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NIKOLSKII INEQUALITY FOR LACUNARY SPHERICAL POLYNOMIALS

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Abstract. We prove that for $d \geq 2$, the asymptotic order of the usual Nikolskii inequality on $\mathbb{S}^d$ (also known as the reverse Hölder inequality) can be significantly improved in many cases, for lacunary spherical polynomials of the form $f = \sum_{j=0}^{m} f_{n_j}$ with f_{n_j} being a spherical harmonic of degree n_j and $n_{j+1} - n_j \geq 3$. As is well known, for $d = 1$, the Nikolskii inequality for trigonometric polynomials on the unit circle does not have such a phenomenon.

1. Introduction

Let $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} \colon |x| = 1\}$ denote the unit sphere of $\mathbb{R}^{d+1}$ equipped with the usual surface Lebesgue measure $d\sigma(x)$, and let ω_d be the surface area of the sphere $\mathbb{S}^d$; that is, $\omega_d := \sigma(\mathbb{S}^d) = 2\pi^{\frac{d+1}{2}}/\Gamma(\frac{d+1}{2})$. Here, $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^{d+1}$. Given $0 < p \leq \infty$, we denote by $L^p(\mathbb{S}^d)$ the usual Lebesgue L^p-space defined with respect to the measure $d\sigma(x)$ on $\mathbb{S}^d$, and $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{S}^d)}$ the quasi-norm of $L^p(\mathbb{S}^d)$; that is,

$$\|f\|_p := \begin{cases} \Big(\int_{\mathbb{S}^d} |f(x)|^p \, d\sigma(x)\Big)^{1/p}, & 0 < p < \infty, \\ \operatorname{ess\,sup}_{x \in \mathbb{S}^d} |f(x)|, & p = \infty. \end{cases}$$

In what follows c, C will denote positive constants whose value may change with each occurrence. The notation $A \asymp B$ means that $c^{-1}A \leq B \leq cA$.

Let Π_n^d denote the space of all spherical polynomials of degree at most n on $\mathbb{S}^d$ (i.e., restrictions on $\mathbb{S}^d$ of polynomials in $d+1$ variables of total degree at most n), and let $\mathcal{H}_n^d$ be the space of all spherical harmonics of degree n on $\mathbb{S}^d$. As is well known (see, e.g., [2, Chap. 1]), both $\mathcal{H}_n^d$ and Π_n^d are finite-dimensional spaces with $\dim \mathcal{H}_n^d \asymp n^{d-1}$ and $\dim \Pi_n^d \asymp n^d$.

The spaces $\mathcal{H}_k^d$ are mutually orthogonal with respect to the inner product of $L^2(\mathbb{S}^d)$, and the orthogonal projection proj_k of $L^2(\mathbb{S}^d)$ onto the space $\mathcal{H}_k^d$ can be expressed as a spherical convolution:

$$\operatorname{proj}_k f(x) = \frac{k+\lambda}{\lambda} \frac{1}{\omega_d} \int_{\mathbb{S}^d} f(y) C_k^\lambda(x \cdot y) \, d\sigma(y), \quad x \in \mathbb{S}^d, \quad \lambda = \frac{d-1}{2}, \tag{1.1}$$

where the C_k^λ denote the Gegenbauer polynomials as defined in [1, Sec. 10.9].

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The classical *Nikolskii inequality* for spherical polynomials reads as follows (see, e.g., [6]):

$$\|f\|_q \le C_d n^{d\,(\frac1p-\frac1q)}\|f\|_p \quad \forall\, f\in \Pi_n^d, \quad 0<p<q\le\infty. \tag{1.2}$$

