



CENTRE DE RECERCA MATEMÀTICA

Title: *Pattern formation in a flux limited reaction–diffusion equation of porous media type*
Journal Information: *Inventiones Mathematicae*,
Author(s): Calvo J., Campos J., Caselles V., Sánchez O., Soler J..
Volume, pages: 206–52, DOI:[10.1007/s00222-016-0649-5]

Pattern formation in a flux limited reaction–diffusion equation of porous media type

J. Calvo¹ · J. Campos² · V. Caselles³ ·
O. Sánchez² · J. Soler²

To Vicent Caselles, in memoriam

Received: 20 July 2014 / Accepted: 10 January 2016 / Published online: 1 February 2016
© Springer-Verlag Berlin Heidelberg 2016

Abstract A non-linear PDE featuring flux limitation effects together with those of the porous media equation (non-linear Fokker–Planck) is presented in this paper. We analyze the balance of such diverse effects through the study

J. Calvo, J. Campos, O. Sánchez and J. Soler have been partially supported by MINECO-Feder (Spain) research grant, project MTM2014-53406-R, and by Junta de Andalucía (Spain), Project FQM 954. J. Calvo is also supported by La Caixa “Collaborative Mathematical Research” programme and was supported by a Juan de la Cierva grant of the Spanish MEC during part of the period in which this work was carried out. V. Caselles was supported in part by MICINN (Spain), project MTM2009-08171, and also acknowledges the partial support by GRC reference 2009 SGR 773, and by “ICREA Acadèmia” prize for excellence in research funded both by the Generalitat de Catalunya.

✉ J. Soler
jsoler@ugr.es

J. Calvo
jcalvo@crm.cat

J. Campos
campos@ugr.es

V. Caselles
vicent.caselles@upf.edu

O. Sánchez
ossanche@ugr.es

¹ Centre de Recerca Matemàtica, Bellaterra, Barcelona, Spain

² Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

³ Departamento de Tecnología, Universitat Pompeu-Fabra, Barcelona, Spain

of the existence and qualitative behavior of some admissible patterns, namely traveling wave solutions, to this singular reaction–diffusion equation. We show the existence and qualitative behavior of different types of traveling waves: classical profiles for wave speeds high enough, and discontinuous waves that are reminiscent of hyperbolic shock waves when the wave speed lowers below a certain threshold. Some of these solutions are of particular relevance as they provide models by which the whole solution (and not just the bulk of it, as it is the case with classical traveling waves) spreads through the medium with finite speed.

Mathematics Subject Classification Primary 35K57 · 35B36 · 35K67 · 34Cxx · 70Kxx; Secondary 35B60 · 37Dxx · 76B15 · 35Q35 · 37D50 · 35Q99

1 Introduction, entropy solutions, main results

Reaction–diffusion equations assume that the behavior of the various populations described is ruled essentially by two processes: local reactions, in which the populations interact between themselves, and diffusion, which makes the populations spread out in the physical space. The concept of population is understood here quite loosely, and several important examples can be found in developmental biology, ecology, geology, combustion theory, physics or computer sciences. Particles, free surface water waves, flames, cells, bacteria or morphogen concentrations in chemical processes may qualify as such, see for instance [6, 18, 32, 37, 42]. Reaction–diffusion equations constitute a usual description for complex systems in all these areas. The prototypical model in this context can be written down as

$$\frac{\partial u}{\partial t} = \operatorname{div} (D \nabla u) + F(u), \quad u(t = 0, x) = u_0(x). \quad (1.1)$$

Here D is a coefficient that could be a constant (in the simplest case of linear diffusion) [31, 36, 45, 46], a function depending on the domain of definition [10, 12, 13], a function depending on u [34, 51, 55, 58], or in general a function $D = D(u, \nabla u)$ [30, 53], which includes the possibility of fractional diffusion associated with Levy processes [16, 17, 52]. The function F represents the reaction term. The different models expressed in Eq. (1.1) have been the object of an intense study in the literature in order to clarify : (1) the qualitative differences when we consider non-linear diffusion operators like the p -Laplacian or the one of the porous media equation in contrast with the behavior associated with a linear diffusion term, (2) what kind of particular solutions (as traveling waves, kinks, or solitons, for instance) can be obtained when different functional forms for $F(u)$ are proposed, (3) the behavior of systems of equations of type (1.1), or even more complicated instances of them—this is a way to

describe pattern formation and cooperative behavior, see [32] for instance, (4) the effect of noise on front propagation, see for example [47], (5) the stability or long time asymptotic properties of the patterns, see [11,29,33], for instance.

Our research in this paper falls into the first and second categories above. We analyze the existence of traveling wave solutions associated to a non-linear diffusion PDE coupled to a reaction term of Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) type [31,36], namely

$$\frac{\partial u}{\partial t} = \nu \operatorname{div}_x \left(\frac{u^m \nabla u}{\sqrt{|u|^2 + \frac{\nu^2}{c^2} |\nabla u|^2}} \right) + F(u), \quad \text{in } \mathcal{Q}_T =]0, T[\times \mathbb{R}^N, \quad (1.2)$$

where $m > 1$ and $F(u)$ is a Lipschitz continuous function such that $F(0) = F(1) = 0$. Here ν is a kinematic viscosity and $c > 0$ is a characteristic speed [53]. Note that (1.2) is a renormalization with respect to the carrying capacity v_0 of the equation

$$\frac{\partial u}{\partial t} = \nu v_0 \operatorname{div}_x \left(\frac{\left(\frac{u}{v_0}\right)^m \nabla_x u}{\sqrt{|u|^2 + \frac{\nu^2}{c^2} |\nabla u|^2}} \right) + F\left(\frac{u}{v_0}\right). \quad (1.3)$$

The reaction term in the FKPP case would be given by $F = K v_0 u (1 - u)$, where K is the growth rate. Although the results and techniques introduced in this paper can be extended to more general cases, we focus our attention on the FKPP case to deal with concrete numerical examples.

Equation (1.2) belongs to the class of flux-limited diffusion equations. Flux limited diffusion ideas were introduced by Rosenau in [26,53] in order to restore the finite speed of propagation of signals in a medium. This property is lost in the classical transport theory that predicts the non-physical divergence of the flux with the gradient, as it happens also with the classical theory of heat conduction (based in Fourier’s law) and with the linear diffusion theory (based in Fick’s law). Besides Rosenau’s derivation [53], the particular case of (1.2) where $m = 1$ was also formally derived by Brenier by means of Monge–Kantorovich’s mass transport theory in [15] (this has been done later in a rigorous way in [44]), where he named it as the relativistic heat equation. More recently, (1.2) has been shown ($m = 1$) to be an effective model to describe the transport of morphogens in cellular communication to induce distinct cell fates in a concentration-dependent manner [59].

The model (1.2) (with $F = 0$) was introduced in [53] (when $m = 3/2$) as an example of flux limited diffusion equation in the context of heat diffusion in a neutral gas. As shown in [53] the acoustic speed is a function of the temperature

and the front is convected nonlinearly. This has been shown mathematically in [22] and it has been proved in [24] that solutions of (1.2) converge to solutions of the classical porous medium equation as $c \rightarrow \infty$. Thus, this model offers a novel dynamical behavior to describe diffusion and propagation phenomena in real media. The finite propagation property is at the basis of this behavior.

Other flux-limited versions of the porous medium equation, namely

$$u_t = v \operatorname{div} \left(\frac{u \nabla u^m}{\sqrt{1 + \frac{v^2}{c^2} |\nabla u^m|}} \right), \quad m > 0, \quad (1.4)$$

were introduced in [26, 53] and further studied in [23, 24] (where convergence to the classical porous medium equation is proved). They can be also derived using transport theory as proposed in [15] (see also [24]). In this case, the acoustic speed is the constant c [23] and thus independent of u . The different behavior between both types of models is not yet fully understood, but numerical evidence [5, 26, 43] shows that model (1.2) may have a richer behavior creating discontinuity fronts starting from smooth initial conditions, while the model (1.4) exhibits a behavior more similar to the corresponding standard porous medium equation [26]. Based on this fruitful dynamical behavior, our purpose here is to concentrate on the study of (1.2) leaving the study of traveling waves for model (1.4) for future research.

The dynamics produced by the combined effects of flux limitation and the non-linearities of porous media type in (1.2) ($F = 0$) (non-linear Fokker–Planck) may be relevant for its potential applicability in the study of other similar operators. From a mathematical point of view, this combination demands the use of different techniques and ideas coming from the fields of non-linear diffusion (non-linear semigroups) and scalar conservation laws, e.g. front propagation and entropy solutions. This concept of entropy solution determines the geometrical features of the admissible solutions because the structure of the singularities that a solution may eventually display is strongly restricted, by virtue of a series of constraints that are ultimately related with the physical principle stating that solutions cannot violate causality. To be more precise, jump discontinuities are characterized by having a vertical profile that moves according to a Rankine–Hugoniot law.

As we will show, the reaction–diffusion equation (1.2) exhibits new properties with respect to the classical reaction terms coupled with linear diffusion mechanisms. The existence of singular traveling waves is one of these new properties, and it is object of study in this paper. The construction of such singular patterns requires the development of novel arguments in dynamical systems. These involve the use of invariant manifolds and blow-up control to

analyze the singular phase diagrams associated to the ODE satisfied by traveling wave solutions of (1.2). In fact, for some choices of the physical constants the classical theory breaks down and we need to use the concept of entropy solution in the dynamical system context and the Rankine–Hugoniot jump condition to construct our profiles, producing discontinuous traveling waves (which, for some particular speed values, may have their support in a half line). This behavior is reminiscent of shock waves in hyperbolic conservation laws. Our analysis gives further insight into the properties of the solutions of (1.2) that were experimentally studied in [5, 26, 43] when $F = 0$, in particular on the existence of solutions which are discontinuous in the interior of their support.

Let us recall that there are several instances of traveling wave solutions not supported in the whole line arising in models with non-linear diffusion mechanisms coupled with reaction terms, among which we mention [7, 30, 48–50, 54–57]. This is an issue of great relevance in several contexts where an infinite speed of propagation of the support is inconsistent with the experimental observations. The traveling waves supported on a half line that are constructed in the previous references are all continuous functions. To our knowledge, only the results in [19, 39, 40] and the ones in this paper are able to produce traveling waves that are not only supported on a half line but also exhibit sharp discontinuity fronts. The papers [39, 40] study traveling waves for models including directly hyperbolic terms of Burger’s type coupled with a diffusion operator of curvature type $\left(\frac{u_x}{\sqrt{1+|u_x|^2}}\right)_x$ and no reaction term. They exhibit the existence of a critical regime above which discontinuous transitions in the traveling wave show up. The research carried in [41] is also related to these issues, as discontinuous steady states, which are not traveling waves, supported in a half line are obtained out of a reaction–diffusion equation whose diffusion mechanism is very similar to that in (1.2), but avoiding the singularity at $u = 0$ (say of curvature type); however the reaction term is not of FKPP type, using cubic and quintic non-linearities, and the techniques based on numerical and asymptotic methods are different. Note that the cubic non-linearity of the Allen–Cahn term produces a bistability effect, which helps in the study of the unique associated traveling wave. The case of the reaction FKPP term is different as regards the stability of traveling waves and their uniqueness. Studying these phenomena may open new perspectives of application of these models to biology or traffic flow frameworks, for instance.

Let us now introduce our assumptions on the reaction term F .

1.1 Assumptions on the reaction term

We will be concerned with the analysis of traveling wave solutions to a family of one-dimensional non-linear flux limited diffusion equations coupled with

a reaction term of FKPP type. Concretely, we are interested in the non-linear diffusion equation which can be written down as (1.2), with $m > 1$. The analysis of such models with $F \equiv 0$ was the object of [4,5]. We assume that $F(u)$ satisfies the following properties:

- $F \in C^1([0, 1])$, $F(0) = F(1) = 0$, and $F(u) > 0$ for every $u \in]0, 1[$.
- $F'(1) < 0$.

Note that we can write $F(u) = uK(u)$ with

- $K \in C^1(]0, 1[) \cap C^0([0, 1])$.
- $K(1) = 0$ and $K(u) > 0$, $\forall u \in]0, 1[$.
- $K(0) = F'(0) \geq 0$, $K'(1) = F'(1) < 0$.

This allows for traveling fronts that connect the constant state $u = 1$ [which, before normalization, would correspond to the state $u = v_0$, see (1.3)] with the zero state. This can be justified by the comparison principle given in Theorem 4.5, which ensures that we can restrict ourselves to the study of solutions between these two constant states. We suggest the reader to keep in mind the prototypical case $F(u) = u^p(1 - u^q)$, where $p, q \geq 1$ (see [46] and references therein for applications). The conditions on function F classify it as a ‘‘Type A’’ reaction function according with the characterization of [14].

The next thing we do is to analyze the structure of discontinuous solutions to (1.2). This will make clear what kind of traveling fronts are to be expected.

1.2 Entropy solutions and the Rankine–Hugoniot condition

Equation (1.2) is a particular instance of the class of flux limited diffusion equations for which the correct concept of solution, allowing to prove existence and uniqueness results, is the notion of entropy solution [4,20,21]. Although somewhat involved, this notion is necessary since (1.2) (as many other flux limited diffusion equations) has a parabolic-hyperbolic behavior, with solutions that may exhibit moving discontinuity fronts [5,26]. In particular, we notice in passing that the right function space to study this class of solutions is the space of functions of bounded variation.

As usual, the notion of entropy solution of (1.2) is described in terms of a set of inequalities of Kruzhkov type [38] that are well adapted to prove uniqueness results. But, as proved in [22] for $F = 0$, we can give a geometric characterization of entropy conditions on the jump set of solutions of (1.2). Indeed, in their jump set, entropy solutions of (1.2) have a vertical graph and this is equivalent to the entropy inequalities there. This permits also to give an explicit form to Rankine–Hugoniot condition that expresses the velocity of moving discontinuity fronts [22]. Both things, the geometric characterization of entropy solutions and the Rankine–Hugoniot condition, are relevant for us

here. Indeed, they will guide us in the search for traveling waves of (1.2), after reducing it to the study of an associated dynamical system (see Sect. 2). Thus, our approach is based on the analysis of that system, taking into account the properties of entropy solutions of (1.2).

Let us briefly recall both the Rankine–Hugoniot condition and the geometric characterization of entropy solutions of (1.2) in a context that is suitable for our purposes here. Since we follow the presentation in [22] we will skip the proofs of the given statements. For continuity of the presentation, the notation and basic background on the functional setting, the definition of entropy solutions and basic existence and uniqueness results for (1.2) are given in the appendix in Sect. 4 (see also [1]). Although the case we are interested in here corresponds to $N = 1$, let us write them in the general case $N \geq 1$.

Let $Q_T =]0, T[\times \mathbb{R}^N$. Assume that $u \in BV_{loc}(Q_T)$. Let us denote by J_u the jump set of u as a function of (t, x) . For any $t > 0$, we denote by $J_{u(t)}$ the jump set of $u(t) \in BV_{loc}(\mathbb{R}^N)$. Let $v := v_u = (v_t, v_x)$ be the unit normal to the jump set of u so that the jump part of the distributional derivative reads $D_{t,x}^j u = [u]v\mathcal{H}^N|_{J_u}$, where \mathcal{H}^N is the N -dimensional Hausdorff measure in \mathbb{R}^N . We denote by $v^{J_{u(t)}}$ the unit normal to the jump set of $u(t)$ so that $D_x^j u(t) = [u(t)]v^{J_{u(t)}}\mathcal{H}^{N-1}|_{J_{u(t)}}$. Here $[u](t, x) := u^+(t, x) - u^-(t, x)$ denotes the jump of u at $(t, x) \in J_u$ and $[u(t)](x) := u(t)^+(x) - u(t)^-(x)$ denotes the jump of $u(t)$ at the point $x \in J_{u(t)}$.

Let us recall the definition of the speed of the discontinuity set of u [22].

Definition 1.1 Let $u \in BV_{loc}(Q_T)$, $F \in L^1_{loc}(\mathbb{R}^N)$, and let $\mathbf{z} \in L^\infty([0, T] \times \mathbb{R}^N, \mathbb{R}^N)$ be such that $u_t = \operatorname{div} \mathbf{z} + F$ in $\mathcal{D}'(Q_T)$. We define the speed of the discontinuity set of u as $v(t, x) = \frac{v_t(t,x)}{|v_x(t,x)|} \mathcal{H}^N$ -a.e. on J_u .

This definition has a sense since, when $u \in BV_{loc}(Q_T)$, $F \in L^1_{loc}(\mathbb{R}^N)$, $\mathbf{z} \in L^\infty([0, T] \times \mathbb{R}^N, \mathbb{R}^N)$ and $u_t = \operatorname{div} \mathbf{z} + F$ in $\mathcal{D}'(Q_T)$, we have (see [22], Lemma 6.4) that

$$\mathcal{H}^N\{(t, x) \in J_u : v_x(t, x) = 0\} = 0.$$

In our next result we state the Rankine–Hugoniot condition in a context that covers the case of Eq. (1.2). The proof follows as in [22] and we omit it.

Proposition 1.2 Assume that $F : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz. Let $u \in BV_{loc}(]0, T[\times \mathbb{R}^N)$ and let $\mathbf{z} \in L^\infty([0, T] \times \mathbb{R}^N, \mathbb{R}^N)$ be such that $u_t = \operatorname{div} \mathbf{z} + F(u)$. For \mathcal{L}^1 almost any $t > 0$ we have

$$[u(t)](x)v(t, x) = [[\mathbf{z} \cdot v^{J_{u(t)}}]]_{+-} \quad \mathcal{H}^{N-1}\text{-a.e. in } J_{u(t)}, \tag{1.5}$$

where $[[\mathbf{z} \cdot v^{J_{u(t)}}]]_{+-}$ denotes the difference of traces from both sides of $J_{u(t)}$.

We call outer side of $J_{u(t)}$ the one to which $v^{J_{u(t)}}$ is pointing. Thus, the outer trace is $u(t) = u(t)^+$. Notice that with this notation, the Rankine–Hugoniot condition (1.5) is expressed in an invariant way. We have denoted as $[\mathbf{z} \cdot \nu^{J_{u(t)}}]$ the weak trace of the normal component of \mathbf{z} on $J_{u(t)}$. This notion is well defined since \mathbf{z} is a bounded vector field whose divergence is a Radon measure [8, 22, 25]. This is covered by the results in [8, 25] if $J_{u(t)}$ is locally a Lipschitz surface. In the present case, we need the further developments in [22].

Assume that $m > 1$. As in [22], the notion of entropy solution of (1.2) (see Sect. 4) can be expressed as a set of inequalities that can be translated into a geometric condition on the jump set of the solution. Informally, one can say that jump discontinuities are fronts with a vertical contact angle moving at the speed given by the Rankine–Hugoniot condition. This can be proved as in [22].

Proposition 1.3 *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz function. Let $u \in C([0, T]; L^1_{loc}(\mathbb{R}^N)) \cap BV_{loc}([0, T] \times \mathbb{R}^N)$. Assume that $Du = D^a u + D^j u$, that is, Du has no Cantor part. Assume that $u_t = \operatorname{div} \mathbf{z} + F(u)$ in $\mathcal{D}'(Q_T)$, where $\mathbf{z} = \mathbf{a}(u, \nabla u)$ is the flux of (1.2). Then u is an entropy solution of (1.2) if and only if for \mathcal{L}^1 -almost any $t > 0$*

$$[\mathbf{z} \cdot \nu^{J_{u(t)}}]_+ = c(u^+(t))^m \quad \text{and} \quad [\mathbf{z} \cdot \nu^{J_{u(t)}}]_- = c(u^-(t))^m \quad (1.6)$$

hold \mathcal{H}^{N-1} a.e. on $J_{u(t)}$. Moreover, from Proposition 1.2, the velocity of the discontinuity fronts is

$$v = c \frac{(u^+(t))^m - (u^-(t))^m}{u^+(t) - u^-(t)}. \quad (1.7)$$

To conclude, let us rephrase the conditions (1.6) in a more geometric way. Under some additional assumptions they amount to a vertical profile of u on its jump set. This is the case if we assume that for \mathcal{H}^N almost all $x \in J_u$ there is a ball B_x centered at x such that either (a) or (b) hold, where

- (a) $u|_{B_x} \geq \alpha > 0$,
- (b) $J_u \cap B_x$ is the graph of a Lipschitz function with $B_x \setminus J_u = B_x^1 \cup B_x^2$, where B_x^1, B_x^2 are open and connected, $u \geq \alpha > 0$ in B_x^1 , while the trace of u on $J_u \cap \partial B_x^2$ computed from B_x^2 is zero.

