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NIKOLSKII CONSTANTS
FOR POLYNOMIALS ON THE UNIT SPHERE

By

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Abstract. This paper studies the asymptotic behavior of the exact constants of the Nikolskii inequalities for the space Π_n^d of spherical polynomials of degree at most n on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ as $n \to \infty$. It is shown that for $0 < p < \infty$,

$$\lim_{n\to\infty} \sup\left\{ \frac{\|P\|_{L^\infty(\mathbb{S}^d)}}{n^{\frac{d}{p}} \|P\|_{L^p(\mathbb{S}^d)}} : P \in \Pi_n^d \right\} = \sup\left\{ \frac{\|f\|_{L^\infty(\mathbb{R}^d)}}{\|f\|_{L^p(\mathbb{R}^d)}} : f \in \mathcal{E}_p^d \right\},$$

where $\mathcal{E}_p^d$ denotes the space of all entire functions of spherical exponential type at most 1 whose restrictions to $\mathbb{R}^d$ belong to the space $L^p(\mathbb{R}^d)$, and it is agreed that $0/0 = 0$. It is also proved that for $0 < p < q < \infty$,

$$\liminf_{n\to\infty} \sup\left\{ \frac{\|P\|_{L^q(\mathbb{S}^d)}}{n^{d(1/p-1/q)} \|P\|_{L^p(\mathbb{S}^d)}} : P \in \Pi_n^d \right\} \ge \sup\left\{ \frac{\|f\|_{L^q(\mathbb{R}^d)}}{\|f\|_{L^p(\mathbb{R}^d)}} : f \in \mathcal{E}_p^d \right\}.$$

These results extend the recent results of Levin and Lubinsky for trigonometric polynomials on the unit circle.
The paper also determines the exact value of the Nikolskii constant for nonnegative functions with $p = 1$ and $q = \infty$:

$$\lim_{n\to\infty} \sup_{0\le P\in\Pi_n^d} \frac{\|P\|_{L^\infty(\mathbb{S}^d)}}{\|P\|_{L^1(\mathbb{S}^d)}} = \sup_{0\le f\in\mathcal{E}_1^d} \frac{\|f\|_{L^\infty(\mathbb{R}^d)}}{\|f\|_{L^1(\mathbb{R}^d)}} = \frac{1}{4^d \pi^{d/2} \Gamma(d/2+1)}.$$

1 Introduction

Let $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$ denote the unit sphere of $\mathbb{R}^{d+1}$ equipped with the usual surface Lebesgue measure $d\sigma(x)$, and ω_d the surface area of the sphere $\mathbb{S}^d$; that is, $\omega_d := \sigma(\mathbb{S}^d) = 2\pi^{\frac{d+1}{2}}/\Gamma(\frac{d+1}{2})$. Here and throughout the paper, $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^{d+1}$. Given $0 < p \le \infty$, we denote by $L^p(\mathbb{S}^d)$ the usual Lebesgue L^p-space defined with respect to the measure $d\sigma(x)$ on $\mathbb{S}^d$, and $\|\cdot\|_p = \|\cdot\|_{L^p(\mathbb{S}^d)}$ the

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quasi-norm of $L^p(\mathbb{S}^d)$; that is,

$$\|f\|_p = \left(\int_{\mathbb{S}^d} |f(x)|^p \, d\sigma(x) \right)^{1/p}, \quad 0 < p < \infty, \quad \|f\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{S}^d} |f(x)|.$$

Let $\rho(x, y) := \arccos(x \cdot y)$ denote the geodesic distance between $x, y \in \mathbb{S}^d$. Throughout this paper, we use the letter e to denote the vector $(0, \dots, 0, 1) \in \mathbb{S}^d$, and the notation $A \asymp B$ means that there exists a positive constant c, called the constant of equivalence, such that $c^{-1}A \le B \le cA$.

Let Π_n^d denote the space of all spherical polynomials of degree at most n on $\mathbb{S}^d$ (i.e., restrictions on $\mathbb{S}^d$ of polynomials in $d+1$ variables of total degree at most n), and $\mathcal{H}_n^d$ the space of all spherical harmonics of degree n on $\mathbb{S}^d$. As is well known (see, e.g., [7, Chap. 1]), both $\mathcal{H}_n^d$ and Π_n^d are finite-dimensional spaces with

$$\dim \mathcal{H}_n^d = \frac{2n+d-1}{d-1} \frac{\Gamma(n+d-1)}{\Gamma(n+1)\Gamma(d-1)} = \frac{2n^{d-1}}{\Gamma(d)}(1 + O(n^{-1}))$$

and

$$\dim \Pi_n^d = \frac{(2n+d)\Gamma(n+d)}{\Gamma(n+1)\Gamma(d+1)} = \frac{2n^d}{\Gamma(d+1)}(1 + O(n^{-1})) \tag{1.1}$$

as $n \to \infty$.

The spaces $\mathcal{H}_k^d$ are mutually orthogonal with respect to the inner product of $L^2(\mathbb{S}^d)$, whereas the orthogonal projection proj_k of $L^2(\mathbb{S}^d)$ onto the space $\mathcal{H}_k^d$ can be expressed as a spherical convolution:

$$\operatorname{proj}_k f(x) = \frac{k+\lambda}{\lambda} \frac{1}{\omega_d} \int_{\mathbb{S}^d} f(y) C_k^\lambda(x \cdot y) \, d\sigma(y), \quad x \in \mathbb{S}^d, \ \lambda = \frac{d-1}{2},$$

where the C_k^λ denote the Gegenbauer polynomials as defined in [23]. As a result, each spherical polynomial $f \in \Pi_n^d$ has an integral representation,

$$f(x) = \int_{\mathbb{S}^d} G_n(x \cdot y) f(y) \, d\sigma(y),$$

where

$$G_n(t) = \frac{1}{\omega_d} \sum_{k=0}^n \frac{k+\lambda}{\lambda} C_k^\lambda(t) = d_n R_n^{(\frac{d}{2}, \frac{d-2}{2})}(t), \tag{1.2}$$

$R_n^{(\alpha,\beta)}(t) = \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)}(1)}$ denotes the normalized Jacobi polynomial, and $d_n := \dim \Pi_n^d / \omega_d$.

The classical **Nikolskii inequality** for spherical polynomials reads as follows (see, e.g., [17]):

$$\|f\|_q \le C_d n^{d(\frac{1}{p} - \frac{1}{q})} \|f\|_p \quad \forall f \in \Pi_n^d, \quad 0 < p < q \le \infty. \tag{1.3}$$

In fact, using (1.2) and the addition formula for spherical harmonics ([7, Chapter 1]), one can easily obtain the following Nikolskii inequality with explicit constant in the case of $0 < p \le 2$ (see, for instance, [2, 10]):

$$\|f\|_q \le (d_n)^{1/p-1/q}\|f\|_p \quad \forall f \in \Pi_n^d,\ 0 < p \le 2,\ 0 < p < q \le \infty. \tag{1.4}$$

Note, however, that the constant $(d_n)^{1/p-1/q}$ here is not optimal unless $p = 2$ and $q = \infty$.

Our main interest in this paper is on the asymptotic behavior of the following sharp Nikolskii constant as $n \to \infty$:

$$C(n, d, p, q) := \sup\{\|f\|_{L^q(\mathbb{S}^d)} : f \in \Pi_n^d \text{ and } \|f\|_{L^p(\mathbb{S}^d)} = 1\}, \quad 0 < p < q \le \infty. \tag{1.5}$$

Note that by log-convexity of the L^p-norm, it is easily seen that if $0 < p < q < q_1 \le \infty$, then

$$C(n, d, p, q) \le C(n, d, p, q_1)^{\frac{1/p-1/q}{1/p-1/q_1}}.$$

Moreover, according to (1.4), if $0 < p \le 2$ and $p < q$, then (see also [9])

$$C(n, d, p, q) \le d_n = \Big(\frac{n^d}{2^d\Gamma(d/2+1)\pi^{d/2}}\Big)^{1/p-1/q}(1 + O(n^{-1})), \quad \text{as } n \to \infty.$$

The asymptotic order of the Nikolskii constant in (1.3) and (1.4) is sharp in the sense that $C(n, d, p, q) \asymp n^{d(\frac{1}{p}-\frac{1}{q})}$ for $0 < p < q \le \infty$ as $n \to \infty$ with the constant of equivalence depending only on d and p when $p \to 0$. However, the exact value of the sharp constant $C(n, d, p, q)$ is known only for $p = 2$ and $q = \infty$, in which case (1.4) gives the best possible Nikolskii constant; that is,

$$C(n, d, 2, \infty) = \sqrt{d_n}. \tag{1.6}$$

Indeed, it is a longstanding open problem to determine the exact value of the Nikolskii constant $C(n, d, p, q)$ for $(p, q) \ne (2, \infty)$ and $0 < p < q \le \infty$. This problem is open even in the case of trigonometric polynomials on the unit circle (i.e., $d = 1$). We refer to [1, 13] for more background information.

Of related interest is a recent result of Arestov and Deikalova [1] showing that the supremum in (1.5) can be in fact achieved by zonal polynomials for $q = \infty$. More precisely, they proved that

$$C(n, d, p, \infty) = \sup_{\deg P \le n} \frac{P(1)}{(\omega_{d-1}\int_{-1}^1 |P(t)|^p(1-t^2)^{(d-2)/2}dt)^{1/p}}, \quad 1 \le p < \infty \tag{1.7}$$

with the supremum being taken over all real algebraic polynomials P of degree at most n on $[-1, 1]$.

