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Calculus of Variations

Weighted gradient inequalities and unique continuation problems

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Abstract
We use Pitt inequalities for the Fourier transform to prove the following weighted gradient inequality

$$\|e^{-\tau\ell(\cdot)}u^{\frac{1}{q}}f\|_q \le c_\tau\|e^{-\tau\ell(\cdot)}v^{\frac{1}{p}}\nabla f\|_p, \quad f \in C_0^\infty(\mathbb{R}^n).$$

This inequality is a Carleman-type estimate that yields unique continuation results for solutions of first order differential equations and systems.

Mathematics Subject Classification Primary: 42B10; Secondary: 35B60

1 Introduction

The main purpose of this paper is to prove that the following weighted Sobolev gradient inequality holds for every linear function $\ell\colon \mathbb{R}^n \to \mathbb{R}$, every $f \in C_0^\infty(\mathbb{R}^n)$ and every $\tau \ge 0$,

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with suitable weights u, v and exponents $1 < p,\ q < \infty$.

$$\|e^{-\tau\ell(\cdot)}u^{\frac{1}{q}} f\|_q \le c_\tau \|e^{-\tau\ell(\cdot)}v^{\frac{1}{p}}\,\nabla f\|_p \tag{1.1}$$

Here, c_τ is a finite constant that may depend on τ but does not depend on ℓ and f. We have denoted with $\|f\|_r = \left(\int_{\mathbb{R}^n} |f(x)|^r\,dx\right)^{\frac{1}{r}}$ the norm in $L^r(\mathbb{R}^n)$ and with $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$ and $|x| = \langle x,\ x\rangle^{\frac{1}{2}}$ the standard inner product and norm in $\mathbb{R}^n$.

When $\tau > 0$, we prove in Theorem 1.1 that $c_\tau = \max(\tau^{-1}, 1)C$; here and throughout the paper, C denotes a generic constant that depends only on non-essential parameters, i.e., $C = C_{u,v,p,q,n}$. In particular, $c_\tau = C$ when $\tau \ge 1$. Inequalities like (1.1) are often called *Carleman inequalities* in literature. In Sects. 3 and 4 we will discuss Carleman inequalities and their connection with unique continuation problems and we will prove new unique continuation results for systems of partial differential equations and inequalities.

When $\tau = 0$ in (1.1), we obtain a standard weighted Sobolev gradient inequality (also called *weighted Poincaré-Sobolev inequality*)

$$\|u^{\frac{1}{q}} f\|_q \le c_0\|v^{\frac{1}{p}}\,\nabla f\|_p, \quad f \in C_0^\infty(\mathbb{R}^n). \tag{1.2}$$

These inequalities have deep applications in partial differential equations. For example, the case $p = 2 < q$ of (1.2) arises in Harnack's inequality and regularity estimates for degenerate second order differential operators in divergence form. They also have applications in the study of the stable solutions of the Laplace and the p-Laplace operators in the Euclidean space, the Laplace–Kohn operator in the Heisenberg group, the sub-Laplace operator in the Engel group, etc.; see e.g. [22,49,58] and the references cited in these papers; see also [10].

Conditions on the weights u and v and the exponents p, q for which (1.2) holds have been investigated by several authors. The most natural approach to study (1.2) is based on the following pointwise inequality (see e.g. [19,46])

$$|f(x)| \le C I_1(|\nabla f|)(x), \quad x \in \mathbb{R}^n,$$

where $I_\alpha\phi(x) = \int_{\mathbb{R}^n} \frac{\phi(y)}{|x-y|^{n-\alpha}}\,dy$, $\alpha < n$, is the *Riesz potential*. This inequality follows from the classical Sobolev integral representation and is proved e.g. in [42].

If the weighted inequality

$$\|u^{\frac{1}{q}} I_1 f\|_q \le C\|v^{\frac{1}{p}} f\|_p \tag{1.3}$$

holds for the weights u and v, we also have

$$\|u^{\frac{1}{q}} f\|_q \le C\|u^{\frac{1}{q}} I_1(|\nabla f|)\|_q \le C\|v^{\frac{1}{p}}|\nabla f|\|_p.$$

Sawyer [48] a complete characterization of the weights u and v for which the gradient inequality (1.3) holds with $p \le q$. However, in some cases, the conditions in [48] are difficult to verify. When $p = q = 2$, a full characterization of the weights for which (1.2) holds is also in [41], but also the conditions in this paper are difficult to verify.

Heinig [25] that weighted norm inequalities for the Fourier transform (or: *Pitt-type inequalities*) in the form of

$$\|\widehat{f}\, u^{\frac{1}{q}}\|_q \le C\|f\, w^{\frac{1}{p}}\|_p, \quad f \in C_0^\infty(\mathbb{R}^n), \tag{1.4}$$

can be used to prove weighted gradient inequalities. The Fourier transform is defined as $\widehat{f}(y) = \int_{\mathbb{R}^n} f(x)e^{-i\langle x,\, y\rangle}dx$.

To prove (1.2) from (1.4), we observe that $\widehat{I_\alpha f}(y) = c_\alpha |y|^{-\alpha} \widehat{f}(y)$, where c_α is an explicit constant; we can see at once that (1.3) is equivalent to

$$\|u^{\frac{1}{q}} (|y|^{-1} \widehat{f})^{\vee}\|_q \leq C \|v^{\frac{1}{p}} f\|_p$$

where $^\vee$ denotes the inverse Fourier transform. We can apply Pitt's inequality twice (with a suitable weight w and an exponent $\gamma \in (1, \infty)$) to obtain

$$\|u^{\frac{1}{q}} (|y|^{-1} \widehat{f})^{\vee}\|_q \leq C \|w^{\frac{1}{\gamma}} |y|^{-1} \widehat{f}\|_\gamma \leq C \|v^{\frac{1}{p}} f\|_p.$$

Taking $w = |y|^\gamma$ and $\gamma = q$ and assuming conditions on the weights that ensure that both Pitt's inequalities hold we obtain the main theorem in [25], which was proved differently; see Theorem 2.1 in Sect. 2.

1.1 Main results

Throughout this paper, we will often write $A \lesssim B$ when $A \leq CB$ with a constant $C > 0$. We will also write $A \asymp B$ when there exists a constant $C > 0$, called the *constant of equivalence*, such that $C^{-1} A \leq B \leq CA$. As usual, we let g^* be the non-increasing rearrangement of g. We let $p' = \frac{p}{p-1}$ be the dual exponent of $p \in (1, \infty)$.

Our main result can be stated as follows.

Theorem 1.1 *Let $u \not\equiv 0$ and $v \not\equiv +\infty$ be weights on $\mathbb{R}^n$, $n \geq 1$.*

(a) *Let $1 < p \leq q < \infty$. If there exists $\gamma > 0$ that satisfies* $\max(p, p') \leq \gamma \leq q$, *for which*

$$\begin{cases} A_u^q(0) := \sup\limits_{s>0} s^{1-q(\frac{1}{\gamma'} - \frac{1}{n})} u^*(s) < \infty, & \dfrac{1}{n} < \dfrac{1}{\gamma'} \leq \dfrac{1}{n} + \dfrac{1}{q}, \\ A_u^q(\tau) := \sup\limits_{s>0} \displaystyle\int_0^s u^*(t)\, dt \Big(\int_0^{\frac{1}{s}} (t + \tau^n)^{-\frac{\gamma'}{n}}\, dt \Big)^{\frac{q}{\gamma'}} < \infty, & \tau > 0, \end{cases} \tag{1.5}$$

and

$$A_v^p := \sup_{s>0} s^{\frac{p}{\gamma'} - 1} (1/v)^*(s) < \infty, \tag{1.6}$$

the inequality

$$\|e^{-\tau \ell(\cdot)} u^{\frac{1}{q}} f\|_q \leq c_\tau \|e^{-\tau \ell(\cdot)} v^{\frac{1}{p}} \nabla f\|_p, \quad f \in C_0^\infty(\mathbb{R}^n), \tag{1.7}$$

holds for every $\tau \geq 0$ and every linear function $\ell(x) = \langle \mathbf{a}, x \rangle + b$, $\mathbf{a} \in \mathbb{R}^n$, $|\mathbf{a}| = 1$, $b \in \mathbb{R}$, with the constant

$$c_\tau = C A_u(\tau) A_v, \tag{1.8}$$

where $C = C_{p,q,\gamma,n}$ is some positive constant. Moreover,

$$A_u(\tau) \leq \max(\tau^{-1}, 1) A_u(1), \quad \tau > 0. \tag{1.9}$$

(b) *Let $1 < q < p < \infty$. If there exists $\gamma > 0$ that satisfies*

$$\begin{cases} \frac{n}{n-1} < \gamma \leq q, & \tau = 0, \\ 1 < \gamma \leq q, & \tau > 0, \end{cases}$$

for which (1.5) *holds and*

$$\tilde{A}_v^r := \int_0^\infty s^{-\frac{r}{\gamma}-1}\Big(\int_0^s (1/v)^*(t)^{\frac{1}{p-1}}\,dt\Big)^{\frac{r}{p'}}\,ds < \infty,$$

with $\frac{1}{r} = \frac{1}{\gamma} - \frac{1}{p}$, *the inequality* (1.7) *holds with the constant*

$$c_\tau = C A_u(\tau)\tilde{A}_v. \tag{1.10}$$

Remark 1.1 When $\tau = 0$ and $\gamma = q$ we obtain Theorem 2.4 in [25] with simplified conditions on u and v. The proof of Theorem 1.1 shows that the assumptions $\frac{1}{\gamma'} \le \frac{1}{n} + \frac{1}{q}$ and $p' \le \gamma$ are to rule out the trivial weights $u \equiv 0$ and $v \equiv +\infty$.

Remark 1.2 For the applications of Theorem 1.1 it is important to have the uniform boundedness of c_τ as $\tau \to \infty$. From (1.8), (1.10) and (1.9), we have $c_\tau \le c_1 \asymp A_u(1)$ whenever $\tau \ge 1$; thus, to prove the boundedness of c_τ, it is sufficient to verify that $A_u(1) < \infty$.

Remark 1.3 It is interesting to compare our weighted gradient inequalities with those proved by Sinnamon [53]. In that paper, a weighted norm inequality in the form of

$$\|fu^{\frac{1}{q}}\|_q \le C\|\langle\nabla f, x\rangle w^{\frac{1}{p}}\|_p, \quad f \in C_0^\infty(\mathbb{R}^n) \tag{1.11}$$

is considered. If we denote with $\partial_r f = \langle\frac{x}{|x|}, \nabla f\rangle$ the radial derivative of f, the inequality (1.11) is equivalent to

$$\|fu^{\frac{1}{q}}\|_q \le C\|\,|x|w^{\frac{1}{p}}\partial_r f\|_p, \quad f \in C_0^\infty(\mathbb{R}^n),$$

and implies (1.2) with $v = |x|^p w$.