In [3], continuing Sogge's investigations [7], we obtained the sharp asymptotic order of the following Nikolskii inequality for spherical harmonics f_n of degree n, that is,

$$\|f_n\|_q \le C n^{c(p,q)}\|f_n\|_p \quad \forall\, f_n\in \mathcal{H}_n^d, \quad 0<p<q\le\infty, \tag{1.3}$$

where the constant $c(p,q)$ is a multivalued function of (p,q); see [3] for details. In many cases, these sharp estimates turn out to be remarkably better than the corresponding estimate for spherical polynomials (1.2). In particular, we have that

$$\|f_n\|_q \le C n^{\frac{d-1}{2}(\frac1p-\frac1q)}\|f_n\|_p \quad \forall\, f_n\in \mathcal{H}_n^d, \quad 1\le p\le 2, \quad p\le q\le p'. \tag{1.4}$$

Furthermore, this estimate is sharp.

These results, in particular, show that there are no exponents $0<p<q\le\infty$ such that the equivalence $\|f_n\|_q\asymp\|f_n\|_p$ holds for any $f_n\in\mathcal{H}_n^d$. This in turn implies that there exists no analogue of the following Zygmund theorem for lacunary trigonometric series [9] for spherical polynomials: *For any trigonometric series of the form*

$$f(x)\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos n_kx+b_k\sin n_kx), \qquad \frac{n_{k+1}}{n_k}\ge\gamma>1,$$

one has

$$\|f\|_{L_p(\mathbb{T})}\asymp\|f\|_{L_2(\mathbb{T})}, \qquad 0<p<\infty, \tag{1.5}$$

where the equivalent constants depend only on p *and* γ.

In this paper, we prove that for $d\ge2$, the asymptotic order of the Nikolskii inequality can be significantly improved when restricted on a wide class of "lacunary" spherical polynomials, although the order is sharp on the whole space of spherical polynomials. To be precise, given positive integers ℓ, m with $m\le n/\ell$, we denote by $\Pi_{n,m,\ell}^d$ the class of all spherical polynomials f that can be represented in the form

$$f=\sum_{j=0}^{m}f_{n_j}, \quad f_{n_j}\in\mathcal{H}_{n_j}^d, \quad j=0,1,\dots,m, \tag{1.6}$$

for some sequence of nonnegative integers $\{n_j\}_{j=0}^m\subset[0,n]$ such that $n_j-n_{j-1}\ge 2\ell+1$ for all $j=1,\dots,m$. Given $f\in\Pi_{n,m,\ell}^d$, we denote by N_f the largest integer j for which $\operatorname{proj}_j f\ne0$. Note that $\Pi_{n,m,\ell}^d$ is not a linear space.

Our main goal in this paper is to show the following.

Theorem 1. *Let* (p,q) *be a pair of exponents satisfying either one of the following two conditions:* (i) $0<p\le1$ *and* $p\le q$ *or* (ii) $1\le p\le 2$ *and* $p\le q\le p'$. *If* $f\in\Pi_{n,m,\ell}^d$*, and* $d\ge2$*, then we have*

$$\|f\|_q\le C_d\big(n^{d-1-\ell_0}m\big)^{\frac1p-\frac1q}\|f\|_p\le C_dn^{(d-\ell_0)(\frac1p-\frac1q)}\|f\|_p,$$

where $\ell_0=\min\{\ell,\frac{d-1}{2}\}$. *In particular, if* $\ell\ge\frac{d-1}{2}$*, then*

$$\|f\|_q\le C_d\big(n^{\frac{d-1}{2}}m\big)^{\frac1p-\frac1q}\|f\|_p\le C_dn^{\frac{d+1}{2}\,(\frac1p-\frac1q)}\|f\|_p. \tag{1.7}$$

It is worth mentioning that the asymptotic order of the Nikolskii exponent for "lacunary" spherical polynomials lies between the classical exponent provided by (1.2) and the one for the spherical harmonics given by (1.3). In particular, we have that for $\ell \geq \frac{d-1}{2}$,

$$\|f_n\|_{p'} \leq Cn^{\frac{d+1}{2}(\frac{1}{p}-\frac{1}{p'})}\|f_n\|_p \quad \forall f \in \Pi^d_{n,m,\ell} \quad 1 \leq p \leq 2.$$

It is also clear that inequality (1.7) generalizes (1.4).

