rho(|x|) \cdot |x|^{-1}x, & \text{if } x \neq 0 \\
0, & \text{if } x = 0.
\end{cases}
\]

Clearly, \( R \) is of class \( LW^p \) and bi-Lipschitz, and maps \( B(0,r_1) \) onto \( B(0,r_2) \).

Putting \( E'_1 := (g \circ R)(E_1) \) and \( E'_2 := ((g \circ R)(E'_2))^c \), Lemma 2.4 implies that
\[
h := g \circ R \circ g^{-1} : E'_1 \cup E'_2 \to (B_1 \cup B_2)^c
\]
is also a homeomorphism of class \( LW^p \). Since \( R|\mathbb{B}^c = id \ |\mathbb{B}^c \), we further obtain
\[
h|\mathbb{B}^c = (g \circ R \circ g^{-1})|\mathbb{B}^c = id \ |\mathbb{B}^c.
\]

So with \( E'_1 \) and \( E'_2 \) in place of \( E_1 \) and \( E_2 \), respectively, \( h \) satisfies Equation (3.1) and the other hypotheses of Lemma 3.2. As a result, there exists a homeomorphism \( H : (E'_2)^c \to B'_2 \) of class \( LW^p \) and a neighborhood \( N' \) of \( \partial E'_2 \) so that
\[
h|(N' \cap (E'_2)^c) = H|(N' \cap (E'_2)^c).
\]
Let $G := H \circ g \circ R^{-1}$. The open set
\[ N := (R \circ g^{-1})(N' \setminus (\bar{B}_1 \cup \bar{B}_2)) \]
contains $\partial E_2$, and by Lemma 2.4, the map $G$ is of class $LW_2^p$. Moreover, for each $x \in N \setminus E_2$, there is a $y \in N' \setminus D_2'$ so that $x = (R \circ g^{-1})(y)$ and therefore
\[
G(x) = (H \circ g \circ R^{-1})((R \circ g^{-1})(y)) = H(y) = h(y) = (g \circ R \circ g^{-1})((g \circ R^{-1})(x)) = g(x).
\]
We thereby obtain $g = G$ on $N \cap E_2^c$, as desired. \qed

4. Extensions of Homeomorphisms of Class $LW_2^p$ between Collars

4.1. Generalized Inversions. To pass to the configurations of domains in Theorem 1.3, we will use generalized inversions. For fixed $a, r > 0$, these are homeomorphisms $I_{a,r} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ of the form
\[
I_{a,r}(x) := r^{a+1}|x|^{-(a+1)}x.
\]
Indeed, the inverse map satisfies $(I_{a,r})^{-1} = I_{1/a, r}$, as well as the estimate
\[
|x|^{a+1} = (r^{1/a+1}|I_{a,r}(x)|^{-1/a})^{a+1} \approx |I_{a,r}(x)|^{-(a+1)}.
\]
For derivatives of $I_{a,r}$, an elementary computation gives
\[
|D^k I_{a,r}(x)| \lesssim r^{a+1}|x|^{-(a+k)}
\]
and similarly, for the Jacobian determinant $JI_{a,r} := |\det(DI_{a,r})|$ we have
\[
JI_{a,r}(x) \leq n r^{n(a+1)}|x|^{-n(a+1)} \approx |I_{a,r}(x)|^{n(a+1)/a}.
\]
If $a = 1$, then $I_{1,r}$ is conformal and maps spheres to spheres. In general, the map $I_{a,r}$ possesses weaker properties which are sufficient for our purposes. For instance, it preserves radial rays, or sets of the form $\{\lambda x : \lambda > 0\}$ for some $x \in \mathbb{R}^n \setminus \{0\}$.

Another property, stated below, is used in the proof of Theorem 1.3 under the following hypotheses. To begin, write $B_1 = B(t, r_1)$ and $B_2 = B(z, r_2)$, where $\bar{B}_1 \subset B_2$. By composing with linear maps, we may assume that
(H1) The $x_n$-coordinate axis crosses through the points $t$ and $z$, with $t_n \leq z_n \leq 0$. As a result, the ‘south poles’ $\tau := t - r_1 \vec{e}_n$ on $\bar{B}_1$ and $\zeta := z - r_2 \vec{e}_n$ on $\bar{B}_2$ satisfy $\zeta_n < \tau_n$ and $|\zeta - \tau| = \text{dist}(\bar{B}_1, \bar{B}_2)$.
(H2) There exists $r \in (0, r_2)$ so that the sphere $\partial B(0, r)$ is tangent to both $\partial B_1$ and $\partial B_2$, with $B(0, r) \subset B_2 \setminus B_1$. In particular, this gives $r_1 < |t_n|$.