In this paper, we will study the asymptotic behavior of the quantity $\frac{C(n,d,p,q)}{n^{d(1/p-1/q)}}$ as $n \to \infty$. Our work was motivated by a recent work of Levin and Lubinsky [19, 20], who proved (using our notation in this paper)[1] that for $d = 1$,

$$\lim_{n\to\infty} \frac{C(n, 1, p, \infty)}{n^{1/p}} = \mathcal{L}(p, \infty), \quad 0 < p < \infty,$$

and

$$\liminf_{n\to\infty} \frac{C(n, 1, p, q)}{n^{1/p-1/q}} \geq \mathcal{L}(p, q), \quad 0 < p < q < \infty.$$

Here the constant $\mathcal{L}(p, q)$ is defined as

$$\mathcal{L}(p, q) := \sup \frac{\|f\|_{L^q(\mathbb{R})}}{\|f\|_{L^p(\mathbb{R})}}, \quad 0 < p < q \leq \infty$$

with the supremum being taken over all entire functions of exponential type at most 1. For more related results in one variable, we also refer to [13, 12].

Our main goal in this paper is to prove a d-dimensional generalization of these results of Levin and Lubinsky. To be more precise, recall that an entire function F in d complex variables is of spherical exponential type at most $\sigma > 0$ if for every $\varepsilon > 0$ there exists a constant $A_\varepsilon > 0$ such that $|F(z)| \leq A_\varepsilon e^{(\sigma+\varepsilon)|\mathrm{Im}(z)|}$ for all $z = (z_1, \dots, z_d) \in \mathbb{C}^d$. Given $0 < p \leq \infty$, we denote by $\mathcal{E}_p^d$ the class of all entire functions of spherical exponential type at most 1 on $\mathbb{C}^d$ whose restrictions to $\mathbb{R}^d$ belong to the space $L^p(\mathbb{R}^d)$. According to the Paley–Wiener theorem ([21, Subsect. 3.2.6]), each function $f \in \mathcal{E}_p^d$ can be identified with a function in $L^p(\mathbb{R}^d)$ whose distributional Fourier transform is supported in the unit ball $\mathbb{B}^d = \{x \in \mathbb{R}^d : |x| \leq 1\}$. Here and throughout this paper, the Fourier transform of $f \in L^1(\mathbb{R}^d)$ is defined by

$$\mathcal{F}f(\xi) \equiv \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx, \quad \xi \in \mathbb{R}^d.$$

Recall also that the inverse Fourier transform is given by

$$\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\xi)e^{ix\cdot\xi}d\xi, \quad f \in L^1(\mathbb{R}^d), \quad x \in \mathbb{R}^d.$$

As is well known, if $0 < p < q \leq \infty$, then $\mathcal{E}_p^d \subset \mathcal{E}_q^d$ and there exists a constant $C = C_{d,p,q}$ such that $\|f\|_q \leq C\|f\|_p$ for all $f \in \mathcal{E}_p^d$. For $0 < p < q \leq \infty$, let $\mathcal{L}(d, p, q)$ denote the sharp Nikolskii constant defined by

$$\mathcal{L}(d, p, q) := \sup\{ \|f\|_{L^q(\mathbb{R}^d)} : f \in \mathcal{E}_p^d \text{ and } \|f\|_{L^p(\mathbb{R}^d)} = 1\}.$$

[1]Trigonometric polynomials in [19, 20] are written in the form $P(e^{it})$ with P being an algebraic polynomial of degree n on $[-1, 1]$. Note that the absolute value $|P(e^{it})|$ corresponds to the absolute value of a trigonometric polynomial of degree at most $(n + 1)/2$.

All the above stated results on properties of functions from the class $\mathcal{E}_p^d$ can be found in [21, Ch. 3] and [15].

In this paper, we will prove the following theorem, which extends the recent result of Levin and Lubinsky [19, 20]:

Theorem 1.1. (i) *For* $0 < p < \infty$*, we have*

$$\lim_{n\to\infty} \frac{C(n,d,p,\infty)}{n^{d/p}} = \mathcal{L}(d,p,\infty).$$

Here we recall that the constant $C(n,d,p,q)$ *is defined in* (1.5).

(ii) *For* $0 < p < q \le \infty$*,*

$$\liminf_{n\to\infty} \frac{C(n,d,p,q)}{n^{d(1/p-1/q)}} \ge \mathcal{L}(d,p,q).$$

Note that as an immediate consequence of (1.6) and Theorem 1.1, we obtain

$$\mathcal{L}(d,2,\infty) = \Big(\frac{2}{\omega_d \Gamma(d+1)}\Big)^{1/2}.$$

Compared with those in [19, 20] and [13, 12] in one variable, the proof of Theorem 1.1 in higher-dimensional case is more difficult because: (1) functions on the sphere can not be identified as periodic functions on Euclidean space; (2) explicit connections between spherical polynomial interpolation $\mathbb{S}^d$ and the Shannon sampling theorem for entire functions of exponential type are not available.

While it remains a very challenging open problem to determine the exact value of the Nikolskii constant $\mathcal{L}(d,1,\infty)$, we are able to find the exact value of the Nikolskii constant for $p = 1$, $q = \infty$ and nonnegative functions $f \in \mathcal{E}_1^d$:

Theorem 1.2. *We have*

$$\lim_{n\to\infty} \sup_{\substack{0\le P\in\Pi_n^d \\ \|P\|_{L^1(\mathbb{S}^d)}=1}} n^{-d}\|P\|_{L^\infty(\mathbb{S}^d)} = \sup_{\substack{0\le f\in\mathcal{E}_1^d \\ \|f\|_{L^1(\mathbb{R}^d)}=1}} \|f\|_{L^\infty(\mathbb{R}^d)} = \frac{1}{(4\sqrt{\pi})^d\Gamma(d/2+1)}.$$

It is worthwhile to point out that the exact Nikolskii constant for nonnegative polynomials with $p = 1$ and $q = \infty$ has interesting applications in metric geometry. For example, it was used to obtain some tight-bounds for spherical designs in [3, 18].

This paper is organized as follows. In Section 2, we briefly review some useful properties of the exponential mapping $\psi : \mathbb{R}^d \to \mathbb{S}^d$ from the tangent space of $\mathbb{S}^d$ to the sphere $\mathbb{S}^d$, which connects functions on $\mathbb{S}^d$ with functions on $\mathbb{R}^d$, and whose dilations can be used to obtain a higher-dimensional analogue of the following useful formula for 2π-periodic functions in one variable:

$$\int_{\mathbb{S}^1} f(x)\,d\sigma(x) = \frac{1}{n}\int_{-n\pi}^{n\pi} f\Big(\sin\frac{\theta}{n}, \cos\frac{\theta}{n}\Big)\,d\theta, \quad n = 1, 2, \dots.$$

Also, in Section 2 we prove that for the de la Vallée–Poussin type kernels $G_{n,\eta}$ associated with a smooth cutoff function $\eta \in C_c^\infty[0,\infty)$ on the sphere $\mathbb{S}^d$,

$$\lim_{n\to\infty} \frac{1}{n^d} G_{n,\eta}\Big(\psi\Big(\frac{x}{n}\Big)\cdot\psi\Big(\frac{y}{n}\Big)\Big) = \widehat{\eta(|\cdot|)}(|x-y|), \quad \forall x,y\in\mathbb{R}^d.$$

These results play an important role in the proof of Theorem 1.1, which is given in Section 3 and Section 4. More precisely, we prove the lower estimate,

$$\liminf_{n\to\infty} \frac{C(n,d,p,q)}{n^{d(1/p-1/q)}} \geq \mathcal{L}(d,p,q), \quad 0<p<q\leq\infty,$$

in Section 3 and the corresponding upper estimate in Section 4. The proofs follow along the same line as those of Levin and Lubinsky [19, 20] for trigonometric polynomials in one variable. Our proof of the upper estimate also relies on a recent deep result of Bondarenko, Radchenko and Viazovska [3, 4] on spherical designs, and an earlier result of Yudin [24, 25] on the distribution of points of spherical designs. Finally, we prove Theorem 1.2, and find the exact value of the Nikolskii constant $\sup_{0\leq P\in\Pi_n^d} \frac{\|P\|_\infty}{\|P\|_1}$ for nonnegative spherical polynomials on $\mathbb{S}^d$ in Section 5. Of crucial importance in the proofs of these results in Section 5 are the Markov type quadrature formulas for even functions of exponential type and the Jacobi–Gauss–Radau quadrature rules for algebraic polynomials.

Throughout the paper, all functions are assumed to be real-valued and Lebesgue measurable unless otherwise stated, and we denote by $B(r)$ the ball in $\mathbb{R}^d$ centered at the origin having radius $r>0$.

2 Preliminary lemmas

In this section, we will present a few preliminary lemmas that will be used in the proof of Theorem 1.1.

We start with the following well-known property of the Geigenbauer polynomials.

Lemma 2.1 ([23, (8.1.1), p. 192]). *For $z\in\mathbb{C}$ and $\mu\geq 0$,*

$$\lim_{k\to\infty} \frac{C_k^\mu(\cos\frac{z}{k})}{C_k^\mu(1)} = j_{\mu-1/2}(z),$$

where $j_\alpha(z) = \Gamma(\alpha+\frac{1}{2})(z/2)^{-\alpha}J_\alpha(z)$, and J_α denotes the Bessel function of the first kind. This formula holds uniformly in every bounded region of the complex z-plane.