In [53, Theorem 4.1], (1.11) is only proved for $p = q$ and $q < p$ under some conditions on u, w; moreover, in [53, Theorem 3.4] it is proved that when $1 \le p < q < \infty$ and the weight w is locally integrable on $\mathbb{R}^n$, the inequality (1.11) holds for every $f \in C_0^\infty(\mathbb{R}^n)$ if and only if $u \equiv 0$ a.e.

When f is radial, $\nabla f(x) = \frac{x}{|x|}\partial_r f(x)$, and so $|\nabla f(x)| = |\partial_r f(x)|$. Thus, our Theorem 1.1 yields (1.11) for radial functions with a nontrivial weight u and with $w = |x|^{-p}e^{-p\tau\ell(x)}v$. We proved in Corollary 1.2 below that we can consider piecewise power weights $v = |x|^{(\beta_1,\beta_2)}$, with $0 \le \beta_1 \le n\big(\frac{p}{\gamma'} - 1\big)$ (see definition (1.13)). For example, if $\beta_1 = n\big(\frac{p}{\gamma'} - 1\big)$, then w is locally integrable for $\frac{1}{n} < \frac{1}{\gamma'}$ because $-p + \beta_1 > -n$. We remark that the counterexample in [53, Theorem 3.4] is not radial.

Remark 1.4 The inequality (1.7) is equivalent to

$$\|u^{\frac{1}{q}} f\|_q \le c_\tau\|v^{\frac{1}{p}}(\tau\mathbf{a}f + \nabla f)\|_p. \tag{1.12}$$

To see this, it is enough to use the substitution $f_1 = e^{-\tau\ell(\cdot)}f$ and $\nabla(e^{\tau\ell(\cdot)}f_1) = e^{\tau\ell(\cdot)}(\tau\mathbf{a}f_1 + \nabla f_1)$.

Let $\beta_1, \beta_2 \in \mathbb{R}$; we define the piecewise power function $t \mapsto t^{(\beta_1,\beta_2)}$ as follows:

$$t^{(\beta_1,\beta_2)} := \begin{cases} t^{\beta_1}, & 0 < t \le 1, \\ t^{\beta_2}, & t \ge 1. \end{cases} \tag{1.13}$$

In the following corollary of Theorem 1.1 we consider the important case of piecewise power weights.

Corollary 1.2 *Let* $1 < p \le q < \infty$*; let* $\gamma > 0$ *that satisfies* $\max(p, p') \le \gamma \le q$ *and* $\frac{1}{n} < \frac{1}{\gamma'} \le \frac{1}{n} + \frac{1}{q}$.

With the notation and the assumptions of Theorem 1.1(a)*, the inequality* (1.7) *holds with* $u(x) = |x|^{(-\alpha_1,-\alpha_2)}$*,* $v(x) = |x|^{(\beta_1,\beta_2)}$*, with* $\alpha_j, \beta_j \ge 0$*, provided that*

$$\alpha_1 \le n\Big(1 - \frac{q}{\gamma'} + \frac{q}{n}\Big), \quad \begin{cases} \alpha_2 \ge n\big(1 - \frac{q}{\gamma'} + \frac{q}{n}\big) & \text{when } \tau = 0, \\ \alpha_2 \ge 0 & \text{when } \tau > 0, \end{cases} \tag{1.14}$$

$$\beta_1 \le n\Big(\frac{p}{\gamma'} - 1\Big), \quad \beta_2 \ge n\Big(\frac{p}{\gamma'} - 1\Big). \tag{1.15}$$

In particular, for power weights $u(x) = |x|^{-\alpha}$*,* $v(x) = |x|^{\beta}$ *the inequality* (1.7) *holds if*

$$\begin{cases} \alpha = n\big(1 - \frac{q}{\gamma'} + \frac{q}{n}\big) \ge 0 & \text{when } \tau = 0, \\ 0 \le \alpha \le n\big(1 - \frac{q}{\gamma'} + \frac{q}{n}\big) & \text{when } \tau > 0, \end{cases} \qquad \beta = n\Big(\frac{p}{\gamma'} - 1\Big) \ge 0.$$

Moreover, the conditions

$$\begin{cases} \frac{\alpha}{q} + \frac{\beta}{p} = n\big(\frac{1}{q} - \frac{1}{p}\big) + 1 & \text{when } \tau = 0, \\ \frac{\alpha}{q} + \frac{\beta}{p} \le n\big(\frac{1}{q} - \frac{1}{p}\big) + 1 & \text{when } \tau > 0, \end{cases} \tag{1.16}$$

are necessary for the validity of (1.7).

Letting $\tau = \alpha = \beta = 0$, $1 < p < n$, and $\gamma = q$ in (1.16), we obtain $q = \frac{np}{n-p}$ and Corollary 1.2 yields the classical Sobolev inequality $\|f\|_q \le C\|\nabla f\|_p$; see also [25, Corollary 2.5].

When $\tau = 0$, we obtain the inequality

$$\Big(\int_{\mathbb{R}^n} |f|^q |x|^{(-\alpha_1,-\alpha_2)}\,dx\Big)^{\frac{1}{q}} \le C\Big(\int_{\mathbb{R}^n} |\nabla f|^p |x|^{(\beta_1,\beta_2)}\,dx\Big)^{\frac{1}{p}},$$

which was proved by Maz'ya [42] and Caffarelli et al. [3] for power weights. In [42, Sect. 2.1.6] it was proved that if $1 < p < n$, $p \le q \le \frac{pn}{n-p}$, and $-\frac{\alpha}{q} = \frac{\beta}{p} - 1 + n\left(\frac{1}{p} - \frac{1}{q}\right) > -\frac{n}{q}$, then

$$\Big(\int_{\mathbb{R}^n} |f|^q |x|^{-\alpha}\,dx\Big)^{\frac{1}{q}} \le C\Big(\int_{\mathbb{R}^n} |\nabla f|^p |x|^{\beta}\,dx\Big)^{\frac{1}{p}}. \tag{1.17}$$

In [28, Lemma 2.1], this inequality was proved for $n \ge 2$, $1 < p < +\infty$, $0 \le \frac{1}{p} - \frac{1}{q} = n\left(1 - \frac{\beta}{p} - \frac{\alpha}{q}\right)$ and $-\frac{n}{q} < -\frac{\alpha}{q} \le \frac{\beta}{p}$. Note that the conditions in [28,42] are the same, except for the extra condition $p < n$ in [42].

From Corollary 1.2 with $\tau = 0$ we have that $\alpha = n\big(1 - \frac{q}{\gamma'} + \frac{q}{n}\big) \ge 0$, $\beta = n\big(\frac{p}{\gamma'} - 1\big) \ge 0$, where $\max(p, p') \le \gamma \le q$ and $\frac{1}{n} < \frac{1}{\gamma'} \le \frac{1}{n} + \frac{1}{q}$. These inequalities imply $\frac{1}{p} - \frac{1}{q} = n\big(1 - \frac{\beta}{p} - \frac{\alpha}{q}\big)$, $-\frac{n}{q} < -\frac{\alpha}{q} \le \frac{\beta}{p}$, but we also have to assume $\alpha \ge 0$, $\beta \ge 0$ because of our method of the proof.

It is interesting to observe that the best constant in the inequality (1.17) has been evaluated in [58] and also in [22] for special values of α and β.

1.2 Unique continuation

Our Theorem 1.1 can be used to prove unique continuation results for weak solutions (also called *solutions in distribution sense*) of systems of differential equations and inequalities; see Sect. 3 for definitions and preliminary results.

We consider solutions in weighted Sobolev spaces of distributions: given a domain $D \subset \mathbb{R}^n$, we let $W_0^{m,p,v}(D)$ be the closure of $C_0^\infty(D)$ with respect to the norm

$$\|f\|_{W^{m,p,v}(D)} = \sum_{|\alpha|=0}^{m} \|v^{\frac{1}{p}} \partial_x^\alpha f\|_{L^p(D)}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and the $\partial_x^\alpha f = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f$ are the partial derivatives of f. In Sect. 3 we prove the following

Theorem 1.3 *Let p, q, γ, u and v be as in Theorem 1.1(a). Let $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Let $f \in W_0^{1,p,v}(\mathbb{R}^n)$ be a solution of the differential inequality*

$$|\nabla f| \le V|f| \tag{1.18}$$

with $V \in L^r(\operatorname{supp} f,\ v^{\frac{r}{p}} u^{-\frac{r}{q}}\, dx)$. If, for some linear function $\ell \colon \mathbb{R}^n \to \mathbb{R}$, we have that $\operatorname{supp} f \subset \{x : \ell(x) \ge 0\}$, *necessarily* $f \equiv 0$.

Note that the condition $V \in L^r(\operatorname{supp} f,\ v^{\frac{r}{p}} u^{-\frac{r}{q}}\, dx)$ follows from either $V \in L^r(\mathbb{R}^n,\ v^{\frac{r}{p}} u^{-\frac{r}{q}}\, dx)$ if $\operatorname{supp} f$ is unbounded, or from $V \in L^r_{\text{loc}}(\mathbb{R}^n,\ v^{\frac{r}{p}} u^{-\frac{r}{q}}\, dx)$ if f has compact support. In particular, for power weights u, v as in Corollary 1.2, the differential inequality (1.18) does not have solutions with compact support if $V \asymp |x|^{-1+\epsilon}$ for some $\epsilon > 0$; see Remark 3.1.

To prove Theorem 1.3 we use a method developed by Carleman [4]. A brief discussion on unique continuation problems and Carleman's method is in Sects. 3 and 4.

When D is measurable and v is a suitable weight we consider the Dirichlet problem

$$\begin{cases} -\operatorname{div}(v\, \nabla f |\nabla f|^{p-2}) = v\, V f |f|^{p-2}, \\ f \in W_0^{1,p,v}(D), \end{cases} \tag{1.19}$$

where $\operatorname{div}((g_1, \ldots, g_n)) = \partial_{x_1} g_1 + \cdots + \partial_{x_n} g_n$ and the potential V is in a suitable L^r space. The operator $\operatorname{div}(v\, \nabla f |\nabla f|^{p-2})$ is known as *weighted p-Laplacian* in the literature (see e.g. [23,34]) and is denoted by Δ_p when $v \equiv 1$. The weighted p-Laplacian is nonlinear when $p \ne 2$ and is linear when $p = 2$.

When $v \equiv 1$, (1.19) can be compared to the Sturm–Liouville problem in the form of $-\Delta_p f = (\lambda m - V) f |f|^{p-2}$ (see e.g. [8]). When $n = 1$ and $p = 2$ we have $-(vf')' = vVf$. This problem is related to the classical Sturm-Liouville problem $-(vf')' = (\lambda w - q) f$. See [40].

We prove the following

Theorem 1.4 *Let $f \in W_0^{1,p,v}(D)$ be a solution of the Dirichlet problem (1.19). Let $V_+ = \max\{V, 0\}$. Assume that $|V|^{\frac{1}{p}} \in L^r(D,\ v^{\frac{r}{p}} u^{-\frac{r}{q}}\, dx)$, where u, v are as in Theorem 1.1 and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Then, either*

$$c_0 \|u^{-\frac{1}{q}} v^{\frac{1}{p}} V_+^{\frac{1}{p}}\|_{L^r(D)} \ge 1,$$

where c_0 is as in (1.2), or $f \equiv 0$ in D.