**Lemma 4.1.** Let $a \in (0, 1)$. If $B_1$ and $B_2$ are balls in $\mathbb{R}^n$ with $\bar{B}_1 \subset B_2$ and which satisfy hypotheses (H1) and (H2), then there exist real numbers $c_1 < c_2$ so that $I_{a,r}(B_1) \subset \{x_n < c_1\}$ and $I_{a,r}(B_2^c) \subset \{x_n > c_2\}$. 
The proof is a computation, and the basic idea is simple. Though the bounded domains $I_{a,r}(B_1)$ and $I_{a,r}(B_2)$ may not be balls, the distance between them is still attained by the images of the ‘north’ and ‘south’ poles of $B_1$ and $B_2$, respectively.

**Proof.** Once again, let $\tau$ and $\zeta$ be the “south poles” of $B_1$ and $B_2$, respectively. From Hypotheses (H1) and (H2), we have

$$\zeta_n = -|\zeta| < -|\tau| = \tau_n.$$  
and putting $I = I_{a,r}$, the image points $\tau' := I(\tau)$ and $\zeta' := I(\zeta)$ therefore satisfy

(4.4)  
$$\tau'_n = -|\tau'| < -|\zeta'| = \zeta'_n.$$  

**Claim 4.2.** For all $y' \in I(B_1)$, we have $y_n < \tau'_n$.

Supposing otherwise, there exists $y \in \partial B_1$ with $y \neq \tau$ and so that $y'$ has the same $n$th coordinate as $\tau'$. Let $\theta$ be the angle between the $x_n$-axis and the line crossing through $y'$ and 0. By our hypotheses, we have $t_n \leq 0$ and $0 < \theta < \frac{\pi}{2}$ and therefore $0 < \cos \theta < 1$. From $|\tau| = r_1 - t_n$, we obtain

$$|y'| = \frac{|\tau'|}{\cos \theta} = \frac{r^{a+1}|\tau|^{-a}}{\cos \theta} = \frac{r^{a+1}}{(r_1 - t_n)^a \cos \theta}$$

so from $|y'| = r^{a+1}|y|^{-a}$ and the above identity, we further obtain

(4.5)  
$$|y| = r^{(a+1)/a} \left[ \frac{r^{a+1}}{(r_1 - t_n)^a \cos \theta} \right]^{1/a} = (\cos \theta)^{1/a}(r_1 - t_n).$$

On the other hand, $I$ preserves radial rays and hence angles between radial rays. As a result, $y \in \partial B_1$ (and the Law of Cosines) imply that

$$r_1^2 = |y|^2 + t_n^2 - 2|y|t_n \cos \theta,$$

so $|y| = -t_n \cos \theta + \sqrt{r_1^2 - t_n^2 \sin^2 \theta}$. From Hypothesis (H2) once again, we obtain $r_1 < |\tau_n|$ and hence

$$|y| < -t_n \cos \theta + \sqrt{r_1^2 - r_1^2 \sin^2 \theta} = (r_1 - t_n) \cos \theta.$$
This is in contradiction with Equation (4.5), since the inequality $\cos \theta \leq (\cos \theta)^{1/a}$ follows from $a \geq 1$. The claim follows.

**Claim 4.3.** For all $w' \in I(B^r_2)$, we have $\zeta_n' < w_n'$.

Suppose there exists $w \in \partial B_2$ so that $w \neq \zeta$ and $w_n' = \zeta_n'$. If $\alpha$ is the angle between $w$ and the $x_n$-axis, then a similar computation as above gives

$$(2r_2 - r) \cos^{1/a} \alpha = |w| = (r_2 - r) \cos \alpha + \sqrt{r_2^2 - (r_2 - r)^2 \sin^2 \theta}$$

Computing further, we obtain $\psi(a) = r_2^2$, where $\psi : (0, \infty) \to (0, \infty)$ is given by

$$\psi(a) := ((2r_2 - r) \cos^{1/a} \alpha - (r_2 - r) \cos \alpha)^2 + (r_2 - r)^2 \sin^2 \alpha$$