In both cases, under the assumptions of Proposition 1.3, by Lemma 5.6 in [22] we can cancel u^m on both sides of the identities in (1.6) and obtain

$$\left[\frac{\nabla u}{\sqrt{u^2 + \frac{v^2}{c^2} |\nabla u|^2}} \cdot \nu^{J_{u(t)}} \right]_+ = \frac{c}{v} \quad \text{on } J_u \cap B(x, r). \quad (1.8)$$

If (a) holds we also have

$$\left[\frac{\nabla u}{\sqrt{u^2 + \frac{v^2}{c^2} |\nabla u|^2}} \cdot \nu^{J_{u(r)}} \right]_- = \frac{c}{v} \quad \text{on } J_u \cap B(x, r). \quad (1.9)$$

In dimension one, assuming that the jump point is isolated and that u is smooth out of the discontinuity, the above conditions mean that the graph of u is vertical at the discontinuity points. The same can be said in any dimension if J_u is a regular surface and u is smooth out of the discontinuity set. In the more general case, the traces in (1.8), (1.9) are interpreted in a weak sense [8, 22, 25].

We conclude this first section by introducing the main results of this paper.

1.3 Statement of the main results

We assume that $m > 1$. We look for one-dimensional incoming wave solutions of (1.2) with range in $[0, 1]$, traveling at constant speed $\sigma > 0$, with their shape being completely unaltered. That is, we look for solutions of the form $u(x - \sigma t) \equiv u(\xi)$. In a first step we will study decreasing traveling profiles, but we will prove that monotonicity is not a real constraint because these are the only piecewise smooth entropy solutions of (1.2) having a traveling wave structure. In fact, these profiles will connect 1 at $-\infty$ and 0 at ∞ and satisfy a null-flux-at-infinity condition (see Definition 4.4 in Appendix 4.3.3), which assures the uniqueness of entropy solution for system (1.2). Let us make precise that when we say that a function is piecewise smooth, up to a finite number of points, we understand that at those singular points there is a jump either of the function or of its first derivative. Let us also remark that the existence of a traveling wave profile implies the existence of a parametric family of traveling wave solutions due to spatial translations. For this type of solutions our main result (see Fig. 1) is the following:

Theorem 1.4 *Let $m > 1$. The following results are verified*

- i) *Existence: There exist two values $0 < \sigma_{ent} < \sigma_{smooth} < mc$, depending on c, v, m and F , such that:*
 - (1) *for $\sigma > \sigma_{smooth}$ there exists a unique smooth traveling wave solution of (1.2),*
 - (2) *for $\sigma = \sigma_{smooth}$ there exists a traveling wave solution of (1.2), which is continuous but not smooth,*
 - (3) *for $\sigma_{smooth} > \sigma \geq \sigma_{ent}$ there exists a traveling wave solution of (1.2), which is discontinuous.*
- ii) *Uniqueness: for any fixed value of $\sigma \in [\sigma_{ent}, +\infty[$, after normalization (modulo spatial translations) there is just one traveling wave solution in*

the class of piecewise smooth solutions (that is, smooth except maybe at a finite number of points) with range in $[0, 1]$ and satisfying the entropy conditions.

iii) *Continuity:* Assume that there is a value $p \geq 1$ such that

$$\liminf_{u \rightarrow 0} \frac{F(u)}{u^p} = k \in]0, +\infty]. \quad (1.10)$$

After suitable normalization (see Definition 3.6 below), there is a family of traveling wave profiles $u_\sigma^{\mathcal{N}}(\xi)$ for $\sigma \in [\sigma_{ent}, +\infty[$ which enjoys the following property:

$$\lim_{\sigma_1 \rightarrow \sigma_2} \|u_{\sigma_1}^{\mathcal{N}} - u_{\sigma_2}^{\mathcal{N}}\|_{L^p(\mathbb{R})} + \|u_{\sigma_1}^{\mathcal{N}} - u_{\sigma_2}^{\mathcal{N}}\|_{L^\infty(\mathbb{R})} = 0,$$

for any $\sigma_1, \sigma_2 \in [\sigma_{ent}, +\infty[$.

Remark 1.5 Note that condition (1.10) has been introduced to complement the uniform convergence with a convergence in some L^p space. Other possible assumptions could involve different spaces. Let us point out that when the condition (1.10) holds for some $p \geq 1$, then it is also verified for any value of p above this one. When F is analytic, we can take p as the order of the zero of F at $u = 0$. In the general case, we may not be able to find a minimal value of p such that (1.10) is fulfilled, an example of this situation being given by $F(u) = u^2 \log\left(\frac{1}{u}\right)$. In any case, the convergence will be at least uniform, see Sect. 3.

From the perspective of applications, the most interesting and novel solutions are those corresponding to $\sigma \in [\sigma_{ent}, \sigma_{smooth}[$, which are discontinuous. In particular, those corresponding to $\sigma = \sigma_{ent}$ are supported on a half line for each t , and they encode processes in which the propagation of information (whatever it may be) takes place at finite speed.

Let us remark that for $m = 1$ the catalog of possible traveling wave solutions is restricted to cases a and d in Fig. 1. This was discussed in [19] by studying the corresponding dynamical system, which has a simpler behavior than for $m > 1$.

1.4 Plan of the paper

Let us finally explain the plan of the rest of the paper. In Sect. 2 we reduce the study of traveling wave solutions of (1.2) to the study of an associated dynamical system. We prove the existence of different types of maximal solutions for this system, depending on the speed of the traveling wave, which will be smooth for high enough wave speeds and discontinuous when the wave speed

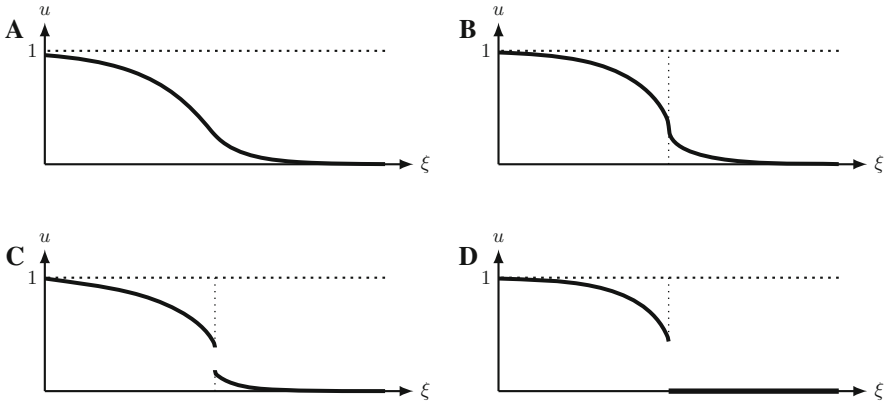


Fig. 1 Traveling wave profiles for (1.2), see Theorem 1.4: **a** $\sigma > \sigma_{smooth}$, **b** $\sigma = \sigma_{smooth}$, **c** $\sigma \in]\sigma_{ent}, \sigma_{smooth}[$, **d** $\sigma = \sigma_{ent}$. Vertical dotted lines show points with infinite slope. These profiles are in correspondence with the orbits depicted in Fig. 2b

lowers below a certain threshold. In Sect. 3 we combine the different wave profiles obtained in Sect. 2 and construct the traveling waves of (1.2). Moreover, we also prove uniqueness of piecewise smooth traveling waves with a given speed, and their continuous dependence on it. We also give some numerical insights about how the patterns can be attractors of time-dependent solutions and how saturation of diffusion in competition with reaction originate shocks. Finally, in Sect. 4 we provide an appendix with the necessary background on entropy solutions of (1.2) in order to give sense to the statements in this Introduction.

2 The associated planar dynamical system

In order to construct traveling wave profiles we substitute the traveling wave ansatz $u(x - \sigma t)$ into (1.2). This leads to the study of the following equation:

$$v \left(\frac{u^m u'}{\sqrt{u^2 + \frac{v^2}{c^2} |u'|^2}} \right)' + \sigma u' + F(u) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (2.1)$$

We can use (2.1) to construct piecewise smooth entropy solutions of (1.2). For that, it suffices to join together smooth solutions of (2.1) defined on intervals of \mathbb{R} fulfilling the following rules:

- (i) If solutions corresponding to two consecutive intervals match in a continuous way, then the first derivative cannot have a jump discontinuity.

Otherwise the term $\left(\frac{u^m u'}{\sqrt{u^2 + \frac{v^2}{c^2} |u'|^2}}\right)'$ would contribute with a Dirac delta

at the matching point, while the terms $\sigma u' + F(u)$ would be in $L^1_{loc}(\mathbb{R})$, and (2.1) could not hold in $\mathcal{D}'(\mathbb{R})$. The same argument shows that when two solutions match in a continuous way and the first derivative is $+\infty$ (resp. $-\infty$) on one side then it must be also $+\infty$ (resp. $-\infty$) on the other side.

- (ii) If solutions corresponding to two consecutive intervals match forming a jump discontinuity, then the speed of the moving front should obey the Rankine–Hugoniot condition (1.7) and the slope of the profile at both sides of the discontinuity must be infinite with the same sign [see (1.8)–(1.9)], except when one of the solutions we are matching with is the zero solution. In that case, when looking for decreasing profiles, we only have to worry about the infinite slope condition on the left side of the discontinuity.

In order to search for smooth solutions of (2.1) in intervals of \mathbb{R} , we write (2.1) as an autonomous planar system. For that we set

$$r(\xi) = -\frac{v}{c} \frac{u'(\xi)}{\sqrt{|u(\xi)|^2 + \frac{v^2}{c^2} |u'(\xi)|^2}}.$$

When looking for decreasing profiles, we observe that $r(\xi) \in [0, 1]$ for all $\xi \in \mathbb{R}$ (while $r(\xi) \in [-1, 1]$ for all $\xi \in \mathbb{R}$ if no monotonicity assumption is made). Moreover, if $u(-\infty) = 1$, $u(+\infty) = 0$ and u is smooth, then $u(\xi) \in [0, 1]$ for all $\xi \in \mathbb{R}$. Then, for smooth solutions, (2.1) is equivalent to the following first order planar dynamical system:

$$\begin{cases} u' = -\frac{c}{v} \frac{ru}{\sqrt{1-r^2}}, \\ r' = \frac{1}{u^{m-1}} \frac{c}{\sqrt{1-r^2}} \left(mu^{m-1} \frac{c}{v} r - \frac{\sigma}{v} \right) + \frac{F(u)}{cu^m}. \end{cases} \tag{2.2}$$

In what follows only decreasing traveling profiles will be studied, for these are the only reasonable traveling waves that can be obtained, as we show in forthcoming Proposition 3.2. Thus, through the present section we restrict the study of (2.2) to the set $[0, 1] \times [0, 1]$; this will be implicitly assumed in every statement referring to (2.2). We notice that the flux related to the previous system is singular at the boundaries $r = 1$ and $u = 0$. The first difficulty that we meet is precisely to give a sense to (2.2) at those points. Indeed, it will turn out that solutions of (2.2) eventually hit either $u = 0$ or $r = 1$. Thus, we will start considering solutions defined in $0 < r < 1$ and $0 < u < 1$, which give

rise to smooth (classical) traveling wave solutions of (2.1) in intervals of \mathbb{R} . Then, entropy solutions of (1.2) can be constructed by pasting those solutions while satisfying rules (i) and (ii) above. If the solutions of (2.1) are defined in all \mathbb{R} , they are smooth entropy solutions.

Remark 2.1 We note that the change of variables above does not coincide with the standard one in this type of problems.

In the next two subsections we analyze the planar system (2.2). The knowledge of the Rankine–Hugoniot relation (1.7) will be crucial to match solutions of (2.2) producing discontinuous profiles that satisfy the entropy conditions.

2.1 The blow-up sets of the planar system

The following characterization of the clustering points of the orbits solving (2.2) constitutes a key result in order to analyze the behavior of such orbits.

Proposition 2.2 *Let $(u, r) :]\omega_-, \omega_+[\rightarrow]0, 1[\times]0, 1[$ be a maximal solution of (2.2) with $\sigma > 0$. Then it satisfies the following:*

- (1) if $\omega_- = -\infty$, then $\lim_{\xi \rightarrow \omega_-} (u(\xi), r(\xi)) = (1, 0)$,
- (2) if $\omega_+ = +\infty$, then $\lim_{\xi \rightarrow \omega_+} (u(\xi), r(\xi)) = (0, r^*)$,
- (3) if $\omega_+ < +\infty$, then $\lim_{\xi \rightarrow \omega_+} (u(\xi), r(\xi)) = (u^+, 1)$ for some $u^+ \in [u^*, 1[$,
- (4) if $\omega_- > -\infty$, then the limit of $(u(\xi), r(\xi))$, as $\xi \rightarrow \omega_-$, belongs to one of the sets $]0, u^*] \times \{1\}$, $]0, 1[\times \{0\}$, or $\{1\} \times]0, 1[$.

The points $(u^*, 1)$ and $(0, r^*)$ are defined by

$$u^* := u^*(\sigma) = \left(\frac{\sigma}{c m} \right)^{\frac{1}{m-1}}$$

and

$$r^* := r^*(\sigma) = \frac{\frac{vK(0)}{c\sigma}}{\sqrt{1 + \left(\frac{vK(0)}{c\sigma} \right)^2}}$$

where, as we mentioned before, $K(0) = F'(0)$.

Proof We start the proof by justifying the existence of the limits $\lim_{\xi \rightarrow \omega_{\pm}} (u(\xi), r(\xi)) = (u^{\pm}, r^{\pm})$ for any solution of (2.2), then we deal with the four specific assertions of the proposition. We do this in a series of steps.

Step 1. Using standard arguments on continuation of solutions of an ODE, it is straightforward to deduce that the pairs (u^{\pm}, r^{\pm}) should belong to the boundary of $[0, 1] \times [0, 1]$ in case they exist. To show that these limits exist we pass to an equivalent system which is absent of singularities. This is achieved formally multiplying both equations in (2.2) by $u^{m-1} \sqrt{1-r^2}$. Thus, we end up with a system on $]0, 1[\times]0, 1[$ which is not singular,

$$\begin{cases} U' = -\frac{c}{v} R U^m, \\ R' = R \left(m U^{m-1} \frac{c}{v} R - \frac{\sigma}{v} \right) + \frac{K(U)}{c} \sqrt{1-R^2}. \end{cases} \quad (2.3)$$

Solutions of (2.3) are related to solutions of (2.2) by means of $r(\xi) = R(\phi(\xi))$, $u(\xi) = U(\phi(\xi))$, where ϕ is a strictly increasing reparametrization governed by $\phi'(\xi) = \frac{1}{u(\xi)^{m-1} \sqrt{1-r^2(\xi)}}$. The analysis of the directions of the flux on the boundaries of the (U, R) -domain is the same as the one we would perform for the (u, r) -system, but having the advantage that the flux is continuous in $[0, 1] \times [0, 1]$.

Step 2. Existence for the initial (and final) value problem for (2.3) is granted in the whole closed set. The regularity of the flow ensures uniqueness in the set $]0, 1] \times [0, 1[$. We also have uniqueness in the set $\{0\} \times [0, 1[$. This is seen as follows: First, if an orbit verifies that $U(\xi_0) = 0$ for some ξ_0 in its domain, then it is easily seen using the first equation of (2.3) that $U = 0$ in its whole domain of existence. Next, we notice that for every such orbit the second equation in (2.3) gives the value of R' as a smooth function of R alone, thus we have uniqueness of solutions for it.

Step 3. Note that r^* and u^* appear when we study the equilibria and bouncing points—see below—of the (U, R) -system. In fact, the points $(0, r^*)$ and $(1, 0)$ are equilibria. In addition, the flow in $]0, 1[\times]0, 1[$ points to the left, except at the boundaries, where we have the following flux analysis (see Fig. 2a):

- (1) If $R = 0$, $U \in]0, 1[$, the flux is completely vertical and pointing inwards.
- (2) If $U = 1$, $R \in]0, 1[$, the flux is always pointing inwards.
- (3) If $R = 1$, there are two possibilities depending on the value of σ . If $\sigma \geq cm$ the flux is always heading SW; notice that $u^* \geq 1$ in such a case. On the other hand, if $\sigma < cm$ and $U \in]0, u^*[$ the flux points SW while for $U \in]u^*, 1[$ the flux points NW.
- (4) If $U = 0$, we have a positively invariant manifold. For $R \in]0, r^*[$ the flux is completely vertical and pointing upwards. While for $R \in]r^*, 1[$ the flux is again completely vertical but pointing downwards. Solutions of (2.3) constrained to this manifold are globally attracted by $(U = 0, R = r^*)$.

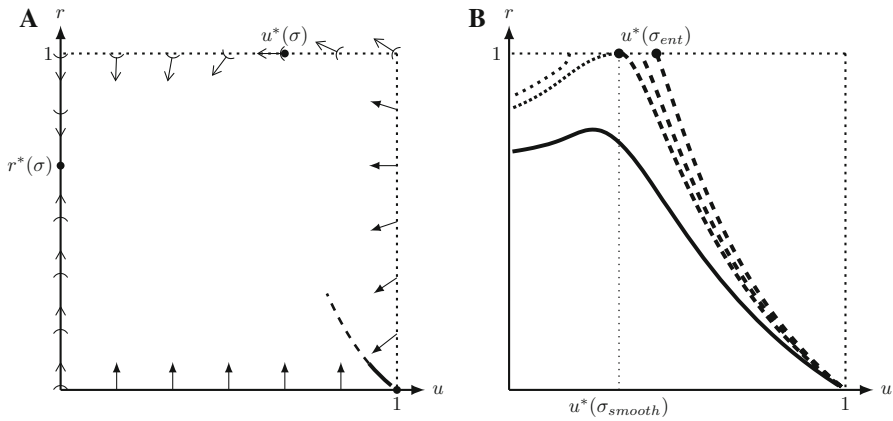


Fig. 2 **a** Normalized direction field of the flux related to (2.2) for $v = c = 1, m = 2, F(u) = u(1 - u)$ and $\sigma = 4/3$. Different types of arrows were used to stress the fact that the actual flux is singular at the boundaries $r = 1$ and $u = 0$. **b** Numerical solutions to Type I (solid), II (dashed) and III (dotted) orbits of (2.2) for several values of σ ; see Definitions 2.7 and 2.9 for an explanation of the terminology. The parameters are the same as those on **a**; with that choice $\sigma_{ent} = 0, 437803$ and $\sigma_{smooth} = 0, 661621$. The lowest Type II and III orbits are those corresponding to σ_{smooth} . The uppermost Type II orbit corresponds to σ_{ent} . The intermediate Type II orbit and the uppermost Type III orbit correspond to a value $\sigma \in]\sigma_{ent}, \sigma_{smooth}[$ and are related by the jump law (1.7). The values σ_{ent} and σ_{smooth} are defined in Propositions 2.16 and 2.18

(5) The point $(u^*, 1)$ is a regular bouncing point in the sense that the vector field is horizontal and pointing to the left. At this level of discussion we do not have tools to precise if (u^\pm, r^\pm) could be identified with $(u^*, 1)$ for some solution. As we will specify later, these possibilities can appear for some types of solutions.

Step 4. If (u, r) is a solution of (2.2) defined in $] \omega_-, \omega_+[$, then the monotone change of variables $\phi :] \omega_-, \omega_+[\rightarrow] \omega_-^{(U,R)}, \omega_+^{(U,R)}[$ allows to obtain $(U(\phi(\xi)), R(\phi(\xi))) = (U(\xi'), R(\xi'))$, which is a solution of (2.3) in the interval $] \omega_-^{(U,R)}, \omega_+^{(U,R)}[$. Now we show that the limits $\lim_{\xi' \rightarrow \omega_\pm^{(U,R)}} (U(\xi'), R(\xi'))$ exist. This is immediate for the component $U(\xi')$ as it is monotone. Thanks to our knowledge of the flux diagram at the boundaries and its continuity on $[0, 1] \times [0, 1]$, we are able to rule out wild oscillations of the orbits close to their hypothetical clustering points, thus the existence of $\lim_{\xi' \rightarrow \omega_\pm^{(U,R)}} R(\xi')$ follows easily.