Thus, the Dirichlet problem (1.19) has the unique solution $f \equiv 0$ if the weighted L^r norm of $V_+^{\frac{1}{p}}$ on D is small enough.

To the best of our knowledge, the method of proof of Theorem 1.4 has been used for the first time in [13]; it is extensively used in [11,17].

2 Proof of Theorem 1.1

In this section we prove our main theorem and a few corollaries.

2.1 Preliminary results

We will use the following theorem due to Heinig [26], Jurkat and Sampson [32] and Muckenhoupt [43].

Theorem 2.1 *Let* $n \geq 1$. *If* $1 < p \leq q < \infty$ *and the weights* u *and* w *satisfy*

$$\sup_{s>0}\Big(\int_0^{\frac{1}{s}} u^*(t)\,dt\Big)^{\frac{1}{q}}\Big(\int_0^{\frac{1}{s}} ((1/w)^*(t))^{\frac{1}{p-1}}\,dt\Big)^{\frac{1}{p'}} =: A_1 < \infty,$$

or if $1 < q < p < \infty$, *and*

$$\sup_{s>0}\bigg(\int_0^{\infty}\Big(\int_0^{\frac{1}{s}} u^*(t)dt\Big)^{\frac{r}{q}}\Big(\int_0^{s} ((1/w)^*(t))^{\frac{1}{p-1}}dt\Big)^{\frac{r}{q'}}((1/w)^*(s))^{\frac{1}{p-1}}\,ds\bigg)^{\frac{1}{r}} =: A_2 < \infty \tag{2.1}$$

where $r = \frac{qp}{q-p}$, *then Pitt's inequality*

$$\|\widehat{f}\,u^{\frac{1}{q}}\|_q \leq C_j\|f\,w^{\frac{1}{p}}\|_p, \quad f \in C_0^{\infty}(\mathbb{R}^n), \quad j = 1, 2,$$

holds with $C_j \leq C_{p,q,j}A_j$.

Recall that the non-increasing rearrangement of a measurable radially decreasing function $f(x) = f_0(|x|)$ is defined as follows: let for $\lambda > 0$

$$\mu_f(\lambda) = \mu\{x\colon |f(x)| > \lambda\} = \mu\{x\colon |x| < f_0^{-1}(\lambda)\} = (f_0^{-1}(\lambda))^n V_n,$$

where V_n is the volume of the unit ball $B^n = \{x \in \mathbb{R}^n\colon |x| \leq 1\}$. Then for $t > 0$

$$f^*(t) = \inf\{\lambda > 0\colon \mu_f(\lambda) < t\} = f_0((t/V_n)^{\frac{1}{n}}).$$

Note that the conditions on u and w are also necessary when u and w are radial, i.e., $u = u_0(|x|)$ and $w(x) = w_0(|x|)$, with $u_0(r)$ non-increasing and $w_0(r)$ non-decreasing. See [26] and also [12, Theorem 1.2] for simpler and more general necessary conditions on the weight u and w. We should also mention [38, Theorem 2.1] where a necessary condition similar to that in [26], with u replaced by a measure $d\mu$, was proved.

We also need the following

Lemma 2.2 *Let* $\psi \not\equiv 0$ *be a non-increasing non-negative function; let* $\beta_1, \beta_2 > 0$ *and let* $\beta_2' = \min(\beta_2, 1)$. *If either*

$$A = \sup_{s>0} s^{(-\beta_1,-\beta_2)}\int_0^s \psi(t)\,dt < \infty,$$

or

$$B = \sup_{s>0} s^{(1-\beta_1, 1-\beta_2')}\psi(s) < \infty,$$

then $\beta_1 \le 1$ *and* $A \asymp B$.

Proof Assume $A < \infty$; then, for every $s > 0$, we have that $\int_0^s \psi(t)\,dt \le As^{(\beta_1,\beta_2)}$. Since ψ is non-increasing, $s\psi(s) \le \int_0^s \psi(t)\,dt$, so $\psi(s) \le As^{(\beta_1-1,\beta_2-1)}$. If $\beta_1 > 1$, then $\lim\limits_{s\to 0^+} \psi(s) = 0$ and consequently $\psi \equiv 0$; since we assumed $\psi \not\equiv 0$, necessarily $\beta_1 \le 1$.

Furthermore, from $\psi(s) \le \psi(1)$ for $s \ge 1$ we can see at once that $\psi(s) \lesssim As^{\beta_2'-1}$ and so $B \lesssim A$.

If we assume $B < \infty$, for every $s > 0$ we have that $\psi(s) \le Bs^{(\beta_1-1,\beta_2'-1)}$ As above we conclude that $\beta_1 \le 1$. For $0 < s \le 1$ we have $\int_0^s \psi(t)\,dt \lesssim Bs^{\beta_1}$. If $s \ge 1$, then

$$\int_0^s \psi(t)\,dt = \int_0^1 \psi(t)\,dt + \int_1^s \psi(t)\,dt \lesssim B + B\int_1^s t^{\beta_2'-1}\,dt \lesssim Bs^{\beta_2'} \le Bs^{\beta_2}.$$

Thus, $\sup_{s\ge 1} s^{-\beta_2}\int_0^s \psi(t)\,dt \lesssim B$ and $A \lesssim B$. □

2.2 Proof of Theorem 1.1

We can assume $\ell(x) = \langle \mathbf{a}, x\rangle$, $|\mathbf{a}| = 1$, without loss of generality.

(a) Let $p \le \gamma \le q$.

Step 1 For $\tau \ge 0$ and $\xi \in \mathbb{R}^n$, define

$$w_\tau(\xi) = |\xi - i\tau\mathbf{a}|^\gamma = (|\xi|^2 + \tau^2)^{\frac{\gamma}{2}}.$$

By Theorem 2.1(a), the inequality

$$\Big(\int_{\mathbb{R}^n} |\widehat{g}(x)|^q u(x)\,dx\Big)^{\frac{1}{q}} \lesssim A_{u,w_\tau}\Big(\int_{\mathbb{R}^n} w_\tau(\xi)|g(\xi)|^\gamma\,d\xi\Big)^{\frac{1}{\gamma}} \tag{2.2}$$

holds with

$$A_{u,w_\tau} = \sup_{s>0}\Big(\int_0^s u^*(t)\,dt\Big)^{\frac{1}{q}}\Big(\int_0^{\frac{1}{s}} ((1/w_\tau)^*(t))^{\frac{1}{\gamma-1}}\,dt\Big)^{\frac{1}{\gamma'}} < \infty.$$

The weight w_τ is radially increasing, so

$$(1/w_\tau)^*(t) = ((t/V_n)^{\frac{2}{n}} + \tau^2)^{-\frac{\gamma}{2}} \asymp (t + \tau^n)^{-\frac{\gamma}{n}}$$

with the constant of equivalence independent of τ. This implies

$$\int_0^{\frac{1}{s}} ((1/w_\tau)^*(t))^{\frac{1}{\gamma-1}}\,dt \asymp \int_0^{\frac{1}{s}} (t+\tau^n)^{-\frac{\gamma'}{n}}\,dt, \quad s > 0.$$

Therefore, for $\tau \ge 0$,

$$A_{u,w_\tau}^q \asymp \sup_{s>0}\int_0^s u^*(t)\,dt\Big(\int_0^{\frac{1}{s}} (t+\tau^n)^{-\frac{\gamma'}{n}}\,dt\Big)^{\frac{q}{\gamma'}} = A_u^q(\tau). \tag{2.3}$$

Since $(t+\tau^n)^{-1} \le \max(\tau^{-n}, 1)(t+1)^{-1}$ for $t, \tau > 0$, from (2.3) we conclude that

$$A_u(\tau) \le \max(\tau^{-1}, 1)A_u(1), \quad \tau > 0.$$

We can give a simple expression for $A_u^q(0)$. Observing that $I := \int_0^{1/s} t^{-\frac{\gamma'}{n}}\,dt$ is finite when $-\frac{\gamma'}{n} > -1$ or, equivalently, $\frac{n}{n-1} < \gamma$, we have that $I \asymp s^{\frac{\gamma'}{n}-1}$. Therefore, (2.3) can be rewritten as

$$A_u^q(0) \asymp \sup_{s>0} s^{-q(\frac{1}{\gamma'}-\frac{1}{n})} \int_0^s u^*(t)\,dt. \tag{2.4}$$

By (2.4) and Lemma 2.2 with $\beta_1 = \beta_2 = q(\frac{1}{\gamma'} - \frac{1}{n})$, there holds that $q(\frac{1}{\gamma'} - \frac{1}{n}) \le 1$ or $\frac{1}{\gamma'} \le \frac{1}{n} + \frac{1}{q}$ and we can redefine $A_u^q(0)$ as follows.

$$A_u^q(0) = \sup_{s>0} s^{-q(\frac{1}{\gamma'}-\frac{1}{n})} \int_0^s u^*(t)\,dt \asymp \sup_{s>0} s^{1-q(\frac{1}{\gamma'}-\frac{1}{n})} u^*(s).$$

Step 2 Let $g(x) = e^{-\langle \tau \mathbf{a}, x\rangle} f(x)$. Then $g \in C_0^\infty(\mathbb{R}^n)$ and

$$\widehat{g}(\xi) = \int_{\mathbb{R}^n} g(x) e^{-i\langle \xi, x\rangle}\,dx = \int_{\mathbb{R}^n} f(x) e^{-i\langle \xi, x\rangle - \langle \tau\mathbf{a}, x\rangle}\,dx = \widehat{f}(\xi - i\tau\mathbf{a}). \tag{2.5}$$

Since for $g \in C_0^\infty(\mathbb{R}^n)$ the Fourier inversion formula holds, (2.2) and (2.3) imply

$$\begin{aligned}\Big(\int_{\mathbb{R}^n} |g(x)|^q u(x)\,dx\Big)^{\frac{1}{q}} &\lesssim A_u(\tau)\Big(\int_{\mathbb{R}^n} |\xi - i\tau\mathbf{a}|^\gamma |\widehat{g}(\xi)|^\gamma\,d\xi\Big)^{\frac{1}{\gamma}}\\ &= A_u(\tau)\Big(\int_{\mathbb{R}^n} \big|(\xi - i\tau\mathbf{a})\widehat{f}(\xi - i\tau\mathbf{a})\big|^\gamma\,d\xi\Big)^{\frac{1}{\gamma}}. \end{aligned} \tag{2.6}$$

Note that $\widehat{f}$ is entire analytic (and so it is defined at $\xi - i\tau\mathbf{a}$) because f has compact support. Since $\widehat{\nabla f}(\xi) = i\xi \widehat{f}(\xi)$, from (2.5) with $h(x) = (h_1(x), \ldots, h_n(x)) = e^{-\langle \tau\mathbf{a}, x\rangle} \nabla f(x)$ we get

$$\widehat{h}(\xi) = \widehat{\nabla f}(\xi - i\tau\mathbf{a}) = i(\xi - i\tau\mathbf{a})\widehat{f}(\xi - i\tau\mathbf{a}).$$

Hence

$$\begin{aligned}&\Big(\int_{\mathbb{R}^n} \big|(\xi - i\tau\mathbf{a})\widehat{f}(\xi - i\tau\mathbf{a})\big|^\gamma\,d\xi\Big)^{\frac{1}{\gamma}}\\ &= \Big(\int_{\mathbb{R}^n} |\widehat{h}(\xi)|^\gamma\,d\xi\Big)^{\frac{1}{\gamma}} = \Big(\int_{\mathbb{R}^n} \Big(\sum_{j=1}^n |\widehat{h}_j(\xi)|^2\Big)^{\frac{\gamma}{2}}\,d\xi\Big)^{\frac{1}{\gamma}}\\ &\le \Big(\int_{\mathbb{R}^n} \Big(\sum_{j=1}^n |\widehat{h}_j(\xi)|\Big)^\gamma\,d\xi\Big)^{\frac{1}{\gamma}} \le \sum_{j=1}^n \Big(\int_{\mathbb{R}^n} |\widehat{h}_j(\xi)|^\gamma\,d\xi\Big)^{\frac{1}{\gamma}},\end{aligned}$$

where the first inequality holds trivially and the second is Minkowski's inequality.