Note that no improvement can be achieved in the order of the Nikolskii inequality for similar "lacunary" trigonometric polynomials on the unit circle; cf. (1.5).

The Nikolskii-type inequalities are closely related to the Remez-type inequalities in a very general setting, as was shown in [8, pp. 601–602]. Moreover, these inequalities play a crucial role in establishing a Sobolev-type embedding result for the Besov spaces: $B^r_q\big(L_p(\mathbb{S}^d)\big) \hookrightarrow L^q(\mathbb{S}^d)$ (see [5, Cor. 4] and [4, Sec.8]). As a result, Theorem 1 can be applied to improve the Remez-type inequalities as well as the limiting smoothness parameter $r = \frac{d+1}{2}\big(\frac{1}{p}-\frac{1}{q}\big)_+$ in place of $r = d\big(\frac{1}{p}-\frac{1}{q}\big)_+$ for "lacunary" spherical polynomials, or "lacunary" spherical functions $f \in \bigcup_n \Pi^d_{n,m,\ell}$ with $\ell \geq \frac{d-1}{2}$.

2. Proof of Theorem 1

We start with some useful definitions. Given $h \in \mathbb{N}$ and a sequence $\{a_n\}_{n=0}^\infty$ of real numbers, define (see, for instance, [3])

$$\triangle_h a_n = a_n - a_{n+h}, \quad \triangle_h^{\ell+1} = \triangle_h \triangle_h^\ell, \quad \ell = 1, 2, \ldots.$$

Next, let

$$R_n(\cos\theta) := \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)}, \quad \theta \in [0, \pi],$$

denote the normalized Gegenbauer polynomial, and for a step $h \in \mathbb{N}$, define

$$\triangle_h^\ell R_n(\cos\theta) := \triangle_h^\ell a_n = \sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}R_{n+hj}(\cos\theta), \quad \ell = 1, 2, \ldots, \quad n = 0, 1, \ldots,$$

with $a_n := R_n(\cos\theta)$. Here and throughout, the difference operator in $\triangle_h^\ell R_n(\cos\theta)$ is always acting on the integer n. In the case when the step $h = 1$, we have the following estimate ([2, Lemma B.5.1]):

$$\big|\triangle_1^\ell R_n(\cos\theta)\big| \leq C\theta^\ell(1+n\theta)^{-\frac{d-1}{2}}, \quad \theta \in [0, \pi/2], \quad \ell \in \mathbb{N}. \tag{2.1}$$

On the other hand, however, the ℓth order difference $\triangle_1^\ell R_n(\cos\theta)$ with step $h = 1$ does not provide a desirable upper estimate when θ is close to π, and as will be seen in our later proof, estimate (2.1) itself will not be enough for our purpose.

To overcome this difficulty, instead of the difference with step 1, we consider the ℓth order difference $\triangle_2^\ell R_n(\cos\theta)$ with step $h = 2$. Since $\triangle_2^\ell a_n = \sum_{j=0}^\ell \binom{\ell}{j}\triangle_1^\ell a_{n+j}$, on one hand, (2.1) implies that

$$\big|\triangle_2^\ell R_n(\cos\theta)\big| \leq C\theta^\ell(1+n\theta)^{-\frac{d-1}{2}}, \quad \theta \in [0, \pi/2].$$