Next, we note that a function on the sphere $\mathbb{S}^d$ in general cannot be identified with a periodic function on $\mathbb{R}^d$, which is different from the one-dimensional case.

In our next lemma, we connect functions on $\mathbb{S}^d$ with functions on $\mathbb{R}^d$ via the following mapping $\psi : \mathbb{R}^d \to \mathbb{S}^d$:

$$\psi(x) := (\xi \sin|x|, \cos|x|) \quad \text{for } x = |x|\xi \in \mathbb{R}^d \text{ and } \xi \in \mathbb{S}^{d-1}.$$

It is easily seen that $\psi\colon B(\pi) \to \mathbb{S}^d$ is a bijective mapping and $\rho(\psi(x), e) = |x|$ for all $x \in B(\pi)$. Furthermore, for each $f \in L^1(\mathbb{S}^d)$,

$$\begin{aligned}\int_{\mathbb{S}^d} f(x)\,d\sigma(x) &= \int_0^\pi \left[\int_{\mathbb{S}^{d-1}} f(\xi\sin\theta, \cos\theta)\,d\sigma_{d-1}(\xi)\right]\left(\frac{\sin\theta}{\theta}\right)^{d-1}\theta^{d-1}\,d\theta\\ &= \int_{B(\pi)} f(\psi(x))\left(\frac{\sin|x|}{|x|}\right)^{d-1}dx,\end{aligned}$$

where $d\sigma_{d-1}$ denotes the usual surface Lebesgue measure on $\mathbb{S}^{d-1}$. As a result, we may identify each function f on the ball $B(n\pi) \subset \mathbb{R}^d$ with a function f_n on the sphere $\mathbb{S}^d$ via dilation and the mapping $y = \psi(x/n)$ for each $x \in B(n\pi)$. Indeed, we have

Lemma 2.2 ([8]). *For $n \in \mathbb{N}$ and $f \in L^1(\mathbb{S}^d)$,*

$$\int_{\mathbb{S}^d} f(x)\,d\sigma(x) = \frac{1}{n^d}\int_{B(n\pi)} f(\psi(x/n))\left(\frac{\sin(|x|/n)}{|x|/n}\right)^{d-1}dx. \tag{2.1}$$

Note that in the case of $d = 1$, (2.1) becomes

$$\int_{\mathbb{S}^1} f(x)\,d\sigma(x) = \frac{1}{n}\int_{-n\pi}^{n\pi} f\left(\sin\frac{\theta}{n}, \cos\frac{\theta}{n}\right)d\theta.$$

Our last preliminary lemma can be stated as follows.

Lemma 2.3. *Let η be a C^∞-function on $[0, \infty)$ that is supported on $[0, 2]$ and is constant near 0. For a positive integer n, define*

$$G_{n,\eta}(t) := \frac{1}{\omega_d}\sum_{j=0}^{2n} \eta(n^{-1}j)\frac{j+\lambda}{\lambda}\,C_j^\lambda(t), \quad t \in [-1, 1],$$

where $\lambda = \frac{d-1}{2}$. Then for any $u, v \in \mathbb{S}^d$, $n \in \mathbb{N}$ and any $\ell > 0$,

$$|G_{n,\eta}(u \cdot v)| \le C_{d,\eta,\ell}\, n^d (1 + n\rho(u, v))^{-\ell}. \tag{2.2}$$

Furthermore,

$$\lim_{n\to\infty} \frac{1}{n^d} G_{n,\eta}\left(\psi\left(\frac{x}{n}\right)\cdot\psi\left(\frac{y}{n}\right)\right) = K_\eta(|x - y|) \tag{2.3}$$

holds uniformly on every compact subset of $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, where $K_\eta(|\cdot|)$ denotes the inverse Fourier transform of the radial function $\eta(|\cdot|)$ on $\mathbb{R}^d$.

Using the formula for the Fourier transforms of radial functions, we have

$$K_\eta(|x|) = \frac{\omega_{d-1}}{(2\pi)^d} \int_0^2 \eta(\rho) j_{d/2-1}(\rho|x|)\rho^{d-1}\, d\rho, \quad x \in \mathbb{R}^d. \tag{2.4}$$

Proof. (2.2) is known (see [5]). We only need to prove (2.3). The proof is very close to that in [8]. But for completeness, we include a detailed proof here. Write

$$n^{-d} G_{n,\eta}\Big(\psi\Big(\frac{x}{n}\Big)\cdot\psi\Big(\frac{y}{n}\Big)\Big) = \int_0^2 b_n(\rho, x, y)\rho^{d-1}\, d\rho, \tag{2.5}$$

where

$$b_n(\rho, x, y) = n^{-d}\frac{1}{\omega_d}\sum_{j=0}^{2n-1} \eta(n^{-1}j)\frac{j+\lambda}{\lambda} C_j^\lambda\Big(\psi\Big(\frac{x}{n}\Big)\cdot\psi\Big(\frac{y}{n}\Big)\Big)\Big(\int_{\frac{j}{n}}^{\frac{j+1}{n}} t^{d-1}\, dt\Big)^{-1}\chi_{[\frac{j}{n},\frac{j+1}{n})}(\rho),$$

where χ_I is the characteristic function of the set I. We first claim that

$$\sup_{x,y\in\mathbb{R}^d} |b_n(\rho, x, y)| \le c_d \quad \forall\, \rho \in [0, 2],\ n = 1, 2, \dots. \tag{2.6}$$

Indeed, if $0 \le \rho < n^{-1}$, then (2.6) holds trivially. Now assume that $0 < \rho \le 2$ and $n > \rho^{-1}$. Let $1 \le j \le 2n-1$ be an integer such that $\frac{j}{n} \le \rho < \frac{j+1}{n}$. Then

$$\int_{\frac{j}{n}}^{\frac{j+1}{n}} t^{d-1}\, dt \ge cn^{-1}\rho^{d-1},$$

and hence

$$\begin{aligned} |b_n(\rho, x, y)| &\le cn^{-d}\frac{j+\lambda}{\lambda}\Big|C_j^\lambda\Big(\psi\Big(\frac{x}{n}\Big)\cdot\psi\Big(\frac{y}{n}\Big)\Big)\Big|\Big(\int_{\frac{j}{n}}^{\frac{j+1}{n}} t^{d-1}\, dt\Big)^{-1} \\ &\le cn^{-(d-1)}\rho^{-(d-1)}j^{d-1} \le c(n\rho)^{-(d-1)}j^{d-1} \le c. \end{aligned}$$

This shows (2.6).

Next, we show that, for any $\rho \in (0, 2]$ and any $M > 1$,

$$\lim_{n\to\infty}\sup_{|x|,|y|\le M}\Big|b_n(\rho, x, y) - \frac{2}{\omega_d\Gamma(d)}\eta(\rho) j_{d/2-1}(\rho|x-y|)\Big| = 0. \tag{2.7}$$

Combining (2.7) with (2.6), (2.5) and (2.4), and observing that

$$\omega_d\omega_{d-1} = \frac{2(2\pi)^d}{\Gamma(d)},$$

we will deduce the desired equation (2.3) by the dominated convergence theorem.

To show (2.7), we assume that $|x|, |y| \le M$. All the constants in the proof below are independent of x, y, but may depend on M. Let $n > \rho^{-1}$ and assume that $\frac{j}{n} < \rho \le \frac{j+1}{n}$ with $1 \le j \le 2n-1$. A straightforward calculation then shows that

$$\left(\int_{\frac{j}{n}}^{\frac{j+1}{n}} t^{d-1}\,dt\right)^{-1} = \frac{n}{\rho^{d-1}}(1+O((n\rho)^{-1})) \quad \text{as } n \to \infty.$$

This implies that for $\frac{j}{n} \le \rho \le \frac{j+1}{n}$ with $1 \le j \le 2n-1$,

$$\begin{aligned} b_n(\rho, x, y) &= \frac{j+\lambda}{(n\rho)^{d-1}\lambda\omega_d}\,\eta(\rho) C_j^{\frac{d-1}{2}}\left(\psi\left(\frac{x}{n}\right)\cdot\psi\left(\frac{y}{n}\right)\right)(1+O((n\rho)^{-1})) \\ &= \frac{2}{\omega_d\Gamma(d)}\,\eta(\rho)\,\frac{C_j^{\frac{d-1}{2}}(\cos\theta_n(x,y))}{C_j^{\frac{d-1}{2}}(1)} + O(1)j^{-1}, \end{aligned}$$

where we used the formula $C_j^{\lambda}(1) = \frac{\Gamma(j+2\lambda)}{\Gamma(j+1)\Gamma(2\lambda)}$ in the last step, and $\theta_n(x, y) \in [0, \pi]$ satisfies

$$\cos\theta_n(x,y) = \psi\left(\frac{x}{n}\right)\cdot\psi\left(\frac{y}{n}\right) = \frac{x\cdot y}{|x|\,|y|}\sin\frac{|x|}{n}\sin\frac{|y|}{n} + \cos\frac{|x|}{n}\cos\frac{|y|}{n}.$$

