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# Weighted norm inequalities for integral transforms

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Abstract. Weighted $(L^p, L^q)$ inequalities are studied for a variety of integral transforms of Fourier type. In particular, weighted norm inequalities for the Fourier, Hankel, and Jacobi transforms are derived from Calderón type rearrangement estimates. The obtained results keep their novelty even in the simplest cases of the studied transforms, the cosine and sine Fourier transforms. Sharpness of the conditions on weights is discussed.

Contents



## 1. Introduction

The problem of characterizing the pairs of the weights governing strong-type norm inequalities for classical integral operators is of considerable importance in analysis. This problem can be formulated as follows. Let $L_v^p := L_{v(\cdot)}^p(Y)$,

$$\|f\|_{p,v} = \left(\int_Y |f(y)|^p v(y)\, dy\right)^{1/p}, \tag{1.1}$$

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where in our considerations $Y = \mathbb{R}^n$, $n \geq 1$, or $\mathbb{R}_+$. Given an operator $F$ and $1 < p \leq q < \infty$, find necessary and sufficient conditions on a pair of the weights $u$ and $v$, i.e., non-negative locally integrable functions, such that

$$\|Ff\|_{q,u} \lesssim \|f\|_{p,v}. \tag{1.2}$$

Here and in the sequel, the expressions $f \lesssim g$, $f \gtrsim g$, $f \asymp g$ mean the inequalities $f \leq Cg$, $f \geq Cg$, $C^{-1}g \leq f \leq Cg$, respectively. Unless otherwise stated, $C$ denotes a positive constant, not necessarily the same at each occurrence. We remark that the expressions $x \lesssim 1$, $x \gtrsim 1$, $x \asymp 1$ mean that $0 < x \leq x_0$, $x \geq x_1$, $x_0 \leq x \leq x_1$, respectively, for some $x_0, x_1 > 0$.

We note that the usual $L_p$ norm is given by

$$\|f\|_p = \|f\|_{p,1}, \quad 1 < p < \infty, \qquad \|f\|_\infty = \operatorname*{ess\,sup}_{y \in Y} |f(y)|.$$

For the classical Fourier transform $Ff = \widehat{f}$, both the Hausdorff-Young inequality

$$\|\widehat{f}\|_{p'} \leq \|f\|_p, \qquad 1 \leq p \leq 2,$$

and the Hardy–Littlewood inequality (see, e.g., [**54**])

$$\left( \int_{\mathbb{R}^n} |\widehat{f}(x)|^p |x|^{p-2} \, dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} |f(y)|^p \, dy \right)^{1/p}, \qquad 1 < p \leq 2, \tag{1.3}$$

are particular cases of (1.2). Moreover, a non-weighted analogue of (1.3) is possible only when $p = 2$, which is the Plancherel inequality.

For the power weights $|x|^{-q\gamma}$ and $|y|^{p\beta}$, inequality (1.2) becomes the classical Pitt inequality [**3, 5, 44, 48**]. The problem of extending it to general weights for particular transforms of Fourier type was intensively studied since the mid 70's after the paper [**40**], where Muckenhoupt stated this problem for the Fourier transform and found some sufficient conditions. Later the study was continued in [**5, 26, 27, 29, 31, 32, 35**] as well as in many other sources. In addition to various applications, inequalities of type (1.2) describe the balance between the relative sizes of a function and its transform at infinity and can be considered as a quantitative expression of the uncertainty principle.

**1.1. Integral transforms of Fourier type.** The main goal of this paper is to study weighted norm inequality (1.2) for the integral operator $F\colon L_v^p \to L_u^q$, given by

$$Ff(x) = \int_0^\infty f(y) K(x,y) s(y) \, dy, \qquad x > 0, \tag{1.4}$$

where $f$ is a locally integrable function, $K$ is a continuous kernel, and $s$ is a continuous non-negative non-decreasing function satisfying the $\Delta_2$-condition. The latter means that

$$s(2y) \lesssim s(y), \quad y > 0. \tag{1.5}$$

In what follows, we are mainly concerned with the transforms (1.4) *of Fourier type* (see, e.g., [**54**, Ch. 7], [**25, 56**]), written $F$-transform, which means that if $f \in L_s^2$, then there exists a non-negative non-decreasing function $w$ satisfying

$$w(x)s(1/x) \asymp 1, \quad x > 0, \tag{1.6}$$

for which Bessel's inequality

$$\|Ff\|_{2,w} \lesssim \|f\|_{2,s}, \quad \text{or} \quad \|w^{1/2}Ff\|_2 \lesssim \|s^{1/2}f\|_2, \tag{1.7}$$

is valid. It is also assumed that the kernel $K$ satisfies the condition

$$|K(x,y)| \lesssim \min\left\{1,\, [w(x)s(y)]^{-1/2}\right\}, \qquad x,y>0. \tag{1.8}$$

It follows from (1.5) and (1.6) that $w$ satisfies the $\Delta_2$-condition as well.

In the sequel, we assume that

$$s(y)f(y) \in L^1_{\mathrm{loc}}.$$

We remark that the functions $s$ and $w$ may also be considered as weights in certain occurrences. However, in our study it is convenient to distinguish between the norm generating weights $u$ and $v$, and the operator generating weights $s$ and $w$.

An important example of a weight $s(\cdot)$ satisfying (1.5) is $s(y) = y^\nu$, $\nu \ge 0$, or more generally, a piecewise power weight, i.e., the one of the form

$$s(y) = y^{\overline{\nu}} := \begin{cases} y^{\nu_1}, & y \le 1, \\ y^{\nu_2}, & y \ge 1, \end{cases} \qquad \nu_1, \nu_2 \ge 0. \tag{1.9}$$

Throughout the paper we write

$$\overline{\nu} = (\nu_1, \nu_2) \in \mathbb{R}^2$$

and $\overline{\nu} \ge 0$ in place of $\nu_1, \nu_2 \ge 0$.

If $s(y) \asymp y^{\overline{\nu}}$, then, by (1.6), we have $w(x) \asymp x^{\overline{\nu}^\circ}$, where

$$\overline{\nu}^\circ := (\nu_2, \nu_1).$$

Power weights will play an important role in our study of the Hankel transform, while piecewise power weights will be important for the analysis of the Jacobi transform. Note that piecewise power weights were considered earlier in the study of weighted Fourier inequalities, see, e.g., [**9, 17**].

The simplest example of an integral transform of Fourier type is the cosine Fourier transform

$$\widehat{f}_c(x) = \int_0^\infty f(y)\cos xy\, dy,$$

for which $K(x,y) = \cos(xy)$, $s(y) \equiv 1$, and $w(x) \equiv 2/\pi$. Likewise, we can deal with the sine Fourier transform

$$\widehat{f}_s(x) = \int_0^\infty f(y)\sin xy\, dy,$$

taking $K(x,y) = (xy)^{-1}\sin(xy)$, $s(y) = y^2$, and $w(x) = (2/\pi)x^2$. Both transforms are particular cases of the Hankel and Mehler–Fock operators, or more generally, Jacobi operators, which will further be considered in detail.