Then we can ensure that

$$(u^\pm, r^\pm) = \lim_{\xi \rightarrow \omega_\pm} (u(\xi), r(\xi)) = \lim_{\xi' \rightarrow \omega_\pm^{(U,R)}} (U(\xi'), R(\xi')).$$

From the previous flux analysis (see Fig. 2a) we know that $(u^-, r^-) \in]0, 1[\times \{0\} \cup]0, u^*] \times \{1\} \cup \{1\} \times]0, 1[$. The event $(u^-, r^-) = (0, r^*)$ cannot take place since u is decreasing. Note also that $(u^+, r^+) \in \{(0, r^*)\} \cup [u^*, 1[\times \{1\}$. In the same way as above $(u^+, r^+) = (1, 0)$ is excluded since u is decreasing. We also observe that no solution starting at any point in $]0, 1[\times]0, 1[$ can reach the point $(1, 1)$ –it is never an exit point. It only can be an entrance point when $\sigma \geq mc$. Then, it will be always considered in the entrance set.

Step 5. Now we are ready to prove the precise assertions of the proposition. We start with the first one. To begin, we note that (u^-, r^-) does not belong to $]0, 1[\times \{0\} \cup \{1\} \times]0, 1[$ and that the flow (2.2) at $(1, 0)$ is regular. To show that $(u^-, r^-) \notin]0, u^*] \times \{1\}$ we will argue by contradiction (having proved that, the first assertion follows). For this purpose, we can use the monotonicity of u and the mean value theorem to construct a sequence $\xi_n \rightarrow -\infty$ for which $u'(\xi_n) \rightarrow 0$. This contradicts the fact that

$$\lim_{n \rightarrow +\infty} -\frac{r(\xi_n)u(\xi_n)}{\sqrt{1 - r(\xi_n)^2}} = -\infty.$$

Thus the first assertion is verified.

To prove the second assertion, it is enough to remark that $(u^+, r^+) \notin [u^*, 1[\times \{1\}$. This can be proved by a similar contradiction argument as in the previous case but taking here a sequence $\xi_n \rightarrow \infty$ for which $u'(\xi_n) \rightarrow 0$.

The third assertion follows if we can prove that $\lim_{\xi \rightarrow \omega_+} (u(\xi), r(\xi)) = (0, r^*)$ cannot hold for $\omega_+ < +\infty$. Integrating for $u(\xi)$ in (2.2) leads us to

$$u(\xi) = u(\bar{\xi}) \exp \left\{ -\frac{c}{v} \int_{\bar{\xi}}^{\xi} \frac{r(s) ds}{\sqrt{1 - r^2(s)}} \right\} \tag{2.4}$$

where $\bar{\xi} \in]\omega_-, \omega_+[$. If we are to have $u(\omega_+) = 0$ for some $\omega_+ < +\infty$, then we need the above integral to be divergent for $\xi = \omega_+$. But this cannot happen as $r(\xi)$ tends to $r^* < 1$ when ξ goes to ω_+ .

Finally, to prove the fourth assertion we have to exclude the case $(u^-, r^-) = (1, 0)$. But this is an easy consequence of the fact that $(1, 0)$ is a regular equilibrium of (2.2). □

Remark 2.3 Note that when $\sigma = 0$ we have that $r^* = 1$ and then the proof of Proposition 2.2 breaks down, namely (2.4) is no longer useful. However, in this case the behavior of the orbits is simpler as we will show in Proposition 2.15 below.

Remark 2.4 In the proof of Proposition 2.2 we have introduced an auxiliary regular system in order to analyze the direction fields in $[0, 1] \times [0, 1]$. Note

that the cases $r = 1$ and $u = 0$ are singular for (2.2), but we can analyze the direction fields on these sets as “limits” of those in the regular regions. These direction fields coincide with those of the regular system (2.3), see Fig. 2a.

2.2 Solutions to the planar system defined on a half line

It will be proved that the construction of piecewise smooth traveling waves to (1.2) uses only maximal orbits of (2.2) (see Proposition 3.2). In Fig. 3 we have compiled the different options given by Proposition 2.2, that defines maximal solutions as well as their corresponding orbits. Only the cases a and b in Fig. 3 are useful to construct piecewise solutions, while the remaining cases cannot be taken into account because they violate that requirement of entropy

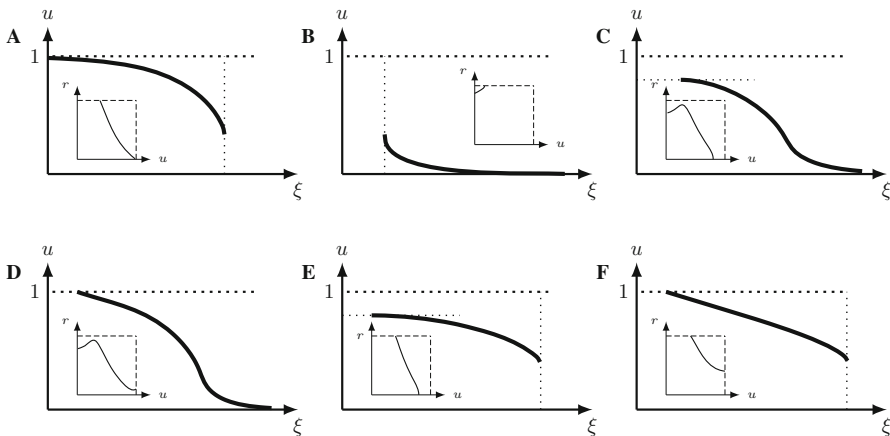


Fig. 3 This figure shows different situations that follow from Proposition 2.2. Each curve represents the decreasing solutions (2.1) in the plane (ξ, u) , where the dashed line is $u \equiv 1$. The small window in each graph represents the orbit, i.e. the inclusion of these solutions as a first component of the system (2.2) in $(0, 1)^2$. Vertical dotted lines in **a**, **b**, **e** and **f** show points with infinite slope. Horizontal dotted lines in **c** and **e** show points with zero slope. These graphs are obtained by combining the options (1) and (4) with (2) and (3) of Proposition 2.2. Option (1) + (2) is skipped in this chart, because it was already shown in Fig. 1a, and corresponds to the classic profile of a traveling wave defined in \mathbb{R} . **a** Stems from the combination of (1) and (3) and is a solution defined on a half-line bounded to the right. **b**, **c** and **d** Correspond to (2) + (4), in which the various options of (4) are combined. All of these cases are solutions defined in a half-line bounded to the left. **e** and **f** Are obtained by combining (3) with the latest options of (4). The first option of (4) cannot appear together with (3) because u is necessarily decreasing and not constant. By adding to these figures the solutions $u = 0$ and $u = 1$ we have the full range of possible admissible (decreasing and constant) solutions of (2.2) in $u \in [0, 1]$. Notice that **c** and **e**, viewed as solutions of (2.2), are not maximal as they can be continued by a downward branch at the lower end point. Also, extending the reaction term F to a neighborhood of $u = 1$, neither **d** nor **f** are maximal. Therefore, there only remain **a** and **b** as maximal solutions not defined globally

solutions consisting of having a vertical graph in the possible matching points (see the Step 2 in the proof of Proposition 3.2).

So, in this section we will deal with solutions globally defined in the whole \mathbb{R} , or with solutions defined in a half line (that is $] - \infty, \omega_+[$ or $] \omega_-, +\infty[$) and such that their slope at ω_{\pm} is infinite.

The following theorem describes all those orbits defined in \mathbb{R} or $] - \infty, \omega_+[$.

Theorem 2.5 *For each value of $\sigma \geq 0$ there exists, up to reparametrization, a unique solution of (2.2) defined in $] - \infty, \omega_+[$ with $\omega_+ \leq \infty$. Furthermore, this solution satisfies:*

$$\lim_{\xi \rightarrow -\infty} (u(\xi), r(\xi)) = (1, 0) \tag{2.5}$$

and $\lim_{\xi \rightarrow -\infty} u'(\xi) = 0$. Finally, we have that

$$\lim_{\xi \rightarrow -\infty} \frac{r'(\xi)}{u'(\xi)} = \lambda_{\sigma} = \frac{1}{c} \frac{2vK'(1)}{\sigma + \sqrt{\sigma^2 - 4K'(1)v}} < 0. \tag{2.6}$$

Proof Let us observe that if any such orbit is to exist, then it has to verify (2.5), thanks to Proposition 2.2. The linearization of the system near $(u = 1, r = 0)$ gives the following Jacobian matrix:

$$\begin{pmatrix} 0 & -\frac{c}{v} \\ \frac{K'(1)}{c} & -\frac{\sigma}{v} \end{pmatrix}.$$

After computing its eigenvalues

$$\lambda_{\sigma}^{\pm} = -\frac{\sigma}{2v} \pm \frac{1}{2} \sqrt{\frac{\sigma^2}{v^2} - \frac{4K'(1)}{v}},$$

we learn that this point is hyperbolic. This allows to apply the unstable manifold theorem (see [35]). Since the eigenvector associated with λ_{σ}^{+} is $(c \frac{\sigma + \sqrt{\sigma^2 - 4vK'(1)}}{2vK'(1)}, 1)$, the solutions starting at $(1, 0)$ enter the diagram, and they do it with a slope given by (2.6). Thanks to the Hartman–Großman theorem, these solutions are uniquely determined for a given value of σ . \square

Remark 2.6 The lower the value of σ , the higher the entrance angle (measured with respect to the u -axis). There is a maximum value for $\sigma = 0$, namely

$$\lambda_0 = -\frac{\sqrt{v|K'(1)|}}{c}.$$

We also stress that λ_{σ} is increasing as a function of σ .

At this point we introduce some terminology that will be useful in the sequel, see Fig. 2. We name the particular types of trajectories we will be interested in.

Definition 2.7 Let $(u(\xi), r(\xi)) \in]0, 1[\times]0, 1[$ be a maximal solution of (2.2)–(2.5) for a given value of σ . We shall say that:

- (u, r) is a Type I orbit if (u, r) is defined in \mathbb{R} and $\lim_{\xi \rightarrow +\infty} (u(\xi), r(\xi)) = (0, r^*(\sigma))$
- (u, r) is a Type II orbit if (u, r) is defined in $] - \infty, \omega_+[$, with $\omega_+ < \infty$, and $\lim_{\xi \rightarrow \omega_+} (u(\xi), r(\xi)) = (u^+, 1)$ for some $0 < u^+ < 1$.

For a given value of σ only one of these two possibilities occurs.

Note that no Type II orbit can show up for $\sigma \geq mc$, since $u^*(\sigma) \geq 1$.

Remark 2.8 Any Type I orbit induces a smooth solution to (2.1) and satisfies $\lim_{\xi \rightarrow +\infty} u(\xi) = 0$. Such profiles are depicted in Fig. 1a. Type II orbits can be found in Fig. 1 too; they correspond to the left branches (with respect to the vertical dotted line) in the cases b, c and d.

We stress that all profiles coming from Type I orbits are regular solutions of (2.2) supported in the whole line. Type II orbits may also give rise to traveling wave solutions, after a suitable matching procedure to extend them to the whole real line is performed. This will be explained in Sect. 3, but before that we need the following:

Definition 2.9 We will say that a maximal solution $(u(\xi), r(\xi))$ of (2.2) for a given value of σ is a Type III orbit if (u, r) is defined in $] \omega_-, +\infty[$, $\lim_{\xi \rightarrow \omega_-} (u(\xi), r(\xi)) = (u^-, 1)$ for some $0 < u^- \leq u^*(\sigma)$ and $\lim_{\xi \rightarrow +\infty} (u(\xi), r(\xi)) = (0, r^*(\sigma))$.

As regards the uniqueness of Type II and III orbits with respect to the beginning or ending point (in the limit sense established in Proposition 2.2) we will describe the orbits of the planar system as graphs $u \mapsto r(u)$ whenever this is possible. This is always the case if we are prepared to allow some derivatives to become infinite eventually (and this may happen only at the boundaries of the domain). Indeed, if a trajectory can be expressed locally as a graph $u \mapsto r(u)$, its derivative is given by

$$\frac{dr(u)}{du} = \frac{r'}{u'} = \frac{\sigma}{cu^m} - m \frac{r}{u} - \frac{K(u)}{u^m} \frac{v}{c^2} \frac{\sqrt{1-r^2}}{r}. \tag{2.7}$$

Lemma 2.10 *The formulations (2.2) and (2.7) are equivalent. Moreover, regular solutions to (2.2)–(2.5) correspond to solutions of (2.7) such that*

$$\lim_{u \rightarrow 1} r(u) = 0. \quad (2.8)$$

In addition, $\lim_{u \rightarrow 1} r'(u) = \lambda_\sigma$ holds.

Proof This is straightforward once we notice that $u' < 0$ in the domain. \square

Once we are able to pass to the formulation given by (2.7), we get the following result.

Lemma 2.11 *Existence and uniqueness for (2.7) holds backwards at any point of the form $(\tilde{u}, 1)$ with $u^*(\sigma) \leq \tilde{u} < 1$. Existence and uniqueness for (2.7) holds forwards at any point of the form $(\tilde{u}, 1)$ with $0 < \tilde{u} \leq u^*(\sigma)$ and $\tilde{u} < 1$.*

Proof The existence problem is addressed by solving the initial value problem for (2.7) with $r(\tilde{u}) = 1$, being $\tilde{u} \neq u^*(\sigma)$. To do this we will use Peano's existence theorem, and for that we need a continuous extension of (2.7) to values $r > 1$.

In the case $\tilde{u} > u^*(\sigma)$ we have $r'(\tilde{u}) < 0$ and, therefore, the function r defined on $]\tilde{u}, \tilde{u} + \epsilon[$, for some small $\epsilon > 0$, maps into $]0, 1[$, solving our original problem. This solution can be extended to $]\tilde{u}, 1[$ and it verifies $r(u) < 1$; otherwise we could find a first value $u_0 > \tilde{u}$ such that $r(u_0) = 1$ and $r'(u_0) < 0$, which would give us a contradiction.

In a similar way, for $\tilde{u} < u^*$ we find a solution of (2.7) on $]\tilde{u} - \epsilon, \tilde{u}[$ such that $r'(\tilde{u}) > 0$, which can be extended to $]0, \tilde{u}[$.

The case $\tilde{u} = u^*(\sigma)$ can be treated by approximation. In fact, taking a sequence $\tilde{u}_n \rightarrow u^*(\sigma)$, $\tilde{u}_n \neq u^*(\sigma)$ and depending on either $\tilde{u}_n < u^*$ or $\tilde{u}_n > u^*$, we find a solution on $]0, u^*(\sigma)[$ or on $]u^*(\sigma), 1[$. Note that the approximating sequence has a partial subsequence which is convergent, via Peano's theorem on continuous dependence with respect to initial conditions and parameters. The limit verifies $r(u) \leq 1$ either on the case of solutions in $]0, u^*(\sigma)[$ or on the case of solutions in $]u^*(\sigma), 1[$. Let us analyze this last case: a solution of the extended problem (2.7) cannot be equal to 1 on an interval having $u^*(\sigma)$ as its left end, since this is not coherent with the values of the flux defined by (2.3) at the boundary. On the other hand, the fact that $r(u) < 1$ for some value u implies that $r(u) < 1$ for greater values and thus for $u \in]u^*(\sigma), 1[$.

In every case, the uniqueness follows from the change of variable $\sqrt{1 - r^2(u)} = s(u)$ leading to the following differential equation:

$$\begin{aligned} s' &= \frac{1}{s} \left(-\frac{\sigma}{cu^m} + \frac{m}{u} \right) - \frac{\sigma(-s^2)}{cu^m s (\sqrt{1 - s^2} + 1)} - \frac{ms}{u} + \frac{k(u)}{u^m} \frac{v}{c^2} \\ &= \frac{1}{s} \left(-\frac{\sigma}{cu^m} + \frac{m}{u} \right) + h(u, s), \end{aligned}$$

where $h(u, s)$ is a Lipschitz function in the second variable in a neighborhood of $(u, s = 0)$, and $s \rightarrow \frac{1}{s} \left(-\frac{\sigma}{cu^m} + \frac{m}{u} \right)$ is a decreasing function if $u > u^*(\sigma)$, resp. increasing if $u < u^*(\sigma)$. We conclude by the classical uniqueness results for equations with right hand side given by a Lipschitz part plus a monotone part [35]. \square

The following result allows to construct Type III orbits which start at any point of $]0, u^*(\sigma)] \times \{1\}$.

Proposition 2.12 *For any $u^- \in]0, u^*(\sigma)]$, there exists a Type III orbit that satisfies (2.2) and such that*

$$\lim_{t \rightarrow \omega_-} (u(t), r(t)) = (u^-, 1). \tag{2.9}$$

Moreover, this orbit is unique up to reparametrizations.

Proof Let $u^- \in]0, u^*(\sigma)]$. Consider the solution of (2.7) such that

$$r(u^-) = 1, \tag{2.10}$$

and let (r_0, u_0) be a point on the graph of r , with $u_0 \in]0, u^-[$. It is easy to see that the solution of

$$u' = -\frac{c}{v} \frac{r(u)u}{\sqrt{1 - r(u)^2}} \tag{2.11}$$

verifying $u(t_0) = u_0$, for any given $t_0 \in \mathbb{R}$, supplemented with $r(t) = r(u(t))$, is a solution of (2.2) which blows up in $\omega_- > -\infty$ (singular in u^-) and which satisfies (2.9). We have also that any Type III orbit verifying (2.9) is a reparametrization of the above one. Thus, a solution (r, u) of (2.2)–(2.9) is a Type III orbit such that $u'(t) < 0$ in $]\omega_-, +\infty[$. Therefore, we can deduce that u is a diffeomorphism with its image, which necessarily is $]0, u^-[$. Then, we can invert it and take $r \circ u^{-1} :]0, u^-[\rightarrow]0, 1[$, which verifies (2.7)–(2.10). Hence, $r \circ u^{-1} \equiv r(t)$ and, as a consequence, u is a solution of (2.11). This allows to assure that u differs from the previous solution just by a reparametrization. The same happens with r , as it is obtained from $u(t)$. \square

2.3 Bifurcation from Type I to Type II orbits

The goal of this section is to analyze the structure of discontinuous solutions of (1.2), in terms of the special classes of orbits defined in Sect. 2.2. Our first aim is to describe the set of values of σ for which these singular solutions can be constructed.

Definition 2.13 Let

$$\Sigma_I = \{\sigma \geq 0 : \text{the associated solution of 2.2–2.5 is a Type I orbit}\}.$$

Also, let

$$\Sigma_{II} = \{\sigma \geq 0 : \text{the associated solution of 2.2–2.5 is a Type II orbit}\}.$$

Note that Σ_{II} is bounded from above by mc .

According to Lemma 2.10, for every $\sigma \in \Sigma_I$ the corresponding solution of (2.2)–(2.5) derives from a solution $r_\sigma :]0, 1[\rightarrow]0, 1[$ of (2.7)–(2.8) satisfying

$$\lim_{u \rightarrow 0} r_\sigma(u) = r^*.$$

When $\sigma \in \Sigma_{II}$, we consider the escape point $u^+(\sigma)$ of (2.2)–(2.5) through $r = 1$ as introduced in Definition 2.7. Then, $r_\sigma :]u^+(\sigma), 1[\rightarrow]0, 1[$ is a solution of (2.7)–(2.8). The function u^+ is, therefore, defined from Σ_{II} to $]0, 1[$ and verifies $u^+(\sigma) \geq u^*(\sigma)$. The way to recover the solutions of (2.2)–(2.5) from the solutions of (2.7)–(2.8) is to integrate the differential equation (2.11) as in Proposition 2.12.

Now we show that the orbits which are candidates for representing traveling wave profiles are ordered with respect to σ .