It is easily seen that

$$\cos\theta_n(x,y) = 1 - \frac{|x-y|^2}{2n^2} + O(n^{-4}).$$

Hence,

$$\begin{aligned} \theta_n(x,y) &= \frac{1}{n}\sqrt{|x-y|^2 + O(n^{-2})} + O(n^{-2}) = \frac{|x-y|}{n} + O(n^{-2}) \\ &= \frac{\rho|x-y| + O(j^{-1})}{j}. \end{aligned}$$

Recalling that $j \asymp n\rho \to \infty$ as $n \to \infty$, using Lemma 2.1, we obtain that

$$\begin{aligned} \lim_{n\to\infty} b_n(\rho, x, y) &= \frac{2}{\omega_d\Gamma(d)}\lim_{n\to\infty}\eta(\rho)\frac{C_j^{\frac{d-1}{2}}(\cos\frac{\rho|x-y|+O(j^{-1})}{j})}{C_j^{\frac{d-1}{2}}(1)} \\ &= \frac{2}{\omega_d\Gamma(d)}\,\eta(\rho) j_{d/2-1}(\rho|x-y|), \end{aligned}$$

which shows (2.7). □

3 Proof of Theorem 1.1: Lower estimate

This section is devoted to the proof of the following lower estimate:

$$\liminf_{n\to\infty} \frac{C(n,d,p,q)}{n^{d(1/p-1/q)}} \geq \mathcal{L}(d,p,q), \quad 0 < p < q \leq \infty. \tag{3.1}$$

The proof requires the use of certain "maximal functions" for entire functions of exponential type given in the following lemma:

Lemma 3.1 ([16]). *If $f \in \mathcal{E}_p^d$, $0 < p < \infty$ and $\ell > d/p$, then*

$$\|f_\ell^*\|_p \leq C_p \|f\|_p,$$

where $f_\ell^*(x) := \sup_{y\in\mathbb{R}^d} \frac{|f(y)|}{(1+|x-y|)^\ell}$ for $x \in \mathbb{R}^d$.

Lemma 3.1 for $d = 1$ is a direct consequence of Lemma 3.5 and Corollary 3.9 of [16, pp. 269–271], where the proof with slight modifications works equally well for the case $d \geq 2$.

Now we turn to the proof of (3.1). Setting

$$L_{pq} := \liminf_{n\to\infty} \frac{C(n,d,p,q)}{n^{d(1/p-1/q)}},$$

we reduce to showing that

$$\|f\|_q \leq L_{pq}\|f\|_p, \quad \forall f \in \mathcal{E}_p^d. \tag{3.2}$$

We first assert that it is enough to prove (3.2) under the additional assumption that $\operatorname{supp}\widehat{f} \subset B(1-\varepsilon)$ for some $\varepsilon \in (0,1)$. Indeed, if $f \in \mathcal{E}_p^d$, then for any $\varepsilon > 0$,

$$\operatorname{supp}\Big(\widehat{f}\Big(\frac{\cdot}{1-\varepsilon}\Big)\Big) = \operatorname{supp}\widehat{f_\varepsilon} \subset B(1-\varepsilon),$$

where $f_\varepsilon(x) = (1-\varepsilon)^d f((1-\varepsilon)x)$. Thus, applying (3.2) to f_ε instead of f yields

$$(1-\varepsilon)^{d/q'}\|f\|_q = \|f_\varepsilon\|_q \leq L_{pq}\|f_\varepsilon\|_p = L_{pq}(1-\varepsilon)^{d/p'}\|f\|_p,$$

where $\frac{1}{q} + \frac{1}{q'} = 1$; (3.2) for general $f \in \mathcal{E}_p^d$ then follows by letting $\varepsilon \to 0$. This proves the assertion.

For the rest of the proof, we assume that $f \in \mathcal{E}_p^d$ and satisfies $\operatorname{supp}\widehat{f} \subset B(1-\varepsilon)$ for some $\varepsilon \in (0,1)$. We will prove (3.2) under this extra condition. Let $\eta \in C^\infty[0,\infty)$ be such that $\eta(t) = 1$ for $t \in [0, 1-\varepsilon]$ and $\eta(t) = 0$ for $t \geq 1$. As in Lemma 2.3, we denote by $K_\eta(|\cdot|)$ the inverse Fourier transform of the function $\eta(|\cdot|)$. Then $K_\eta(|\cdot|)$ is a Schwartz function on $\mathbb{R}^d$, and since $\widehat{f}(\xi) = \widehat{f}(\xi)\eta(|\xi|)$, we have

$$f(x) = f * K_\eta(x) = \int_{\mathbb{R}^d} f(y)K_\eta(|x-y|)\,dy, \quad x \in \mathbb{R}^d. \tag{3.3}$$

Let $m > 1$ be a temporarily fixed parameter. For $n > m$, define $f_{n,m}$ to be a function on $\mathbb{S}^d$ supported on the spherical cap $\{x \in \mathbb{S}^d : \rho(x, e) \le \frac{m}{n}\}$ and such that

$$f_{n,m}(\psi(x/n)) = f(x)\chi_{B(m)}(x) \quad \forall\, x \in B(n\pi).$$

Consider the spherical polynomial $P_{n,m} \in \Pi_n^d$ given by

$$P_{n,m}(x) := \int_{\mathbb{S}^d} f_{n,m}(y) G_{n,\eta}(x \cdot y)\, d\sigma(y), \quad x \in \mathbb{S}^d \tag{3.4}$$

with

$$G_{n,\eta}(t) = \frac{1}{\omega_d} \sum_{j=0}^{n} \eta\Big(\frac{j}{n}\Big) \frac{j + (d-1)/2}{(d-1)/2} C_j^{\frac{d-1}{2}}(t).$$

By Nikolskii's inequality (1.5),

$$\|P_{n,m}\|_{L^q(\mathbb{S}^d)} \le C(n, d, p, q) \|P_{n,m}\|_{L^p(\mathbb{S}^d)}. \tag{3.5}$$

Moreover, using (2.1), we have that for any $x \in B(n\pi)$

$$\begin{aligned} P_{n,m}(\psi(x/n)) &= \int_{\mathbb{S}^d} f_{n,m}(u) G_{n,\eta}(\psi(x/n) \cdot u)\, d\sigma(u) \\ &= \frac{1}{n^d} \int_{B(m)} f(y) G_{n,\eta}(\psi(x/n) \cdot \psi(y/n)) \Big(\frac{\sin(|y|/n)}{|y|/n}\Big)^{d-1} dy. \end{aligned} \tag{3.6}$$

We now break the proof of (3.2) into several parts:

Step 1. We show that for any $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \sup_{x \in B(2m)} \Big| P_{n,m}\Big(\psi\Big(\frac{x}{n}\Big)\Big) - \int_{B(m)} f(y) K_\eta(|x - y|)\, dy \Big| = 0. \tag{3.7}$$

This combined with (3.3), in particular, implies that

$$\lim_{m\to\infty} \limsup_{n\to\infty} |P_{n,m}(\psi(x/n)) - f(x)| = 0 \quad \forall\, x \in \mathbb{R}^d. \tag{3.8}$$

To show (3.7), we use (3.6) to obtain

$$P_{n,m}(\psi(x/n)) = \int_{B(m)} K_\eta(|x - y|) f(y)\, dy + R_{n,1}(x) + R_{n,2}(x),$$

where

$$\begin{aligned} |R_{n,1}(x)| &\le \int_{B(m)} |f(y)| \Big| \frac{1}{n^d} G_{n,\eta}\Big(\psi\Big(\frac{x}{n}\Big) \cdot \psi\Big(\frac{y}{n}\Big)\Big) - K_\eta(|x - y|) \Big|\, dy, \\ |R_{n,2}(x)| &\le C \int_{B(m)} |f(y)| |K_\eta(|x - y|)| \Big[1 - \Big(\frac{\sin(|y|/n)}{|y|/n}\Big)^{d-1}\Big]\, dy. \end{aligned}$$

By either Nikolskii's inequality, $\|f\|_1 \le C_p\|f\|_p$ for $0 < p < 1$, or Hölder's inequality if $p \ge 1$,

$$|R_{n,2}(x)| \le C_m\|f\|_p \sup_{|y|\le m}\Big[1-\Big(\frac{\sin(|y|/n)}{|y|/n}\Big)^{d-1}\Big],$$

which goes to zero uniformly as $n \to \infty$. On the other hand, it follows by Lemma 2.3 and the dominated convergence theorem that

$$\limsup_{n\to\infty}\sup_{x\in B(2m)}|R_{n,1}(x)|$$
$$\le \Big(\int_{B(m)}|f(y)|dy\Big)\lim_{n\to\infty}\Big(\sup_{u,v\in B(2m)}\Big|\frac{1}{n^d}G_{n,\eta}\Big(\psi\Big(\frac{u}{n}\Big)\cdot\psi\Big(\frac{v}{n}\Big)\Big)-K_\eta(|u-v|)\Big|\Big)=0.$$

This proves (3.7).