From the general point of view, the kernel $K$ of the integral transform (1.4) can be considered as an eigenfunction of the following Sturm–Liouville problem:

$$\frac{d}{dy}\left(s(y)\,\frac{d}{dy}\,K(x,y)\right) + x^2 s(y)K(x,y) = 0, \qquad y \ge 0,$$

$$K(x,0) = 1, \qquad \frac{d}{dy}\,K(x,0) = 0,$$

with spectrum $x \ge 0$. Here the function $x \mapsto K(x,y)$, $y>0$, is an even entire function of exponential type $y$. It follows from the spectral theory of the Sturm–Liouville problem (see, e.g., [**36**]) that under certain additional assumptions on $s$ an associated $w$ exists. Moreover, for an arbitrary $f \in L_s^2$, we have $Ff \in L_w^2$ along with the inversion formula

$$f(y) = \int_0^\infty Ff(x)K(x,y)w(x)\,dx$$

and Parseval's identity

$$\|Ff\|_{2,w} = \|f\|_{2,s}.$$

Note that condition (1.8), needed in our study, follows from general properties of eigenfunctions of the Sturm–Liouville problem. First, the condition $|K(x,y)| \le K(x,0) = 1$, $x, y \ge 0$, holds provided that, for example, $s$ is non-decreasing (see [**52**, Th. 7.31.1]). On the other hand, if the kernel $K(x,y)$ can be represented as the following Mehler integral

$$K(x,y) = \int_0^y A(t,y)\cos(xt)\,dt, \qquad K(0,y) = 1$$

with a continuous non-negative function $A(t,y)$, then we have again $|K(x,y)| \le 1$, $x, y \ge 0$ as well as $K(x,y) \asymp 1$, $xy \lesssim 1$. Second, the condition $|K(x,y)| \lesssim [w(x)s(y)]^{-1/2}$ can be derived from the asymptotic behavior of eigenfunctions for large $x$, $y$; say, for eigenvalues of (1.18) type.

We will now outline the obtained Pitt's inequalities for various integral transforms.

**1.2. Weighted norm inequalities for the Fourier transforms.** It is known that for the power weights $u(x) = |x|^{-q\gamma}$ and $v(y) = |y|^{p\beta}$, the corresponding Pitt inequality for the Fourier transform

$$\widehat{f}(x) = \int_{\mathbb{R}^n} f(y)e^{-ixy}\,dx \tag{1.10}$$

manifests itself as

$$\left(\int_{\mathbb{R}^n} |\widehat{f}(x)|^q |x|^{-q\gamma} dx\right)^{1/q} \lesssim \left(\int_{\mathbb{R}^n} |f(y)|^p |y|^{p\beta}\,dy\right)^{1/p}, \qquad 1 < p \le q < \infty, \tag{1.11}$$

that is valid if and only if

$$\max\left\{0, n\Big(\frac{1}{p} + \frac{1}{q} - 1\Big)\right\} \le \gamma < \frac{n}{q} \tag{1.12}$$

and

$$\beta - \gamma = n\left(1 - \frac{1}{p} - \frac{1}{q}\right).$$

Since this inequality is the one of the basic results in Fourier analysis (in particular, it contains Plancherel's theorem ($p = q = 2$, $\gamma = \beta = 0$), the Hardy–Littlewood theorem ($1 < p = q \le 2$, $\beta = 0$ or $p = q \ge 2$, $\gamma = 0$), and the Hausdorff–Young theorem ($q = p' \ge 2$, $\gamma = \beta = 0$)), the problem of extending the range of $\gamma$ under additional regularity of $f$ has been intensively studied (see, e.g., [**4, 16, 38, 45, 51**]). In particular, it turns out that for the Fourier transform of a radial function the sharp range for $\gamma$ is given by

$$\frac{n}{q} - \frac{n+1}{2} + \max\left\{\frac{1}{p}, \frac{1}{q'}\right\} \le \gamma < \frac{n}{q}, \tag{1.13}$$

see discussion in Section 6 and Appendix. Moreover, for the Fourier transform of a radial function which, in addition, is monotone (general monotone) the sharp range for $\gamma$ is given by

$$\frac{n}{q} - \frac{n+1}{2} < \gamma < \frac{n}{q},$$

see [**23**].

For the cosine Fourier transform $\widehat{f_c}$, our results applied to piecewise power weights $\overline{\gamma} = (\gamma_1, \gamma_2)$ (see Theorem 7.1 below) yield the following estimates:
*Let $1 < p \le q < \infty$, $\overline{\gamma} = (\gamma_1, \gamma_2)$, $\overline{\beta} = (\beta_1, \beta_2)$, $\gamma_1 - \beta_1 = \gamma_2 - \beta_2$. Pitt's inequality*

$$\|x^{-\overline{\gamma}^{\circ}} \widehat{f_c}\|_q \lesssim \|y^{\overline{\beta}} f\|_p$$

*holds if and only if*

$$\overline{\beta} = \overline{\gamma} + \frac{1}{p'} - \frac{1}{q}$$

*and*

$$\max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\} \le \gamma_i < \frac{1}{q}, \quad i = 1, 2.$$

In particular, for $\gamma_1 = \gamma_2 = \gamma$, we obtain

$$\max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\} \le \gamma < \frac{1}{q},$$

which corresponds to (1.12) and (1.13) with $n = 1$. It is interesting that for the sine Fourier transform $\widehat{f_s}$ the sharp range for $\gamma$ is given by

$$\max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\} \le \gamma < \frac{1}{q} + 1,$$

see Theorem 7.2 below. For monotone type functions, this result has earlier been proved in [**37**], [**38**, (3.9)]; related results can be found in, e.g., [**10, 11, 24, 28, 46, 53**].

It is worth mentioning that the conditions on $\gamma$ which guarantee Pitt's inequality to hold for the cosine Fourier transform are the same as the ones for the general Fourier transform (1.10) in the case $n = 1$ (cf. (1.12)). On the other hand, for the sine Fourier transform the range on $\gamma$ is wider, see above. It is interesting that this result supplements the one of Sadosky and Wheeden [**45**], which shows that if a function $f$ satisfies

$$\int_{\mathbb{R}} f(x)\, dx = 0,$$

Pitt's inequality (1.11) holds if $\gamma$ satisfies either (1.12) or $1/q < \gamma < 1/q + 1$ but not when $\gamma = 1/q$. Therefore, within the scope of Pitt's inequality, considering odd functions differs from dealing with all functions with mean zero.

Concerning the Fourier inequalities with general weights, the following result was proved in 1983–84 by Heinig [**26**], Jurkat–Sampson [**31**] and Muckenhoupt [**41, 42**]: *If the weight $u$ is non-increasing and the weight $v$ is non-decreasing, then*

$$\|u^{1/q} \widehat{f}\|_q \lesssim \|v^{1/p} f\|_p \tag{1.14}$$

*holds, for $1 < p \le q < \infty$, if and only if*

$$\sup_{r>0} \left( \int_0^{1/r} u(t)\, dt \right)^{1/q} \left( \int_0^r v(t)^{1-p'}\, dt \right)^{1/p'} < \infty.$$

Another approach to obtain (1.14) with general weights has recently been considered in [**17**]. It is based on restriction inequalities for the Fourier transform on the unit sphere of $\mathbb{R}^n$ (see, e.g., [**9**]).