Clearly $\psi$ is smooth and an elementary computation shows that it attains a minimum at a unique point in $(0, 1)$. We observe that

$$\psi(1) = r_2^2 \cos^2 \alpha + (r_2 - r)^2 \sin^2 \alpha < r_2^2.$$

Since $0 < \cos \alpha < 1$, we see that $\cos^{1/a} \alpha \to 0$ as $a \to 0$. It follows that

$$\lim_{a \to 0} \psi(a) = (0 + (r_2 - r) \cos \alpha)^2 + (r_2 - r)^2 \sin^2 \alpha = (r_2 - r)^2 < r_2^2$$

and therefore $\psi(a) < r_2^2$ holds for all $(0, 1)$. This is a contradiction, which proves Claim 4.3. Combining both claims and Equation (4.4), the lemma follows. □

4.2. From Doubly-Punctured Domains to Collars. We now prove Theorem 1.3. The argument requires several lemmas.

**Lemma 4.4.** Let $a > 0$ and let $D_1$, $D_2$, $B_1$, $B_2$, and $f$ be given as in Theorem 1.3. If there exists $r > 0$ so that $\bar{B}(0, r) \subset D_2 \setminus D_1$ and $\bar{B}(0, r) \subset B_2 \setminus B_1$, and if $f(0) = 0$, then $I_{a,r} \circ f \circ I_{a,r}^{-1}$ is a homeomorphism of class $LW^p_2$.

**Proof.** Since $\Omega := I_{a,r}(D_2 \setminus (\bar{D}_1 \cup \{0\}))$ and $I_{a,r}(B_2 \setminus (\bar{B}_1 \cup \{0\}))$ lie in $\mathbb{R}^n \setminus B(0, \epsilon)$, for some $\epsilon > 0$, the restricted maps $I_{a,r}^{-1}|\Omega$ and $I_{a,r}|\Omega'$ are diffeomorphisms. By Lemma 2.4, it follows that $g := I_{a,r} \circ f \circ I_{a,r}^{-1} : \Omega \to \Omega'$ is of class $LW^p_2$. □

**Lemma 4.5.** Let $E_1$, $E_2$, $C_1$, $C_2$, $B$, and $g$ be given as in Lemma 3.3, and let $G$ be given as in Equation (3.6). If $0 \in E_2$, if $0 \in C_2$, and if there exists $r > 0$ so that $B = B(0, r)$, then for each $a > 0$, the map

$$F(x) := \begin{cases} (I^{-1} \circ G \circ I)(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is a locally bi-Lipschitz homeomorphism.

**Proof.** Without loss of generality, let $r = 1$ and put $I = I_{a,r}$ and $b = 1/a$. By Equation (3.6), we have $|G(x)| \to \infty$ as $|x| \to \infty$, so $F$ is a well-defined homeomorphism. For each $\epsilon > 0$, put $B_\epsilon := B(0, \epsilon)$. The restrictions $I|B_\epsilon^c$ and $I^{-1}|B_\epsilon^c$ are diffeomorphisms, so $F|B_\epsilon^c$ is already locally bi-Lipschitz for each $\epsilon > 0$. 

To show that $F|B_\epsilon$ is bi-Lipschitz, recall that $DG \in L^\infty(E_2^c)$ follows from Lemma 3.4. So from Equations (2.1), (4.1), and (4.2), it follows that, for a.e. $x \in I^{-1}(E_2^c)$,

$$|DF(x)| \leq |DI^{-1}((G \circ I)(x))||DG(I(x))||DI(x)|$$

$\lesssim \frac{\|DG\|_\infty |(G \circ I)(x)|^{b+1}}{|(G \circ I)(x)|^{b+1}} \approx \frac{\|DG\|_\infty |I(x)|^{b+1}}{|(G \circ I)(x)|^{b+1}}.$