Step 2. Prove that for any $\ell > 1$,

$$n^d\int_{\rho(x,e)\ge\frac{2m}{n}}|P_{n,m}(x)|^p\,d\sigma(x) \le Cm^{-\ell p}\|f\|_p^p. \tag{3.9}$$

For $x \in \mathbb{S}^d$ such that $\rho(x,e) \ge \frac{2m}{n}$, write $x = \psi(u/n)$ with $2m \le |u| \le n\pi$. Since $f_{n,m}$ is supported in the spherical cap $\{y \in \mathbb{S}^d : \rho(y,e) \le \frac{m}{n}\}$, using (3.4) and Lemma 2.2 with $\ell > d(1+1/p)$, we obtain that

$$\begin{aligned}|P_{n,m}(x)| &\le \int_{\rho(y,e)\le\frac{m}{n}}|f_{n,m}(y)||G_{n,\eta}(x\cdot y)|\,d\sigma(y)\\ &\le Cn^d(1+n\rho(x,e))^{-2\ell-d-1}\int_{\mathbb{S}^d}|f_{n,m}(y)|\,d\sigma(y)\\ &\le Cm^{-\ell}\int_{|v|\le m}|f(v)|(1+|u-v|)^{-\ell-d-1}\,dv \ \le Cm^{-\ell}f_\ell^*(u).\end{aligned}$$

Integrating over the domain $\{x \in \mathbb{S}^d : \rho(x,e) \ge \frac{2m}{n}\}$ then yields

$$\begin{aligned}n^d\int_{\rho(x,e)\ge\frac{2m}{n}}|P_{n,m}(x)|^p\,d\sigma(x) &\le C\int_{2m\le|u|\le n\pi}|P_{n,m}(\psi(u/n))|^p\,du\\ &\le Cm^{-\ell p}\int_{\mathbb{R}^d}|f_\ell^*(u)|^p\,du \le Cm^{-\ell p}\|f\|_p^p,\end{aligned}$$

where the last step uses Lemma 3.1.

Step 3. Show that for each fixed $m \ge 1$ and any $\ell > 1$,

$$\begin{aligned}&\limsup_{n\to\infty}\Big(n^d\int_{\rho(y,e)\le\frac{2m}{n}}|P_{n,m}(y)|^p\,d\sigma(y)\Big)^{p_1/p}\\ &\qquad\le (1+Cm^{-\ell})\|f\|_p^{p_1}+C\Big(\int_{|x|\ge m/2}|f_\ell^*(x)|^p dx\Big)^{p_1/p}\end{aligned} \tag{3.10}$$

where $p_1 = \min\{p, 1\}$.

Indeed, using (2.1), we have

$$\begin{aligned}\Big(n^d\int_{\rho(y,e)\le\frac{2m}{n}}|P_{n,m}(y)|^p d\sigma(y)\Big)^{p_1/p}&\\
&=\Big(\int_{B(2m)}|P_{n,m}(\psi(x/n))|^p\Big(\frac{\sin(|x|/n)}{|x|/n}\Big)^{d-1}dx\Big)^{p_1/p}\\
&\le\Big(\int_{B(2m)}\Big|P_{n,m}(\psi(x/n))-\int_{B(m)}f(y)K_\eta(|x-y|)\,dy\Big|^p dx\Big)^{p_1/p}\\
&\quad+\Big(\int_{B(2m)}\Big|\int_{B(m)}f(y)K_\eta(|x-y|)\,dy\Big|^p dx\Big)^{p_1/p}=:I_{n,m}+J_{n,m}.\end{aligned}$$

For the first term $I_{n,m}$, we have

$$I_{n,m}\le Cm^{p_1d/p}\sup_{x\in B(2m)}\Big|P_{n,m}(\psi(x/n))-\int_{B(m)}f(y)K_\eta(|x-y|)\,dy\Big|^{p_1},$$

which, using (3.7), goes to zero as $n\to\infty$. For the second term $J_{n,m}$, we use (3.3) to obtain

$$\begin{aligned}J_{n,m}&=\Big(\int_{B(2m)}\Big|f(x)-\int_{|y|\ge m}f(y)K_\eta(|x-y|)dy\Big|^p dx\Big)^{p_1/p}\\
&\le\|f\|_p^{p_1}+\Big(\int_{m/2\le|x|\le 2m}\Big|\int_{|y|\ge m}|f(y)||K_\eta(|x-y|)|dy\Big|^p dx\Big)^{p_1/p}\\
&\quad+\Big(\int_{|x|\le m/2}\Big|\int_{|y|\ge m}|f(y)||K_\eta(|x-y|)|dy\Big|^p dx\Big)^{p_1/p}\\
&=:\|f\|_p^{p_1}+J_{n,m,1}+J_{n,m,2}.\end{aligned}$$

Since $K_\eta(|\cdot|)$ is a Schwartz function, it is easily seen that for any $\ell>1$,

$$J_{n,m,1}\le C\Big(\int_{|x|\ge m/2}|f_\ell^*(x)|^p\,dx\Big)^{p_1/p}$$

and

$$J_{n,m,2}\le Cm^{-\ell}\Big|\int_{|y|\ge m}|f(y)|(1+|y|)^{-\ell-d-1}\,dy\Big|^{p_1}\le Cm^{-\ell}\|f\|_p^{p_1},$$

where the last step uses Hölder's inequality if $p\ge 1$, and Nikolskii's inequality if $p<1$. Putting the above together, we obtain (3.10).

Step 4. Conclude the proof of (3.2).

Setting $P_{n,m}^*(x)=P_{n,m}(\psi(x/n))\chi_{B(n\pi)}(x)$, we have

$$\begin{aligned}\|f\|_{L^q(\mathbb{R}^d)}&\le\liminf_{m\to\infty}\liminf_{n\to\infty}\|P_{n,m}^*\|_{L^q(\mathbb{R}^d)}=\liminf_{m\to\infty}\liminf_{n\to\infty}n^{d/q}\|P_{n,m}\|_{L^q(\mathbb{S}^d)}\\
&\le\liminf_{m\to\infty}\liminf_{n\to\infty}n^{d/q}C(n,d,p,q)\|P_{n,m}\|_{L^p(\mathbb{S}^d)}\\
&\le\Big[\liminf_{n\to\infty}\frac{C(n,d,p,q)}{n^{d(1/p-1/q)}}\Big][\liminf_{m\to\infty}\limsup_{n\to\infty}n^{d/p}\|P_{n,m}\|_{L^p(\mathbb{S}^d)}],\end{aligned}$$

where we used (3.8) and Fatou's lemma in the first step, (2.1) in the second step, and (3.5) in the third step. However, combining (3.9) with (3.10), we get

$$\limsup_{n\to\infty} (n^{d/p}\|P_{n,m}\|_p)^{p_1} \le (1+Cm^{-\ell})\|f\|_p^{p_1} + C\Big(\int_{|x|\ge m/2} |f_\ell^*(x)|^p\,dx\Big)^{p_1/p}$$

which, according to Lemma 3.1, goes to $\|f\|_p^{p_1}$ as $m\to\infty$. This proves (3.2). □

4 Proof of Theorem 1.1: Upper estimate

In this section, we will prove that for $0<p<\infty$,

$$\limsup_{n\to\infty} \frac{C(n,d,p,\infty)}{n^{d/p}} \le \mathcal{L}(d,p,\infty). \tag{4.1}$$

Let $P_n\in\Pi_n^d$ be such that $\frac{\|P_n\|_\infty}{\|P_n\|_p} = C(n,d,p,\infty)$. Without loss of generality, we may assume that $P_n(e)=\|P_n\|_\infty=1$. For the proof of (4.1), it is then sufficient to prove that

$$\liminf_{n\to\infty} n^{d/p}\|P_n\|_p \ge \mathcal{L}(d,p,\infty)^{-1}. \tag{4.2}$$

The proof (4.2) relies on several lemmas. The first lemma is on optimal asymptotic bounds for well-separated spherical designs, proved recently by Bondarenko, Radchenko and Viazovska [3, 4].

Lemma 4.1 ([3, 4]). *Let $A=A_d$ be a large parameter depending only on d. Then for each integer $N\ge A_d n^d$, there exists a set $\{z_{n,j}\}_{j=1}^N$ of N points on $\mathbb{S}^d$ such that*

$$\frac{1}{\omega_d}\int_{\mathbb{S}^d} P(x)d\sigma(x) = \frac{1}{N}\sum_{j=1}^N P(z_{n,j}),\quad \forall P\in\Pi_{4n}^d \tag{4.3}$$

and $\min_{1\le i\ne j\le N}\rho(z_{n,i},z_{n,j})\ge c_d N^{-1/d}$.

The second lemma is on the distribution of nodes of spherical designs. Denote by $B(x,\theta)$ the spherical cap $\{y\in\mathbb{S}^d : \rho(x,y)\le\theta\}$ with center $x\in\mathbb{S}^d$ and radius $\theta\in(0,\pi]$.

Lemma 4.2 ([24, 25, 11]). *If the formula* (4.3) *holds for all $P\in\Pi_n^d$, then*

$$\mathbb{S}^d = \bigcup_{j=1}^N B(z_{n,j},\theta_n),$$

where $\theta_n = \arccos t_n \sim \frac{1}{n}$ *and* t_n *is the largest root of the following algebraic polynomial on* $[-1, 1]$*:*

$$Q_n(t) := \begin{cases} P_k^{(\frac{d-2}{2}, \frac{d-2}{2})}(t), & \text{if } n = 2k-1, \\ P_k^{(\frac{d-2}{2}, \frac{d}{2})}(t), & \text{if } n = 2k. \end{cases}$$

The third lemma reveals a connection between positive cubature formulas and the Marcinkiewitcz–Zygmund inequality on the sphere.