**1.3. Weighted norm inequalities for the Hankel and Jacobi transforms.** One of the most important instances of the considered transforms, the Hankel transform, is defined by setting

$$Hf(x) = \int_0^\infty f(y)K(x,y)s(y)\,dy, \qquad K(x,y) = j_\alpha(xy),$$

where

$$s(y) = y^\nu, \qquad w(x) = b_\alpha x^\nu, \qquad b_\alpha^{-1} = 2^{2\alpha}\Gamma(\alpha+1)^2, \qquad \nu = 2\alpha+1 \geq 0,$$

Here $\alpha \geq -1/2$ and the normalized Bessel function $j_\alpha(t)$ is given by

$$j_\alpha(t) = \Gamma(\alpha+1)(t/2)^{-\alpha}J_\alpha(t) = \sum_{k=0}^\infty \frac{(-1)^k\Gamma(\alpha+1)(t/2)^{2k}}{k!\,\Gamma(k+\alpha+1)}, \tag{1.15}$$

where $J_\alpha$ is the classical Bessel function (see, e.g., [**50**, Ch. IV]). Note that $H^{-1} = b_\alpha H$,

$$w(x)s(1/x) = b_\alpha, \quad x > 0, \quad \text{and} \quad \|H_\nu f\|_{2,w}^2 = b_\alpha^{-1}\|f\|_{2,s}^2.$$

In this case, for $\nu \geq 0$ and for the power weights $u(x) = x^{-q\gamma}$ and $v(y) = y^{p\beta}$, Pitt's inequality

$$\left(\int_0^\infty |Hf(x)|^q x^{-\gamma q} w(x)\,dy\right)^{1/q} \lesssim \left(\int_0^\infty |f(y)|^p y^{\beta p} s(y)\,dy\right)^{1/p}, \quad 1 < p \leq q < \infty, \tag{1.16}$$

holds if and only if

$$\beta = \gamma + (\nu+1)\left(\frac{1}{p'} - \frac{1}{q}\right) \quad \text{and} \quad \frac{\nu+1}{q} - \frac{\nu+2}{2} + \max\left\{\frac{1}{p}, \frac{1}{q'}\right\} \leq \gamma < \frac{\nu+1}{q};$$

see Section 6.

As for the Jacobi transforms, their operator generating weight $s$ (see (1.4)) is of piecewise power type. The Jacobi functions are defined by

$$\varphi_\lambda^{(\alpha,\beta)}(t) = F\left(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \alpha+1; -(\sinh t)^2\right), \qquad t \geq 0, \quad \alpha,\beta,\lambda \in \mathbb{C},$$

where $\rho = \alpha+\beta+1$ and

$$F(a,b;c;z) = \sum_{k=0}^\infty \frac{(a)_k(b)_k}{(c)_k k!}\, z^k \tag{1.17}$$

is the hypergeometric Gauss function.

We consider the case $\alpha \geq \beta \geq -1/2$. The direct and inverse Jacobi transforms are defined by the identities

$$Jf(\lambda) = \int_0^\infty f(t)\varphi_\lambda^{(\alpha,\beta)}(t)m(t)\,dt,$$

$$J^{-1}f(t) = \int_0^\infty f(\lambda)\varphi_\lambda^{(\alpha,\beta)}(t)n(\lambda)\,d\lambda,$$

respectively, where

$$m(t) = (2\pi)^{-1/2}\Delta(t), \qquad \Delta(t) = 2^{2\rho}(\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1},$$

$$n(\lambda) = (2\pi)^{-1/2}|c(\lambda)|^{-2}, \qquad c(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma\left(\frac{\rho+i\lambda}{2}\right)\Gamma\left(\frac{\rho+i\lambda}{2}-\beta\right)}.$$

For both transforms Parseval's identities are true:

$$\|Jf\|_{L^2_n} = \|f\|_{L^2_m}, \qquad \|f\|_{L^2_n} = \|J^{-1}f\|_{L^2_m}.$$

In the case $\alpha = \beta$, the Jacobi transform is also known as the Mehler–Fock transform, see [**39**], [**55**].

For the weight $n(\lambda)$, $\lambda > 0$, we have (cf. (1.9))

$$n(\lambda) \asymp \begin{cases} \lambda^2, & 0 < \lambda < 1, \\ \lambda^{2\alpha+1}, & \lambda \ge 1. \end{cases}$$

This shows that the use of piecewise power weights is essential. They appear in a natural way and are within the framework of our theory. The weight $m(x)$ is not reciprocal for $n(\lambda)$ in the sense of (1.6) (it is of exponential behavior); therefore, we modify the Jacobi transforms in an appropriate way and then formulate Pitt's inequalities for them (see Theorem 8.1). For details, see Section 8.

As is mentioned, in the case of the Hankel transform we consider radial functions in the Euclidean space $\mathbb{R}^n = M(n)/SO(n)$ [**55**]. The Jacobi transform [**34**] and the Mehler–Fock transform as its partial case are, in turn, related to radial functions on hyperbolic spaces, in particular on the hyperboloid $\mathbb{H}^n = SO_0(n,1)/SO(n)$. The obtained results give an opportunity to establish Pitt's inequality in this multidimensional case.

**1.4. Structure of the paper.** The paper is organized as follows. First, we obtain results for the general Fourier type integral operators (Sections 2–5). We install the needed machinery and we apply it to get weighted norm inequalities. Second, we make use of the obtained results to analyze specific operators, namely, the Fourier, Hankel, and Jacobi operators.

In Section 2, we give a Hausdorff–Young estimate for the operator (1.2) and prove Calderón type characterization for a sublinear operator to be of type $(1,\infty)$ and $(a,a')$, $1 < a < \infty$. Section 3 contains sufficient conditions on weights to ensure Pitt's inequality for a Fourier type operator. For operators with kernels $K(x,y) \asymp 1$, $0 \le xy \lesssim 1$, a necessary condition is obtained as well. In Section 4, we prove necessary conditions on weights to have Pitt's inequality for the operators with oscillating kernels, i.e., which satisfy, for $xy \gtrsim 1$,

$$[w(x)s(y)]^{1/2} K(x,y) = \begin{cases} C\left[\cos(xy-c) + O(x^{-1})\right], & x \gtrsim 1, \quad y \asymp 1, \\ D\left[\cos(xy-d(x)) + O(y^{-1})\right], & y \gtrsim 1, \quad x \asymp 1. \end{cases} \tag{1.18}$$

Note that such asymptotic relations ensure (1.8).

In Section 5, we concentrate on necessary and sufficient conditions for Pitt's inequality, where weights are power weights with piecewise exponent: $u(x) = x^{-q\overline{\gamma}}$ and $v(y) = y^{p\overline{\beta}}$.