Now fix $y_0 \in E_2^c$. Putting $L := L(G^{-1}|B^c)$, for all $x \in B_\epsilon$ we have

$$|G(I(x)) - G(y_0)| \geq L^{-1}(|I(x) - y_0|) \geq L^{-1}(|I(x)| - |y_0|).$$

Applying the triangle inequality to the right-hand side, we obtain

$$|G(I(x))| \geq L^{-1}(|I(x)| - |y_0|) - |G(y_0)|$$

and taking reciprocals, we further obtain

$$\left\{ \begin{array}{l}
\frac{|I(x)|}{|(G \circ I)(x)|} \leq \frac{L |I(x)|}{|I(x)| - |y_0| - L |G(y_0)|} \\
= \frac{L r^{a+1}}{r^{a+1} - |x|^a |y_0| - |x|^a L |G(y_0)|} \to L
\end{array} \right.$$ 

as $x \to 0$. Combining the previous estimates, for sufficiently small $\epsilon > 0$

$$|DF(x)| \lesssim \frac{\|DG\|_\infty |I(x)|^{b+1}}{|(G \circ I)(x)|^{b+1}} \lesssim (2L)^{b+1}\|DG\|_\infty < \infty$$

holds for a.e. $x \in B_\epsilon$, and therefore $|DF| \in L^\infty_{loc}(I^{-1}(E_2^c))$. By [EG92, Thm 4.2.3.5], it follows that $F$ is locally Lipschitz on $B(0, \epsilon)$. By symmetry, the same holds for $F^{-1}$, so $F$ is locally bi-Lipschitz on all of $I^{-1}(E_2^c)$.

In the remaining proofs, we will require explicit forms of the extensions from Lemma 3.2 and from Theorem 3.1.

**Lemma 4.6.** Let $E_1$, $E_2$, $C_1$, $C_2$, $g$, and $B = B(0, r)$ be given as in Lemma 4.5, let $G$ be given as in Equation (3.6), and let $p \in [1, n)$. If $a < n/p - 1$, then the homeomorphism $I_{a,r}^{-1} \circ G \circ I_{a,r}$ is of class $LW^p_2$.

**Proof.** For convenience, we reuse the notation from the proof of Lemma 4.5. As before, $I|B_\epsilon^c$ and $I^{-1}|B_\epsilon^c$ are diffeomorphisms, so by Lemma 2.4, the map $F|B_\epsilon^c$ is of class $LW^p_2$. It suffices to show that $F \in W^{2p}_{loc}(B_\epsilon; \mathbb{R}^n)$ and $F^{-1} \in W^{2p}_{loc}(F(B_\epsilon); B_\epsilon)$, for each $\epsilon > 0$. 

To estimate second derivatives, we use Equations (2.1), (4.1), (4.2), and (4.6) once again. As a shorthand, put \( y := I(x) \) and \( z := (G \circ I)(x) \). We then obtain (4.7)

\[
|D^2 F(x)| = |D^2 (I^{-1} \circ G \circ I)(x)| \\
\leq |D^2 I^{-1}(z)||DG(y)|^2|Dx_1|^2 \\
+ |D^1 I(z)||D^2 G(y)||Dx_1|^2 + |D^2 G(y)||Dx_1||D^2 I(x)| \\
< \frac{\|DG\|^2_\infty}{|z|^{b+2}|x|^{2(a+1)}} + \frac{1}{|z|^{b+1}} \left( \frac{|D^2 G(y)|}{|x|^{2(a+1)}} + \frac{\|DG\|_\infty}{|x|^{a+2}} \right) \\
< \frac{|I(x)|^{2(b+1)}}{G(I(x))} + \frac{|I(x)|^{2(b+1)}|D^2 G(I(x))|}{G(I(x))} + \frac{|I(x)|^{b+1}}{G(I(x))} \\
< |I(x)|^b + |I(x)|^{b+1}|D^2 G(I(x))| + |x|^{-1}
\]

for a.e. \( x \in B_c \). Since \( p < n \) and \( b = 1/a \), the function \( x \mapsto |I(x)|^b = |x|^{-1} \) lies in \( L_p(B_c) \). For the remaining term, Equations (4.1) and (4.3) imply that

\[
1 = JJ^{-1}(I(x))JI(x) \lesssim |I^{-1}(I(x))|^{n(a+1)}JI(x) = |I(x)|^{-n(b+1)}JI(x)
\]

so by a change of variables [Zie89, Thm 2.2.2] and Equation (4.3), we have

\[
\int_{B_c} |I(x)|^{p(b+1)}|D^2 G(I(x))|^p \, dx \lesssim \int_{B_c} \frac{|D^2 G(I(x))|^pJI(x)}{|I(x)|^{(n-p)(b+1)}} \, dx \\
= \int_{B_c} \frac{|D^2 G(y)|^p}{|y|^{(n-p)(b+1)}} \, dy.
\]