Let us use Pitt's inequality with $p \le \gamma$:

$$\|\widehat{f}\|_\gamma \le C_{p,\gamma} A_{1,v} \|f\, v^{\frac{1}{p}}\|_p, \tag{2.7}$$

where

$$A_{1,v} := \sup_{s>0} \Big(\int_0^{\frac{1}{s}} dt\Big)^{\frac{1}{\gamma}} \Big(\int_0^s (1/v)^*(t)^{\frac{1}{p-1}}\,dt\Big)^{\frac{1}{p'}} < \infty. \tag{2.8}$$

As in Step 1, we apply Lemma 2.2 with $\beta_1 = \beta_2 = \frac{p'}{\gamma}$. We obtain $\frac{p'}{\gamma} \le 1$ and

$$A_{1,v}^{p'} \asymp \sup_{s>0} s^{-\frac{p'}{\gamma}} \int_0^s (1/v)^*(t)^{\frac{1}{p-1}}\, dt \asymp \sup_{s>0} s^{1-\frac{p'}{\gamma}} (1/v)^*(s)^{\frac{1}{p-1}}.$$

It follows that

$$A_{1,v}^{p} = A_{1,v}^{p'(p-1)} \asymp \sup_{s>0} s^{\frac{p}{\gamma'}-1}(1/v)^*(s) = A_v^p < \infty.$$

Applying (2.7) with f replaced by h_j, $j = 1, \dots, n$, we gather

$$\begin{aligned}\Big(\int_{\mathbb{R}^n} |\widehat{h}_j(\xi)|^\gamma\, d\xi\Big)^{\frac{1}{\gamma}} &\lesssim A_v\Big(\int_{\mathbb{R}^n} |h_j(x)|^p v(x)\, dx\Big)^{\frac{1}{p}}\\ &\lesssim A_v\Big(\int_{\mathbb{R}^n}\Big(\sum_{k=1}^n |h_k(x)|^2\Big)^{\frac{p}{2}} v(x)\, dx\Big)^{\frac{1}{p}}\\ &= A_v\Big(\int_{\mathbb{R}^n} |e^{-\langle \tau\mathbf{a}, x\rangle}\nabla f(x)|^p v(x)\, dx\Big)^{\frac{1}{p}}.\end{aligned} \tag{2.9}$$

This, together with (2.6) proves part (a) of the theorem.

(b) Let $1 < \gamma \le q < p$. We proceed as in the proof of part (a) to obtain (2.2), provided that (2.3) holds. We note that we assume $\frac{n}{n-1} < \gamma$ when $\tau = 0$.

Analogously, we get (2.9), but instead of (2.8) we use (2.1) with $u = 1$, $w = v$ and $\gamma < p$. Then we have

$$\begin{aligned}A_{1,v}^r &= \int_0^\infty s^{-\frac{r}{\gamma}}\Big(\int_0^s (1/v)^*(t)^{\frac{1}{p-1}}\, dt\Big)^{\frac{r}{\gamma'}} (1/v)^*(s)^{\frac{1}{p-1}}\, ds\\ &\asymp \int_0^\infty s^{-\frac{r}{\gamma}} \frac{d}{ds}\Big(\int_0^s (1/v)^*(t)^{\frac{1}{p-1}}\, dt\Big)^{\frac{r}{\gamma'}+1} ds,\end{aligned}$$

where $\frac{1}{r} = \frac{1}{\gamma} - \frac{1}{p}$. After integrating by parts, we get

$$A_{1,v}^r \asymp \int_0^\infty s^{-\frac{r}{\gamma}-1}\Big(\int_0^s (1/v)^*(t)^{\frac{1}{p-1}}\, dt\Big)^{\frac{r}{p'}} ds = \tilde{A}_v^r < \infty.$$

This proves part (b) of the theorem.

2.3 Corollaries and remarks

Let us first discuss the conditions on γ in Theorem 1.1. We recall that in part (a) of Theorem 1.1 we assume $1 < p \le q < \infty$ and $\max(p, p') \le \gamma \le q$; when $\tau = 0$ we assume also $\frac{1}{n} < \frac{1}{\gamma'}$.

Note that this extra assumption on γ is not necessary when $n \ge 3$. Indeed, from $\max(p, p') \le \gamma \le q$ follows that $2 \le \gamma \le q$ and $q' \le \gamma' \le 2$; thus, $\frac{1}{n} < \frac{1}{\gamma'}$ whenever $n \ge 3$.

When $n = 1$, the inequality $\frac{1}{n} < \frac{1}{\gamma'}$ (or: $\gamma > \frac{n}{n-1}$) can never be satisfied by γ' and only the case $\frac{1}{n} \ge \frac{1}{\gamma'}$ is possible. In fact, the condition $\max(p, p') \le \gamma \le q$ always implies $\frac{1}{2} \le \frac{1}{\gamma'} < 1$.

If $n = 2$ we can either have $\frac{1}{n} < \frac{1}{\gamma'}$ or $\frac{1}{n} \ge \frac{1}{\gamma'}$. Note that $\frac{1}{2} \ge \frac{1}{\gamma'}$ implies that $p = \gamma = 2$.

For applications, it is important to simplify the expression for $A_u^q(1)$ in (1.5). Recall that, when $\tau > 0$, $A_u(\tau) \le \max(\tau^{-1}, 1)A_u(1)$ (see Remark 1.2). We prove the following

Corollary 2.3 *Let* $1 < p \le q < \infty$ *and let* $\max(p, p') \le \gamma \le q$.

(i) *If* $n \ge 2$ *and* $\frac{1}{n} < \frac{1}{\gamma'}$, *then* $\frac{1}{\gamma'} \le \frac{1}{n} + \frac{1}{q}$ *and*

$$A_u^q(1) \asymp \sup_{s>0} s^{(1-q(\frac{1}{\gamma'}-\frac{1}{n}),\,0)} u^*(s).$$

(ii) *If* $n = 2$ *and* $p = \gamma = 2$, *then*

$$A_u^q(1) = \sup_{s>0} \left(\ln(s^{-1}+1)\right)^{q/2} \int_0^s u^*(t)\,dt.$$

(iii) *If* $n = 1$, *then*

$$A_u^q(1) \asymp \sup_{s>0} s^{(0,-\frac{q}{\gamma'})} \int_0^s u^*(t)\,dt.$$

Proof (i) Recall that

$$A_u^q(\tau) = \sup_{s>0} \int_0^s u^*(t)\,dt \Big(\int_0^{\frac{1}{s}} (t+\tau^n)^{-\frac{\gamma'}{n}}\,dt\Big)^{\frac{q}{\gamma'}}, \quad \tau > 0.$$

For $\tau = 1$ and $\frac{1}{n} < \frac{1}{\gamma'}$, we have

$$\Big(\int_0^{\frac{1}{s}} (t+1)^{-\frac{\gamma'}{n}}\,dt\Big)^{\frac{q}{\gamma'}} \asymp \big((s^{-1}+1)^{1-\frac{\gamma'}{n}} - 1\big)^{\frac{q}{\gamma'}}$$
$$\asymp s^{(-q(\frac{1}{\gamma'}-\frac{1}{n}),\,-\frac{q}{\gamma'})}, \quad s > 0.$$

Hence

$$A_u^q(1) \asymp \sup_{s>0} s^{(-q(\frac{1}{\gamma'}-\frac{1}{n}),\,-\frac{q}{\gamma'})} \int_0^s u^*(t)\,dt,$$

where $q(\frac{1}{\gamma'} - \frac{1}{n}) > 0$ and $\frac{q}{\gamma'} \ge 1$, since $\gamma' \le 2 \le q$.

Now we can apply Lemma 2.2 with $\beta_1 = q(\frac{1}{\gamma'} - \frac{1}{n})$, $\beta_2 = \frac{q}{\gamma'} \ge 1$. We obtain $q(\frac{1}{\gamma'} - \frac{1}{n}) \ge 1$ or $\frac{1}{\gamma'} \le \frac{1}{n} + \frac{1}{q}$ and

$$A_u^q(1) \asymp \sup_{s>0} s^{(1-q(\frac{1}{\gamma'}-\frac{1}{n}),\,0)} u^*(s).$$

Part (ii) is obvious. To prove part (iii), we note that $\gamma' > 1$, which gives $\int_0^{\frac{1}{s}} (t+1)^{-\gamma'}\,dt \asymp s^{(0,-1)}$. □

Proof of Corollary 1.2 Recall that in this corollary $u(x) = |x|^{(-\alpha_1,-\alpha_2)}$, $v(x) = |x|^{(\beta_1,\beta_2)}$ with $\alpha_j, \beta_j \ge 0$. We consider the case when $1 < p \le q < \infty$ and $\gamma \in [\max(p, p'), q]$, with $\frac{1}{n} < \frac{1}{\gamma'} \le \frac{1}{n} + \frac{1}{q}$, and we let $\tau = 0$ or $\tau = 1$.

Since $w^*(s) \asymp w_0(s^{\frac{1}{n}})$, $s > 0$, for any non-increasing radial weight function $w(x) = w_0(|x|)$ we have

$$u^*(s) \asymp s^{(-\frac{\alpha_1}{n},-\frac{\alpha_2}{n})}, \quad (1/v)^*(s) \asymp s^{(-\frac{\beta_1}{n},-\frac{\beta_2}{n})}.$$

whenever $\alpha_j, \beta_j \ge 0$.