As applications of general results in Sections 2–5, we are mainly concerned with Pitt's inequalities for the following Fourier type operators: Fourier, Hankel, and Jacobi transforms. In Section 6, we deal with the Hankel transforms. In Section 7, we discuss special cases of the general result: the radial Fourier transform, the cosine and sine Fourier transforms. Jacobi type transforms are studied in Section 8.

In a somewhat technical Appendix, we consider the conditions that ensure the Pitt inequality (1.2) if one applies special Hardy's inequalities for monotone functions rather than general Hardy's inequality.

## 2. Hausdorff–Young and Calderón's inequalities

The results of this section provide us with a needed machinery for establishing weighted norm estimates, and they are of interest by themselves.

**2.1. Hausdorff–Young type results.** The next lemma shows that the $F$-transforms possess an important property of the Fourier transforms: the Hausdorff–Young inequality, i.e., these operators are of $(a,a')$ type, where $1\le a\le 2$.

Lemma 2.1. *There holds*

$$\|Ff\|_{a',w}\lesssim \|f\|_{a,s},\qquad 1\le a\le 2,$$

*with* $1/a+1/a'=1$.

Proof. It follows from (1.4) and (1.8) that

$$\|Ff\|_{\infty}=\sup_{x>0}\left|\int_0^\infty f(y)K(x,y)s(y)\,dy\right|\lesssim \int_0^\infty |f(y)\,s(y)|\,dy=\|f\|_{1,s}.$$

Interpolating this inequality and Bessel's inequality (1.7) (see [**49**]), we arrive at the assertion of the lemma. □

The following lemma asserts that the $F$-transforms are of $(1,\infty)$ type with respect to weights $s$ and $w$.

Lemma 2.2. *Let* $1\le a\le 2$. *The inequality holds*

$$\|w^{1/a'}Ff\|_\infty\lesssim \|s^{1/a}f\|_1.$$

Proof. It follows from (1.4) that, for $x>0$,

$$\begin{aligned} Ff(x) &= \int_0^\infty f(y)K(x,y)s(y)\,dy\\ &= \int_0^{1/x} f(y)K(x,y)s(y)\,dy+\int_{1/x}^\infty f(y)K(x,y)s(y)\,dy=:I_1+I_2.\end{aligned}$$

This and the first estimate in (1.8) yield

$$|I_1|\lesssim \int_0^{1/x}|f(y)|s(y)\,dy=\int_0^{1/x}|f(y)|s(y)^{1/a'}s(y)^{1/a}\,dy\lesssim s(1/x)^{1/a'}\int_0^{1/x}|f(y)|s(y)^{1/a}\,dy.$$

where we used that the function $s(\cdot)$ is non-decreasing. Therefore, taking into account (1.6), we obtain

$$w(x)^{1/a'}|I_1|\lesssim w(x)^{1/a'}s(1/x)^{1/a'}\int_0^{1/x}|f(y)|s(y)^{1/a}\,dy\lesssim \int_0^{1/x}|f(y)|s(y)^{1/a}\,dy.$$

Similarly, using the second estimate in (1.8) and that the function $s(\cdot)^{1/a'-1/2}$, with $1/a' - 1/2 \le 0$, is non-increasing, we get

$$w(x)^{1/a'}|I_2| \lesssim w(x)^{1/a'}w(x)^{-1/2}\int_{1/x}^{\infty}|f(y)|s(y)^{1/2}\,dy$$
$$\lesssim w(x)^{1/a'-1/2}s(1/x)^{1/a'-1/2}\int_{1/x}^{\infty}|f(y)|s(y)^{1/a}\,dy \lesssim \int_{1/x}^{\infty}|f(y)|s(y)^{1/a}\,dy.$$

Finally,

$$|w(x)^{1/a'}Ff(x)| \lesssim \int_0^{1/x}|f(y)|s(y)^{1/a}\,dy + \int_{1/x}^{\infty}|f(y)|s(y)^{1/a}\,dy$$

implies $\|w^{1/a'}Ff\|_\infty \lesssim \|s^{1/a}f\|_1$, which completes the proof. □

**2.2. Hardy's inequalities.** In this work we frequently use Hardy's inequalities with general weights [**12**]:

$$\|P_x g\|_{q,u} \lesssim \|g\|_{p,v} \tag{2.1}$$

and

$$\|Q_x g\|_{q,u} \lesssim \|g\|_{p,v}. \tag{2.2}$$

Here $1 \le p \le q < \infty$, $u$ and $v$ are weights, that is, non-negative locally integrable functions, $g \ge 0$, and

$$P_x g = \int_0^x g(y)\,dy, \qquad Q_x g = \int_x^\infty g(y)\,dy$$

are the Hardy and Bellman operators, respectively. Inequality (2.1) holds if and only if, for each $r > 0$,

$$(Q_r u)^{1/q}\left(P_r v^{1-p'}\right)^{1/p'} \lesssim 1, \tag{2.3}$$

where here and in similar conditions the constant on the right does not depend on $r$. For the $p = 1$ and $q < \infty$ see also [**43**, (5.12)]. Similarly, (2.2) holds if and only if, for each $r > 0$,

$$(P_r u)^{1/q}\left(Q_r v^{1-p'}\right)^{1/p'} \lesssim 1.$$

**2.3. Calderón type results.** Let us now proceed to Calderón type rearrangement inequalities for the general sublinear operators $T$. As usual, the non-increasing rearrangement of a function $g$ is denoted by $g^*$ (see, e.g., [**50**]).

THEOREM 2.1. *Let $T$ be a sublinear operator. $T$ is of type $(1,\infty)$ and $(a,a')$, $1 < a < \infty$, if and only if for each $f$, which belongs to both $L^1(\mathbb{R}^n)$ and $L^a(\mathbb{R}^n)$,*

$$\left(\int_0^x (Tf)^*(t)^{a'}\,dt\right)^{1/a'} \lesssim \left(\int_0^x\left(\int_0^{1/t} f^*(s)\,ds\right)^a t^{a-2}\,dt\right)^{1/a}, \quad x > 0. \tag{2.4}$$

Proof. The case $a = 2$ can be found in [**29**, Th. 4.6], while another proof of the "if" part for the Fourier transform can be found in [**32**, Th. 1 and (3.5)].

Let (2.4) hold. Since $(Tf)^*$ is non-increasing, we have

$$x(Tf)^*(x)^{a'} \le \int_0^x ((Tf)^*(t)^{a'}\,dt \lesssim \left(\int_0^x \left(\int_0^{1/t} f^*(s)\,ds\right)^a t^{a-2}\,dt\right)^{a'/a}$$
$$\le \left(\int_0^x \left(\int_0^{\infty} f^*(s)\,ds\right)^a t^{a-2}\,dt\right)^{a'/a} \lesssim x\|f\|_1^{a'}.$$

By this, $\|Tf\|_\infty = \|(Tf)^*\|_\infty \lesssim \|f\|_1$.