Lemma 4.3 ([6, Theorem 4.2])**.** *Suppose that* Λ *is a finite subset of* $\mathbb{S}^d$, $\lambda_\omega \ge 0$ *for all* $\omega \in \Lambda$*, and the formula,* $\int_{\mathbb{S}^d} f(x)d\sigma(x) = \sum_{\omega\in\Lambda} \lambda_\omega f(\omega)$*, holds for all* $f \in \Pi_{3n}^d$*. Then for* $0 < p < \infty$ *and all* $f \in \Pi_n^d$*,*

$$\|f\|_p \asymp \left(\sum_{\omega\in\Lambda} \lambda_\omega |f(\omega)|^p \right)^{1/p}$$

with the constant of equivalence depending only on d and p when $p \to 0$*.*

Now we turn to the proof of (4.2). Let $\varepsilon \in (0, 1)$ be an arbitrarily given positive parameter, and let $\eta_1 = \eta_{1,\varepsilon} \in C_c^\infty[0, \infty)$ be such that $\eta_1(x) = 1$ for $x \in [0, 1]$ and $\eta_1(x) = 0$ for $x \ge 1 + \varepsilon$. Let G_{n,η_1} denote the polynomial on $[-1, 1]$ as defined in Lemma 2.3. Invoking Lemma 4.1 with $N = N_n = An^d$, we have that for $x \in \mathbb{S}^d$,

$$P_n(x) = \int_{\mathbb{S}^d} P_n(y) G_{n,\eta_1}(x\cdot y) d\sigma(y) = \frac{\omega_d}{N_n} \sum_{j=1}^{N_n} P_n(z_{n,j}) G_{n,\eta_1}(x\cdot z_{n,j}). \tag{4.4}$$

According to Lemma 4.2 and Lemma 4.1, the set of nodes $\{z_{n,i}\}_{i=1}^{N_n} \subset \mathbb{S}^d$ satisfies

$$\min_{1\le i\ne j\le N_n} \rho(z_{n,i}, z_{n,j}) \ge \frac{\delta_d}{n} \quad \text{and} \quad \max_{x\in\mathbb{S}^d} \min_{1\le j\le N_n} \rho(x, z_{n,j}) \le \frac{c_d}{n}. \tag{4.5}$$

Without loss of generality, we may assume that $z_{n,1} = e$. By Lemma 4.3,

$$\left(\sum_{j=1}^{N_n} |P_n(z_{n,j})|^p \right)^{1/p} \le Cn^{d/p} \|P_n\|_p = \frac{Cn^{d/p}}{C(n, d, p, \infty)} \le C_d < \infty. \tag{4.6}$$

Next, write $z_{n,j} = \psi(y_{n,j}/n)$ for $1 \le j \le N_n$ with $y_{n,j} \in B(n\pi)$. Since

$$\rho(\psi(u), \psi(v)) \le \frac{\pi}{\sqrt{2}} |u - v|$$

for any $u, v \in B(\pi)$, we obtain from (4.5) that

$$\min_{1\le i\ne j\le N_n} |y_{n,i} - y_{n,j}| \ge \delta_d' > 0. \tag{4.7}$$

Rearrange the order of the codes $z_{n,j}$ of the spherical design so that

$$0 = |y_{n,1}| \le |y_{n,2}| \le \cdots \le |y_{n,N_n}|.$$

Set $\Lambda_n := \{y_{n,j}\colon 1 \le j \le N_n\}$. We claim that there exists a constant $\gamma_d > 0$ depending only on d such that for $m = 1, \dots, n$,

$$B(\gamma_d^{-1} m) \cap \Lambda_n \subset \{y_{n,1}, \dots, y_{n,m^d}\} \subset B(\gamma_d m) \cap \Lambda_n. \tag{4.8}$$

Indeed, noticing that for any $0 < t \le n$,

$$\Big\{j : 1 \le j \le N_n,\ rho(z_{n,j}, e) \le \frac{t\pi}{n}\Big\} = \{j : 1 \le j \le N_n,\ |y_{n,j}| \le t\pi\},$$

we deduce from (4.5) that for any $1 \le t \le n\pi$,

$$\#\{j : 1 \le j \le N_n,\ |y_{n,j}| \le t\} \asymp t^d,$$

which together with the monotonicity of $\{\,|y_{n,j}|\}_{j=0}^{N_n}$ implies the claim (4.8).

Now the rest of the proof follows along the same line as that of [19]. For simplicity, we set $P_n^*(x) := P_n(\psi(x/n))$ for $x \in \mathbb{R}^d$. Let $\mathcal{A}$ be a sequence of positive integers such that

$$\lim_{n\to\infty,\ n\in\mathcal{A}} n^{d/p}\|P_n\|_p = \liminf_{n\to\infty} n^{d/p}\|P_n\|_p.$$

By (4.6) and (4.8), for each fixed $m \ge 1$, we may find a subsequence $\mathcal{T}_m$ of $\mathcal{A}$ such that

$$\lim_{n\to\infty,\ n\in\mathcal{T}_m} P_n(z_{n,j}) = \lim_{n\to\infty,\ n\in\mathcal{T}_m} P_n^*(y_{n,j}) = \alpha_j \in \mathbb{R}, \quad j = 1, \dots, m^d, \tag{4.9}$$

and

$$\lim_{n\to\infty,\ n\in\mathcal{T}_m} y_{n,j} = y_j \in B(\gamma_d m), \quad j = 1, \dots, m^d.$$

We may also assume that

$$\mathcal{A} \supset \mathcal{T}_1 \supset \mathcal{T}_2 \supset \cdots \supset \mathcal{T}_m \supset \mathcal{T}_{m+1} \supset \cdots,$$

so that both the sequences $\{\alpha_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ are independent of m. Note that $\alpha_1 = P_n(e) = 1$ and $y_1 = 0$.

By (4.6) and (4.9), for each $m \ge 1$,

$$\sum_{j=1}^{m^d} |\alpha_j|^p = \lim_{n\to\infty, n\in\mathcal{T}_m} \sum_{j=1}^{m^d} |P_n(z_{n,j})|^p \le C_d^p.$$

Hence,

$$\sum_{j=1}^{\infty} |\alpha_j|^p \le C_d^p < \infty. \tag{4.10}$$

Now we define

$$f(x) := \frac{\omega_d}{A}\sum_{j=1}^{\infty}\alpha_j K_{\eta_1}(|x-y_j|), \quad x\in\mathbb{R}^d, \tag{4.11}$$

where $K_{\eta_1}(|\cdot|)$ is the inverse Fourier transform of $\eta_1(|\cdot|)$ on $\mathbb{R}^d$. According to (4.7), $\inf_{j\neq j'}|y_j - y_{j'}| \ge \delta' > 0$. Since K_{η_1} is a Schwartz function on $\mathbb{R}^d$, $\sup_{x\in\mathbb{R}^d}\sum_{j=1}^{\infty}|K_{\eta_1}(|x-y_j|)| \le C_{\eta_1} < \infty$. By (4.10), this implies that the series (4.11) converges to f both uniformly on $\mathbb{R}^d$ and in the norm of $L^p(\mathbb{R}^d)$. Moreover, the function f satisfies that

$$\|f\|_p \le CA^{-1}\Big(\sum_{j=1}^{\infty}|\alpha_j|^p\Big)^{1/p} < \infty,$$

and

$$\widehat{f}(\xi) = \frac{\omega_d}{A}\eta_1(|\xi|)\Big(\sum_{j=1}^{\infty}\alpha_j e^{-iy_j\cdot\xi}\Big),$$

where the infinite series converges in a distributional sense. According to the Paley–Wiener theorem, f extends to an entire function on $\mathbb{C}^d$ of spherical exponential type $1+\varepsilon$. In particular, this implies that (4) implies that the function

$$f_\varepsilon(x) := (1+\varepsilon)^{-d} f\Big(\frac{x}{1+\varepsilon}\Big)$$

belongs to the space $\mathcal{E}_p^d$.

To complete the proof of (4.2), we need the following technical lemma:

Lemma 4.4. *Let γ_d denote the constant in* (4.8)*. If $r \ge 1$ and $m \ge 2\gamma_d r$, then for any $\ell \ge 1$,*

$$\limsup_{n\to\infty, n\in\mathcal{T}_m}\sup_{x\in B(r)}\Big|P_n^*(x) - A^{-1}\omega_d\sum_{1\le j\le m^d}\alpha_j K_{\eta_1}(|x-y_j|)\Big| \le C_{\eta_1,\ell,r}m^{-\ell}. \tag{4.12}$$

In particular, this implies that

$$f(0) = A^{-1}\omega_d\sum_{j=0}^{\infty}\alpha_j K_{\eta_1}(|y_j|) = 1. \tag{4.13}$$

For the moment, we assume Lemma 4.4 and proceed with the proof of (4.2).