For each \( k \in \mathbb{N} \), Equation (3.6) implies that \( G|\tau_k(E_2) = \text{id} \) and \( G|\tau_k(E_1) = \tau_1 \), and therefore \( D^2 G|\tau_k(E_1 \cup E_2) = 0 \). The rightmost integral in Equation (4.8) can therefore be restricted to the subset

\[
\Omega := \mathbb{B}^c \setminus \bigcup_{k=1}^\infty \tau_k(E_1 \cup E_2).
\]

As defined in the proof of Lemma 3.2 the maps \( g_*, G_*, \) and \( G \) satisfy (4.9)

\[
|D^2 G(y)| \lesssim |D^2 g_*^{-1}((S \circ g_*)(y))| + |D^2 S(g_*(y))| + |D^2 g_*(y)|
\]

for a.e. \( y \in I^{-1}(E_2) \), and where \( \lesssim \) includes the constants \( L(g_*) \), \( L(g_*^{-1}) \), \( L(S) \), and \( L(\tau_1) \). Using the second derivative bound for \( S \) (Equation (3.3)), we obtain

\[
\int_{\Omega} \frac{|D^2 S(g_*(y))|^p}{|y|^{(n-p)(b+1)}} \, dy \lesssim \int_{\Omega} \frac{2c^{2p}}{|y|^{(n-p)(b+1)}} \, dy \lesssim \int_{1}^{\infty} \frac{\rho^{n-1}}{\rho^{(n-p)(b+1)}} \, d\rho.
\]

The rightmost integral is finite, since \( a < n/p - 1 \) implies that \( b > p/(n-p) \) and

\[
(n - 1) - (n - p)(b + 1) < (n - 1) - (n - p)\left(\frac{p}{n-p} - 1\right) = -1.
\]
another change of variables, we further estimate

\[
\int_{\Omega} \frac{|D^2 g_*^{-1}(S \circ g_*)(y)|^p}{|y|^{(n-p)(b+1)}} dy = \sum_{k=1}^{\infty} \int_{\tau_k((S \circ g_*)^{-1}(B)) \cap \Omega} \frac{|D^2 g_*^{-1}(S \circ g_*)(y)|^p}{|y|^{(n-p)(b+1)}} dy \\
\approx \sum_{k=1}^{\infty} \int_{g_*^{-1}(\Omega) \cap \tau_k(B)} \frac{|D^2 g_*^{-1}(z)|^p}{|y|^{(n-p)(b+1)}} dz
\]

Equation (3.2) implies that \(|S^{-1}(y)| \geq |y|\) holds, for each \(y \in \mathbb{R}^n\), and therefore

\[
|(S \circ g_*)^{-1}(z)| \geq 3k - 1 > k
\]

holds, for each \(z \in \tau_k(B)\) and each \(k \in \mathbb{N}\). From the above inequalities and another change of variables, we further estimate

\[
\int_{g_*^{-1}(\Omega) \cap \tau_k(B)} \frac{|D^2 g_*^{-1}(z)|^p}{|(S \circ g_*)^{-1}(z)|^{(n-p)(b+1)}} dz \lesssim \int_{g_*^{-1}(\Omega) \cap \tau_k(B)} \frac{|D^2 g_*^{-1}(z)|^p}{k^{(n-p)(b+1)}} dz \\
\leq \frac{\int_{B \setminus (C_1 \cup C_2)} |D^2 g_*^{-1}(z)|^p dz}{k^{(n-p)(b+1)}},
\]

so

\[
\int_{\Omega} \frac{|D^2 g_*^{-1}(S \circ g_*)(y)|^p}{|y|^{(n-p)(b+1)}} dy \lesssim \sum_{k=1}^{\infty} \frac{\|D^2 g_*^{-1}\|_{L^p(B \setminus (C_1 \cup C_2))}}{k^{(n-p)(b+1)}}.
\]

The rightmost sum is finite, since \((n-p)(b+1) > 1\) follows from the hypothesis that \(a < n/p - 1\). A similar estimate gives \(|y|^{(p-n)(b+1)}|D^2 g_*(y)| \in L^p(B_*)\), so by Equations (4.7)-(4.9), we obtain \(|D^2 F| \in L^p(B_*)\), as desired.