Let now $f \in L^a$. Since (2.4) holds for all $x$, we get

$$\left(\int_0^\infty (Tf)^*(t)^{a'}\,dt\right)^{1/a'} \lesssim \left(\int_0^\infty \left(\int_0^{1/t} f^*(s)\,ds\right)^a t^{a-2}\,dt\right)^{1/a}$$
$$= \left(\int_0^\infty \left(\int_0^{t} f^*(s)\,ds\right)^a t^{-a}\,dt\right)^{1/a} = \left(\int_0^\infty \left(\frac{1}{t}\int_0^{t} f^*(s)\,ds\right)^a dt\right)^{1/a}.$$

By Hardy's inequality (2.1), the last integral does not exceed

$$\frac{a}{a-1}\left(\int_0^\infty f^*(t)^a\,dt\right)^{1/a} = \frac{a}{a-1}\,\|f\|_a.$$

This yields $T\colon L^a \to L^{a'}$.

Conversely, let now $T$ be such that $T\colon L^1 \to L^\infty$ and $T : L^a \to L^{a'}$. Fixing $s > 0$ and decomposing $f = f_1 + f_2$ so that (cf. [**32**])

$$f_1^*(t) = \begin{cases} f^*(t), & 0 < t \le s,\\ 0, & t > s,\end{cases}$$

and $f_2^*(t) = f^*(t+s)$, $t > 0$, we obtain for any measurable set $E$ with $|E| = x$

$$\left(\int_{\mathbb{R}^n} |(Tf)(\xi)|^{a'}\chi_E(\xi)\,d\xi\right)^{1/a'} \lesssim \left(\int_{\mathbb{R}^n} |(Tf_1)(\xi)|^{a'}\chi_E(\xi)\,d\xi\right)^{1/a'}$$
$$+ \left(\int_{\mathbb{R}^n} |(Tf_2)(\xi)|^{a'}\chi_E(\xi)\,d\xi\right)^{1/a'}$$
$$\le x^{1/a'}\|Tf_1\|_\infty + \left(\int_{\mathbb{R}^n} |(Tf_2)(\xi)|^{a'}\chi_E(\xi)\,d\xi\right)^{1/a'}$$
$$\lesssim x^{1/a'}\int_0^\infty f_1^*(t)\,dt + \left(\int_0^\infty f_2^*(t)^a\,dt\right)^{1/a}$$
$$\le x^{1/a'}\int_0^s f^*(t)\,dt + \left(\int_s^\infty f^*(t)^a\,dt\right)^{1/a}.$$

Taking $s = 1/x$, we get

$$\left(\int_0^x (Tf)^*(t)^{a'}\,dt\right)^{1/a'} \le \sup_{E\colon |E|=x} \left(\int_{\mathbb{R}^n} |(Tf)(\xi)|^{a'} \chi_E(\xi)\,d\xi\right)^{1/a'}$$
$$\lesssim x^{1/a'} \int_0^{1/x} f^*(t)\,dt + \left(\int_{1/x}^\infty f^*(t)^a\,dt\right)^{1/a}.$$

The last term can be rewritten as

$$\left(\int_{1/x}^\infty f^*(t)^a\,dt\right)^{1/a} = \left(\int_0^x \left[\frac{1}{t}\,f^*\Big(\frac{1}{t}\Big)\right]^a t^{a-2}\,dt\right)^{1/a}.$$

In virtue of the monotonicity of $f^*$ it is dominated by the right-hand side of (2.4).

For the first term, the monotonicity of $f^*$ and Hölder's inequality yield

$$x^{1/a'} \int_0^{1/x} f^*(t)\,dt \le x^{-1/a} \int_0^x \left[\int_0^{1/t} f^*(s)\,ds\, t^{1-2/a}\right] t^{-1+2/a}\,dt$$
$$\le x^{-1/a} \left(\int_0^x \left(\int_0^{1/t} f^*(s)\,ds\right)^a t^{a-2}\,dt\right)^{1/a} \left(\int_0^x t^{(-1+2/a)a'}\,dt\right)^{a'}$$
$$\lesssim \left(\left(\int_0^{1/t} f^*(s)\,ds\right)^a t^{a-2}\,dt\right)^{1/a},$$

the desired estimate. □

In particular, this result implies the following Calderón's inequality ([**15**]; see also [**26**]).

THEOREM 2.2. *Let $T$ be a sublinear operator. If $T$ is of type $(1,\infty)$ and $(a,a')$, $1 < a < \infty$, then*

$$(Tf)^*(x) \lesssim \int_0^{1/x} f^*(y)\,dy + x^{-1/a'} \int_{1/x}^\infty y^{-1/a'} f^*(y)\,dy. \tag{2.5}$$

PROOF. Using a weaker form of (2.4), we have

$$(Tf)^*(x) \lesssim x^{-1/a'} \left(\int_{1/x}^\infty \left(\int_0^t f^*(s)\,ds\right)^a t^{-a}\,dt\right)^{1/a}. \tag{2.6}$$

The right-hand side is bounded by

$$x^{-1/a'} \left(\int_{1/x}^\infty \left(\int_0^{1/x} f^*(s)\,ds\right)^a t^{-a}\,dt\right)^{1/a}$$
$$+ x^{-1/a'} \left(\int_{1/x}^\infty \left(\int_{1/x}^t f^*(s)\,ds\right)^a t^{-a}\,dt\right)^{1/a}.$$

The first summand is just the first integral on the right-hand side of (2.5). To show that the second summand is controlled by

$$x^{-1/a'} \int_{1/x}^\infty y^{-1/a'} f^*(y)\,dy,$$

we use Hardy's inequality (2.1). Note that in this case condition (2.3) reads as $\sup_{1/x<t<\infty}\left[\left(\int_t^\infty s^{-a}ds\right)^{1/a}\sup_{1/x<\xi<t}\xi^{1/a'}\right]$, which is finite. □

## 3. Pitt's inequality

In this section, we obtain sufficient and necessary conditions on weights that guarantee that a Pitt type inequality for the $F$-transform holds.

**3.1. Sufficient conditions for Pitt's inequality.** Define $v_* = [(1/v)^*]^{-1}$.

Theorem 3.1. (A) *Let* $1 < p \le q < \infty$, $1 < a \le 2$, $(p, q, a) \ne (2, 2, 2)$. *Let also* $u$ *and* $v$ *be the weights, which satisfy*

$$\left(P_{1/r}u^*\right)^{1/q}\left(P_r v_*^{1-p'}\right)^{1/p'} \lesssim 1, \qquad r > 0, \tag{3.1}$$

$$\left[Q_{1/r}(x^{-q/a'}u^*)\right]^{1/q}\left[Q_r(y^{-p'/a'}v_*^{1-p'})\right]^{1/p'} \lesssim 1, \qquad r > 0. \tag{3.2}$$

*Then the following Pitt inequality holds*

$$\|w^{1/a'}Ff\|_{q,u} \lesssim \|s^{1/a}f\|_{p,v},$$

*where* $s$ *and* $w$ *come from the definition of the* $F$*-transform* (1.4).