On the other hand, however, since

$$\triangle_2^\ell R_n(\cos\theta) = \sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}R_{n+2j}(\cos\theta),$$

and since $R_{n+2j}(-z)=(-1)^n R_{n+2j}(z)$ ([1, Sec. 10.9]), we have $\triangle_2^\ell R_n(\cos(\pi-\theta)) = (-1)^n \triangle_2^\ell R_n(\cos\theta)$. It follows that

$$\left|\triangle_2^\ell R_n(\cos\theta)\right| \le C \begin{cases} \theta^\ell (1+n\theta)^{-\frac{d-1}{2}}, & \theta\in[0,\pi/2],\\ (\pi-\theta)^\ell(1+n(\pi-\theta))^{-\frac{d-1}{2}}, & \theta\in[\pi/2,\pi]. \end{cases} \tag{2.2}$$

By (1.1), we obtain that for every $P\in\mathcal{H}_n^d$,

$$P(x)=c_n\frac{1}{\omega_d}\int_{\mathbb{S}^d} P(y)R_n(x\cdot y)\,d\sigma(y),\quad x\in\mathbb{S}^d,$$

where

$$c_n := \frac{\Gamma(\frac{d+1}{2})}{2\pi^{(d+1)/2}}\,\frac{d+2n-1}{d+n-1}\,\frac{\Gamma(d+n)}{\Gamma(n+1)\Gamma(d)} \asymp n^{d-1},$$

and $x\cdot y$ denotes the dot product of $x,y\in\mathbb{R}^d$. Since $R_j(x\cdot\,)\in\mathcal{H}_j^d$ for any fixed $x\in\mathbb{S}^d$, it follows by the orthogonality of spherical harmonics that for any $P\in\mathcal{H}_n^d$ and any $\ell\in\mathbb{N}$,

$$\begin{aligned} P(x) &= c_n\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\frac{1}{\omega_d}\int_{\mathbb{S}^d}P(y)R_{n+2j}(x\cdot y)\,d\sigma(y)\\ &= c_n\frac{1}{\omega_d}\int_{\mathbb{S}^d}P(y)\triangle_2^\ell R_n(x\cdot y)\,d\sigma(y). \end{aligned}$$

By (2.2), for any positive integer ℓ,

$$\left|\triangle_2^\ell R_n(\cos\theta)\right| \le C\min\{n^{-\ell}, n^{-\frac{d-1}{2}}\}. \tag{2.3}$$

Let $f\in\Pi_{n,m,\ell}^d$ be given in (1.6) with $n_m=n$. Define the operator

$$Tg(x) := \frac{1}{\omega_d}\int_{\mathbb{S}^d} g(y)H(x\cdot y)\,d\sigma(y),\quad x\in\mathbb{S}^d,\quad g\in L^1(\mathbb{S}^d),$$

where

$$H(\cos\theta)=\sum_{k=0}^m c_{n_k}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}R_{n_k+2j}(\cos\theta).$$

Clearly,

$$Tg=\sum_{k=0}^m\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\frac{c_{n_k}}{c_{n_k+2j}}\operatorname{proj}_{n_k+2j}(g),$$

and hence $Tf=f$. Since, by (2.3),

$$\|H\|_\infty \le C\sum_{k=0}^m n_k^{d-1-\ell_0} \le Cmn^{d-1-\ell_0},$$

it follows that

$$\|Tg\|_\infty \le Cmn^{d-1-\ell_0}\|g\|_1 \quad \forall\, g\in L^1(\mathbb{S}^d).$$

On the other hand, by Plancherel's formula, we have

$$\|Tg\|_2\le C\|g\|_2 \quad \forall\, g\in L^2(\mathbb{S}^d).$$

Thus, applying the Riesz–Thorin interpolation theorem, we deduce that for $1\le p\le 2$,

$$\|Tg\|_{p'} \le C(mn^{d-1-\ell_0})^{\frac{1}{p}-\frac{1}{p'}}\|g\|_p \quad \forall\, g\in L^p(\mathbb{S}^d).$$

Taking $Tf = f$ we arrive at

$$\|f\|_{p'} \le C(mn^{d-1-\ell_0})^{\frac{1}{p}-\frac{1}{p'}}\|f\|_p, \quad 1 \le p \le 2. \tag{2.4}$$

Further, log-convexity of L^p norms, namely $\|f\|_q \le \|f\|_p^\theta \|f\|_{p'}^{1-\theta}$ with $\frac{\theta}{p} + \frac{1-\theta}{p'} = \frac{1}{q}$ and $0 \le \theta \le 1$, implies

$$\|f\|_q \le C_d\big(mn^{d-1-\ell_0}\big)^{\frac{1}{p}-\frac{1}{q}}\|f\|_p,$$

where $1 \le p \le 2$ and $p \le q \le p'$.