Since $f_\varepsilon \in \mathcal{E}_p^d$, we have

$$(1+\varepsilon)^{-d} = |f_\varepsilon(0)| \le \|f_\varepsilon\|_\infty \le \mathcal{L}(d,p,\infty)\|f_\varepsilon\|_p = \mathcal{L}(d,p,\infty)(1+\varepsilon)^{-d/p'}\|f\|_p.$$

Thus,

$$\|f\|_p \ge \mathcal{L}(d,p,\infty)^{-1}(1+\varepsilon)^{-d/p}. \tag{4.14}$$

On the other hand, setting $p_1 = \min\{p, 1\}$, we obtain that for $n \in \mathcal{T}_m$,

$$\begin{aligned}(n^{d/p}\|P_n\|_p)^{p_1} &\geq \Big(n^d \int_{B(e,\frac{r}{n})} |P_n(x)|^p d\sigma(x)\Big)^{p_1/p} \\ &= \Big(\int_{B(r)} |P_n^*(x)|^p \Big(\frac{\sin(|x|/n)}{|x|/n}\Big)^{d-1} dx\Big)^{p_1/p} \\ &\geq \Big(\int_{B(r)} \Big|\frac{\omega_d}{A}\sum_{j=1}^{m^d} \alpha_j K_{\eta_1}(|x-y_j|)\Big|^p \Big(\frac{\sin(|x|/n)}{|x|/n}\Big)^{d-1} dx\Big)^{p_1/p} \\ &\quad - C_r \sup_{x\in B(r)} \Big|P_n^*(x) - (A^{-1}\omega_d)\sum_{j=1}^{m^d} \alpha_j K_{\eta_1}(|x-y_j|)\Big|^{p_1}.\end{aligned}$$

It then follows from Lemma 4.4 that for any $\ell > 1$,

$$\begin{aligned}\liminf_{n\to\infty} (n^{d/p}\|P_n\|_p)^{p_1} &= \lim_{n\to\infty,\ n\in\mathcal{T}_m} (n^{d/p}\|P_n\|_p)^{p_1} \\ &\geq \Big(\int_{B(r)} \Big|\frac{\omega_d}{A}\sum_{j=1}^{m^d} \alpha_j K_{\eta_1}(|x-y_j|)\Big|^p dx\Big)^{p_1/p} - C_r m^{-\ell}.\end{aligned}$$

Letting $m \to \infty$, we obtain from (4.11) and the dominated convergence theorem that

$$\liminf_{n\to\infty} n^{d/p}\|P_n\|_p \geq \Big(\int_{B(r)} |f(x)|^p dx\Big)^{1/p}.$$

Letting $r \to \infty$, and using (4.14), we then deduce

$$\liminf_{n\to\infty} n^{d/p}\|P_n\|_p \geq \|f\|_p \geq \mathcal{L}(d,p,\infty)^{-1}(1+\varepsilon)^{-d/p}.$$

Now the desired estimate (4.2) follows by letting $\varepsilon \to 0$. □

It remains to prove Lemma 4.4.

Proof of Lemma 4.4. Note first that by Lemma 2.3,

$$\lim_{n\to\infty} \frac{1}{n^d} G_{n,\eta_1}\Big(\psi\Big(\frac{x}{n}\Big)\cdot\psi\Big(\frac{y}{n}\Big)\Big) = K_{\eta_1}(|x-y|)$$

holds uniformly for $x, y \in B(\gamma_d m)$. Note also that for $1 \leq j \leq m^d$,

$$\begin{aligned}&\Big|n^{-d}G_{n,\eta_1}\Big(\psi\Big(\frac{x}{n}\Big)\cdot\psi\Big(\frac{y_{n,j}}{n}\Big)\Big) - K_{\eta_1}(|x-y_j|)\Big| \\ &\qquad \leq \sup_{z\in B(\gamma_d m)}\Big|n^{-d}G_{n,\eta_1}\Big(\psi\Big(\frac{x}{n}\Big)\cdot\psi\Big(\frac{z}{n}\Big)\Big) - K_{\eta_1}(|x-z|)\Big| \\ &\qquad\quad + \Big|K_{\eta_1}(|x-y_{n,j}|) - K_{\eta_1}(|x-y_j|)\Big|.\end{aligned}$$

Letting $n \to \infty$ and $n \in \mathcal{T}_m$, we conclude that for $1 \le j \le m^d$,

$$\lim_{n\to\infty, n\in\mathcal{T}_m} \sup_{x\in B(\gamma_d m)} \left| \frac{1}{n^d} G_{n,\eta_1}\Big(\psi\Big(\frac{x}{n}\Big)\cdot\psi\Big(\frac{y_{n,j}}{n}\Big)\Big) - K_{\eta_1}(|x-y_j|)\right| = 0. \tag{4.15}$$

Next, using (4.4), we obtain that for $x \in B(r)$ and $m \ge 2r\gamma_d$,

$$\begin{aligned} P_n^*(x) &= \frac{1}{An^d}\sum_{j=1}^{An^d} P_n(z_{n,j}) G_{n,\eta_1}\Big(\psi\Big(\frac{x}{n}\Big)\cdot\psi\Big(\frac{y_{n,j}}{n}\Big)\Big) \\ &= \sum_{1\le j\le m^d} + \sum_{m^d<j\le An^d} =: I_{n,m}(x) + J_{n,m}(x). \end{aligned}$$

To estimate the second term $J_{n,m}(x)$, we note that $\rho(\psi(\frac{y_{n,j}}{n}), e) \ge \frac{\gamma_d^{-1} m}{n} \ge \frac{2r}{n}$ for $m^d \le j \le An^d$, and $\rho(\psi(\frac{x}{n}), e) \le \frac{r}{n}$ for $x \in B(r)$. Thus, $\rho(\psi(\frac{x}{n}), \psi(\frac{y_{n,j}}{n})) \ge \frac{c_d m}{n}$ for $x \in B(r)$ and $m^d \le j \le An^d$. It follows by (4.6) that for any $\ell \ge 1$ and $x \in B(r)$,

$$|J_{n,m}(x)| \le Cm^{-\ell}\Big(\sum_{j=1}^{An^d} |P_n(z_{n,j})|^p\Big)^{1/p} \le Cm^{-\ell} n^{d/p}\|P_n\|_p \le Cm^{-\ell}. \tag{4.16}$$

For the term $I_{n,m}$, we use (4.15) and (4.9) to obtain

$$\lim_{n\to\infty,\ n\in\mathcal{T}_m} I_{n,m}(x) = A^{-1}\omega_d \sum_{1\le j\le m^d} \alpha_j K_{\eta_1}(|x-y_j|) \quad \text{uniformly for } x\in B(r). \tag{4.17}$$

Combining (4.16) with (4.17), we conclude that

$$\limsup_{n\to\infty, n\in\mathcal{T}_m} \sup_{x\in B(r)} \left| P_n^*(x) - A^{-1}\omega_d \sum_{1\le j\le m^d} \alpha_j K_{\eta_1}(|x-y_j|)\right| \le Cm^{-\ell}.$$

This proves (4.12).

Finally, invoking (4.12) with $x = 0$, and recalling that $P_n^*(0) = P_n(e) = 1$, we obtain

$$\left|1 - A^{-1}\omega_d \sum_{0\le j\le m^d} \alpha_j K_{\eta_1}(|y_j|)\right| \le Cm^{-2} \quad \forall\, m \ge 1.$$

Letting $m \to \infty$, we obtain (4.13). This completes the proof. □

5 Proof of Theorem 1.2

We break the proof of Theorem 1.2 into two parts. In the first part, we prove the following proposition, which gives the exact value of the Nikolskii constant for nonnegative functions from the class $\mathcal{E}_1^d$ on $\mathbb{R}^d$.

Proposition 5.1. *We have*

$$\sup_{0\le f\in\mathcal{E}_1^d}\frac{\|f\|_{L^\infty(\mathbb{R}^d)}}{\|f\|_{L^1(\mathbb{R}^d)}}=\frac{1}{4^d\pi^{d/2}\Gamma(d/2+1)}. \tag{5.1}$$

In the second part, we compute the exact value of the Nikolskii constant for nonnegative polynomials on $\mathbb{S}^d$:

Proposition 5.2. *For $n=1,2,\dots,$*

$$\sup_{0\le P\in\Pi_n^d}\frac{\|P\|_\infty}{\|P\|_1}=\omega_d^{-1}\begin{cases}\frac{(2k+d)(k+d-1)!}{k!d!}, & n=2k,\\ 2\binom{d+k}{d}, & n=2k+1.\end{cases} \tag{5.2}$$

Note that (5.2), in particular, implies

$$\lim_{n\to\infty}\sup_{0\le P\in\Pi_n^d}\frac{\|P\|_\infty}{n^d\|P\|_1}=\frac{1}{4^d\pi^{d/2}\Gamma(d/2+1)}. \tag{5.3}$$

Theorem 1.2 is a direct consequence of (5.3) and (5.1).

We point out that (5.2) for algebraic polynomials on intervals was obtained in [18]. Proofs of Propositions 5.1 and 5.2 are given in the next two subsections respectively.