The expression (1.5), Corollary 2.3(i), and (1.6) imply that for $s > 0$

$$u^*(s) \lesssim \begin{cases} s^{q(\frac{1}{\gamma'}-\frac{1}{n})-1}, & \tau = 0, \\ s^{(q(\frac{1}{\gamma'}-\frac{1}{n})-1,0)}, & \tau = 1, \end{cases} \qquad (1/v)^*(s) \lesssim s^{1-\frac{p}{\gamma'}}.$$

It is easy to see that when $a_j, b_j \geq 0$, the inequality $s^{(-a_1,-a_2)} \lesssim s^{(-b_1,-b_2)}$, holds if and only if $a_1 \leq b_1$, $a_2 \geq b_2$. It follows that

$$\alpha_1 \leq n\Big(1 - \frac{q}{\gamma'} + \frac{q}{n}\Big), \quad \begin{cases} \alpha_2 \geq n\big(1 - \frac{q}{\gamma'} + \frac{q}{n}\big), & \tau = 0, \\ \alpha_2 \geq 0, & \tau = 1, \end{cases}$$

and

$$0 \leq \beta_1 \leq n\Big(\frac{p}{\gamma'} - 1\Big), \quad \beta_2 \geq n\Big(\frac{p}{\gamma'} - 1\Big)$$

which proves (1.14) and (1.15).

To prove (1.16) we use a standard homogeneity argument. Let us consider (1.12) (which by Remark 1.4 is equivalent to (1.7)) with $f = f_\lambda(x) = f(\lambda x)$ for some $f \in C_0^\infty(\mathbb{R}^n)$ and $\lambda > 0$. We obtain

$$\||x|^{-\frac{\alpha}{q}} f_\lambda\|_q \leq c_\tau \||x|^{\frac{\beta}{p}} (\tau \mathbf{a} f_\lambda + \lambda(\nabla f)_\lambda\|_p.$$

After the change of variables $x \mapsto \lambda^{-1}x$, we get

$$\lambda^{\frac{\alpha}{q}-\frac{n}{q}+\frac{\beta}{p}+\frac{n}{p}-1} \||x|^{-\frac{\alpha}{q}} f\|_q \leq c_\tau \||x|^{\frac{\beta}{p}} (\lambda^{-1}\tau \mathbf{a} f + \nabla f)\|_p. \tag{2.10}$$

The limits of the two sides of the inequality (2.10), as $\lambda \to 0$ or as $\lambda \to \infty$, must be the same. If $\tau = 0$ the right-hand side of (2.10) does not depend on λ, so we must have $\frac{\alpha}{q} - \frac{n}{q} + \frac{\beta}{p} + \frac{n}{p} - 1 = 0$.

If $\tau > 0$, we must have

$$\lambda^{\frac{\alpha}{q}-\frac{n}{q}+\frac{\beta}{p}+\frac{n}{p}-1} \lesssim \begin{cases} \lambda^{-1}, & \lambda \to 0, \\ 1, & \lambda \to \infty. \end{cases}$$

so necessarily $\frac{\alpha}{q} - \frac{n}{q} + \frac{\beta}{p} + \frac{n}{p} - 1 \leq 0$. □

3 Uniqueness problems

In this section and in Sect. 4 we use the inequality (1.1) to prove uniqueness questions for solutions of partial differential equations and systems. First, we state some definitions and preliminary results.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a vector with non-negative integer components; we use the notation $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial_x^\alpha f = \frac{\partial^{\alpha_1}}{\partial_{x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial_{x_n}^{\alpha_n}} f$.

Let $D \subset \mathbb{R}^n$ open and connected and let $1 \leq p < \infty$. Recall that $W_0^{m,p}(D)$ is the closure of $C_0^\infty(D)$ with respect to the Sobolev norm $\|f\|_{W_0^{m,p}(D)} = \sum_{|\alpha|=0}^m \|\partial_x^\alpha f\|_p$, When $m = 1$, and D is bounded in at least one direction, the classical Poincare' inequality states that $\|f\|_{L^p(D)} \leq C\|\nabla f\|_{L^p(D)}$ (see e.g. [2]); thus, the Sobolev norm in $W_0^{1,p}(D)$ is equivalent to $\|\nabla f\|_{L^p(D)}$.

Given the weight $v\colon D \to [0,\infty]$ and $1 \le p < \infty$, we let $W_0^{m,p,v}(D)$ be the closure of $C_0^\infty(D)$ with respect to the norm $\|f\|_{W_0^{1,p,v}(D)} = \sum_{|\alpha|=0}^m \|v^{\frac{1}{p}} \partial_x^\alpha f\|_p$. We use the standard notation $L^{p,v}(D)$ or $L^p(D,\, v\,dx)$ for the closure of $C_0^\infty(D)$ with respect to the norm $\|v^{\frac{1}{p}} f\|_p$.

Let $P(\partial) = \sum_{|\alpha|=0}^m a_\alpha \partial_x^\alpha$ be a linear partial differential operator of order $m > 0$ with complex constant coefficients. We let $P(-\partial)u = \sum_{|\alpha|=0}^m \overline{a_\alpha}\,(-1)^{|\alpha|}\partial_x^\alpha u$.

A *weak solution* (or: a *solution in distribution sense*) of the equation $P(\partial)f = 0$ on a domain $D \subset \mathbb{R}^n$ is a distribution $f \in W^{m,p}(D)$ that satisfies $\int_D f(x)\,P(-\partial)\phi(x)\,dx = 0$ for every $\phi \in C_0^\infty(D)$. Weak solutions for non linear partial differential operators can be defined on a case-by-case basis. See e.g. [18] or other standard textbooks on partial differential equations for details. We will often consider differential inequalities in the form of $|P(\partial)f| \le |Vf|$ on a given domain D; by that we mean that the inequality $|P(\partial)f(x)| \le |Vf(x)|$ is satisfied a.e. in D, i.e., it is satisfied pointwise with the possible exception of a set of measure zero.

3.1 Unique continuation and Carleman method

Let $P(\partial)$ be a homogeneous partial differential operator of order $m \ge 1$. Clearly, $f \equiv 0$ is a solution of the equation $P(\partial)f = 0$ on any domain $D \subset \mathbb{R}^n$. It is natural to ask whether this equation has also nontrivial solutions, i.e., distributions in some suitable Sobolev space that satisfy the equation in distribution sense and are not identically $= 0$. In particular it is natural to ask whether (1), (2) or (3) below are satisfied or not on a given domain D.

(1) *Uniqueness for the Dirichlet problem.* The only solution of the Dirichlet problem $\begin{cases} P(\partial)f = 0, \\ f \in W_0^{m,p}(D) \end{cases}$ is $f \equiv 0$.

(2) *Weak unique continuation property* (or: unique continuation from an open set). Every solution of the equation $P(\partial)f = 0$ which is $\equiv 0$ on an open subset of D is $\equiv 0$.

(3) *Strong continuation property* (or: unique continuation from a point). Let $x_0 \in D$. Every solution of the equation $P(\partial)f = 0$ that satisfies

$$\lim_{r\to 0} r^{-N} \int_{|x-x_0|<r} |f(x)|^2\,dx = 0$$

for every $N > 0$ is $\equiv 0$.

For other relevant unique continuation problems see the survey paper [55].

Historically, the study of unique continuation originated from the uniqueness for the Cauchy problem; an equally strong motivation arose from some fundamental questions in mathematical physics, with the study of the eigenvalues of the time-independent Schrödinger operator $H = -\Delta + V$ as a notable example. See [50,51] and also [36] and the references cited there.

Carleman [4] a new weighted Sobolev inequality to show that the Schrödinger operator $H = -\Delta + V$ has the strong unique continuation property when $n = 2$ and V is bounded. Carleman's original idea has permeated the large majority of results on unique continuation. The weighted Sobolev inequality that he used in his proof has been widely generalized and applied to a vast array of problems in unique continuation and control theory.

A *Carleman-type inequality* for a differential operator $P(\partial)$ is a weighted inequality of the form of

$$\|\eta^{\tau_k} f\|_q \le C\|\eta^{\tau_k} P(\partial) f\|_p, \qquad f \in C_0^\infty(D), \tag{3.1}$$

where $\eta\colon D \to [0, 1)$, the sequence $\{\tau_k\}_{k\in\mathbb{N}} \subset (0, \infty)$ increases to $+\infty$, the constant C is independent of the sequence of the τ_k and of f, and $1 \le p \le q < \infty$. If (3.1) holds with a suitable function η, a version of the argument used in the proof of Theorem 1.3 can be applied to show that the operator $Q(\partial) = P(\partial) - V(x)$ has the unique continuation property (2) or (3) (or some variation of these properties) whenever $V \in L^{\frac{pq}{q-p}}(D)$.

The literature on Carleman inequalities and unique continuation is very extensive. A sample of references on unique continuation problems for second order elliptic operators include the important [30,31,35,52] and the survey papers [37,55,57].

The inequality (1.7) in Theorem 1.1 can be viewed as a weighted Carleman-type inequality for the operator $P(\partial)f = |\nabla f|$. To the best of our knowledge, the inequality (1.7) is new in the literature, even when $u(x) \asymp v(x) \asymp 1$.

3.2 Proof of Theorem 1.3

In this section we prove Theorem 1.3 and some corollary.

Proof of Theorem 1.3 Assume for simplicity that $f \equiv 0$ when $x_n < 0$ (the proof is similar in the general case). It is enough to show that $f \equiv 0$ also on the strip $S_\epsilon = \{x\colon 0 < x_n < \epsilon\}$, where $\epsilon > 0$ will be determined during the proof. Using Theorem 1.1(a) with $\mathbf{a} = (0, \dots, 0, 1)$, $\tau \ge 1$ and $c_\tau \le c_1$ (see Remark 1.2), the differential inequality (1.18) and Hölder's inequality with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, we can write the following chain of inequalities:

$$\begin{aligned}
\|e^{-\tau x_n} f u^{\frac{1}{q}}\|_{L^q(S^\epsilon)} &\le c_1\|e^{-\tau x_n}\nabla f v^{\frac{1}{p}}\|_{L^p(\mathbb{R}^n)}\\
&\le c_1\|e^{-\tau x_n}\nabla f v^{\frac{1}{p}}\|_{L^p(S_\epsilon)} + c_1\|e^{-\tau x_n}\nabla f v^{\frac{1}{p}}\|_{L^p(\{x_n>\epsilon\})}\\
&\le c_1\|e^{-\tau x_n} f V v^{\frac{1}{p}}\|_{L^p(S_\epsilon)} + c_1 e^{-\tau\epsilon}\|\nabla f v^{\frac{1}{p}}\|_{L^p(\{x_n>\epsilon\})}\\
&\le c_1\|V v^{\frac{1}{p}} u^{-\frac{1}{q}}\|_{L^r(S_\epsilon\cap \operatorname{supp} f)}\|e^{-\tau x_n} f u^{\frac{1}{q}}\|_{L^q(S^\epsilon)} + C' e^{-\tau\epsilon}.
\end{aligned}$$

Here, $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ and we have let $C' = c_1\|\nabla f v^{\frac{1}{p}}\|_{L^p(\{x_n>\epsilon\})}$. Note that C' does not depend on τ.