Lemma 2.14 *If $\sigma_1 < \sigma_2$, then $r_{\sigma_1}(u) > r_{\sigma_2}(u)$ in their common domain of definition.*

Proof Recall that λ_σ increases strictly as σ increases (Remark 2.6). Then $r_{\sigma_1}(u) > r_{\sigma_2}(u)$ in a neighborhood of $u = 1$. If our thesis is false then there exists a first value $0 < \tilde{u} < 1$ such that $r_{\sigma_1}(\tilde{u}) = r_{\sigma_2}(\tilde{u})$. Then, the hypothesis $\sigma_1 < \sigma_2$ implies by (2.7) that $\frac{dr_{\sigma_1}}{du}(\tilde{u}) < \frac{dr_{\sigma_2}}{du}(\tilde{u})$, which constitutes a contradiction. \square

The above result implies that Σ_I and Σ_{II} are intervals, i.e., given a value of σ such that the corresponding orbit is a Type I (resp. Type II) orbit, then this is also the case for upper (resp. lower) values of σ ; we have also that $u^+(\sigma)$ is a decreasing function of σ . Moreover, $\Sigma_I \cup \Sigma_{II} = [0, +\infty[$ (it is in fact a Dedekind cut) and in the sense of sets $\Sigma_{II} < \Sigma_I$, i.e., if $\sigma \in \Sigma_{II}$, then Σ_{II} contains all values below this σ , $[0, \sigma] \subset \Sigma_{II}$ and if $\sigma \in \Sigma_I$, then Σ_I contains all values above this σ , $[\sigma, +\infty[\subset \Sigma_I$; moreover all the elements of Σ_{II} are below those of Σ_I . Note that at this stage we have not ruled out the possibility of having $\Sigma_{II} = \emptyset$ yet. The following result takes care of this issue.

Proposition 2.15 *The maximal orbit associated with $\sigma = 0$ verifying (2.7)–(2.8) exits the phase diagram by a point $(u^+(0), 1)$ with $u^+(0) > 0$.*

Proof The Eq. (2.7) can be recast after multiplication by u^m as

$$u^m r'(u) + mu^{m-1}r(u) = -\frac{\nu}{c^2}K(u)\frac{\sqrt{1-r^2}}{r}.$$

This implies

$$\frac{d}{du}(u^m r(u)) \leq 0.$$

Assume now that $r(u)$ is defined for $u \in]0, 1[$. Then $\lim_{u \rightarrow 0} u^m r(u) = 0$ and thus $u^m r(u)$ is identically equal to zero, which contradicts the fact that the slope at $u = 1$ is known to be strictly negative. This shows that $r(u)$ is defined only in an interval $]u^+(0), 1[$ and that $\lim_{u \rightarrow u^+(0)} r(u) = 1$. \square

The previous result shows that Σ_{II} is not empty, at least it contains the value zero. In the next Proposition we prove that it contains a non-trivial interval of values and we characterize its supremum.

Proposition 2.16 *The value $\sigma_{smooth} = \sup\{\sigma : \sigma \in \Sigma_{II}\}$ verifies $mc > \sigma_{smooth} > 0$. Moreover:*

- (1) *any maximal solution satisfying (2.2)–(2.5) with $\sigma > \sigma_{smooth}$ is a Type I orbit,*
- (2) *any maximal solution satisfying (2.2)–(2.5) with $\sigma \leq \sigma_{smooth}$ is a Type II orbit.*

Furthermore, σ_{smooth} is the unique value of σ having the property that the associated solution, which is a Type II orbit, terminates at the point $(u^(\sigma), 1)$. Then $u^+(\sigma_{smooth}) = u^*(\sigma_{smooth})$.*

Proof We notice that for $\sigma > \sigma_{smooth}$ we get Type I orbits, while for $\sigma < \sigma_{smooth}$ we get Type II orbits. By Remark 2.8 we also know that $\sigma_{smooth} < mc$. It only remains to prove that $\sigma_{smooth} > 0$, that this value belongs to Σ_{II} , and to verify that $u^+(\sigma_{smooth}) = u^*(\sigma_{smooth})$. These claims follow from the continuous dependence of solutions of (2.2)–(2.5) w.r.t σ . More precisely, \square

Lemma 2.17 *Consider the maximal solutions of (2.7) extended to the right end by means of $r_{\sigma_n}(1) = 0$. The following assertions are satisfied:*

- (1) *Let $\{\sigma_n\}_{n \geq 0}$ be a monotonically decreasing sequence such that $\sigma_n \rightarrow \sigma_\infty$, with $\sigma_\infty \in \Sigma_{II}$. Then the sequence $\{r_{\sigma_n}\}$ of solutions to (2.7)–(2.8) converges uniformly on compact sets of $]u^+(\sigma_\infty), 1[$ to r_{σ_∞} . Moreover, if $\sigma_n \in \Sigma_I$ for all n , then $u^+(\sigma_\infty) = u^*(\sigma_\infty)$, and if $\sigma_n \in \Sigma_{II}$ for advanced n , then $u^+(\sigma_n) \rightarrow u^+(\sigma_\infty)$.*

(2) Let $\{\sigma_n\}_{n \geq 0}$ be a monotonically increasing sequence such that $\sigma_n \rightarrow \sigma_\infty$, with $\sigma_n \in \Sigma_{II}$. Then, $\sigma_\infty \in \Sigma_{II}$ and $u^+(\sigma_n) \rightarrow u^+(\sigma_\infty)$. In addition, the sequence $r_{\sigma_n} :]u^+(\sigma_n), 1] \rightarrow]0, 1]$ converges on compact sets of $]u^+(\sigma_\infty), 1]$ to r_{σ_∞} .

Proof As we pointed in the statement of the lemma, in this proof we will consider the functions r_σ to be extended by continuity to their value at $u = 1$, even if the differential equation (2.7) is defined only on the open interval, being singular at $r = 0$. We split the proof into two steps.

Step 1. Let us start by proving the second assertion. We stress that, due to Lemma 2.14 and to the monotonicity of σ_n , the sequence r_{σ_n} has increasing intervals of definition $]u^+(\sigma_n), 1]$. Define $\alpha = \inf_{n \in \mathbb{N}} \{u^+(\sigma_n)\}$; our aim is to prove that $u^+(\sigma_\infty)$ is well defined and coincides with α .

As a consequence of Lemma 2.14 we have

$$\begin{cases} \text{either } r_{\sigma_\infty}(u) \text{ is defined on } u \in]0, 1] \\ \text{or } u^+(\sigma_\infty) \leq \alpha. \end{cases} \tag{2.12}$$

Moreover, given $u \in]\alpha, 1]$, from Lemma 2.14 we deduce that the value $r_{\sigma_n}(u)$ is defined for $n \in \mathbb{N}$ large enough. Furthermore, these values constitute a decreasing sequence. Then, we define

$$\begin{cases} \tilde{r} :]\alpha, 1] \rightarrow]0, 1] \\ \tilde{r}(u) = \lim_{n \rightarrow \infty} r_{\sigma_n}(u). \end{cases}$$

The alternative (2.12) implies that the domain of any r_{σ_n} is contained in that of \tilde{r} . Using again Lemma 2.14 we obtain

$$r_{\sigma_\infty}(u) \leq \tilde{r}(u), \quad \forall u \in]\alpha, 1]. \tag{2.13}$$

Now, the inequality $u^+(\sigma_n) \geq u^*(\sigma_n)$ leads to

$$\alpha \geq u^*(\sigma_\infty) > 0. \tag{2.14}$$

We also have the following estimate which is independent of n

$$\frac{K(u)}{r_{\sigma_n}(u)} \leq \sup_{u \in]\alpha, 1[} \frac{K(u)}{r_{\sigma_\infty}(u)}, \quad \forall u \in]u^+(\sigma_n), 1[. \tag{2.15}$$

Note that the function $u \in]\alpha, 1[\mapsto \frac{K(u)}{r_{\sigma_\infty}(u)}$ is bounded at both endpoints –recall that $K'(1)$ exists– and thus on its whole domain.

Combining (2.14), (2.15) and (2.7) we deduce

$$|r'_{\sigma_n}(u)| \leq M, \quad \forall u \in]u^+(\sigma_n), 1[, \tag{2.16}$$

where M is independent of n . Then, Ascoli’s theorem ensures that r_{σ_n} converges to \tilde{r} uniformly on compact sets of $] \alpha, 1]$; the bound (2.16) is also valid at the boundary once all the objects are properly extended. In particular, \tilde{r} is a continuous function verifying (2.8). At the same time, (2.13) ensures that r'_{σ_n} also converges uniformly on compact sets of $] \alpha, 1 [$. Hence, \tilde{r} satisfies (2.7) and thanks to Lemma 2.10 we have $\tilde{r} \equiv r_{\sigma_\infty}$ on $] \alpha, 1]$.

Finally, let us prove that $\alpha = u^+(\sigma_\infty)$. For that, it suffices to check that

$$\lim_{u \rightarrow \alpha} \tilde{r}(u) = 1. \tag{2.17}$$

To do this, let $\epsilon > 0$ and let $u \in] \alpha, \alpha + \epsilon [$. Then for n large enough we have that $|r_{\sigma_n}(u) - \tilde{r}(u)| < \epsilon$. Since also $u^+(\sigma_n) > \alpha$ for n large enough, we have that $|u - u^+(\sigma_n)| \leq \epsilon$. Then, the mean value theorem and estimate (2.16) yield that $|r_{\sigma_n}(u) - 1| \leq M\epsilon$. Since $u \in] \alpha, \alpha + \epsilon [$ we find $|\tilde{r}(u) - 1| \leq \epsilon + M\epsilon$, and (2.17) follows. This finishes the proof of the second assertion.

Step 2. We are now concerned with the proof of the first assertion. Due to Lemma 2.14 the sequence of functions $\{r_{\sigma_n}\}_{n \in \mathbb{N}}$ is defined on a common interval: either $] \alpha, 1]$ with $\alpha = \sup_{\sigma_n \in \Sigma_{II}} \{u^+(\sigma_n)\} > 0$ or $] 0, 1]$ when $\sigma_n \in \Sigma_I$ for any $n \in \mathbb{N}$.

In the first case, the same argument as in the previous step leads to

$$\frac{K(u)}{r_{\sigma_n}(u)} \leq \sup_{u \in] \alpha, 1 [} \frac{K(u)}{r_{\sigma_1}(u)},$$

which allows to prove the uniform convergence of r_{σ_n} to a function \tilde{r} on $] \alpha, 1]$. Note that the uniform convergence on compact sets of the sequence r'_{σ_n} allows to deduce that \tilde{r} verifies (2.7); this is possible because $\tilde{r}(u) \geq r_{\sigma_1}(u) > 0$, $\forall u \in] \alpha, 1 [$. This function \tilde{r} coincides (because it is a pointwise limit) with r_{σ_∞} on $] u^+(\sigma_\infty), 1 [$ since in this case Lemma 2.14 ensures that

$$\alpha \leq u^+(\sigma_\infty). \tag{2.18}$$

The case $\alpha \neq u^+(\sigma_\infty)$ can be excluded by a contradiction argument. In fact, \tilde{r} is a C^1 function such that $\tilde{r}(u^+(\sigma_\infty)) = 1$ and $\tilde{r} \leq 1$. Thus, this value is a maximum and, therefore, $\tilde{r}'(u^+(\sigma_\infty)) = 0$. Taking into account (2.7) and

$$\tilde{r}'(u^+(\sigma_\infty)) = \lim_{u \rightarrow u^+(\sigma_\infty)} \tilde{r}'(u)$$

we find $u^+(\sigma_\infty) = u^*(\sigma_\infty)$. Therefore, $\lim_{n \rightarrow \infty} u^*(\sigma_n) \leq \alpha = \lim_{n \rightarrow \infty} u^+(\sigma_n) < u^+(\sigma_\infty) = u^*(\sigma_\infty)$, which is in contradiction with (2.18).

It remains to study the case $\sigma_n \in \Sigma_I$, for any $n \in \mathbb{N}$. Then, we use a value $\alpha \in]0, u^+(\sigma_\infty)[$ to argue in a similar way as we did above and we finish once we find that $u^+(\sigma_\infty) = u^*(\sigma_\infty)$. \square

(Proof of Proposition 2.16, continued)

Assertion 1) in Lemma 2.17 together with Proposition 2.15 prove that $\sigma_{smooth} > 0$, because $u^*(\sigma = 0) < u^+(\sigma = 0)$. Then we can apply Lemma 2.17.(2) to any sequence that converges to σ_{smooth} and we conclude that this value belongs to Σ_{II} . Finally, σ_{smooth} is the only value σ such that $u^*(\sigma) = u^+(\sigma)$. This follows from Lemma 2.17.(1) and the facts that $\sigma \mapsto u^+(\sigma)$ is strictly decreasing (see the paragraph after Lemma 2.14) and $\sigma \mapsto u^*(\sigma)$ is strictly increasing. At this point we can use Proposition 2.12 to show that the orbit associated to σ_{smooth} can be extended as a continuous curve further to the right (matching with a Type III orbit). More is true, as we show in our next result, which is paramount in order to characterize completely the discontinuous traveling wave solutions of (1.2).

Proposition 2.18 *There exists a value $0 < \sigma_{ent} < \sigma_{smooth}$ such that the following assertions hold true in the range $\sigma_{ent} \leq \sigma \leq \sigma_{smooth}$:*

- (1) *Any Type II orbit can be extended to the whole \mathbb{R} matching it with a Type III orbit.*
- (2) *There is only one way to perform the aforementioned matching. It is given by the following formula:*

$$\sigma = c \frac{(u^+(\sigma))^m - (u^-(\sigma))^m}{u^+(\sigma) - u^-(\sigma)}. \quad (2.19)$$

Here $(u^+(\sigma), 1)$ is the arrival point for the Type II orbit and $(u^-(\sigma), 1)$ is the departure point for the Type III orbit.

- (3) *Moreover $\sigma \mapsto u^-(\sigma)$ is a continuous, strictly increasing mapping, and the value σ_{ent} is defined as the value of $\sigma \geq 0$ for which*

$$\lim_{\sigma \rightarrow \sigma_{ent}} u^-(\sigma) = 0.$$

In addition, we have

$$\lim_{\sigma \rightarrow \sigma_{smooth}} u^-(\sigma) = u^+(\sigma_{smooth}) = u^*(\sigma_{smooth}).$$

The proof of this result is just a consequence of the previous ideas, together with Proposition 1.3 and the following statement.

Lemma 2.19 *Let $\sigma \leq \sigma_{smooth}$. Then, if (2.19) is fulfilled, there must hold that $\sigma > \sigma_{smooth}/m$. Whenever (2.19) holds, the pair $(u^+(\sigma), u^-(\sigma))$ is unique and the mapping $\sigma \rightarrow u^-(\sigma)$ is strictly increasing. Finally, (2.19) holds at least for a neighborhood $]\sigma_{smooth} - \epsilon, \sigma_{smooth}]$ of σ_{smooth} .*

Proof For the sake of clarity we will denote $u^+(\sigma), u^-(\sigma)$ by u^+, u^- whenever this creates no confusion. Given the value u^+ , we want to figure out the value of u^- in order that (2.19) holds. It may happen that no such value exists. To deal with this issue, we consider the continuous function

$$\psi(u^+, x) = \begin{cases} \frac{(u^+)^m - x^m}{u^+ - x} & \text{if } x \neq u^+ \\ m(u^+)^{m-1} & \text{if } x = u^+ \end{cases}$$

defined for $x \in [0, \infty[$. The first thing to note is that (2.19) is trivially satisfied for $\sigma = \sigma_{smooth}$ with $u^+ = u^-$. Note that once u^+ is fixed ψ is a strictly increasing function, since

$$\frac{\partial \psi(u^+, x)}{\partial x} = \frac{(u^+)^m}{(u^+ - x)^2} \left((m - 1) \left(\frac{x}{u^+} \right)^m - m \left(\frac{x}{u^+} \right)^{m-1} + 1 \right) > 0 \text{ for } x \neq u^+.$$

Thus, $\psi(u^+, \cdot)$ is a bijection, $\psi(u^+, \cdot) : [0, u^+] \rightarrow [(u^+)^{m-1}, m(u^+)^{m-1}]$. We must check if σ/c belongs to the latter interval. As $\sigma < \sigma_{smooth}$, $u^* = u^+ > (\sigma/(mc))^{1/(m-1)}$ we deduce that

$$\frac{\sigma}{c} < m(u^+)^{m-1}.$$

It remains to be determined when do we have that $(u^+)^{m-1} \leq \sigma/c$. Notice that for $\sigma = \sigma_{smooth}$ the above inequality is strict. Thus, it continues to hold for some neighborhood $]\sigma_{smooth} - \epsilon, \sigma_{smooth}]$, thanks to Lemma 2.17 (more precisely, we know that the value of u^+ increases as σ decreases). The previous arguments ensure that in such a case there is a unique pair (u^+, u^-) verifying (2.19). We can also prove that the mapping $\sigma \mapsto u^-(\sigma)$ is strictly increasing, because if this is not the case the existence of values $\sigma_1 < \sigma_2$ such that $u^-(\sigma_1) \geq u^-(\sigma_2)$ leads to the following contradiction:

$$\frac{\sigma_1}{c} = \psi(u^+(\sigma_1), u^-(\sigma_1)) > \psi(u^+(\sigma_2), u^-(\sigma_2)) = \frac{\sigma_2}{c},$$

where we have used that the function ψ is increasing in both variables (by symmetry) and the fact that $u^+(\sigma_1) > u^+(\sigma_2)$.

Now we show that $u^- \leq u^*$, so that $(u^-, 1)$ can be a departure point for a Type III orbit. More precisely, either $u^+ = u^* = u^-$ or $u^+ > u^* > u^-$. To show that, we write

$$(u^+)^m - (u^-)^m = \int_{u^-}^{u^+} ms^{m-1} ds.$$

Under any of the events $u^+ \geq u^- > u^*$ or $u^+ > u^- \geq u^*$ we have

$$(u^+)^m - (u^-)^m > m(u^*)^{m-1}(u^+ - u^-).$$

Then we learn that $\sigma/c > m(u^*)^{m-1} = \sigma/c$, which constitutes a contradiction. This implies that $u^+ > u^* > u^-$ or $u^+ = u^* = u^-$. Finally, the necessary condition $\sigma > \sigma_{smooth}/m$ shows up at once, since $\sigma_{smooth}/mc = (u^+(\sigma_{smooth}))^{m-1}$ and thus the relation $u^+(\sigma)^{m-1} \leq \sigma/c$ (which was seen to be required in order that an admissible choice of u^- exists) cannot hold for $\sigma = \sigma_{smooth}/m$, being the map $\sigma \mapsto u^+(\sigma)$ strictly decreasing. \square

Remark 2.20 When $m = 2$ the condition (2.19) reduces to

$$u^- = \frac{\sigma}{c} - u^+.$$

Remark 2.21 Estimates so far show that

$$\sigma_{smooth}/m < \sigma_{ent} < c \quad \text{and} \quad \sigma_{ent} < \sigma_{smooth} < mc.$$

This is coherent with the case $m = 1$ [19].

3 Construction of traveling wave solutions

The purpose of this section is to prove Theorem 1.4. Let us first precise that our solutions satisfy the property of having null flux at infinity. First of all, we have proved in Theorem 2.5 that $\lim_{\xi \rightarrow -\infty} u'(\xi) = 0$, but it is also true that $\lim_{\xi \rightarrow \infty} u'(\xi) = 0$ because $\frac{u'(\xi)}{u(\xi)} \rightarrow -\frac{K(0)}{\sigma}$ holds, as $\xi \rightarrow \infty$, and $r^* < 1$. Then

$$\lim_{|\xi| \rightarrow \infty} \frac{u'(\xi)u^m(\xi)}{\sqrt{|u(\xi)|^2 + \frac{v^2}{c^2}|u'(\xi)|^2}} = 0,$$

and our claim follows. Thanks to our study of dynamical system (2.2) we have all the tools required to describe the traveling wave solutions of (1.2). This is the object of our next results.