5.1 Proof of Proposition 5.1.

For simplicity, we set

$$\mathcal{L}^+:=\sup\{\|f\|_\infty : 0\le f\in\mathcal{E}_1^d,\ \|f\|_1=1\}.$$

To show the lower estimate,

$$\mathcal{L}^+\ge\frac{1}{4^d\Gamma(d/2+1)\pi^{d/2}},$$

we consider the function $f(x):=(j_{d/2}(|x|/2))^2$. Note that

$$G(x):=\frac{\omega_{d-1}}{d(2\pi)^d}j_{d/2}(|x|)$$

is the inverse Fourier transform of $\chi_{\mathbb{B}^d}$, where $\mathbb{B}^d$ denotes the unit ball centered at the origin in $\mathbb{R}^d$. It follows that $0\le f\in\mathcal{E}_1^d$. Furthermore, by Plancherel's theorem,

$$\begin{aligned}\|f\|_1&=\Big(\frac{d(2\pi)^d}{\omega_{d-1}}\Big)^2 2^d\int_{\mathbb{R}^d}|G(x)|^2dx=\frac{d^2 2^d(2\pi)^d}{(\omega_{d-1})^2}\int_{\mathbb{B}^d}1dx\\&=\frac{2^d d(2\pi)^d}{\omega_{d-1}}=2^{d-1}\omega_d d!.\end{aligned}$$

This yields the stated lower estimate:

$$\mathcal{L}^+ \geq \frac{f(0)}{\|f\|_1} = \frac{1}{2^{d-1}d!\omega_d} = \frac{1}{4^d\Gamma(d/2+1)\pi^{d/2}}.$$

To show the upper estimate,

$$\mathcal{L}^+ \leq \frac{1}{4^d\Gamma(d/2+1)\pi^{d/2}}, \tag{5.4}$$

we need the following Markov type quadrature formula, which was established in [14]:

Lemma 5.1 ([14]). *Assume that $\alpha \geq -\frac{1}{2}$ and $\tau > 0$. Let $\mathcal{B}_{\alpha,\tau}$ denote the set of all even entire functions f of exponential type $\leq 2\tau$ such that $\int_0^\infty |f(t)|t^{2\alpha+1}dt < \infty$. Then there exists a sequence $\{\rho_k\}_{k=0}^\infty$ of positive numbers with*

$$\rho_0 = 2^{2\alpha}(\Gamma(\alpha+1))^2(2\alpha+2)/\tau^{2\alpha+2}$$

such that

$$\int_0^\infty f(t)t^{2\alpha+1}dt = \rho_0 f(0) + \sum_{k=1}^\infty \rho_k f(q_{\alpha+1,k}/\tau), \quad \forall f \in \mathcal{B}_{\alpha,\tau},$$

where the infinite series converges absolutely, and $\{q_{\alpha+1,k}\}_{k=1}^\infty$ is the sequence of all positive zeros of the Bessel function $J_{\alpha+1}(x)$ arranged in increasing order.

Now we turn to the proof of the estimate (5.4). Given $\varepsilon \in (0,1)$, let $f \in \mathcal{E}_1^d$ be a nonnegative function such that $\|f\|_1 = 1$ and $\|f\|_\infty \geq \mathcal{L}^+ - \varepsilon$. Without loss of generality, we may assume that $\|f\|_\infty = f(0)$. Define a nonnegative radial function g by

$$g(x) = g_0(|x|) := \frac{1}{\omega_{d-1}}\int_{\mathbb{S}^{d-1}} f(|x|\xi)d\sigma(\xi), \quad x \in \mathbb{R}^d.$$

It is easily seen that

$$\widehat{g}(x) := \frac{1}{\omega_{d-1}}\int_{\mathbb{S}^{d-1}} \widehat{f}(|x|\xi)d\sigma(\xi), \quad x \in \mathbb{R}^d,$$

and that $g(0) = f(0)$, and $\|g\|_1 = \|f\|_1 = 1$. By the Paley–Wiener theorem, this in particular implies $g \in \mathcal{E}_1^d$. Thus, we may apply Lemma 5.1 to the function g_0 with $\tau = 1/2$ and $\alpha = d/2 - 1$. Taking into account the facts that $\rho_j \geq 0$ for $j = 0, 1, \ldots$ and g_0 is nonnegative, we then obtain

$$1 = \|g\|_1 = \omega_{d-1}\int_0^\infty g_0(t)t^{d-1}dt \geq \omega_{d-1}\rho_0 g(0) = \omega_{d-1}2^{2d-2}d(\Gamma(d/2))^2 f(0).$$

Thus,

$$\mathcal{L}^+ - \varepsilon \leq f(0) \leq \frac{1}{\omega_{d-1}2^{2d-2}d(\Gamma(d/2))^2} = \frac{1}{d!2^{d-1}\omega_d} = \frac{1}{4^d\Gamma(d/2+1)\pi^{d/2}}.$$

Letting $\varepsilon \to 0$ yields the desired estimate (5.4).

5.2 Proof of Proposition 5.2. Without loss of generality, we may assume $n = 2k$. (The case $n = 2k + 1$ can be treated similarly.) The proof follows along the same line as that of Proposition 5.1.

To show the lower estimate,

$$\sup_{0 \le P \in \Pi_{2k}^d} \frac{\|P\|_\infty}{\|P\|_1} \ge \omega_d^{-1} \frac{(2k+d)(k+d-1)!}{k!d!}, \tag{5.5}$$

we consider the polynomial

$$f(x) := [R_k^{(\frac{d}{2}, \frac{d-2}{2})}(x \cdot e)]^2, \quad x \in \mathbb{S}^d,$$

where $e \in \mathbb{S}^d$ is a fixed point on $\mathbb{S}^d$ and $R_k^{(\alpha,\beta)}(t) = P_k^{(\alpha,\beta)}(t)/P_k^{(\alpha,\beta)}(1)$. Clearly, $f \in \Pi_n^d$, and $\|f\|_\infty = f(e) = 1$. Moreover, using (1.2), we have

$$\|f\|_1 = \int_{\mathbb{S}^d} |R_k^{(\frac{d}{2}, \frac{d-2}{2})}(x \cdot e)|^2 d\sigma(x) = \frac{1}{d_k^2} \sum_{j=0}^{k} \frac{\dim \mathcal{H}_j^d}{\omega_d} = \frac{\omega_d}{\dim \Pi_k^d}.$$

It then follows from (1.1) that

$$\frac{\|f\|_\infty}{\|f\|_1} = \frac{\dim \Pi_k^d}{\omega_d} = \frac{1}{\omega_d} \frac{(2k+d)\Gamma(k+d)}{k!d!},$$

which shows the lower estimate (5.5).

The proof of the upper estimate,

$$\sup_{0 \le P \in \Pi_{2k}^d} \frac{\|P\|_\infty}{\|P\|_1} \le \omega_d^{-1} \frac{(2k+d)(k+d-1)!}{k!d!}, \tag{5.6}$$

relies on the following Jacobi–Gauss–Radau quadrature rules, which can be found in [22, p. 81]:

Lemma 5.2 ([22])**.** *Let* $\{x_j\}_{j=1}^N$ *be the zeros of the Jacobi polynomial* $P_N^{(\alpha+1,\beta)}$ *with* $\alpha, \beta > -1$. *Then for every algebraic polynomial* P *of degree at most* $2N$,

$$\int_{-1}^{1} P(x)(1-x)^\alpha (1+x)^\beta dx = \lambda_0 P(1) + \sum_{j=1}^{N} \lambda_j P(x_j),$$

where

$$\lambda_0 = \frac{2^{\alpha+\beta+1}(\alpha+1)(\Gamma(\alpha+1))^2 N!\Gamma(N+\beta+1)}{\Gamma(N+\alpha+2)\Gamma(N+\alpha+\beta+2)},$$

$$\lambda_j = \frac{2^{\alpha+\beta+4}\Gamma(N+\alpha+2)\Gamma(N+\beta+1)}{(1+x_j)(1-x_j)^2[P_{N-1}^{(\alpha+2,\beta+1)}(x_j)]^2(N+\alpha+\beta+2)\Gamma(N+\alpha+\beta+3)}.$$

To show (5.6), let f be an arbitrary nonnegative spherical polynomial of degree at most $2k$ such that $\|f\|_\infty = f(x_0) = 1$ for some $x_0 \in \mathbb{S}^d$. Without loss of generality, we may assume that $x_0 = (1, 0, \dots, 0)$. Define

$$g(t) := \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} f(t, \sqrt{1-t^2}\xi)\, d\sigma(x), \quad t \in [-1, 1].$$

It is easily seen that g is an algebraic polynomial of degree at most $2k$ on $[-1, 1]$, $g(1) = f(x_0) = 1$, and

$$\begin{aligned}\int_{-1}^{1} g(t)(1-t^2)^{\frac{d-2}{2}}\, dt &= \frac{1}{\omega_{d-1}} \int_{-1}^{1} \int_{\mathbb{S}^{d-1}} f(t, \sqrt{1-t^2}\xi)\, d\sigma(\xi)(1-t^2)^{\frac{d-2}{2}}\, dt \\ &= \frac{1}{\omega_{d-1}} \|f\|_{L^1(\mathbb{S}^d)}.\end{aligned}$$

Using Lemma 5.2 with $\alpha = \beta = \frac{d-2}{2}$, and taking into account the facts that g is nonnegative and $\lambda_j > 0$ for $j = 0, 1, \dots$, we deduce

$$\begin{aligned}\|f\|_{L^1(\mathbb{S}^d)} &= \omega_{d-1} \int_{-1}^{1} g(t)(1-t^2)^{\frac{d-2}{2}}\, dt \geq \lambda_0 \omega_{d-1} g(1) \\ &= \omega_{d-1} \frac{2^{d-2} d (\Gamma(d/2))^2 k!}{(k+d/2)\Gamma(k+d)} = \omega_d \frac{k! d!}{(2k+d)\Gamma(k+d)}.\end{aligned}$$

Thus,

$$\frac{\|f\|_\infty}{\|f\|_{L^1(\mathbb{S}^d)}} \leq \omega_d \frac{(2k+d)\Gamma(k+d)}{k! d!},$$

and the upper estimate (5.6) then follows.

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