To complete the proof we have to show that this inequality is valid for $0 < p < 1$ and $p \le q$. First let $q = \infty$. Using (2.4) with $p = 1$, we have

$$\begin{aligned}\|f\|_1 = \||f|^{1-p}|f|^p\|_1 &\le \|f\|_\infty^{1-p}\|f\|_p^p \\ &\le C_d\big(mn^{d-1-\ell_0}\big)\|f\|_1\|f\|_\infty^{-p}\|f\|_p^p.\end{aligned}$$

This yields that

$$\|f\|_\infty \le C_d\big(mn^{d-1-\ell_0}\big)^{\frac{1}{p}}\|f\|_p. \tag{2.5}$$

If $q < \infty$, we write

$$\|f\|_q = \||f|^{1-\frac{p}{q}}|f|^{\frac{p}{q}}\|_q \le \|f\|_\infty^{1-\frac{p}{q}}\|f\|_p^{\frac{p}{q}}.$$

Applying (2.5) implies

$$\|f\|_q \le C_d\big(mn^{d-1-\ell_0}\big)^{\frac{1}{p}-\frac{1}{q}}\|f\|_p^{1-\frac{p}{q}}\|f\|_p^{\frac{p}{q}} = C_d\big(mn^{d-1-\ell_0}\big)^{\frac{1}{p}-\frac{1}{q}}\|f\|_p,$$

completing the proof.

References

[1] G. Bateman, A. Erdélyi, et al., *Higher Transcendental Functions*, Vol. II, McGraw Hill Book Company, New York, 1953.

[2] Feng Dai and Yuan Xu, *Approximation theory and harmonic analysis on spheres and balls*, Springer Monographs in Mathematics, Springer, New York, 2013. MR3060033

[3] Feng Dai, Han Feng, and Sergey Tikhonov, *Reverse Hölder's inequality for spherical harmonics*, Proc. Amer. Math. Soc. **144** (2016), no. 3, 1041–1051, DOI 10.1090/proc/12986. MR3447658

[4] Feng Dai and Sergey Tikhonov, *Weighted fractional Bernstein's inequalities and their applications*, J. Anal. Math. **129** (2016), 33–68, DOI 10.1007/s11854-016-0014-z. MR3540592

[5] K. Hesse, H. N. Mhaskar, and I. H. Sloan, *Quadrature in Besov spaces on the Euclidean sphere*, J. Complexity **23** (2007), no. 4-6, 528–552, DOI 10.1016/j.jco.2006.10.004. MR2372012

[6] A. I. Kamzolov, *Approximation of functions on the sphere S^n* (Russian), Serdica **10** (1984), no. 1, 3–10. MR764160

[7] Christopher D. Sogge, *Oscillatory integrals and spherical harmonics*, Duke Math. J. **53** (1986), no. 1, 43–65, DOI 10.1215/S0012-7094-86-05303-2. MR835795

[8] V. Temlyakov and S. Tikhonov, *Remez-type and Nikol'skii-type inequalities: general relations and the hyperbolic cross polynomials*, Constr. Approx. **46** (2017), no. 3, 593–615, DOI 10.1007/s00365-017-9370-x. MR3735702

[9] A. Zygmund, *Trigonometric series. Vol. I, II*, 3rd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002. With a foreword by Robert A. Fefferman. MR1963498

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