Proposition 3.1 *The following statements hold true:*

- (1) *Any Type I orbit induces a smooth traveling wave $u(x - \sigma t)$ which is an entropy solution of (1.2) with null flux at infinity (see Definition 4.4 in Appendix 4.3.3). Hence they are unique in the sense of the initial value problem, with initial condition $u(x)$. This is the case for $\sigma > \sigma_{smooth}$.*
- (2) *For any $\sigma_{ent} \leq \sigma \leq \sigma_{smooth}$, there exists a traveling wave solution $u(x - \sigma t)$ with null flux at infinity. These traveling waves are unique entropy solutions in the sense of the initial value problem, with initial condition $u(x)$. Moreover:*
 - *When $\sigma_{ent} < \sigma < \sigma_{smooth}$ the traveling wave is discontinuous at the junction $x - \sigma t = 0$ and smooth off of it. The slope is infinite at both sides of this point.*
 - *When $\sigma = \sigma_{smooth}$ the traveling wave is continuous in the whole line, $x - \sigma t = \xi \in \mathbb{R}$, and smooth off of the junction at $x - \sigma t = 0$. The slope is infinite at both sides of this point.*
 - *If $\sigma = \sigma_{ent}$, then $u^-(\sigma) = 0$ and the corresponding solution is supported on a half line $x - \sigma t = \xi \in \mathbb{R}^-$.*

Proof (1) is a consequence of Remark 2.8 and Proposition 2.16. The uniqueness result follows from Theorem 4.5 (see Appendix 4.3.3).

(2) When $\sigma < \sigma_{smooth}$ we have that $u \in C([0, T]; L^1_{loc}(\mathbb{R}^N)) \cap BV_{loc}([0, T] \times \mathbb{R}^N)$ and Du has no Cantor part. Since by Proposition 2.18 the speed of the discontinuity fronts satisfies (1.7), then Proposition 1.3 implies that $u(x - \sigma t)$ is an entropy solution of (1.2). As a concatenation of Type II and Type III orbits, it is smooth out of the discontinuity set and has a null flux at infinity. When $\sigma = \sigma_{smooth}$, the traveling wave satisfies $u \in C([0, T]; L^1_{loc}(\mathbb{R}^N)) \cap W^{1,1}_{loc}([0, T] \times \mathbb{R}^N)$. Hence, by Proposition 1.3, it is an entropy solution. As a concatenation of Type II and Type III orbits, it has a null flux at infinity. Uniqueness follows from Theorem 4.5 (see Appendix 4.3.3). The additional statements are consequences of Proposition 2.18. □

Now we wonder about the number of traveling waves that can be constructed with a given speed. Let us recall that when we say that a function is piecewise smooth, up to a finite number of points, we understand that at those singular points there is a jump either of the function or of its first derivative.

Proposition 3.2 *Given any $\sigma \in [\sigma_{ent}, +\infty[$, the only non-trivial entropy solution of (1.2) with the form $u(x - \sigma t)$, having its range in $[0, 1]$ and being piecewise smooth –up to a finite number of points– is (up to spatial shifts) the one provided by Proposition 3.1.*

Proof The proof is divided into a series of steps.

Step 1. Precise setting of the problem. Let $\sigma \in [\sigma_{ent}, +\infty[$ and let $u(x - \sigma t)$ be a traveling wave which is piecewise smooth, up to a finite number of

points, and satisfies the entropy conditions. Recall that, as it was shown in Sect. 2, traveling wave solutions of (1.2) with range in $[0, 1]$ are in close correspondence with solutions of the system (2.2) considered over the range $0 \leq u \leq 1$, $-1 \leq r \leq 1$ (here we are not making any monotonicity assumption). Thus, during this proof we consider the system (2.2) to be defined on $[0, 1] \times [-1, 1]$.

Let $I_i =]\xi_i, \xi_{i+1}[$, $i = 1, \dots, p$, be maximal intervals of smoothness of u , so that $\xi_1 = -\infty$, $\xi_{p+1} = +\infty$, and either u or u' has a jump point at $\xi = \xi_i$ for all $i = 2, \dots, p$. Since Du has no Cantor part, entropy solutions are characterized by Proposition 1.3 and so observations (i) and (ii) in Sect. 2 hold. Moreover, u is a solution of (2.1) in $\mathcal{D}'(\mathbb{R})$ and the pair $(u(\xi), r(\xi))$ is a solution of (2.2) in each interval I_i .

Step 2. We show that each of the intervals I_i is a maximal interval of existence for the system (2.2). Let i be a fixed value. As I_i is a maximal interval of smoothness, then it is a subset of a maximal interval of existence of (2.2). Assume for instance that the maximal interval of existence has the form $]\xi_i, \bar{\xi}[$ for some $\bar{\xi} > \xi_{i+1}$, the other possibilities can be handled in a similar way. Then, there exists a smooth pair $(\bar{u}(\xi), \bar{r}(\xi))$ defined on $]\xi_i, \bar{\xi}[$ as a maximal solution to (2.2), such that (u, r) and (\bar{u}, \bar{r}) coincide over I_i , but $(u, r)(\xi_{i+1}^+) \neq (\bar{u}, \bar{r})(\xi_{i+1}^+)$. As $r(\xi_{i+1}^-) = \bar{r}(\xi_{i+1}^-)$ we get that $u'(\xi_{i+1}^-)$ is finite. Being $u(x - \sigma t)$ an entropy solution of (1.2), in case that $u(\xi_{i+1}^+) = \bar{u}(\xi_{i+1}^+)$ we must have that $u'(\xi_{i+1}^+) = \bar{u}'(\xi_{i+1}^+) \in \mathbb{R}$ thanks to observation (i) at the beginning of Sect. 2; thus $u(\xi)$ could be extended smoothly to the right of I_i , which would be a contradiction. Then, this means that $u(\xi_{i+1}^+) \neq \bar{u}(\xi_{i+1}^+)$. Knowing that $u(x - \sigma t)$ is an entropy solution of (1.2), observation (ii) at the beginning of Sect. 2 shows that $|u'(\xi_{i+1}^-)| = \infty$, but this is again a contradiction as we already knew that this value was finite. Thus, the only way out is to conclude that I_i is a maximal interval of existence.

Step 3. Let us prove that $(u(\xi), r(\xi)) \rightarrow (1, 0)$ as $\xi \rightarrow -\infty$. Proceeding as in Proposition 2.2, the analysis of the flow given by (2.2) at the boundaries of $[0, 1] \times [-1, 1]$ is straightforward. This can be combined with arguments similar to those in Proposition 2.2 to show that either $(u(\xi), r(\xi))$ tends to $\{u = 0\} \times [-1, 1]$ when $\xi \rightarrow -\infty$ or $(u(\xi), r(\xi)) \rightarrow (1, 0)$ as $\xi \rightarrow -\infty$.

The first possibility yields only the zero solution: note that the set $\{u = 0\} \times [-1, 1]$ is positively invariant under the flow (2.2). So, no attempt to try to construct a non-trivial solution such that $(u(\xi), r(\xi))$ tends to $\{u = 0\} \times [-1, 1]$ as $\xi \rightarrow -\infty$ is successful. Indeed, any such solution would be equal to zero in I_1 , with $\xi_2 < +\infty$, and being not identically zero we have to extend it further to the right in a non-trivial way. Being $\{u = 0\} \times [-1, 1]$ positively invariant under the flow, the only way to do this is performing a discontinuous matching with some other orbit defined in I_2 . The matching to be performed has to satisfy the requirements set up in Proposition 1.3, which implies that

the profile must be traveling from right to left, i.e. $\sigma < 0$. This contradicts the assumptions of the current proposition. Thus, the only chance that is left is to have $(u(\xi), r(\xi)) \rightarrow (1, 0)$ as $\xi \rightarrow -\infty$.

Step 4. By Theorem 2.5 and our assumption on the range of the traveling wave, the solution (u, r) in I_1 is unique and satisfies that $(u(\xi), r(\xi)) \rightarrow (1, 0)$ as $\xi \rightarrow -\infty$. The solution has a decreasing profile in I_1 and a limit $u(\xi_2^-)$ as $\xi \rightarrow \xi_2^-$. By Proposition 2.16, if $\sigma > \sigma_{smooth}$, then $\xi_2 = +\infty$, and u is smooth in all \mathbb{R} and coincides with the solution constructed in Proposition 3.1. If $\sigma \in [\sigma_{ent}, \sigma_{smooth}]$, then u is a Type II orbit in I_1 . Let us prove that $p = 2$ and the statement of the proposition holds. We distinguish three cases.

- a) If $\sigma \in]\sigma_{ent}, \sigma_{smooth}[$, then by Proposition 2.18 we have that $u(\xi_2^-) = u^+(\sigma) > u^*(\sigma)$. As in Lemma 2.19, it holds that $0 < u(\xi_2^+) < u^*(\sigma)$ and then, by Proposition 2.12 (see also Lemma 2.11), $I_2 =]\xi_2, +\infty[$ and uniqueness of (2.2) holds in I_2 . Thus, the solution u of (2.1) in I_2 coincides with the solution of (2.2) in that interval.
- b) If $\sigma = \sigma_{smooth}$, then by Proposition 2.18 we have that $u(\xi_2^-) = u^+(\sigma) = u^*(\sigma)$ and $r(\xi_2^-) = 1$. By the Rankine–Hugoniot condition (2.19) and observation (i) in Sect. 2, respectively, we have $u(\xi_2^+) = u^-(\sigma) = u^*(\sigma)$ and $r(\xi_2^+) = 1$. Again, by Proposition 2.12 (see also Lemma 2.11), $I_2 =]\xi_2, +\infty[$ and uniqueness of (2.2) holds in I_2 . Thus, the solution u of (2.1) in I_2 coincides with the solution of (2.2) in that interval. In this case, u has an infinite slope at both sides of ξ_2 but r matches continuously there.
- c) If $\sigma = \sigma_{ent}$, then $u(\xi_2^-) = u^+(\sigma) > u^*(\sigma)$ (see the proof of Lemma 2.19). Recall that, by our definition of σ_{ent} , we have $u^-(\sigma_{ent}) = 0$. Since u is an entropy solution, then $r(\xi_2^-) = 1$ and Rankine–Hugoniot condition (2.19) holds. Finally, we can ensure that $u(\xi) = 0$ for $\xi \in]\xi_2, \infty[$ as the set $\{0\} \times [-1, 1]$ is a positively invariant manifold of the dynamical system. The solution coincides with the traveling wave found in Proposition 3.1.

This concludes the proof. □

Remark 3.3 We stress that as a consequence of Proposition 3.2, no traveling structures (soliton-like solutions in particular) with range in $[0, 1]$ and speed $\sigma \geq 0$ other than the ones given by Proposition 3.1 can be constructed (Fig. 4). Regarding the case $\sigma < 0$, a similar analysis could be carried to show that the only admissible traveling profiles in our framework are mirror images of those constructed for $\sigma > 0$.

Remark 3.4 Let $u(t, x)$ be the entropy solution of (1.2) corresponding to an initial condition u_0 with compact support and such that $\|u_0\|_\infty < 1$. By Proposition 3.1, we know that there is an entropic singular traveling wave, moving with velocity σ_{ent} , bounding u_0 from above. As a consequence of the comparison principle in Theorem 4.5, there exists a positive constant β , depending on

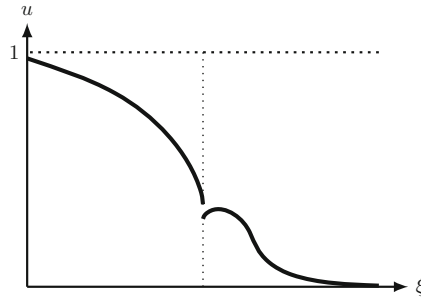


Fig. 4 An example of a piecewise smooth traveling wave profile which is not monotone that arises from a solution of (2.2) and for which the wave speed obeys (1.7) for $v = c = 1, m = 2, F(u) = u(1 - u)$ and $\sigma = 0, 641233$. Vertical dotted lines show points with infinite slope. Note that the observation (ii) at the beginning of Sect. 2 is not fulfilled, thus this profile does not solve (1.2) in the entropic sense

the support of u_0 and on the shape of the entropic traveling wave, such that the support of $u(t, \cdot)$ is contained in $(-\infty, \beta + \sigma_{ent} t)$. Using a similar argument based on waves traveling to the left, the value β can be chosen so that the support of $u(t, \cdot)$ is contained in $(-\beta - \sigma_{ent} t, \beta + \sigma_{ent} t)$. Thus $u(t, x)$ has compact support for any $t > 0$.

Note also that the traveling waves with support in a half line can be used to prove that solutions of

$$\frac{\partial u}{\partial t} = v \left(\frac{u^m u_x}{\sqrt{u^2 + \frac{v^2}{c^2} |u_x|^2}} \right)_x, \quad \text{in }]0, T[\times \mathbb{R}, \tag{3.1}$$

corresponding to initial data with compact support are compactly supported. Let us sketch the proof of this fact. Let $u_0 \in (L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))^+$ and assume that u_0 is supported in $[a, b]$. Let $u(t, x)$ be the entropy solution of (3.1) with $u(0, x) = u_0(x)$. Observe first that the homogeneity (of degree $m > 1$) of the operator in (3.1) implies that for any $\lambda > 0, u_\lambda(t, x) = \lambda^{1/(m-1)} u(\lambda t, x)$ is the entropy solution of (3.1) with initial datum $u_\lambda(0, x) = \lambda^{1/(m-1)} u_0(x)$. By an appropriate choice of λ depending on $\|u_0\|_\infty$ and after a suitable translation of the initial profile of $u_{\sigma_{ent}}$ (eventually with $[a, b]$ in the interior of the support of $u_{\sigma_{ent}}$) we may ensure that $u_\lambda(0, x) \leq u_{\sigma_{ent}}(x), x \in \mathbb{R}$. Since $u_{\sigma_{ent}}(x - \sigma_{ent} t)$ is a super-solution of (3.1), by the comparison principle in Theorem 4.5.(ii) we have that $u_\lambda(t, x) \leq u_{\sigma_{ent}}(x - \sigma_{ent} t)$ for any $t > 0$. Writing this inequality in terms of $u(t, x)$ we have $u(t, x) \leq \lambda^{-1/(m-1)} u_{\sigma_{ent}}(x - \sigma_{ent} \frac{t}{\lambda})$. By comparing with a traveling wave moving to the left with speed σ_{ent} we deduce that for any $t > 0$ the support of $u(t)$ is contained in $[a - \epsilon - \sigma_{ent} \frac{t}{\lambda}, b + \epsilon + \sigma_{ent} \frac{t}{\lambda}]$ for some $\epsilon, \lambda > 0$ determined by u_0 .

3.1 Numerical insights about traveling waves viewed as attractors

Studying the stability of traveling wave solutions and their dynamic ability to attract other solutions is a very interesting problem that is beyond the scope of this paper. Another problem that will surely open new lines of research is to understand how the saturation of diffusion produces shocks (and the role played by the reaction terms, if any). The idea of this paragraph is to give some insights of how these two problems raise new challenges in this context. To do that we use the numerical solutions of the dynamical system associated with traveling waves (2.2) together with those associated to the partial differential equation (1.2). In Fig. 5 we have represented both curves associated to different settings. The traveling wave profiles have been displaced by matching its discontinuities with those of the time dependent solutions. For the numerical solution of the time-dependent problem we have used a WENO solver together with a Runge–Kutta scheme.

Figure 5a describes the evolution of an initial data with compact support and how does it evolve into an entropic jump which locally around the front behaves like a traveling wave. By using the comparison principle for solutions in Theorem 4.5, we deduce that the traveling wave will be above the time dependent solution of the system (1.2). The entropic traveling wave provides an upper estimate of the growth rate of the support. Let us precise in the following result the order of the singularity of the traveling wave solution near the jump ξ_2 , where we use the notation of Step 4 in Proposition 3.2.

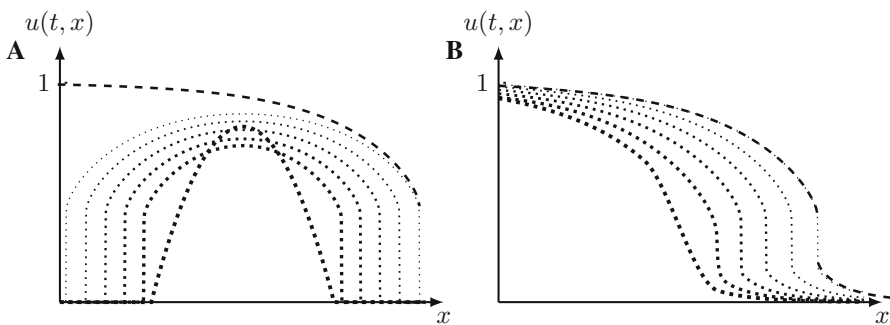


Fig. 5 Dotted lines represent time dependent solutions for (1.2) (the smaller the dots, the more advanced the times). Dashed lines describe traveling wave profiles obtained from (2.2) for different values of σ . In all cases we have used $v = c = 1$, $F(u) = u(1 - u)$ and $m = 2$. **a** Time evolution of a compactly supported initial condition with null Dirichlet conditions. The traveling wave profile depicted, corresponding to $\sigma_{ent} = 0, 437803$, constitutes a supersolution for the time-dependent solution. **b** Time evolution of a regular initial condition, with Neumann boundary conditions. The traveling wave profile depicted corresponds to $\sigma = 0, 57 \in]\sigma_{ent}, \sigma_{smooth}[$

Lemma 3.5 *Let u be an entropic traveling wave for $\sigma \in [\sigma_{ent}, \sigma_{smooth}]$. Then, the vertical angle near the jump ξ_2 is of order $|\xi - \xi_2|^{-\frac{1}{3}}$.*

Proof Using (2.7), we obtain that the points u_{\pm} at which $r(u)$ touches the edge $r = 1$ verify

$$\lim_{u \rightarrow u_{\pm}} \left| \frac{r(u) - 1}{u - u_{\pm}} \right| = \lim_{u \rightarrow u_{\pm}} |r'(u)| = \alpha,$$

where $\alpha = \frac{1}{cu_{\pm}} |mc - \sigma u_{\pm}^{1-m}|$. Hence, combining (2.11) together with the approximation $\sqrt{1 - r^2} \sim \sqrt{1 - r} \sqrt{2}$ we deduce for $\sigma_{ent} < \sigma < \sigma_{smooth}$, which implies $u_{\pm} \neq u^*$, the following equality

$$\lim_{\xi \rightarrow \xi_2^{\pm}} |u(\xi) - u_{\mp}|^{\frac{1}{2}} |u'(\xi)| = \frac{cu_{\mp}}{\sqrt{2\alpha} v} = \frac{(cu_{\mp})^{\frac{3}{2}}}{\sqrt{2} v |mc - \sigma u_{\mp}^{1-m}|^{\frac{1}{2}}}.$$

Then, we have

$$\lim_{\xi \rightarrow \xi_2^{\pm}} \frac{\frac{2}{3} |u(\xi) - u_{\mp}|^{\frac{3}{2}}}{|\xi - \xi_2|} = \frac{(cu_{\mp})^{\frac{3}{2}}}{\sqrt{2} v |mc - \sigma u_{\mp}^{1-m}|^{\frac{1}{2}}}$$

or equivalently

$$\lim_{\xi \rightarrow \xi_2^{\pm}} \frac{|u(\xi) - u_{\mp}|}{|\xi - \xi_2|^{\frac{2}{3}}} = \left(\frac{3(cu_{\mp})^{\frac{3}{2}}}{2\sqrt{2} v |mc - \sigma u_{\mp}^{1-m}|^{\frac{1}{2}}} \right)^{\frac{2}{3}}.$$

Letting $\beta_{\pm} = \left(\frac{3(cu_{\pm})^{\frac{3}{2}}}{2\sqrt{2} v |mc - \sigma u_{\pm}^{1-m}|^{\frac{1}{2}}} \right)^{\frac{2}{3}}$, we find

$$\begin{aligned} u(\xi) &\sim u_+ + \beta_+ |\xi - \xi_2|^{\frac{2}{3}}, & \xi < \xi_2, \\ u(\xi) &\sim u_- - \beta_- |\xi - \xi_2|^{\frac{2}{3}}, & \xi > \xi_2. \end{aligned}$$

For $\sigma = \sigma_{ent}$, since $u(\xi) = 0 = u_- = 0$ for $\xi > \xi_2$, taking into account that $\sigma = cu_+^{m-1}$, we analogously obtain

$$u(\xi) \sim \left(\frac{\sigma}{c}\right)^{\frac{1}{m-1}} + \frac{1}{2} \left(\frac{3c}{v(m-1)^{\frac{1}{2}}}\right)^{\frac{2}{3}} \left(\frac{\sigma}{c}\right)^{\frac{1}{m-1}} |\xi - \xi_2|^{\frac{2}{3}}, \quad \xi < \xi_2, \quad (3.2)$$

which provides an estimate of the order of approximation of the traveling wave profile near the front. □

The numerical time dependent solutions given in Fig. 5a has the same power law behavior near the front than the corresponding traveling wave (3.2).