(B) *Let* $p = q = a = 2$, $u$ *and* $v$ *be the weights, which satisfy*

$$\left(P_{1/r}u^*\right)\left(P_r v_*^{-1}\right) \lesssim 1, \qquad r > 0. \tag{3.3}$$

*Then the following Pitt's type inequality holds*

$$\|w^{1/2}Ff\|_{2,u} \lesssim \|s^{1/2}f\|_{2,v}. \tag{3.4}$$

Recall that condition (3.1) assumes that $u^*$, $v_*^{1-p'} \in L^1_{\mathrm{loc}}$, cf. Subsection 2.2.

To prove Theorem 3.1, in addition to (2.1), we will make use of Hardy's inequality for rearrangements (see, e.g., [**5**])

$$\|f\|_{p,u} \le \|f^*\|_{p,u^*}, \qquad \|f^*\|_{p,v_*} \le \|f\|_{p,v}, \tag{3.5}$$

and the following auxiliary result.

Lemma 3.1. *If* $f$ *and* $g$ *are two non-negative monotone functions for which*

$$\int_0^x f(t)\,dt \le \int_0^x g(t)\,dt$$

*for all* $x > 0$, *then for any non-increase function* $u^*$ *there holds*

$$\int_0^\infty f(t)u^*(t)\,dt \le \int_0^\infty g(t)u^*(t)\,dt.$$

Let us now proceed to the proof of Theorem 3.1.

Proof. (A) Let

$$Tg = w^{1/a'}Ff, \qquad g = s^{1/a}f.$$

Taking into account Theorem 2.2, we get that

$$(Tg)^*(x) \lesssim \int_0^{1/x} g^*(y)\,dy + x^{-1/a'}\int_{1/x}^\infty y^{-1/a'}g^*(y)\,dy = P_{1/x}g^* + x^{-1/a'}Q_{1/x}(y^{-1/a'}g^*).$$

The latter relation, the first Hardy inequality in (3.5), and Minkowski's inequality yield

$$\|Tg\|_{q,u} \le \|(Tg)^*\|_{q,u^*} \le \left\|P_{1/x}g^*\right\|_{q,u^*} + \left\|x^{-1/a'}Q_{1/x}(y^{-1/a'}g^*)\right\|_{q,u^*} =: I_1 + I_2.$$

Let us begin with $I_1$. Substituting $1/x \to x$, we obtain

$$I_1 = \left(\int_0^\infty u^*(x)(P_{1/x}g^*)^q\,dx\right)^{1/q} = \left(\int_0^\infty \overline{u^*}(x)(P_x g^*)^q\,dx\right)^{1/q} = \|P_x g^*\|_{q,\overline{u^*}},$$

where $\overline{u^*}(x) := x^{-2}u^*(1/x)$. Applying inequality (2.1) with the weights $\overline{u^*}$ and $v_*$ and the second inequality in (3.5), we get

$$I_1 = \|P_x g^*\|_{q,\overline{u^*}} \lesssim \|g^*\|_{p,v_*} \lesssim \|g\|_{p,v}.$$

This is true provided

$$(Q_r\overline{u^*})^{1/q}(P_r v_*^{1-p'})^{1/p'} \lesssim 1, \qquad r>0. \tag{3.6}$$

Observing that

$$Q_r\overline{u^*} = \int_r^\infty x^{-2}u^*(1/x)\,dx = \int_0^{1/r} u^*(x)\,dx = P_{1/r}u^*,$$

we rewrite (3.6) as

$$(P_{1/r}u^*)^{1/q}(P_r v_*^{1-p'})^{1/p'} \lesssim 1, \qquad r>0. \tag{3.7}$$

Let us now turn to $I_2$. Using the obtained estimates for $I_1$, we get

$$I_2 = \left\|x^{-1/a'}Q_{1/x}(y^{-1/a'}g^*)\right\|_{q,u^*} = \left\|Q_{1/x}(y^{-1/a'}g^*)\right\|_{q,x^{-q/a'}u^*} = \left\|Q_x(y^{-1/a'}g^*)\right\|_{q,\overline{x^{-q/a'}u^*}}.$$

We then apply the second inequality in (2.1) with the weights $\overline{x^{-q/a'}u^*}$ and $y^{p/a'}v_*$:

$$I_2 \lesssim \left\|y^{-1/a'}g^*\right\|_{p,y^{p/a'}v_*} = \|g^*\|_{p,v_*} \lesssim \|g\|_{p,v}.$$

This holds true provided

$$\left[P_r(\overline{x^{-q/a'}u^*})\right]^{1/q}\left[Q_r(y^{p/a'}v_*)^{1-p'}\right]^{1/p'} \lesssim 1,$$

or, substituting $x \to 1/x$,

$$\left[Q_{1/r}(x^{-q/a'}u^*)\right]^{1/q}\left[Q_r(y^{-p'/a'}v_*^{1-p'})\right]^{1/p'} \lesssim 1. \tag{3.8}$$

We thus have

$$\|Tg\|_{q,u} \lesssim I_1 + I_2 \lesssim \|g\|_{p,v}$$

under conditions (3.7) and (3.8). Hence, the part (A) of the theorem is proved.

(B) For $p=q=a=2$, inequality (2.4) from Theorem 2.1 is

$$\int_0^x (Tg)^*(t)^2\,dt \lesssim \int_0^x \left(\int_0^{1/t} g^*(s)\,ds\right)^2 dt, \qquad x>0. \tag{3.9}$$

This gives

$$\int_0^\infty (Tg)^*(t)^2 u^*(t)\,dt \lesssim \int_0^\infty \left(\int_0^{1/t} g^*(s)\,ds\right)^2 u^*(t)\,dt, \qquad x>0, \tag{3.10}$$

which follows from Lemma 3.1. To conclude the proof, we use the first Hardy inequality (2.1). □

COROLLARY 3.1. *If $a' < \max(q,p')$ or $a=p=q=2$, then the assertion of Theorem 3.1 is valid only under condition (3.1). In particular, it is so for $a=2$ and any $1<p\le q<\infty$.*

PROOF. The case $a=p=q=2$ is part (B) of Theorem 3.1. We have to prove that if $a'<\max(q,p')$, then (3.1) implies (3.2). Let first $p<a$. Then

$$\left(P_{1/r}u^*\right)^{1/q}\left(P_r v_*^{1-p'}\right)^{1/p'}\lesssim 1$$

implies, by monotonicity,

$$u^*(1/r)\lesssim r\left(\int_0^r v_*^{1-p'}\right)^{-q/p'}\lesssim r^{1-q/p'}v_*(r)^{q/p}. \tag{3.11}$$

Therefore,

$$\begin{aligned}&\left[Q_{1/r}(x^{-q/a'}u^*)\right]^{1/q}\left[Q_r(y^{-p'/a'}v_*^{1-p'})\right]^{1/p'}\\ &\lesssim\left(\int_{1/r}^\infty x^{-q/a'}(1/x)^{1-q/p'}v_*(1/x)^{q/p}\,dx\right)^{1/q}\left(\int_r^\infty y^{-p'/a'}v_*(y)^{-p'/p}\,dy\right)^{1/p'}.\end{aligned}$$

Using the monotonicity of $v_*$, we continue this estimate as follows:

$$\lesssim v_*^{1/p}(r)\left(\int_{1/r}^\infty x^{-q(1/a'-1/p')}\frac{dx}{x}\right)^{1/q}v_*^{-1/p}(r)\left(\int_r^\infty y^{1-p'/a'}\frac{dy}{y}\right)^{1/p'}\lesssim 1,$$

since $a'<p'$.