Since $V \in L^r(\operatorname{supp} f,\ v^{\frac{r}{p}} u^{-\frac{r}{q}}\,dx)$ we can chose $\epsilon > 0$ so that $c_1\|V v^{\frac{1}{p}} u^{-\frac{1}{q}}\|_{L^r(S_\epsilon\cap \operatorname{supp} f)} < \frac{1}{2}$. From the chain of inequalities above, follows that

$$\|e^{-\tau x_n} f u^{\frac{1}{q}}\|_{L^q(S^\epsilon)} \le \frac{1}{2}\|e^{-\tau x_n} f u^{\frac{1}{q}}\|_{L^q(S^\epsilon)} + C' e^{-\tau\epsilon}.$$

We gather

$$\frac{1}{2}\|e^{\tau(\epsilon - x_n)} f u^{\frac{1}{q}}\|_{L^q(S^\epsilon)} \le C'.$$

Since $\epsilon - x_n > 0$ on S^ϵ, if $f \not\equiv 0$ the left-hand side of this inequality goes to infinity when τ goes to infinity; this is a contradiction because C' does not depend on τ and so necessarily $f \equiv 0$ in S_ϵ. □

Corollary 3.1 *Let p, q and γ be as in Theorem 1.1(a). Let $u = |x|^{(-\alpha_1,-\alpha_2)}$ and $v = |x|^{(\beta_1,\beta_2)}$, with $0 \le \alpha_1 \le n\big(1 - \frac{q}{\gamma'} + \frac{q}{n}\big)$, $\alpha_2 \ge 0$ and $0 \le \beta_1 \le n\big(\frac{p}{\gamma'} - 1\big)$, $\beta_2 \ge n\big(\frac{p}{\gamma'} - 1\big)$. Let $V = |x|^{(s_1,s_2)}$, with*

$$s_1 > -\frac{n}{r} - \frac{\alpha_1}{q} - \frac{\beta_1}{p}, \tag{3.2}$$

and, if supp f *is unbounded,*

$$s_2 < -\frac{\alpha_2}{q} - \frac{\beta_2}{p} - \frac{n}{r}. \tag{3.3}$$

Then, every solution of the differential inequality $|\nabla f| \le V|f|$ is $\equiv 0$.

Proof The weights u and v are as in Corollary 1.2, so the inequality (1.7) holds with $\tau > 0$. By Theorem 1.3, every solution of the differential inequality $|\nabla f| \le V|f|$ is $\equiv 0$ whenever $V v^{\frac{1}{p}} u^{-\frac{1}{q}} \in L^r(\operatorname{supp} f)$. We can see at once that $V v^{\frac{1}{p}} u^{-\frac{1}{q}} = |x|^{(t_1,t_2)} \in L^r(\operatorname{supp} f)$ if and only if $t_1 = s_1 + \frac{\alpha_1}{q} + \frac{\beta_1}{p} > -\frac{n}{r}$ and, if supp f is unbounded, $t_2 = s_2 + \frac{\alpha_2}{q} + \frac{\beta_2}{p} < -\frac{n}{r}$, which is equivalent to (3.2) and (3.3). This concludes the proof. □

Remark 3.1 From the inequalities above and the assumptions on α_j, β_j, and γ' (see Corollary 1.2) follows that

$$t_1 \le s_1 + \frac{n}{q}\Big(1 - \frac{q}{\gamma'} + \frac{q}{n}\Big) + \frac{n}{p}\Big(\frac{p}{\gamma'} - 1\Big) = s_1 - \frac{n}{r} + 1.$$
$$t_2 \ge s_2 + \frac{n}{p}\Big(\frac{p}{\gamma'} - 1\Big) = s_2 + \frac{n}{\gamma'} - \frac{n}{p} > s_2 - \frac{n}{p} + 1.$$

The condition $t_1 > -\frac{n}{r}$ yields $s_1 > -1$. We can see at once that $t_2 < -\frac{n}{r}$ yields $s_2 < \frac{n}{q} - 1$. In particular, $V = |x|^{-1+\epsilon}$ with $0 < \epsilon < \frac{n}{q}$, satisfies the assumptions of Corollary 3.1. If f has compact support, then we can omit the condition on t_2 and assume only $\epsilon > 0$.

Potentials $V(x) = C|x|^{-s}$, with $s,\ C > 0$ are known as *Hardy potentials* in the literature. They appear in the relativistic Schrödinger equations and in problem of stability of relativistic matter in magnetic fields. See e.g. [27] and the introduction to [20,21], just to cite a few.

It is proved in [16] that when $\mathcal{L}$ is the Dirac operator in dimension $n \ge 2$ (see Sect. 4.2) the differential inequality $|\mathcal{L}f| \le C|x|^{-1}|f|$ has the strong unique continuation property from the point $x_0 = 0$ whenever $C \le 1$. We conjecture that also the differential inequalities $|\nabla f| \le C|x|^{-1}|f|$ has the strong unique continuation property from the origin when C is sufficiently small.

3.3 Proof of Theorem 1.4

Recall that the solution f of the Dirichlet problem (1.19) is intended in distribution sense, i.e., f satisfies

$$\int_D \langle \nabla\psi, \nabla f\rangle |\nabla f|^{p-2}\, v\, dx = \int_D \psi\, V f |f|^{p-2}\, v\, dx \tag{3.4}$$

for every $\psi \in C_0^\infty(D)$. To prove Theorem 1.4 we need two important lemmas:

Lemma 3.2 *Suppose that the weighted gradient inequality*

$$\|u^{\frac{1}{q}} f\|_q \le c_0 \|v^{\frac{1}{p}} \nabla f\|_p, \quad f \in C_0^\infty(D) \tag{3.5}$$

holds with exponents $1 \leq p,\ q < \infty$. *Then the space* $W_0^{1,p,v}(D)$ *embeds into* $L^q(D, u\,dx)$ *and* $\|f\|_{L^q(D,\,u\,dx)} \leq c_0\|\nabla f\|_{L^p(D,\,v\,dx)}$.

Proof Fix $f \in W_0^{1,p,v}(D)$; let $\{f_n\}_{n\in\mathbb{N}} \subset C_0^\infty(D)$ be a sequence that converges to f in the Sobolev norm $\|\cdot\|_{W_0^{1,p,v}(D)}$. Thus, $\{f_n\}$ is a Cauchy sequence in $W_0^{1,p,v}(D)$; for every $\epsilon > 0$ we can chose $N > 0$ such that

$$\|f_n - f_m\|_{W^{1,p,v}(D)} = \|v^{\frac{1}{p}}(f_n - f_m)\|_{L^p(D)} + \|v^{\frac{1}{p}}\nabla(f_n - f_m)\|_{L^p(D)} < \epsilon$$

whenever $n,\ m > N$; thus, $\|v^{\frac{1}{p}}\nabla(f_n - f_m)\|_{L^p(D)} < \epsilon$. By (3.5),

$$\|u^{\frac{1}{q}}(f_n - f_m)\|_{L^q(D)} \leq c_0\|v^{\frac{1}{p}}\nabla(f_n - f_m)\|_{L^p(D)} < c_0\epsilon.$$

We have proved that $\{f_n\}$ is a Cauchy sequence in $L^q(D,\ u\,dx)$ (which is complete) and so it converges to f also in $L^q(D,\ u\,dx)$. We gather

$$\begin{aligned}\|f\|_{L^q(D,\,u\,dx)} &= \lim_{n\to\infty}\|f_n\|_{L^q(D,\,u\,dx)} \leq c_0 \lim_{n\to\infty}\|\nabla f_n\|_{L^p(D,\,v\,dx)}\\ &= c_0\|\nabla f\|_{L^p(D,\,v\,dx)}\end{aligned}$$

as required. □

Lemma 3.3 *Suppose that the weighted gradient inequality* (3.5) *holds with* $1 < p < q$. *Let* f *be a solution to the Dirichlet problem* (1.19), *with* $|V|^{\frac{1}{p}} \in L^r(D,\ v^{\frac{r}{p}}u^{-\frac{r}{q}}\,dx)$. *We have*

$$\int_D |\nabla f|^p v\,dx = \int_D V|f|^p v\,dx.$$

Proof Let $\{\psi_n\}$ be a sequence of functions in $C_0^\infty(D)$ that converges to $\overline{f}$, the complex conjugate of f, in $W_0^{1,p,v}(D)$. We show first that $\lim_{n\to\infty}\int_D\langle\nabla\psi_n, \nabla f\rangle|\nabla f|^{p-2}v\,dx = \int_D|\nabla f|^p v\,dx$. Indeed,

$$\begin{aligned}&\int_D\big(\langle\nabla\psi_n, \nabla f\rangle|\nabla f|^{p-2} - |\nabla f|^p\big)v\,dx\\ &\quad= \int_D\big(\langle\nabla\psi_n, \nabla f\rangle|\nabla f|^{p-2} - \langle\nabla\overline{f}, \nabla f\rangle|\nabla f|^{p-2}\big)v\,dx\\ &\quad= \int_D\langle\nabla\psi_n - \nabla\overline{f},\ \nabla f|\nabla f|^{p-2}\rangle\, v\,dx\\ &\quad\leq \|(\nabla\psi_n - \nabla\overline{f})v^{\frac{1}{p}}\|_p\, \||\nabla f|^{p-1}\, v^{\frac{1}{p'}}\|_{p'}\\ &\quad= \|\nabla(\psi_n - \overline{f})v^{\frac{1}{p}}\|_p\, \|\,|\nabla f|\, v^{\frac{1}{p}}\|_p^{\frac{p}{p'}}\end{aligned}$$

and $\lim_{n\to\infty}\|\nabla(\psi_n - \overline{f})v^{\frac{1}{p}}\|_p = 0$, as required.