In Fig. 5b the numerical calculations show spontaneous singularization of solutions and the convergence of an initial data towards a traveling wave solution to the type described in Fig. 1c.

3.2 The L^p -continuity w.r.t. the wave speed

The purpose of this paragraph is to prove the continuity of the traveling profiles constructed in Proposition 3.1 with respect to the wave velocity. In order to do that it is convenient to choose a privileged normalization for the traveling profiles, so that we get a family $u^{\mathcal{N}}(\sigma)$ defined in a unique way. We do this as follows:

Definition 3.6 Let $u_{\sigma}^{\mathcal{N}}$ with $\sigma \in [\sigma_{ent}, +\infty[$ be the family of traveling wave solutions constructed in Proposition 3.1 and enjoying the following additional properties:

- If $\sigma > \sigma_{smooth}$ we set $u_{\sigma}^{\mathcal{N}}(0) = u^*(\sigma_{smooth}) = u^+(\sigma_{smooth}) = u^-(\sigma_{smooth})$,
- If $\sigma_{smooth} \geq \sigma > \sigma_{ent}$ we set $\lim_{\xi \rightarrow 0^+} u_{\sigma}^{\mathcal{N}}(\xi) = u^{\pm}(\sigma)$.
- If $\sigma = \sigma_{ent}$ we set $\lim_{\xi \uparrow 0} u_{\sigma}^{\mathcal{N}}(\xi) = u^+(\sigma)$ and $u_{\sigma}^{\mathcal{N}}(\xi) = 0$, for $\xi > 0$.

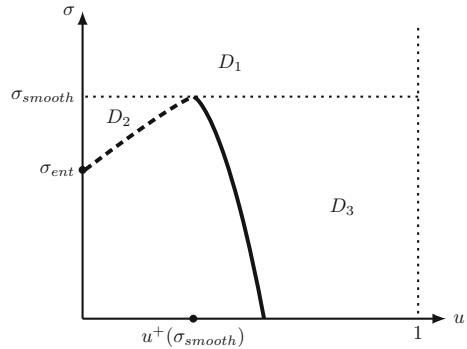
We assume in this section that F satisfies

$$\text{there is some } p \geq 1 \text{ such that } \liminf_{u \rightarrow 0} \frac{F(u)}{u^p} = k \in]0, +\infty]. \quad (3.3)$$

Now we are going to prove that this specially parameterized family of traveling wave profiles has the property of continuous dependence with respect to the wave speed stated in Theorem 1.4, that is,

$$\lim_{\sigma_1 \rightarrow \sigma_2} \|u_{\sigma_1}^{\mathcal{N}} - u_{\sigma_2}^{\mathcal{N}}\|_{L^p(\mathbb{R})} + \|u_{\sigma_1}^{\mathcal{N}} - u_{\sigma_2}^{\mathcal{N}}\|_{L^{\infty}(\mathbb{R})} = 0, \quad \forall \sigma_1, \sigma_2 \in [\sigma_{ent}, +\infty[. \quad (3.4)$$

Fig. 6 Numerical representation of the set D for $v = c = 1, m = 2, F(u) = u(1 - u)$. Continuous and dashed lines represent, respectively, the inverse of the mappings $u^+(\sigma)$ and $u^-(\sigma)$



To prove the previous result we will need to argue with pairs of the form $(u, \sigma) \in]0, 1[\times]0, +\infty[$ (Fig. 6). We consider the following subsets:

$$\begin{aligned}
 D_1 &=]0, 1[\times]\sigma_{smooth}, +\infty[, \\
 D_2 &= \{(u, \sigma) \in]0, 1[\times]0, +\infty[: \sigma_{ent} < \sigma \leq \sigma_{smooth}, 0 < u < u^-(\sigma)\}, \\
 D_3 &= \{(u, \sigma) \in]0, 1[\times]0, +\infty[: 0 < \sigma \leq \sigma_{smooth}, u^+(\sigma) < u < 1\},
 \end{aligned}$$

and finally we let $D = D_1 \cup D_2 \cup D_3$.

We observe that the map $(u, \sigma) \mapsto r_\sigma(u)$ is defined at least on $D_1 \cup D_3$. To extend it to D_2 we choose $r_\sigma(u)$ as the solution of (2.7) defined over $]0, u^-(\sigma)[$ and such that $r_\sigma(u^-(\sigma)) = 1$. This extension is justified by Lemma 2.11.

Lemma 3.7 *The function $(u, \sigma) \in D \mapsto r_\sigma(u)$ is continuous. Moreover, it has the following properties:*

- (i) *Let $\sigma_{ent} < \sigma_1 < \sigma_2 \leq \sigma_{smooth}$, then $r_{\sigma_1}(u) > r_{\sigma_2}(u)$, for any $u \in]0, u^-(\sigma_1)[$ (which is their common interval of definition).*
- (ii) *Let $\sigma_n \rightarrow \sigma_0 \geq 0$, then there exist $u_0 \in]0, 1[$ and $L > 0$ such that*

$$\frac{(1 - u)\sqrt{1 - r_{\sigma_n}^2(u)}}{u r_{\sigma_n}(u)} \geq L, \quad \forall u \in]u_0, 1[\text{ and } n \in \mathbb{N}. \tag{3.5}$$

Let $\sigma_n, \sigma_0 > \sigma_{ent}$ and $\sigma_n \rightarrow \sigma_{ent}$. Then, there exist $u_0 \in]0, 1[$ and $L > 0$ such that

$$\frac{\sqrt{1 - r_{\sigma_n}^2(u)}}{r_{\sigma_n}(u)} \geq L, \quad \forall u \in]u_0, 1[\text{ and } n \in \mathbb{N}. \tag{3.6}$$

Proof Note that the continuity in $D_1 \cup D_3$ follows from the uniform convergence on compact sets given in Lemma 2.17. On the other hand, the continuity on D_2 follows from Peano’s theorem on continuous dependence with respect to initial conditions, which we apply to any continuous extension of (2.7).

Thus, our claim will be proved if we are able to ensure that the values of $r_\sigma(u)$ in $]0, u^-(\sigma_{smooth})[\times \{\sigma_{smooth}\}$ obtained by its extension to D_2 coincide with the values obtained by performing limits of points in D_1 .

We pick sequences $\sigma_n \downarrow \sigma_{smooth}$ and $u_n \rightarrow u_0 \in]0, u^-(\sigma_{smooth})[$. Coming back to Lemma 2.11, after bounding r'_{σ_n} and using Ascoli’s theorem, we notice that the limit of r_{σ_n} in $]0, u^-(\sigma_{smooth})[$ (recall that $r_{\sigma_n}(\cdot)$ is defined over $]0, 1[$) gives a Type III orbit such that $r(u^+(\sigma_{smooth}) = u^*(\sigma_{smooth})) = 1$. Noting that the convergence is uniform over compact sets, this allows to conclude.

To prove assertion (i), we argue by contradiction. For that we consider that there exists $\tilde{u} \in]0, u^-(\sigma_1)[$ such that

$$r_{\sigma_1}(\tilde{u}) \leq r_{\sigma_2}(\tilde{u}). \tag{3.7}$$

As $r_{\sigma_2}(u^-(\sigma_1)) < 1 = r_{\sigma_1}(u^-(\sigma_1))$, we can assume that \tilde{u} is the last point for which (3.7) is verified. Then $r_{\sigma_1}(\tilde{u}) = r_{\sigma_2}(\tilde{u})$ and $r'_{\sigma_2}(\tilde{u}) \leq r'_{\sigma_1}(\tilde{u})$. By Eq. (2.7), this contradicts the fact that $\sigma_1 < \sigma_2$.

Next we move to assertion (ii); we consider the upper bound (3.5) in first place. Let $u_0 \in]0, 1[$ such that the left hand side of (3.5) is defined in $[u_0, 1[$, for any n . To do that, let $\bar{\sigma} = \inf_{n \in \mathbb{N}} \sigma_n$ and we choose $u_0 \in]u^+(\bar{\sigma}), 1[$, or $u_0 \in]0, 1[$ if $\bar{\sigma} > \sigma_{smooth}$. Now we use the monotonicity of r_σ with respect to σ together with the monotonicity of the function $r \mapsto \frac{\sqrt{1-r^2}}{r}$ to devise the following estimate

$$\frac{(1-u)\sqrt{1-r_{\sigma_n}^2(u)}}{u r_{\sigma_n}(u)} \geq \frac{1-u}{u} \frac{\sqrt{1-r_{\bar{\sigma}}^2(u)}}{r_{\bar{\sigma}}(u)}, \quad \forall u \in]u_0, 1[. \tag{3.8}$$

The right hand side of the (3.8) is positive and bounded from below by a constant $L > 0$, since it is bounded at both ends of the interval $]u_0, 1[$ (note that $\lim_{u \rightarrow 1} \frac{1-u}{u} \frac{\sqrt{1-r_{\bar{\sigma}}^2(u)}}{r_{\bar{\sigma}}(u)} = -\frac{1}{r'_{\bar{\sigma}}(1)} > 0$).

To deal with the second estimate (3.6), let $u_0 \in]0, 1[$ in such a way that the left hand side of (3.6) is well-defined in $]0, u_0[$. To do that, we consider again $\bar{\sigma} = \inf_{n \in \mathbb{N}} \sigma_n$ and $u_0 \in]0, u^-(\bar{\sigma})[$, or $u_0 \in]0, 1[$ if $\bar{\sigma} > \sigma_{smooth}$, which is consistent with the fact that $\bar{\sigma} > \sigma_{ent}$. Then, as in the previous case we can estimate from below

$$\frac{\sqrt{1-r_{\sigma_n}^2(u)}}{r_{\sigma_n}(u)} \geq \frac{\sqrt{1-r_{\bar{\sigma}}^2(u)}}{r_{\bar{\sigma}}(u)}, \quad \forall u \in]0, u_0[. \tag{3.9}$$

Now, taking into account

$$\lim_{u \rightarrow 0} \frac{\sqrt{1 - r_\sigma^2(u)}}{r_\sigma(u)} = \begin{cases} \frac{\sqrt{1 - (r^*)^2}}{r^*}, & \text{if } r^* > 0, \\ +\infty, & \text{if } r^* = 0, \end{cases}$$

we can deduce that the right hand side of the (3.9) is positive and bounded from below by a constant $L > 0$, which concludes the proof. \square

Making use of the graphs $u \mapsto r_\sigma(u)$ we are able to introduce the following function:

$$G(u, \sigma) = \begin{cases} - \int_{u^*(\sigma_{smooth})}^u \frac{\sqrt{1 - r_\sigma^2(v)}}{vr_\sigma(v)} dv, & \text{in } D_1, \\ - \int_{u^-(\sigma)}^u \frac{\sqrt{1 - r_\sigma^2(v)}}{vr_\sigma(v)} dv, & \text{in } D_2, \\ - \int_{u^+(\sigma)}^u \frac{\sqrt{1 - r_\sigma^2(v)}}{vr_\sigma(v)} dv, & \text{in } D_3. \end{cases}$$

We can use this function to recover the traveling wave profiles $u_\sigma(\xi)$.

Lemma 3.8 *For any $\sigma > \sigma_{ent}$ and $\xi \neq 0$ we have that*

$$\frac{c}{v} G(u_\sigma^N(\xi), \sigma) = \xi.$$

Proof We argue first for $\xi > 0$. Choose $0 < \xi_1 < \xi$, and integrate (2.11) between ξ_1 and ξ to get

$$-\frac{c}{v} \int_{\xi_1}^\xi \frac{\sqrt{1 - r_\sigma^2(u_\sigma^N(\eta))}}{u_\sigma^N(\eta)r_\sigma(u_\sigma^N(\eta))} (u_\sigma^N)'(\eta) d\eta = \xi - \xi_1.$$

Now, after the change of variables $v = u_\sigma^N(\eta)$, we arrive to

$$-\frac{c}{v} \int_{u_\sigma^N(\xi_1)}^{u_\sigma^N(\xi)} \frac{\sqrt{1 - r_\sigma^2(v)}}{vr_\sigma(v)} dv = \xi - \xi_1.$$

Finally, we let $\xi_1 \rightarrow 0$; observe that the integrand is positive and in particular the integral exists. We can argue in a similar way if $\xi < 0$. \square

Remark 3.9 Note that for $\sigma_{ent} \leq \sigma < \sigma_{smooth}$ the traveling wave solutions u_σ^N are not well defined for $\xi = 0$.

Let us focus now on the properties of G .

Proposition 3.10 *The following properties are satisfied:*

- (1) *The function G is continuous over D .*
- (2) *The following assertions give the behavior of G at the boundary of D :*
 - i) *G tends to zero when we approach any point of the set*

$$\{(u^+(\sigma), \sigma) : 0 < \sigma \leq \sigma_{smooth}\} \cup \{(u^-(\sigma), \sigma) : \sigma_{ent} < \sigma \leq \sigma_{smooth}\},$$

- ii) *G tends to $-\infty$ when we approach any point of the set $\{1\} \times [0, \infty[$,*
- iii) *G tends to $+\infty$ when we approach any point of the set $\{0\} \times]\sigma_{ent}, \infty[$.*

Proof Since the function $(u, \sigma) \in D \mapsto r_\sigma(u)$ is continuous (Lemma 3.7), then r_σ can be extended continuously by 1 to $]0, 1[\times]0, \infty[\setminus D$. Thus, G is also continuous because it is given by the integral of a continuous function depending continuously on σ , the integral being extended to intervals that also depend continuously on (u, σ) . This proves assertions (1). and (2).i).

To prove (2). ii), let $\sigma_n \rightarrow \sigma_0$ and $u_n \rightarrow 1$. By Lemma 3.7.(ii)

$$\frac{\sqrt{1 - r_{\sigma_n}^2(u)}}{ur_{\sigma_n}(u)} \geq \frac{L}{1 - u}$$

is satisfied on some interval $]u_0, 1[$. This leads us to

$$G(u, \sigma_n) \leq h + L \ln(1 - u),$$

which holds for $u_0 < u < 1$, where h, L are positive constants not depending on n . Then, (2). ii) follows.

Let now $\sigma_n \rightarrow \sigma_0 > \sigma_{ent}$ and $u_n \rightarrow 0$. Using Lemma 3.7.(ii) again we find an interval $]0, u_0[$ for which

$$\frac{\sqrt{1 - r_{\sigma_n}^2(u)}}{ur_{\sigma_n}(u)} \geq \frac{L}{u}.$$

After integration in $[u^*(\sigma_{smooth}), u]$ if $\sigma_0 \geq \sigma_{smooth}$ or in $[u^-(\sigma), u]$ if $\sigma_0 < \sigma_{smooth}$ (both intervals coincide if $\sigma_0 = \sigma_{smooth}$) we obtain

$$G(u, \sigma_n) \geq h - L \ln(u),$$

for some positive constants h, L which do not depend on n . This proves (2). iii). □

Lemma 3.11 *The map $\sigma \mapsto u_\sigma^N(\xi)$ is monotonically increasing for $\xi > 0$ and monotonically decreasing for $\xi < 0$.*

Proof Thanks to Lemma 3.7. (i) we have that the mapping $\sigma \mapsto \frac{\sqrt{1-r_\sigma^2(v)}}{r_\sigma(v)}$ is monotonically increasing for any fixed $v \in [0, 1]$. Next, we note that $u < u^-(\sigma)$ in D_2 , with $\sigma \mapsto u^-(\sigma)$ increasing. We also have that $u > u^+(\sigma)$ in D_3 , with $\sigma \mapsto u^+(\sigma)$ decreasing. We combine the previous information with the representation formula for G given by Lemma 3.8 to obtain the result. \square

Proposition 3.12 *Let $\{\sigma_n\} \subset [\sigma_{ent}, +\infty[$ and $\sigma_n \rightarrow \sigma_0 > \sigma_{ent}$. Then for any $T > 0$ the sequence $u_{\sigma_n}^N$ converges uniformly on $[-T, T] \setminus \{0\}$.*

Proof To check the uniform convergence of $u_{\sigma_n}^N$ on $]0, T]$ we argue on $[0, T]$, after extending the functions $u_{\sigma_n}^N$ to $\xi = 0$ using either $u^*(\sigma_{smooth})$, in case that $\sigma_0 \geq \sigma_{smooth}$, or $u^-(\sigma_n)$, in case that $\sigma_0 \in]\sigma_{ent}, \sigma_{smooth}[$.

We use the characterization of the uniform convergence by sequences: given any fixed sequence $\xi_n \geq 0$ that converges to some $\xi_0 > 0$, we are to show that $u_{\sigma_n}^N(\xi_n) - u_{\sigma_0}(\xi_n)$ converges to zero. Since

$$\frac{c}{\nu} G(u_{\sigma_n}^N(\xi_n), \sigma_n) = \xi_n$$

is bounded, then $u_{\sigma_n}^N(\xi_n)$ stays in $]0, 1[$ thanks to Proposition 3.10. Any convergent subsequences of $u_{\sigma_n}^N(\xi_n)$ will converge to a point $u_0 \in]0, 1[$, which may in principle depend on the subsequence. Taking the limit along any such subsequence we get

$$\frac{c}{\nu} G(u_0, \sigma_0) = \xi_0. \tag{3.10}$$

This relation is solved only by $u_0 = u_{\sigma_0}^N(\xi_0)$, no matter if we are on $0 < u \leq u^-(\sigma_0)$ or on $0 < u \leq u^*(\sigma_{smooth})$ – note that G does not change sign within D_2 nor in D_3 , $D_1 \cap \{u > u^*\}$ or $D_1 \cap \{u < u^*\}$. This shows that in fact $u_{\sigma_n}^N(\xi_n) \rightarrow u_{\sigma_0}^N(\xi_0)$ for the whole sequence, and our claim follows.

The case $\xi_n \rightarrow \xi_0 = 0$ requires a more detailed analysis because in this case (3.10) may have two solutions: $u^+(\sigma_0)$ and $u^-(\sigma_0)$. However, in this case $u_{\sigma_n}^N(\xi_n) \leq \lim_{\xi \downarrow 0} u_{\sigma_n}^N(\xi) = u^-(\sigma_n)$ and, as a consequence, the corresponding limit \bar{u}_0 verifies $\bar{u}_0 \leq u^-(\sigma_0)$. Using (3.10) for $\xi_0 = 0$, we deduce that $\bar{u}_0 = u^-(\sigma_0)$, which coincides with the extension we made at the beginning of this proof.

The proof of the uniform convergence over $[-T, 0[$ is similar and we omit the details. \square

Remark 3.13 The above result is still valid in the case $\sigma_n \rightarrow \sigma_0 = \sigma_{ent}$ in the interval $] - T, 0[$, for any $T > 0$.

Let us now prove the uniform continuity of the traveling wave profiles with respect to σ . Consider a sequence $\sigma_n \rightarrow \sigma_0 \geq \sigma_{ent}$. In a first step we study the

case $\sigma_0 > \sigma_{ent}$. By Proposition 3.12, it is enough to prove that $u_{\sigma_n}^{\mathcal{N}}(\xi_n) \rightarrow 0$ as $\xi_n \rightarrow +\infty$, and $u_{\sigma_n}^{\mathcal{N}}(\xi_n) \rightarrow 1$ as $\xi_n \rightarrow -\infty$, since this obviously implies that $|u_{\sigma_n}^{\mathcal{N}}(\xi_n) - u_{\sigma_0}^{\mathcal{N}}(\xi_n)| \rightarrow 0$. Being both assertions similar, let us prove only the first one. For that, we note that given $\epsilon > 0$, there exists $\bar{\xi}$ such that $u_{\sigma_0}^{\mathcal{N}}(\bar{\xi}) < \epsilon$. Since, by Proposition 3.12, $u_{\sigma_n}^{\mathcal{N}}(\bar{\xi}) \rightarrow u_{\sigma_0}^{\mathcal{N}}(\bar{\xi})$, there is a value n_0 such that $u_{\sigma_n}^{\mathcal{N}}(\bar{\xi}) < \epsilon$ for $n > n_0$. Let $n_1 \in \mathbb{N}$ be such that $\xi_n \geq \bar{\xi}$ for any $n > n_1$. Then, choosing $n > \max\{n_0, n_1\}$, we find that $u_{\sigma_n}^{\mathcal{N}}(\xi_n) \leq u_{\sigma_n}^{\mathcal{N}}(\bar{\xi}) < \epsilon$.