The same argument, with \(G^{-1}\) for \(G\), shows that the map \(F^{-1} = I^{-1} \circ G^{-1} \circ I\) also lies in \(W^{2,p}_{loc}(F(B_*)\setminus B_*)\). This proves the lemma. \(\square\)

Using the previous lemmas, we now prove the main theorem.

**Proof of Theorem 1.3.** Let \(a < n/p - 1\) be given.

By post-composing \(f\) with linear maps, we may assume that the balls \(B_1\) and \(B_2\) satisfy hypotheses (H1) and (H2) from Section 4.1, so in particular we have \(B(0, r) \subset B_2 \setminus \overline{B}_1\). We further assume that \(B(0, r) \subset D_2 \setminus D_1\) and \(f(0) = 0\).

By Lemma 4.1, there exist \(c_1 < c_2\) so that \(B_1 \subset \{x_n < c_1\}\) and \(B_2 \subset \{x_n > c_2\}\). For \(I := I_{a,r}\) and \(g := I \circ f \circ I^{-1}\), Lemma 4.4 implies that \(g\) is of class \(LW^p\).

Put \(E_1 := I(D_1), E_2 := I(D_2^c), C_1 := I(B_1),\) and \(C_2 := I((B_2)^c)^c\). By Lemma 3.3 and the proof of Theorem 3.1, there exists a homeomorphism \(G\) of class \(LW^p\) and a neighborhood \(N\) of \(\partial E_2\) so that

\[
g|(N \cap E_2^c) = G|(N' \cap E_2^c).
\]

As a result, the homeomorphism \(F\), as defined in Lemma 4.5, and the open set \(N := I^{-1}(N')\), a neighborhood of \(\partial D_2\), therefore satisfy the identity

\[
f|(N \cap \overline{D}_2) = F|(N \cap \overline{D}_2).
\]
Recalling the proof of Theorem 3.1, we have \( G = H \circ g \circ R^{-1} \), where

(H3) \( R \) is a diffeomorphism that agrees with the identity map on \( \mathbb{R}^c \);  
(H4) \( H \) is a homeomorphism of class \( LW_2^p \), as given from Lemma 3.3, that 
  agrees with \( h = g \circ R \circ g^{-1} \) on the open set \( (g \circ R)(N') \).

Putting \( H_* := I^{-1} \circ H \circ I \) and \( R_* := I^{-1} \circ R \circ I \), we rewrite 
\[
F = I^{-1} \circ (H \circ g \circ R^{-1}) \circ I = H_* \circ f \circ R_*^{-1}.
\]

From property (H3) and properties of \( I \) and \( I^{-1} \), we see that \( R_*^{-1} \) is a diffeo-
  morphism from \( \mathbb{R}^n \setminus \{0\} \) onto itself. In particular, for each \( r > 0 \) the restriction 
\( R_*^{-1}B(0,r)^c \) is bi-Lipschitz. On the other hand, for sufficiently small \( r > 0 \) we have 
\( R^{-1} \circ I = I \) on \( B(0,r) \). Letting \( \text{Id}_n \) be the \( n \times n \) identity matrix,
\[
D R_*^{-1} \big| B(0,r) = D(I^{-1} \circ R^{-1} \circ I) \big| B(0,r) = D(I^{-1} \circ I) \big| B(0,r) = \text{Id}_n
\]
\[
D^2 R_*^{-1} \big| B(0,r) = D^2(I^{-1} \circ R^{-1} \circ I) \big| B(0,r) = D^2(I^{-1} \circ I) \big| B(0,r) = 0.
\]

This implies that \( R_*^{-1} \in W^{2,p}_{loc}(\mathbb{R}^n; \mathbb{R}^n) \) and by Lemma 2.2, that \( R_*^{-1} \) is bi-
Lipschitz. By symmetry the same holds for \( R_* = I^{-1} \circ R \circ I \), so \( R_*^{-1} \) is of 
class \( LW_2^p \).

Property (H4) and Lemma 4.6 imply that \( H_* \) is of class \( LW_2^p \). By hypothesis, 
\( f \) is of class \( LW_2^p \), so by Lemma 2.4, \( F \) is of class \( LW_2^p \). The theorem follows. \( \square \)

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Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260

E-mail address: jasun@pitt.edu