In view of (3.4), we have that

$$\int_D\langle\nabla\psi_n, \nabla f\rangle|\nabla f|^{p-2}v\,dx = \int_D \psi_n\, Vf|f|^{p-2}\, v\,dx;$$

to complete the proof it suffices to show that

$$\lim_{n\to\infty}\int_D \psi_n\, Vf|f|^{p-2}\, v\,dx = \int_D V|f|^p\, v\,dx$$

when $|V|^{\frac{1}{p}} \in L^r(D,\ v^{\frac{r}{p}} u^{-\frac{r}{q}} dx)$. By Lemma 3.2, ψ_n converges to $\bar{f}$ in $L^q(D,\ u\,dx)$. Using Holder's inequality with $\frac{p}{r} + \frac{p}{q} = 1$, we gather

$$\begin{aligned}\int_D \big(\psi_n V f |f|^{p-2} - V|f|^p\big)\, v\, dx &\le \int_D |V v u^{-\frac{p}{q}}|\, |f|^{p-1}\, |\psi_n - \bar{f}| u^{\frac{p}{q}}\ dx \\ &\le \Big(\int_D |v V u^{-\frac{p}{q}}|^{\frac{r}{p}}\, dx\Big)^{\frac{p}{r}} \Big(\int_D |f|^{(p-1)\frac{q}{p}} |\psi_n - \bar{f}|^{\frac{q}{p}} u\, dx\Big)^{\frac{p}{q}} \\ &= \Big(\int_D (|V|^{\frac{1}{p}} v^{\frac{1}{p}} u^{-\frac{1}{q}})^r\, dx\Big)^{\frac{p}{r}} \Big(\int_D |f|^{\frac{q}{p'}} |\psi_n - \bar{f}|^{\frac{q}{p}} u\, dx\Big)^{\frac{p}{q}}.\end{aligned}$$

We let $C = \big(\int_D (|V|^{\frac{1}{p}} v^{\frac{1}{p}} u^{-\frac{1}{q}})^r\, dx\big)^{\frac{p}{r}}$ and we apply Hölder's inequality (with $\frac{1}{p} + \frac{1}{p'} = 1$) to the remaining integral. We obtain

$$\begin{aligned}&\int_D \big(V f |f|^{p-2} \psi_n - V|f|^p\big)\, v\, dx \\ &\le C\Big(\int_D |f|^q u\, dx\Big)^{\frac{p}{qp'}} \Big(\int_D |\psi_n - \bar{f}|^q u\, dx\Big)^{\frac{1}{q}} \\ &= C\| f u^{\frac{1}{q}} \|_q^{p-1} \|(\psi_n - \bar{f}) u^{\frac{1}{q}}\|_q. \qquad (3.6)\end{aligned}$$

By assumption, $\lim_{n\to\infty} \|(\psi_n - \bar{f}) u^{\frac{1}{q}}\|_q = 0$; by Lemma 3.2, $\|f u^{\frac{1}{q}}\|_q < \infty$, and so the right-hand side of (3.6) goes to zero when $n \to \infty$ as required. □

Proof of Theorem 1.4 Since the weights u and v are as in Theorem 1.1, the weighted gradient inequality (3.5) holds. By Lemma 3.3 and Hölder's inequality (with $\frac{p}{q} + \frac{p}{r} = 1$) we have the following chain of inequalities

$$\begin{aligned}\|f u^{\frac{1}{q}}\|^p_{L^q(D)} &\le c_0^p \|\nabla f\, v^{\frac{1}{p}}\|^p_{L^p(D)} = c_0^p \int_D v|\nabla f|^p dx \\ &= c_0^p \int_D V v |f|^p\, dx \le c_0^p \int_D V_+ v u^{-\frac{p}{q}}\, |f|^p u^{\frac{p}{q}}\ dx \\ &\le c_0^p \Big(\int_D V_+^{\frac{r}{p}} v^{\frac{r}{p}} u^{-\frac{r}{q}}\ dx\Big)^{\frac{p}{r}} \Big(\int_D |f|^q u\, dx\Big)^{\frac{p}{q}} \\ &\le c_0^p \|V_+^{\frac{1}{p}}\|^p_{L^r(D,\, v^{\frac{r}{p}} u^{-\frac{r}{q}}\ dx)} \|f u^{\frac{1}{q}}\|^p_{L^q(D)}.\end{aligned}$$

We obtain $\|f u^{\frac{1}{q}}\|_{L^q(D)} \big(1 - c_0^p \|V_+^{\frac{1}{p}}\|_{L^r(D,\, v^{\frac{r}{p}} u^{-\frac{r}{q}}\ dx)}\big) \le 0$; this inequality is possible only if either $c_0^p\ \|V_+^{\frac{1}{p}}\|_{L^r(D,\, v^{\frac{r}{p}} u^{-\frac{r}{q}}\ dx)} \ge 1$ or $f \equiv 0$ in D. □

4 Linear systems of PDE and the Dirac operator

We use the following notation: If $\vec{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m$, we let $|\vec{p}\,| = (p_1^2 + \cdots + p_m^2)^{\frac{1}{2}}$. If $\mathbf{A}$ is a matrix with rows $A_1, \ldots, A_N$, we will let $|\mathbf{A}| = (|A_1|^2 + \cdots + |A_N|^2)^{\frac{1}{2}}$. Note that, by Cauchy Schwartz inequality,

$$\|\mathbf{A}\vec{p}\,| = (\langle A_1,\ \vec{p}\,\rangle^2 + \cdots + \langle A_N,\ \vec{p}\,\rangle^2)^{\frac{1}{2}} \le (|A_1|^2 + \cdots + |A_N|^2)^{\frac{1}{2}} |\vec{p}\,| = |\mathbf{A}|\,|\vec{p}\,|.$$

Let $\vec{F} = (f_1, \ldots, f_N) \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$. We denote with $\nabla \vec{F}$ the $N \times n$ matrix whose rows are $\nabla f_1, \ldots, \nabla f_N$.

Unless otherwise specified, we assume that p, q, u and v are as in Theorem 1.1(a) and that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$.

In this section we use the Carleman inequality (1.1) to prove unique continuation properties of systems of linear partial differential equations of the first order.

4.1 Linear systems of PDE

Most of the first order systems considered in the literature are in the form of

$$\sum_{j=1}^{n} \mathbf{L}_j(x)\partial_{x_j}\vec{F} = V(x)\vec{F}, \tag{4.1}$$

where $\vec{F} = (f_1, \ldots, f_N)$ and the $\mathbf{L}_j(x)$ and V are $M \times N$ matrices defined in a domain $D \subset \mathbb{R}^n$. We let $\mathbf{L}(x)(\vec{F}) = \sum_{j=1}^n \mathbf{L}_j(x)\partial_{x_j}\vec{F}$. Differential inequalities in the form of

$$|\mathbf{L}(x)\vec{F}| \leq |\mathbf{V}(x)\vec{F}| \tag{4.2}$$

are also considered. In some of early papers on the subject, it is proved that solutions of elliptic systems in the form of (4.1) that vanish of sufficiently high order at the origin are $\equiv 0$; see [7,15,47] and the references cited in these papers for definitions of elliptic systems. A classical method of proof is to reduce the systems to (quasi-) diagonal form; this approach requires conditions on the regularity and the multiplicity of the eigenvalues of the system that are often difficult to check; see [9,24,29,56]. The strong continuation properties of systems of complex analytic vector fields in the form of $\vec{L}u = 0$ defined on a real-analytic manifold is proved in [1].

We have found only a few papers in the literature where the Carleman method is used to prove unique continuation properties of first-order systems. The Carleman method often allows to prove unique continuation results for the differential inequality (4.2), often with a singular potential V. In [14, Theorem 4.1] Carleman estimates are used to prove that (4.2) has the weak unique continuation property when $\vec{L}$ is a system of vector fields on a pseudoconcave Cauchy–Riemann (CR) with some specified conditions and V is bounded. Okaji [44,45] considers systems in two independent variables, Maxwell's equations, and the Dirac operator; he proved that the differential inequalities (4.2) with $|V(x)| \asymp |x|^{-1}$ has the strong unique continuation property using sophisticated $L^2 \to L^2$ Carleman estimates. See also [54], which improves results in [44].

We prove the following

Theorem 4.1 *Let $\vec{F} \in W_0^{1,p,v}(\mathbb{R}^n, \mathbb{R}^N)$ be a solution of the differential inequality* (4.2). *Assume that $\vec{F}$ satisfies also*

$$|\nabla \vec{F}| \lesssim |\mathbf{L}(x)\vec{F}|. \tag{4.3}$$

If $|\mathbf{V}| \in L^r(\operatorname{supp} \vec{F}, u^{-\frac{r}{q}} v^{\frac{r}{p}}\, dx)$, with $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ and $\vec{F}$ vanishes on one side of a hyperplane, then $\vec{F} \equiv 0$.

In particular, for power weights u, v as in Remark 3.1, the differential inequality (4.2) does not have solutions with compact support support that satisfy also (4.3) if $V \asymp |x|^{-1+\epsilon}$ for some $\epsilon > 0$.

Our unique continuation result is weaker than other results in the literature, but it applies to first-order systems of linear partial differential equations that satisfy only the assumptions (4.3). Furthermore, we consider solutions in weighted Sobolev spaces and potential in weighted L^r spaces that, to the best of our knowledge, have not been considered in other papers.

Before proving Theorem 4.1 we prove the following Lemma, which is an easy consequence of Theorem 1.1.

Lemma 4.2 *Let* $\mathbf{A}$ *be a* $N \times N$ *invertible matrix. Under the assumptions of Theorem* 1.1(a)*, the following inequality holds for all* $\vec{F} \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$ *and* $\tau \geq 0$

$$\|e^{-\tau\ell(x)} u^{\frac{1}{q}} \vec{F}\|_q \leq c_{\tau,N,\mathbf{A}} \|e^{-\tau\ell(x)} v^{\frac{1}{p}} \mathbf{A}\nabla\vec{F}\|_p, \tag{4.4}$$

where $c_{\tau,N,\mathbf{A}} = NC_{\mathbf{A}}c_\tau$ *and* c_τ *is the constant in* (1.7).

Proof Using Theorem 1.1(a), the elementary inequalities

$$|\vec{F}| = (f_1^2 + \cdots + f_N^2)^{\frac{1}{2}} \leq |f_1| + \cdots + |f_N|, \quad |f_j| \leq |\vec{F}|,$$

and Minkowsky's inequality, we obtain

$$\begin{aligned}\|e^{-\tau\ell(x)} u^{\frac{1}{q}} \vec{F}\|_q &\leq \sum_{j=1}^N \|e^{-\tau\ell(x)} u^{\frac{1}{q}} f_j\|_q \leq c_\tau \sum_{j=1}^N \|e^{-\tau\ell(x)} v^{\frac{1}{p}} \nabla f_j\|_p \\ &\leq c_\tau N \|e^{-\tau\ell(x)} v^{\frac{1}{p}} \nabla\vec{F}\|_p.\end{aligned}$$

If $\mathbf{A}$ is invertible, then, for every $\xi \in \mathbb{R}^n$, we have that $|\mathbf{A}\vec{\xi}| \geq C_{\mathbf{A}}^{-1}|\xi|$ for some $C_{\mathbf{A}} > 0$; thus,

$$\|e^{-\tau\ell(x)} u^{\frac{1}{q}} \vec{F}\|_q \leq c_\tau N C_{\mathbf{A}} \|e^{-\tau\ell(x)} v^{\frac{1}{p}} \mathbf{A}\nabla\vec{F}\|_p$$

as required. □

Proof of Theorem 4.1 We argue as in the proof of Theorem 1.3. Without loss of generality, we can assume that $\vec{F} \equiv 0$ when $x_n < 0$ and $\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the $N \times N$ identity matrix. For simplicity of notation, we denote with c_1 the constant $c_{1,N,\mathbf{I}}$ in Lemma 4.2. We show that $\vec{F} \equiv 0$ also on the strip $S_\epsilon = \{x : 0 < x_n < \epsilon\}$, for some $\epsilon > 0$ to be determined during the proof.