In case that $\sigma_n \rightarrow \sigma_{ent}$ we have to distinguish between $]-\infty, 0[$ and $]0, \infty[$. If $\xi \in]-\infty, 0[$ we use Remark 3.13 and Lemma 3.11 to deduce the same result. For $\xi \in]0, \infty[$ we conclude by using the bound $u_{\sigma_n}^{\mathcal{N}}(\xi) \leq u^-(\sigma_n)$.

Finally we end the proof of (3.4) by proving the continuity of the traveling waves with respect to σ in $L^p(\mathbb{R})$, where p is given by (3.3). This will conclude the proof of the continuity assertions of Theorem 1.4.

First, let us prove that

$$u_{\sigma}^{\mathcal{N}} \in L^p(\mathbb{R}^+) \quad \text{and} \quad 1 - u_{\sigma}^{\mathcal{N}} \in L^1(\mathbb{R}^-). \tag{3.11}$$

For that we notice that (3.11) holds if $F(u_{\sigma}^{\mathcal{N}}) \in L^1(\mathbb{R})$. This is a consequence of the fact

$$\lim_{\xi \rightarrow \infty} \frac{F(u_{\sigma}^{\mathcal{N}}(\xi))}{(u_{\sigma}^{\mathcal{N}}(\xi))^p} = \liminf_{u \rightarrow 0} \frac{F(u)}{(u)^p} = k \in]0, +\infty],$$

$$\lim_{\xi \rightarrow -\infty} \frac{F(u_{\sigma}^{\mathcal{N}}(\xi))}{1 - u_{\sigma}^{\mathcal{N}}(\xi)} = \lim_{u \rightarrow 1} \frac{F(u)}{1 - u} = -F'(1) > 0.$$

Now we prove the integrability of $F(u_{\sigma}^{\mathcal{N}})$ over the whole real line. For that we rewrite (2.1) as

$$F(u(\xi)) = (cu^m(\xi)r(\xi) - \sigma u(\xi))'.$$

Integrating the previous relation and using the boundedness of F , and the finiteness of $\lim_{\pm\infty} u$ and $\lim_{\pm\infty} r$, we get that $F(u_{\sigma}^{\mathcal{N}}) \in L^1(\mathbb{R})$. Hence, (3.11) holds.

Since $1 - u_{\sigma}^{\mathcal{N}} \in L^1(\mathbb{R}^-)$ and $|1 - u_{\sigma}^{\mathcal{N}}| \leq 1$, we also have that

$$1 - u_{\sigma}^{\mathcal{N}} \in L^p(\mathbb{R}^-). \tag{3.12}$$

This allows us to conclude the convergence of any sequence $u_{\sigma_n}^{\mathcal{N}}$ to $u_{\sigma_0}^{\mathcal{N}}$ in $L^p(\mathbb{R})$ as $\sigma_n \rightarrow \sigma_0 \geq \sigma_{ent}$. Indeed, by Proposition 3.12 and (3.12) the sequence $|u_{\sigma_n}^{\mathcal{N}} - u_{\sigma_0}^{\mathcal{N}}|^p$ is dominated by a function in $L^p(\mathbb{R})$ and converges pointwise

to 0. The result follows as a consequence of the Dominated Convergence Theorem thanks to Lemma 3.11.

Remark 3.14 Note that in the proof of the uniform convergence of $u_{\sigma_n}^N$ we have not used any hypothesis on the asymptotic behavior of F at 0. Note also that under the hypothesis $K(0) > 0$ the $L^1(\mathbb{R})$ convergence holds, since this hypothesis implies that (3.3) is fulfilled for $p = 1$.

4 Appendix: Entropy solutions

Our purpose in this Appendix is to give the necessary background in order to introduce the notion of entropy solutions to (1.2), to state some existence and uniqueness results for them, and to give sense to the properties stated in Sect. 1.2.

Equation (1.2) belongs to the more general class of flux limited diffusion equations, which has been extensively studied in [2,4,5,21,22]. As shown in those papers, the notion of entropy solution is the right one in order to prove existence and uniqueness results and to describe the qualitative features of solutions. In particular, and closely related to this work, the so-called relativistic heat equation [which corresponds to $m = 1$ in (1.2)] coupled with a Fisher–Kolmogorov type reaction term has been studied in [3,19]. Existence and uniqueness results for that model were proved in [3], the construction of traveling waves being the object of [19].

Thus, our first purpose is to give a brief review of the concept of entropy solution for flux limited diffusion equations. Although we are only concerned with the case $N = 1$, we state the results in the more general context where $N \geq 1$ since this may be useful for future reference. For a more detailed treatment we refer to [4,21]. We consider parabolic equations of the form

$$\begin{cases} \frac{\partial u}{\partial t} = \operatorname{div} \mathbf{a}(u, \nabla u) + F(u), & \text{in } Q_T =]0, T[\times \mathbb{R}^N \\ u(0, x) = u_0(x), & \text{in } x \in \mathbb{R}^N \end{cases} \tag{4.1}$$

where $F(u)$ is a Lipschitz continuous function such that $F(0) = 0$ and $\mathbf{a}(z, \zeta) = \nabla_{\zeta} f(z, \zeta)$ is associated to a Lagrangian f satisfying a set of technical assumptions. Let us give a brief account of them, referring to [4,21] for a thorough presentation. Thus, we assume that

(H) f is continuous on $[0, \infty[\times \mathbb{R}^N$ and is a convex differentiable function of ζ such that $\nabla_{\zeta} f(z, \zeta) \in C([0, \infty[\times \mathbb{R}^N)$. Further, we require f to satisfy the coercivity and linear growth conditions

$$C_0(z)|\zeta| - D_0(z) \leq f(z, \zeta) \leq M_0(z)(|\zeta| + 1), \tag{4.2}$$

for any $(z, \zeta) \in [0, \infty[\times \mathbb{R}^N$, and some positive and continuous functions $C_0, D_0, M_0 \in C([0, \infty[)$ with $C_0(z) > 0$ for any $z \neq 0$. Notice that $|\zeta|$ denotes the Euclidian norm of $\zeta \in \mathbb{R}^N$. We assume that

$$C_0(z) \geq c_0 z^m, \quad \text{for some } c_0 > 0, \quad m \geq 1, \quad z \in [0, \infty[.$$

Let $\mathbf{a}(z, \zeta) = \nabla_\zeta f(z, \zeta)$, $(z, \zeta) \in [0, \infty[\times \mathbb{R}^N$. We assume that there is a vector field $\mathbf{b}(z, \zeta)$ and a constant $M > 0$ such that

$$\mathbf{a}(z, \zeta) = z^m \mathbf{b}(z, \zeta) \quad \text{with} \quad |\mathbf{b}(z, \zeta)| \leq M, \quad \forall (z, \zeta) \in [0, \infty[\times \mathbb{R}^N. \quad (4.3)$$

We consider the function $h : [0, \infty[\times \mathbb{R}^N \rightarrow \mathbb{R}$ defined by

$$h(z, \zeta) := \mathbf{a}(z, \zeta) \cdot \zeta. \quad (4.4)$$

From the convexity of f in ζ , (4.2) and (4.3), it follows that

$$C_0(z)|\zeta| - D_1(z) \leq h(z, \zeta) \leq Mz^m|\zeta|,$$

for any $(z, \zeta) \in [0, \infty[\times \mathbb{R}^N$, where $D_1(z) = D_0(z) + f(z, 0)$. We also assume that the recession functions f^0, h^0 exist, being the recession function g^0 of a function g defined as

$$g^0(x, z, \zeta) = \lim_{t \rightarrow 0^+} tg \left(x, z, \frac{\zeta}{t} \right).$$

It is convex and homogeneous of degree 1 in ζ .

Other technical assumptions on f, h are required:

- $f^0(z, \zeta) = h^0(z, \zeta)$ for all $z \in \mathbb{R}$ and $\zeta \in \mathbb{R}^N$
- $h(z, \zeta) = h(z, -\zeta)$ for all $z \in \mathbb{R}$ and $\zeta \in \mathbb{R}^N$
- $\frac{\partial \mathbf{a}_i}{\partial \zeta_i}(z, \zeta) \in C(\mathbb{R} \times \mathbb{R}^N)$ for any $i = 1, \dots, N$
- $\mathbf{a}(z, \zeta) \cdot \eta \leq h^0(z, \eta)$ for all $z \in \mathbb{R}$ and $\zeta, \eta \in \mathbb{R}^N$
- $h^0(z, \zeta)$ can be written as $h^0(z, \zeta) = \varphi(z)\psi^0(\zeta)$, where φ is a Lipschitz continuous function such that $\varphi(z) > 0$ for any $z \neq 0$ and ψ^0 is a convex function which is homogeneous of degree one
- For any $R > 0$ there exists a constant $C > 0$ such that

$$|(\mathbf{a}(z, \zeta) - \mathbf{a}(\hat{z}, \zeta)) \cdot (\zeta - \hat{\zeta})| \leq C|z - \hat{z}|\|\zeta - \hat{\zeta}\|$$

for any $(z, \zeta), (\hat{z}, \hat{\zeta}) \in \mathbb{R} \times \mathbb{R}^N, |z|, |\hat{z}| \leq R$.

When we say that assumption (H) holds, we refer to the complete set of assumptions above.

For the generalized relativistic heat equation (1.2) the function

$$f(z, \zeta) = \frac{c^2}{v} z^m \sqrt{z^2 + \frac{v^2}{c^2} |\zeta|^2} \quad (4.5)$$

satisfies all the assumptions that allow to work in the context of entropy solutions (see [2,4]). In this case

$$\mathbf{a}(z, \zeta) = v \frac{z^m \zeta}{\sqrt{z^2 + \frac{v^2}{c^2} |\zeta|^2}} \quad \text{and} \quad h(z, \zeta) = \mathbf{a}(z, \zeta) \cdot \zeta = v \frac{z^m |\zeta|^2}{\sqrt{z^2 + \frac{v^2}{c^2} |\zeta|^2}}.$$

Due to the linear growth condition on the Lagrangian, the natural energy space to study the solutions of (4.1) is the space of functions of bounded variation, or BV functions. In Sect. 4.1 we recall some basic facts about them.

The notion of entropy solutions is based on a set of Kruzkov's type inequalities and it requires to define a functional calculus for functions whose truncations are in BV . We briefly review in Sect. 4.2 this functional calculus which is based on the works [27,28], which prove lower semicontinuity results for functionals on BV . After this, in Sect. 4.3 we state without proof an existence and uniqueness result for entropy solutions of (4.1). The proof can be obtained by a suitable adaptation of the techniques in [3]. Since the traveling wave solutions we construct are functions in $L^\infty(\mathbb{R}^N)^+$, we give a uniqueness result for solutions in that space (see Sect. 4.3.3). A similar result was proved in [3] for the case $m = 1$.

This section gives the necessary background for the characterization of entropy conditions given in Sect. 1.2.

4.1 Functions of bounded variation and some generalizations

Denote by \mathcal{L}^N and \mathcal{H}^{N-1} the N -dimensional Lebesgue measure and the $(N - 1)$ -dimensional Hausdorff measure in \mathbb{R}^N , respectively. Given an open set Ω in \mathbb{R}^N we denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support in Ω . The space of continuous functions with compact support in \mathbb{R}^N will be denoted by $C_c(\mathbb{R}^N)$.

Recall that if Ω is an open subset of \mathbb{R}^N , a function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a vector valued Radon measure with finite total variation in Ω is called a *function of bounded variation*. The class of such functions will be denoted by $BV(\Omega)$. For $u \in BV(\Omega)$, the vector measure Du decomposes into its absolutely continuous and singular parts $Du = D^a u + D^s u$. Then $D^a u = \nabla u \mathcal{L}^N$, where ∇u is the Radon–Nikodym derivative of the measure Du with respect to the Lebesgue measure

\mathcal{L}^N . We also split $D^s u$ in two parts: the *jump* part $D^j u$ and the *Cantor* part $D^c u$. It is well known (see for instance [1]) that

$$D^j u = (u^+ - u^-) \nu_u \mathcal{H}^{N-1} \llcorner J_u,$$

where $u^+(x), u^-(x)$ denote the upper and lower approximate limits of u at x , J_u denotes the set of approximate jump points of u (i.e. points $x \in \Omega$ for which $u^+(x) \neq u^-(x)$), and $\nu_u(x) = \frac{Du}{|Du|}(x)$, being $\frac{Du}{|Du|}$ the Radon–Nikodym derivative of Du with respect to its total variation $|Du|$. For further information concerning functions of bounded variation we refer to [1].

We need to consider the following truncation functions. For $a < b$, let $T_{a,b}(r) := \max(\min(b, r), a)$. We denote

$$\mathcal{T}_r := \{T_{a,b} : 0 < a < b\}.$$

Given any function w and $a, b \in \mathbb{R}$ we shall use the notation $\{w \geq a\} = \{x \in \mathbb{R}^N : w(x) \geq a\}$, $\{a \leq w \leq b\} = \{x \in \mathbb{R}^N : a \leq w(x) \leq b\}$, and similarly for the sets $\{w > a\}, \{w \leq a\}, \{w < a\}$, etc.

We need to consider the following function space

$$TBV_r^+(\mathbb{R}^N) := \left\{ w \in L^1(\mathbb{R}^N)^+ : T_{a,b}(w) - a \in BV(\mathbb{R}^N), \forall T_{a,b} \in \mathcal{T}_r \right\}.$$

Notice that $TBV_r^+(\mathbb{R}^N)$ is closely related to the space $GBV(\mathbb{R}^N)$ of generalized functions of bounded variation introduced by Di Giorgi and Ambrosio (see [1]) Using the chain rule for BV-functions (see for instance [1]), one can give a sense to ∇u for a function $u \in TBV^+(\mathbb{R}^N)$ as the unique function v which satisfies

$$\nabla T_{a,b}(u) = v \chi_{\{a < u < b\}} \quad \mathcal{L}^N - \text{a.e.}, \quad \forall T_{a,b} \in \mathcal{T}_r.$$

We refer to Lemma 2.1 of [9] or [1] for details.

4.2 Functionals defined on BV

In order to define the notion of entropy solutions of (4.1) and give a characterization of them, we need a functional calculus defined on functions whose truncations are in BV .

Let Ω be an open subset of \mathbb{R}^N . Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow [0, \infty[$ be a Borel function such that

$$C(x)|\zeta| - D(x) \leq g(x, z, \zeta) \leq M'(x) + M|\zeta|$$

for any $(x, z, \zeta) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, $|z| \leq R$, and any $R > 0$, where M is a positive constant and $C, D, M' \geq 0$ are bounded Borel functions which may depend on R . Assume that $C, D, M' \in L^1(\Omega)$.

Following Dal Maso [27] we consider the functional:

$$\begin{aligned} \mathcal{R}_g(u) := & \int_{\Omega} g(x, u(x), \nabla u(x)) \, dx + \int_{\Omega} g^0 \left(x, \tilde{u}(x), \frac{Du}{|Du|}(x) \right) |D^c u| \\ & + \int_{J_u} \left(\int_{u_-(x)}^{u_+(x)} g^0(x, s, \nu_u(x)) \, ds \right) d\mathcal{H}^{N-1}(x), \end{aligned}$$

for $u \in BV(\Omega) \cap L^\infty(\Omega)$, being \tilde{u} is the approximated limit of u [1].

In case that Ω is a bounded set, and under standard continuity and coercivity assumptions, Dal Maso proved in [27] that $\mathcal{R}_g(u)$ is L^1 -lower semi-continuous for $u \in BV(\Omega)$. More recently, De Cicco, Fusco, and Verde [28] have obtained a very general result about the L^1 -lower semi-continuity of \mathcal{R}_g in $BV(\mathbb{R}^N)$.

Assume that $g : \mathbb{R} \times \mathbb{R}^N \rightarrow [0, \infty[$ is a Borel function such that

$$C|\zeta| - D \leq g(z, \zeta) \leq M(1 + |\zeta|) \quad \forall (z, \zeta) \in \mathbb{R}^N, |z| \leq R, \tag{4.6}$$

for any $R > 0$ and for some constants $C, D, M \geq 0$ which may depend on R . Observe that both functions f, h defined in (4.5), (4.4) satisfy (4.6).

Assume that

$$\chi_{\{u \leq a\}} (g(u(x), 0) - g(a, 0)), \chi_{\{u \geq b\}} (g(u(x), 0) - g(b, 0)) \in L^1(\mathbb{R}^N),$$

for any $u \in L^1(\mathbb{R}^N)^+$. Let $u \in TBV_r^+(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $T = T_{a,b} \in \mathcal{T}_r$. For each $\phi \in C_c(\mathbb{R}^N)$, $\phi \geq 0$, we define the Radon measure $g(u, DT(u))$ by

$$\begin{aligned} \langle g(u, DT(u)), \phi \rangle := & \mathcal{R}_{\phi g}(T_{a,b}(u)) + \int_{\{u \leq a\}} \phi(x) (g(u(x), 0) - g(a, 0)) \, dx \\ & + \int_{\{u \geq b\}} \phi(x) (g(u(x), 0) - g(b, 0)) \, dx. \end{aligned} \tag{4.7}$$

If $\phi \in C_c(\mathbb{R}^N)$, we write $\phi = \phi^+ - \phi^-$ with $\phi^+ = \max(\phi, 0)$, $\phi^- = -\min(\phi, 0)$, and we define $\langle g(u, DT(u)), \phi \rangle := \langle g(u, DT(u)), \phi^+ \rangle - \langle g(u, DT(u)), \phi^- \rangle$.

Recall that, if $g(z, \zeta)$ is continuous in (z, ζ) , convex in ζ for any $z \in \mathbb{R}$, and $\phi \in C^1(\mathbb{R}^N)^+$ has compact support, then $\langle g(u, DT(u)), \phi \rangle$ is lower semi-continuous in $TBV^+(\mathbb{R}^N)$ with respect to $L^1(\mathbb{R}^N)$ -convergence [28]. This property is used to prove existence of solutions of (4.1).

We can now define the required functional calculus. We follow [21] and note that it represents an extension of the functional calculus in [2,4] that uses a more restrictive class of test functions.

Let us denote by \mathcal{P} the set of Lipschitz continuous functions $p : [0, +\infty[\rightarrow \mathbb{R}$ satisfying $p'(s) = 0$ for s large enough. We write $\mathcal{P}^+ := \{p \in \mathcal{P} : p \geq 0\}$.

Let $S \in C([0, \infty[)$ and $p \in \mathcal{P} \cap C^1([0, \infty[)$. We denote

$$f_{S;p}(z, \zeta) = S(z)p'(z)f(z, \zeta), \quad h_{S;p}(z, \zeta) = S(z)p'(z)h(z, \zeta).$$

If $Sp' \geq 0$, then the function $f_{S;p}(z, \zeta)$ satisfies the assumptions implying the lower semicontinuity of the associated energy functional [28].

Assume that $p(r) = p(T_{a,b}(r))$, $0 < a < b$. We assume that $u \in TBV_r^+(\mathbb{R}^N)$ and

$$\chi_{\{u \leq a\}}S(u)(f(u(x), 0) - f(a, 0)), \chi_{\{u \geq b\}}S(u)(f(u(x), 0) - f(b, 0)) \in L^1(\mathbb{R}^N).$$

Since $h(z, 0) = 0$, the last assumption clearly holds for h .

Finally, we define $f_{S;p}(u, DT_{a,b}(u))$, $h_{S;p}(u, DT_{a,b}(u))$ as the Radon measures given by (4.7) with $g(z, \zeta) = f_{S;p}(z, \zeta)$ and $g(z, \zeta) = h_{S;p}(z, \zeta)$, respectively.