If $a'<q$, then (3.1) implies

$$r^{1/p'}v_*^{-1/p}(r)\lesssim\left(\int_0^{1/r}u^*\right)^{-1/q}\lesssim r^{1/q}u^*(1/r)^{-1/q}$$

or, equivalently,

$$v_*(r)^{-1}\lesssim u^*(1/r)^{p/q}\,r^{p/q-p/p'}.$$

Hence,

$$\begin{aligned}&\left[Q_{1/r}(x^{-q/a'}u^*)\right]^{1/q}\left[Q_r(y^{-p'/a'}v_*^{1-p'})\right]^{1/p'}\\ &\lesssim\left(\int_{1/r}^\infty x^{-q/a'}u^*(x)\,dx\right)^{1/q}\left(\int_r^\infty y^{-p'/a'}\left(u^*(1/y)^{p/q}\,y^{p/q-p/p'}\right)^{-p'/p}dy\right)^{1/p'}\\ &\lesssim u^*(1/r)^{1/q}\left(\int_{1/r}^\infty x^{-q/a'}\,dx\right)^{1/q}u^*(1/r)^{-1/q}\left(\int_r^\infty y^{p'(1/q-1/a')}\frac{dy}{y}\right)^{1/p'}\lesssim 1,\end{aligned}$$

since $a'<q$. □

Let the positive functions $u$ and $v$ be non-increasing and non-decreasing, respectively. Then we have

$$u=u^*,\qquad v_*=[(1/v)^*]^{-1}=v. \tag{3.12}$$

Thus, Theorem 3.1 and Corollary 3.1 lead to the following statement, cf. [**26**].

Corollary 3.2. *Let* $1 < p \le q < \infty$, $1 < a \le 2$, $u$ *be non-increasing, and* $v$ *be non-decreasing. If*

$$\left(P_{1/r}u\right)^{1/q}\left(P_r v^{1-p'}\right)^{1/p'} \lesssim 1, \tag{3.13}$$

*and for* $a' \ge \max\{q, p'\}$

$$\left[Q_{1/r}(x^{-q/a'}u)\right]^{1/q}\left[Q_r(y^{-p'/a'}v^{1-p'})\right]^{1/p'} \lesssim 1, \tag{3.14}$$

*then the following Pitt's type inequality holds*

$$\|w^{1/a'}Ff\|_{q,u} \lesssim \|s^{1/a}f\|_{p,v}. \tag{3.15}$$

*If* $p = q = a = 2$, *then the condition* (3.13) *alone implies* (3.15).

**3.2. Necessary conditions for Pitt's inequality.** Conditions (3.13) and (3.14) suffice to ensure Pitt's inequality. It is natural to ask whether these conditions are also necessary.

Recall that like in Subsection 2.2, (3.13) assumes that

$$u \in L^1_{\mathrm{loc}}, \qquad v^{1-p'} \in L^1_{\mathrm{loc}}. \tag{3.16}$$

In what follows both conditions are assumed. Their importance is shown in [**4, 6**].

We now present the following conditions, which are necessary for Pitt's inequality to hold.

Theorem 3.2. *Assume that the kernel* $K$ *satisfies*

$$K(x,y) \asymp 1, \qquad 0 \le xy \lesssim 1, \tag{3.17}$$

*and for every* $a \in (1,2]$ *Pitt's inequality* (3.15) *is valid. Then condition* (3.13) *holds.*

Proof. Setting

$$f(y) = v(y)^{1-p'}s(y)^{-1}\chi_{(0,r]}(y), \qquad r > 0,$$

we derive from (3.17) that for $x \lesssim 1/r$

$$Ff(x) = \int_0^r f(y)K(x,y)s(y)\,dy \asymp \int_0^r v^{1-p'}(y)dy < \infty,$$

cf. (3.16). Taking into account that $u, v^{1-p'} \in L^1_{\mathrm{loc}}$, we derive from Pitt's inequality (3.15)

$$\left(\int_0^{1/r} w(x)^{q/a'}u(x)\,dx\right)^{1/q}\int_0^r v(y)^{1-p'}dy\left(\int_0^r s(y)^{-p/a'}v(y)^{1-p'}dy\right)^{-1/p} \lesssim 1, \quad r > 0.$$

If $a \to 1$ the left-hand side of this inequality tends to $\left(P_{1/r}u\right)^{1/q}\left(P_r v^{1-p'}\right)^{1/p'}$, which implies (3.13). This completes the proof. $\square$

## 4. $F$-transforms with oscillating kernels: necessary conditions for Pitt's inequality

In this section, we will show that oscillation of kernels of the $F$-transforms allows us to get necessary conditions on weights in Pitt's inequality which are different than those in Theorem 3.2.

THEOREM 4.1. *Let $1 < p \le q < \infty$ and let $u$ and $v$ be non-increasing and non-decreasing, respectively. Given $a \in (1,2]$, let Pitt's inequality*

$$\|w^{1/a'}Ff\|_{q,u} \lesssim \|s^{1/a}f\|_{p,v}.$$

*hold.*

(A) *If condition (3.17) is valid, and the asymptotic equality*

$$K(x,y) = \frac{C}{[w(x)s(y)]^{1/2}}\left[\cos(xy-c)+O(x^{-1})\right], \qquad x \gtrsim 1, \quad y \asymp 1, \tag{4.1}$$

*holds, where $C>0$ and $c \in \mathbb{R}$ are constants, then for any $\mu > 1/p'$*

$$\int_0^1 w(x)^{q/a'}u(x)\,dx + \int_1^\infty w(x)^{q(1/a'-1/2)}x^{-q\mu}u(x)\,dx < \infty. \tag{4.2}$$

(B) *Let the asymptotic equality*

$$K(x,y) = \frac{D}{[w(x)s(y)]^{1/2}}\left[\cos(xy-d(x))+O(y^{-1})\right], \qquad y \gtrsim 1, \quad x \asymp 1, \tag{4.3}$$

*hold, where $D>0$ is a constant and $d(x)$ is a continuous function. Then for $\mu \ge 1/q$*

$$\int_1^\infty s(y)^{p(1/a-1/2)}y^{(\mu-1)p}v(y)\,dy = \infty. \tag{4.4}$$

REMARK 4.1. If one sets $a=2$ in Theorem 4.1, then (4.2) and (4.4) yield

$$\int_1^\infty x^{-q\mu}u(x)\,dx < \infty \quad \text{and} \quad \int_1^\infty y^{(\mu-1)p}v(y)\,dy = \infty,$$

respectively. It is important that these integrals are independent of the weights $w$ and $s$.