Using (4.4) with $\ell(x) = x_n$ and $\tau \geq 1$, the differential inequality (4.3), Hölder's inequality and Remark 1.2, we obtain

$$\begin{aligned}&\|e^{-\tau x_n} \vec{F} u^{\frac{1}{q}}\|_{L^q(S^\epsilon)} \\ &\leq c_1 \|e^{-\tau x_n} \nabla\vec{F}\, v^{\frac{1}{p}}\|_{L^p(\mathbb{R}^n)} \\ &\leq c_1 \|e^{-\tau x_n} \nabla\vec{F}\, v^{\frac{1}{p}}\|_{L^p(S_\epsilon)} + c_1 \|e^{-\tau x_n} \nabla\vec{F}\, v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})} \\ &\leq c_1 C \|e^{-\tau x_n} \mathbf{L}(x)(\nabla\vec{F})\, v^{\frac{1}{p}}\|_{L^p(S_\epsilon)} + c_1 e^{-\tau\epsilon} \|\nabla\vec{F} v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})} \\ &\leq c_1 C \|e^{-\tau x_n} \mathbf{V}\vec{F}\, v^{\frac{1}{p}}\|_{L^p(S_\epsilon)} + c_1 e^{-\tau\epsilon} \|\nabla\vec{F} v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})} \\ &\leq c_1 C \||\mathbf{V}|\, v^{\frac{1}{p}} u^{-\frac{1}{q}}\|_{L^r(S_\epsilon \cap \operatorname{supp} \vec{F})} \|e^{-\tau x_n} \vec{F} u^{\frac{1}{q}}\|_{L^q(S^\epsilon)} + C' e^{-\tau\epsilon},\end{aligned}$$

where we have let $C' = c_1 \|\nabla\vec{F}\, v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})}$.

Since $|\mathbf{V}| \in L^r(\operatorname{supp}\vec{F}, u^{-\frac{r}{q}} v^{\frac{r}{p}}\, dx)$ we can chose $\epsilon > 0$ so that $c_1 C \| |\mathbf{V}|\, v^{\frac{1}{p}} u^{-\frac{1}{q}} \|_{L^r(S_\epsilon \cap \operatorname{supp}\vec{F})} < \frac{1}{2}$. We have obtained

$$\|e^{-\tau x_n} \vec{F} u^{\frac{1}{q}}\|_{L^q(S^\epsilon)} \le \frac{1}{2} \|e^{-\tau x_n} \vec{F} u^{\frac{1}{q}}\|_{L^q(S^\epsilon)} + C' e^{-\tau\epsilon}.$$

In view of $\epsilon - x_n > 0$ on S^ϵ, the left-hand side of this inequality goes to infinity with τ unless $\vec{F} \equiv 0$ on S_ϵ; this is a contradiction because C' does not depend on τ, and so $\vec{F} \equiv 0$ in S_ϵ. □

Let $\mathbf{G}_1(x), \ldots, \mathbf{G}_n(x)$ be $N \times n$ matrices defined on a domain $D \subset \mathbb{R}^n$. We consider the operator

$$\mathbf{G}(\vec{F}) = \mathbf{G}(f_1, \ldots, f_N) = \sum_{j=1}^{N} \mathbf{G}_j(x) f_j$$

with $f_j \in C_0^\infty(D)$.

In [39], systems in the form of $\nabla F = \mathbf{G}\vec{F}$ are considered. These systems can be used to model linear elasticity (in curvilinear coordinates) of linearly elastic shells. See [5] and the references cited there. We prove the following

Theorem 4.3 *Let $\vec{F} \in W_0^{1,p,v}(D, \mathbb{R}^N)$ be a solution of the differential inequality*

$$|\nabla \vec{F}| \lesssim |\mathbf{G}\vec{F}|. \tag{4.5}$$

If $|\mathbf{G}| \in L^r(\operatorname{supp}\vec{F}, u^{-\frac{r}{q}} v^{\frac{r}{p}}\, dx)$, with $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$, and $\vec{F}$ vanishes on one side of a hyperplane, then $\vec{F} \equiv 0$.

Proof Assume for simplicity that $\vec{F} \equiv 0$ when $x_n < 0$ (the proof is similar in the general case). We show that $\vec{F} \equiv 0$ also on the strip $0 < x_n < \epsilon$, for some $\epsilon > 0$ to be determined during the proof. As in the proof of Theorem 4.1, we use (4.4) with $\mathbf{A} = \mathbf{I}$, $\ell(x) = x_n$ and $\tau \ge 1$. For each $j = 1, \ldots, N$, we use the differential inequality (4.5) and Hölder's inequality in the following chain of inequalities

$$\begin{aligned}
&\|e^{-\tau x_n} \vec{F} u^{\frac{1}{q}}\|_{L^q(S^\epsilon)} \\
&\le c_1 \|e^{-\tau x_n} \nabla \vec{F}\, v^{\frac{1}{p}}\|_{L^p(\mathbb{R}^n)} \\
&\le c_1 \|e^{-\tau x_n} \nabla \vec{F}\, \vec{v}^{\frac{1}{p}}\|_{L^p(S_\epsilon)} + c_1 \|e^{-\tau x_n} \nabla \vec{F}\, v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})} \\
&\le c_1 C \|e^{-\tau x_n} \mathbf{G}\vec{F}\, v^{\frac{1}{p}}\|_{L^p(S_\epsilon)} + c_1 e^{-\tau\epsilon} \|\nabla \vec{F} v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})} \\
&\le c_1 C \sum_{j=1}^{N} \|e^{-\tau x_n} |\mathbf{G}_j| f_j\, v^{\frac{1}{p}}\|_{L^p(S_\epsilon)} + c_1 e^{-\tau\epsilon} \|\nabla \vec{F} v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})} \\
&\le c_1 C \sum_{j=1}^{N} \| |\mathbf{G}_j|\, v^{\frac{1}{p}} u^{-\frac{1}{q}} \|_{L^r(S_\epsilon \cap \operatorname{supp}\vec{F})} \|e^{-\tau x_n} f_j u^{\frac{1}{q}}\|_{L^q(S^\epsilon)} + C' e^{-\tau\epsilon} \\
&\le c_1 C N \| |\mathbf{G}|\, v^{\frac{1}{p}} u^{-\frac{1}{q}} \|_{L^r(S_\epsilon \cap \operatorname{supp}\vec{F})} \|e^{-\tau x_n} |\vec{F}| u^{\frac{1}{q}}\|_{L^q(S^\epsilon)} + C' e^{-\tau\epsilon},
\end{aligned}$$

where we have let $C' = c_1 \|\nabla \vec{F}\, v^{\frac{1}{p}}\|_{L^p(\{x_n > \epsilon\})}$.

We chose $\epsilon > 0$ so that $c_1 C N \| |\mathbf{G}| \, v^{\frac{1}{p}} u^{-\frac{1}{q}} \|_{L^r(S_\epsilon \cap \text{supp}\, \vec{F})} < \frac{1}{2}$. We gather

$$\|e^{-\tau x_n} \vec{F} u^{\frac{1}{q}} \|_{L^q(S^\epsilon)} \leq \frac{1}{2} \|e^{-\tau x_n} \vec{F} u^{\frac{1}{q}} \|_{L^q(S^\epsilon)} + C' e^{-\tau \epsilon}$$

which gives

$$\frac{1}{2} \|e^{\tau(\epsilon - x_n)} \vec{F} u^{\frac{1}{q}} \|_{L^q(S^\epsilon)} \leq C',$$

and we can conclude the proof as in Theorem 4.1. □

Remark 4.1 It is shown in [39] that the $W^{1,1}(D, \mathbb{R}^n)$ solutions of the system $\nabla \vec{F} = \mathbf{G} \vec{F}$, with $\mathbf{G} \in L^1(D, \mathbb{R}^{(n \times n) \times n})$, cannot vanish on an open set. The proof in [39] does not use Carleman inequalities.

4.2 The Dirac operator

Let α_j, $j = 0, \ldots, n$, be $N \times N$ matrices which satisfy the following relations.

$$\alpha_j^* = \alpha_j, \quad \alpha_j^2 = I, \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 0, \quad j \neq k \tag{4.6}$$

(we also say that the α_j form a basis of a Clifford algebra). It is known that for (4.6) to hold, N must be in the form $2^{[\frac{n+1}{2}]} m$, with $m > 0$ integer

The (n-dimensional) Dirac operator associated to the matrices α_j is a matrix value operator, initially defined on $C_0^\infty(\mathbb{R}^n, \mathbb{R}^{N \times N})$ as follows.

$$\mathcal{L} U = -i \sum_{j=1}^{n} \alpha_j \partial_{x_j} U.$$

Here, $\partial_{x_i} U$ is a matrix whose entries are the partial derivative of the entries of U. We can use (4.6) to show that $\mathcal{L} \circ \mathcal{L} U = -\Delta U \, I$, where I is the identity matrix. When $U = f I$, where $f \in C_0^\infty(\mathbb{R}^n)$, we can see at once that $(\mathcal{L}(f I))^2 = -I |\nabla f|^2$, Thus, a Dirac operators can be viewed as a generalization of the gradient operator and a square root of the Laplacian.

There is a lot of literature on the Dirac operator and its role in several domains of mathematics and physics See e.g. [6]. For example, the Dirac equation which describes free relativistic electrons is represented by

$$i \hbar \partial_t \psi(t, x) = H_0 \psi(t, x),$$

where H_0 is given explicitly by the 4×4 matrix-valued differential expression

$$H_0 = -i \hbar c \sum_{j=1}^{3} \alpha_j \partial_{x_j} + \alpha_0 m c^2.$$

Here, c is the speed of light, m is a mass of a particle and $\hbar$ is the Planck's constant.

In [16] is proved that the the differential inequality

$$|\mathcal{L} U| \leq |V U| \tag{4.7}$$

where $V(x)$ is a $N \times N$ matrix, has the strong unique continuation property from the origin whenever $V(x) \leq C|x|^{-1}$, with $0 \leq C \leq 1$. It is also proved in [16] that the condition $C \leq 1$ cannot be improved. See also [33] and the references cited there. We prove the following

Theorem 4.4 *Let $f \in W_0^{1,p,v}(D)$ be a solution of the differential inequality (4.7). If $|\mathbf{V}| \in L^r(\text{supp } f,\ u^{-\frac{r}{q}} v^{\frac{r}{p}}\, dx)$ with $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$ and f vanishes on one side of a hyperplane, then $f \equiv 0$.*

Proof Since $\mathcal{L}(fI) \cdot \mathcal{L}(fI) = -I|\nabla f|^2$, we can see at once that

$$|\nabla f| = |\mathcal{L}(fI) \cdot \mathcal{L}(fI)| \le |\mathcal{L}(fI)|^2$$

With this observation, the proof of Theorem 4.4 is almost a line-by-line repetition of the proof of Theorem 4.1. We leave the details to the reader. □

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