4.3 Existence and uniqueness of entropy solutions

4.3.1 The class of test functions

Let us introduce the class of test functions required to define entropy sub- and super-solutions. If $u \in TBV_r^+(\mathbb{R}^N)$, we define $TSUB$ (resp. $TSUPER$) as the class of functions $S, T \in \mathcal{P}$ such that

$$S \geq 0, S' \geq 0 \quad \text{and} \quad T \geq 0, T' \geq 0,$$

$$(\text{resp. } S \leq 0, S' \geq 0 \quad \text{and} \quad T \geq 0, T' \leq 0)$$

and $p(r) = \tilde{p}(T_{a,b}(r))$ for some $0 < a < b$, where \tilde{p} is differentiable in a neighborhood of $[a, b]$ and p represents either S or T .

Although the proof of uniqueness and the development of the theory requires only the use of test functions $S, T \in \mathcal{T}^+$ and this was the family used in [4], the analysis of the entropy conditions is facilitated by the use of more general test functions in $TSUB$ and $TSUPER$.

4.3.2 Entropy solutions in $L^1 \cap L^\infty$

Let $L^1_w(0, T, BV(\mathbb{R}^N))$ be the space of weakly* measurable functions $w : [0, T] \rightarrow BV(\mathbb{R}^N)$ (i.e., $t \in [0, T] \rightarrow \langle w(t), \phi \rangle$ is measurable for every ϕ in the predual of $BV(\mathbb{R}^N)$) such that $\int_0^T \|w(t)\|_{BV} dt < \infty$. Observe that, since $BV(\mathbb{R}^N)$ has a separable predual (see [1]), it follows easily that the map $t \in [0, T] \rightarrow \|w(t)\|_{BV}$ is measurable. By $L^1_{loc,w}(0, T, BV(\mathbb{R}^N))$ we denote the space of weakly* measurable functions $w : [0, T] \rightarrow BV(\mathbb{R}^N)$ such that the map $t \in [0, T] \rightarrow \|w(t)\|_{BV}$ is in $L^1_{loc}([0, T])$.

Definition 4.1 Assume that $u_0 \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$. A measurable function $u :]0, T[\times \mathbb{R}^N \rightarrow \mathbb{R}$ is an *entropy sub-solution* (resp. *super-solution*) of (4.1) in $Q_T =]0, T[\times \mathbb{R}^N$ if $u \in C([0, T]; L^1(\mathbb{R}^N))$, $T_{a,b}(u(\cdot)) - a \in L^1_{loc,w}(0, T, BV(\mathbb{R}^N))$ for all $0 < a < b$, and

- (i) $u(0) \leq u_0$ (resp. $u(0) \geq u_0$), and
- (ii) the following inequality is satisfied

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} \phi h_{S:T}(u, DT_{a,b}(u)) dt + \int_0^T \int_{\mathbb{R}^N} \phi h_{T:S}(u, DS_{c,d}(u)) dt \\ & \leq \int_0^T \int_{\mathbb{R}^N} \left\{ J_{TS}(u(t))\phi'(t) - \mathbf{a}(u(t), \nabla u(t)) \cdot \nabla \phi T(u(t))S(u(t)) \right\} dx dt \\ & + \int_0^T \int_{\mathbb{R}^N} \phi(t)T(u(t))S(u(t))F(u(t)) dx dt, \end{aligned} \tag{4.8}$$

for truncation functions $(S, T) \in \mathcal{TSUB}$ (resp. $(S, T) \in \mathcal{TSUPER}$) with $T = \tilde{T} \circ T_{a,b}$, $S = \tilde{S} \circ S_{c,d}$, $0 < a < b$, $0 < c < d$, and any smooth function ϕ of compact support, in particular those of the form $\phi(t, x) = \phi_1(t)\rho(x)$, $\phi_1 \in \mathcal{D}([0, T])$, $\rho \in \mathcal{D}(\mathbb{R}^N)$.

We say that $u :]0, T[\times \mathbb{R}^N \rightarrow \mathbb{R}$ is an *entropy solution* of (4.1) if it is an entropy sub- and super-solution.

Notice that if u is an entropy sub-solution (resp. super-solution), then $u_t \leq \operatorname{div} \mathbf{a}(u(t), \nabla u(t)) + F(u(t))$ (resp. \geq) in $\mathcal{D}'(Q_T)$. We notice also that u is an entropy solution if $u_t = \operatorname{div} \mathbf{a}(u(t), \nabla u(t)) + F(u(t))$ in $\mathcal{D}'(Q_T)$, $u(0) = u_0$ and the inequalities (4.8) hold for truncations $(S, T) \in \mathcal{TSUB}$ and any test functions as in (ii) [21].

We have the following existence and uniqueness result, which is an extension of those in [3].

Theorem 4.2 *Let the set of assumptions (H) be satisfied and let F be Lipschitz continuous with $F(0) = 0$. Then, for any initial datum $0 \leq u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ there exists a unique entropy solution u of (4.1) in Q_T for every $T > 0$ such that $u(0) = u_0$, satisfying $u \in C([0, T]; L^1(\mathbb{R}^N))$ and $F(u(t)) \in$*

$L^1(\mathbb{R}^N)$ for almost all $0 \leq t \leq T$. Moreover, if $u(t), \bar{u}(t)$ are entropy solutions corresponding to initial data $u_0, \bar{u}_0 \in (L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))^+$, respectively, then

$$\|u(t) - \bar{u}(t)\|_1 \leq e^{t\|F\|_{Lip}} \|u_0 - \bar{u}_0\|_1 \quad \text{for all } t \geq 0.$$

4.3.3 Entropy solutions in L^∞

In order to cover the case of bounded traveling waves, we extend the notion of entropy solutions to functions in $L^\infty(\mathbb{R}^N)^+$. We follow the presentation in [3].

Definition 4.3 Given $0 \leq u_0 \in L^\infty(\mathbb{R}^N)$, we say that a measurable function $u :]0, T[\times \mathbb{R}^N \rightarrow \mathbb{R}$ is an *entropy sub-solution* (respectively, *entropy super-solution*) of the Cauchy problem (4.1) in $Q_T =]0, T[\times \mathbb{R}^N$ if $u \in C([0, T]; L^1_{loc}(\mathbb{R}^N))$, $u(0) \leq u_0$ (resp. $u(0) \geq u_0$), $F(u(t)) \in L^1_{loc}(\mathbb{R}^N)$ for almost every $0 \leq t \leq T$, $T_{a,b}(u(\cdot)) - a \in L^1_{loc,w}(0, T, BV_{loc}(\mathbb{R}^N))$ for all $0 < a < b$, $\mathbf{a}(u(\cdot), \nabla u(\cdot)) \in L^\infty(Q_T)$, and the inequalities (4.8) are satisfied for truncations $(S, T) \in \mathcal{TSUB}$ (resp. $(S, T) \in \mathcal{TSUPER}$) with $T = \tilde{T} \circ T_{a,b}$, $S = \tilde{S} \circ S_{c,d}$, $0 < a < b$, $0 < c < d$, and any smooth function ϕ of compact support, in particular those of the form $\phi(t, x) = \phi_1(t)\rho(x)$, $\phi_1 \in \mathcal{D}(]0, T[)$, $\rho \in \mathcal{D}(\mathbb{R}^N)$.

We say that $u :]0, T[\times \mathbb{R}^N \rightarrow \mathbb{R}$ is an entropy solution of (4.1) if u is an entropy sub-solution and super-solution.

Definition 4.4 Let u be a sub- or a super-solution of (4.1) in Q_T . We say that u has a null flux at infinity if

$$\lim_{R \rightarrow +\infty} \int_0^T \int_{\mathbb{R}^N} |\mathbf{a}(u(t), \nabla u(t))| |\nabla \psi_R(x)| dx dt = 0$$

for all $\psi_R \in \mathcal{D}(\mathbb{R}^N)$ such that $0 \leq \psi_R \leq 1$, $\psi_R \equiv 1$ on B_R , $\text{supp}(\psi_R) \subset B_{R+2}$ and $\|\nabla \psi_R\|_\infty \leq 1$.

We have uniqueness of entropy solutions for initial data in $L^\infty(\mathbb{R}^N)$ when they have null flux at infinity.

Theorem 4.5 Let the set of assumptions (H) be satisfied and let F be Lipschitz continuous with $F(0) = 0$.

- (i) Let $u(t), \bar{u}(t)$ be two entropy solutions of (4.1) with initial data $u_0, \bar{u}_0 \in L^\infty(\mathbb{R}^N)^+$, respectively. Assume that $u(t)$ and $\bar{u}(t)$ have null flux at infinity. Then

$$\|u(t) - \bar{u}(t)\|_1 \leq e^{t\|F\|_{Lip}} \|u_0 - \bar{u}_0\|_1, \quad \text{for all } t \geq 0.$$

- (ii) Assume that $u_0 \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$, $\bar{u}_0 \in L^\infty(\mathbb{R}^N)^+$. Let $u(t)$ be the entropy solution of (4.1) with initial datum u_0 . Let $\bar{u}(t)$ be an entropy super-solution of (4.1) with initial datum $\bar{u}_0 \in L^\infty(\mathbb{R}^N)^+$ having a null flux at infinity. Assume in addition that $\bar{u}(t) \in BV_{\text{loc}}(\mathbb{R}^N)$ for almost every $0 < t < T$. Then

$$\|(u(t) - \bar{u}(t))^+\|_1 \leq e^{t\|F\|_{Lip}} \|(u_0 - \bar{u}_0)^+\|_1, \quad \text{for all } t \geq 0.$$

References

1. Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems. Oxford Mathematical Monographs (2000)
2. Andreu, F., Caselles, V., Mazón, J.M.: A Strongly Degenerate Quasilinear Elliptic Equation. *Nonlinear Anal. TMA* **61**, 637–669 (2005)
3. Andreu, F., Caselles, V., Mazón, J.M.: A Fisher-Kolmogorov-Petrovskii-Piskunov equation with finite speed of propagation. *J. Differ. Equ.* **248**, 2528–2561 (2010)
4. Andreu, F., Caselles, V., Mazón, J.M.: The Cauchy Problem for a Strongly Degenerate Quasilinear Equation. *J. Eur. Math. Soc.* **7**, 361–393 (2005)
5. Andreu, F., Caselles, V., Mazón, J.M., Soler, J., Verbeni, M.: Radially symmetric solutions of a tempered diffusion equation. A porous media flux-limited case. *SIAM J. Math. Anal.* **44**, 1019–1049 (2012)
6. Aronson, D.G., Weinberger, H.F.: Multidimensional nonlinear diffusions arising in population genetics. *Adv. Math.* **30**, 33–76 (1978)
7. Aronson, D.G.: Density dependent interaction diffusion systems. In: Proceedings of the Advanced Seminar on Dynamics and Modeling of Reactive Systems, Academic Press, New York (1980)
8. Anzellotti, G.: Pairings Between Measures and Bounded Functions and Compensated Compactness. *Ann. di Matematica Pura et Appl. IV* **135**, 293–318 (1983)
9. Bénilan, Ph, Boccardo, L., Gallouet, T., Gariepy, R., Pierre, M., Vázquez, J.L.: An L^1 -Theory of Existence and Uniqueness of Solutions of Nonlinear Elliptic Equations. *Ann. Scuola Normale Superiore di Pisa, IV* **XXII**, 241–273 (1995)
10. Berestycki, H., Hamel, F.: Front propagation in periodic excitable media. *Comm. Pure Appl. Math.* **55**, 949–1032 (2002)
11. Berestycki, H., Hamel, F., Matano, H.: Bistable travelling waves around an obstacle. *Comm. Pure Appl. Math.* **62**, 729–788 (2009)
12. Berestycki, H., Hamel, F., Nadirashvili, N.: The speed of propagation for KPP type problems. I - Periodic framework. *J. Eur. Math. Soc.* **7**, 173–213 (2005)
13. Berestycki, H., Hamel, F., Nadirashvili, N.: The speed of propagation for KPP type problems. II: General domains. *J. Am. Math. Soc.* **23**, 1–34 (2010)
14. Berestycki, H., Nirenberg, L.: Travelling fronts in cylinders. *Ann. Inst. H. Poincaré, Anal. Non Lin.* **9**, 497–572 (1992)
15. Brenier, Y.: Extended Monge-Kantorovich Theory. In: “Optimal Transportation and Applications”, Lectures given at the C.I.M.E. Summer School held in Martina Franca, L.A. Caffarelli and S. Salsa (eds.), Lecture Notes in Math. 1813, Springer-Verlag, 91–122 (2003)
16. Cabré, X., Roquejoffre, J.-M.: Front propagation in Fisher-KPP equations with fractional diffusion. *C. R. Math. Acad. Sci. Paris* **347**, 1361–1366 (2009)
17. Caffarelli, L., Soria, F., Vázquez, J.L.: Regularity of solutions of the fractional porous medium. *J. Eur. Math. Soc. (JEMS)* **15**, 1701–1746 (2013)
18. Constantin, A., Escher, J.: Analyticity of periodic traveling free surface water waves with vorticity. *Ann. Math.* **173**, 559–568 (2011)

19. Campos, J., Guerrero, P., Sánchez, O., Soler, J.: On the analysis of traveling waves to a nonlinear flux limited reaction-diffusion equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **30**, 141–155 (2013)
20. Calvo, J., Mazón, J., Soler, J., Verbeni, M.: Qualitative properties of the solutions of a nonlinear flux-limited equation arising in the transport of morphogens. *Math. Mod. Meth. Appl. Sci.* **21**, 893–937 (2011)
21. Caselles, V.: An existence and uniqueness result for flux limited diffusion equations. *Discret. Contin. Dyn. Syst.* **31**, 1151–1195 (2011)
22. Caselles, V.: On the entropy conditions for some flux limited diffusion equations. *J. Diff. Eqs.* **250**, 3311–3348 (2011)
23. Caselles, V.: Flux limited generalized porous media diffusion equations. *Publicacions Matemàtiques* **57**, 155–217 (2013)
24. Caselles, V.: Convergence of flux limited porous media diffusion equations to its classical counterpart. *Ann. Scuola Norm. Pisa* **XIV**(2), 481–505 (2015)
25. Chen, G.Q., Frid, H.: Divergence-Measure Fields and Hyperbolic Conservation Laws. *Arch. Ration. Mech. Anal.* **147**, 89–118 (1999)
26. Chertock, A., Kurganov, A., Rosenau, P.: Formation of discontinuities in flux-saturated degenerate parabolic equations. *Nonlinearity* **16**, 1875–1898 (2003)
27. Dal, G.: Maso, Integral representation on $BV(\Omega)$ of Γ -limits of variational integrals. *Manuscr. Math.* **30**, 387–416 (1980)
28. De Cicco, V., Fusco, N., Verde, A.: On L^1 -lower semicontinuity in BV . *J. Convex Anal.* **12**, 173–185 (2005)
29. Fife, P.C., McLeod, J.B.: The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Ration. Mech. Anal.* **65**, 335–361 (1977)
30. Enguica, R., Gavioli, A., Sánchez, L.: A class of singular first order differential equations with applications in reaction-diffusion. *Discret. Contin. Dyn. Syst.* **33**, 173–191 (2013)
31. Fisher, R.A.: The wave of advance of advantageous genes. *Ann. Eugenics* **7**, 335–369 (1937)
32. Gatenby, R.A., Gawlinski, E.T.: A Reaction-Diffusion Model of Cancer Invasion. *Cancer Res.* **56**, 5745–5753 (1996)
33. Jones, C.K.R.T., Gardner, R., Kapitula, T.: Stability of travelling waves for non-convex scalar viscous conservation laws. *Comm. Pure Appl. Math.* **46**, 505–526 (1993)
34. Hadeler, K.P., Rothe, F.: Travelling fronts in nonlinear diffusion equations. *J. Math. Biol.* **2**, 251–263 (1975)
35. Hartman, P.: Ordinary Differential Equations. Wiley, New York (1964)
36. Kolmogorov, N., Petrovsky, I.G., Piskunov, N.S.: Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Bulletin Université d'Etat à Moscou (Bjul. Moskowskogo Gos. Univ.)*, Série internationale A 1, 1–26 (1937)
37. Kondo, S., Miura, T.: Reaction-Diffusion model as a framework for understanding biological pattern formation. *Science* **329**, 1616–1620 (2010)
38. Kruzhkov, S.N.: First order quasilinear equations in several independent variables. *Math. USSR-Sb.* **10**, 217–243 (1970)
39. Kurganov, A., Rosenau, P.: Effects of a saturating dissipation in Burgers-type equations. *Commun. Pure Appl. Math.* **50**, 753–771 (1997)
40. Kurganov, A., Levy, D., Rosenau, P.: On Burgers-type equations with nonmonotonic dissipative fluxes. *Commun. Pure Appl. Math.* **51**, 443–473 (1998)
41. Kurganov, A., Rosenau, P.: On reaction processes with saturating diffusion. *Nonlinearity* **19**, 171–193 (2006)
42. Majda, A.J., Souganidis, P.E.: Flame fronts in a turbulent combustion model with fractal velocity fields. *Comm. Pure Appl. Math.* **51**, 1337–1348 (1998)

43. Marquina, A.: Diffusion front capturing schemes for a class of FokkerPlanck equations: Application to the relativistic heat equation. *J. Comput. Phys.* **229**, 2659–2674 (2010)
44. McCann, R., Puel, M.: Constructing a relativistic heat flow by transport time steps. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**, 2539–2580 (2009)
45. Meinhardt, H., Prusinkiewicz, P., Fowler, D.R.: *The algorithmic beauty of sea shells*. Springer-Verlag, NY (1998)
46. Murray, J.D.: *Mathematical Biology*. Springer-Verlag, NY (1996)
47. Mueller, C., Mytnik, L., Quastel, J.: Effect of noise on front propagation in reaction-diffusion equations of KPP type. *Inv. Math.* **184**, 405–453 (2011)
48. Newman, W.I.: Some Exact Solutions to a Non-linear Diffusion Problem in Population Genetics and Combustion. *J. Theor. Biol.* **85**, 325–334 (1980)
49. Newman, W.I., Sagan, C.: Galactic Civilizations: Populations Dynamics and Interstellar Diffusion. *Icarus* **46**, 293–327 (1981)
50. Ngamsaad, W., Khomphurngson, K.: Self-similar solutions to a density-dependent reaction-diffusion model. *Phys. Rev. E* **85**, 066120 (2012)
51. de Pablo, A., Vázquez, J.L.: Travelling waves and finite propagation in a reaction-diffusion equation. *J. Differ. Equ.* **93**, 19–61 (1991)
52. de Pablo, A., Quirós, F., Rodríguez, A., Vázquez, J.L.: A general fractional porous medium equation. *Comm. Pure Appl. Math.* **65**, 1242–1284 (2012)
53. Rosenau, P.: Tempered Diffusion: A Transport Process with Propagating Front and Inertial Delay. *Phys. Rev. A* **46**, 7371–7374 (1992)
54. Rosenau, P.: Reaction and concentration dependent diffusion model. *Phys. Rev. Lett.* **88**, 194501 (2002)
55. Sánchez-Garduño, F., Maini, P.K.: Existence and uniqueness of a sharp traveling wave in degenerate non-linear diffusion Fisher-KPP equations. *J. Math. Biol.* **33**, 163–192 (1994)
56. Sánchez-Garduño, F., Maini, P.K.: Traveling wave phenomena in Some Degenerate Reaction-Diffusion Equations. *J. Diff. Eq.* **177**, 281–319 (1995)
57. Sánchez-Garduño, F., Maini, P.K., Kappos, M.E.: A shooting argument approach to a sharp-type solution for nonlinear degenerate Fisher-KPP equations. *IMA J. Appl. Math.* **57**, 211–221 (1996)
58. Vázquez, J.L.: *The Porous Medium Equation. Mathematical Theory*. Oxford Univ. Press, Oxford (2006)
59. Verbeni, M., Sánchez, O., Mollica, E., Siegl-Cachedenier, I., Carleton, A., Guerrero, I., Ruiz i Altaba, A., Soler, J.: Modeling morphogenetic action through flux-limited spreading. *Phys. Life Rev.* **10**, 457–475 (2013)