REMARK 4.2. For symmetric kernels of Hankel type (see Subsection 6.1), which satisfy $w(x)s(y) \asymp w(y)s(x)$, asymptotic equality (4.3) follows from (4.1). In this case (B) is a consequence of (A). It is not the case for the non-symmetric Jacobi kernels (see Subsection 8.3).

Let us begin with the following auxiliary statement.

LEMMA 4.1. *Let $0<\mu<1$. Define*

$$G(x) := G_\mu(x,b(x)) = \int_0^x t^{\mu-1}\cos(t-b(x))\,dt, \qquad x>0, \tag{4.5}$$

*where $b(\cdot)$ is a continuous function. Then*

$$\begin{aligned} |G(x)| &= O(x^\mu), \qquad x \lesssim 1,\\ G(x) &= \Gamma(\mu)\cos\left(b(x)-\frac{\pi\mu}{2}\right)+O(x^{\mu-1}), \qquad x \gtrsim 1. \end{aligned} \tag{4.6}$$

*In particular, $|G(x)| = O(1)$ for $x>0$.*

PROOF. The first equality in (4.6) is obvious. In order to prove the second one, let us express $G$ via the Bohmer integrals (generalized Frenel functions) [**2**, Ch. 9]

$$C(x,\mu)=\int_x^\infty t^{\mu-1}\cos t\,dt,\qquad S(x,\mu)=\int_x^\infty t^{\mu-1}\sin t\,dt.$$

We have

$$\begin{aligned}G(x)&=\cos(b(x))\int_0^x t^{\mu-1}\cos t\,dt+\sin(b(x))\int_0^x t^{\mu-1}\sin t\,dt\\&=\cos(b(x))\left[C(0,\mu)-C(x,\mu)\right]+\sin(b(x))\left[S(0,\mu)-S(x,\mu)\right].\end{aligned}$$

Since $C(0,\mu)=S(0,\mu)=\Gamma(\mu)\cos(\pi\mu/2)$, we get

$$G(x)=\Gamma(\mu)\cos\left(b(x)-\frac{\pi\mu}{2}\right)-\left[C(x,\mu)\cos(b(x))+S(x,\mu)\sin(b(x))\right].$$

What remains is to take into account the asymptotic equalities [**2**, Ch. 9]

$$|C(x,\mu)|=O(x^{\mu-1}),\quad |S(x,\mu)|=O(x^{\mu-1}),\qquad x\gtrsim 1.$$

The lemma is proved. □

PROOF OF THEOREM 4.1. (A) Assume that (3.17) and (4.1) hold. Consider the interval $y\in[r,2r]$, $r>0$, on which $v(y)\asymp 1$. For example, one can take $r=1$. We introduce the function

$$f_\mu(y)=s(y)^{-1/2}(y-r)^{\mu-1}\chi_{(r,2r]}(y),\qquad 0<\mu<1,$$

and estimate its transform

$$Ff_\mu(x)=\int_0^\infty f_\mu(y)K(x,y)s(y)\,dy=\int_r^{2r}(y-r)^{\mu-1}K(x,y)s(y)^{1/2}\,dy.$$

Because of (3.17) and (1.5), we have for $x\lesssim 1/r$,

$$Ff_\mu(x)\asymp\int_r^{2r}(y-r)^{\mu-1}s(y)^{1/2}\,dy\asymp s(r)^{1/2}\int_r^{2r}(y-r)^{\mu-1}\,dy\asymp 1. \tag{4.7}$$

Let $x\gtrsim 1/r$. Then, by (4.1),

$$Ff_\mu(x)=\frac{C}{w(x)^{1/2}}\int_r^{2r}(y-r)^{\mu-1}\left[\cos(xy-c)+O(x^{-1})\right]dy=\frac{C}{w(x)^{1/2}}\left[I+O(x^{-1})\right],$$

where

$$I:=\int_r^{2r}(y-r)^{\mu-1}\cos(xy-c)\,dy=x^{-\mu}\int_0^{rx}t^{\mu-1}\cos(t+rx-c)\,dt=x^{-\mu}G_\mu(rx,c-rx).$$

This and Lemma 4.1 yield

$$I=x^{-\mu}\left[\Gamma(\mu)\cos(c-rx-\pi\mu/2)+O((rx)^{\mu-1})\right],\qquad x\gtrsim 1.$$

Hence, we have

$$Ff_\mu(x)=\frac{C_0}{x^\mu w(x)^{1/2}}\left[\cos(rx-c_0)+O(x^{\mu-1})\right],\qquad x\gtrsim 1/r. \tag{4.8}$$

Now, we are going to check whether Pitt's inequality $\|w^{1/a'}Ff_\mu\|_{q,u} \lesssim \|s^{1/a}f_\mu\|_{p,v}$ holds. First, if $1/p' < \mu < 1$ or, equivalently, $-1 < (\mu-1)p < 0$, we calculate

$$\|s^{1/a}f_\mu\|_{p,v} = \left(\int_r^{2r} s(y)^{-p/2}(y-r)^{(\mu-1)p}v(y)\,dy\right)^{1/p} \lesssim \left(\int_r^{2r}(y-r)^{(\mu-1)p}\,dy\right)^{1/p} \lesssim 1.$$

On the other hand,

$$\|w^{1/a'}Ff_\mu\|_{q,u} = \left(\int_0^{1/r}|w^{1/a'}Ff_\mu|^q u\,dx + \int_{1/r}^{\infty}|w^{1/a'}Ff_\mu|^q u\,dx\right)^{1/q} = (I_1+I_2)^{1/q},$$

where, by (4.7), we have $I_1 \asymp \int_0^{1/r} w^{q/a'}u\,dx$. This implies that the function $w^{q/a'}u$ should be integrable in a neighborhood of zero.

It follows from (4.8) that

$$I_2 \gtrsim \int_{1/r}^{\infty} w(x)^{q(1/a'-1/2)}x^{-q\mu}\left|\cos(rx-c_0)+O(x^{\mu-1})\right|^q u(x)\,dx.$$

Taking into account that both $u(x)$ and $x^{-q\mu}$ are non-increasing, the weight $w(x)$ satisfies the $\Delta_2$-condition, and the inequality

$$|\cos(rx-c_0)| \ge \cos 1, \qquad x \in A = \bigcup_{k=-\infty}^{\infty}\left[\frac{\pi k - 1 + c_0}{r}, \frac{\pi k + 1 + c_0}{r}\right],$$

holds, we can write

$$I_2 \gtrsim \int_{[1/r,\infty)\cap A} w(x)^{q(1/a'-1/2)}x^{-q\mu}u(x)\,dx \gtrsim \int_{1/r}^{\infty} w(x)^{q(1/a'-1/2)}x^{-q\mu}u(x)\,dx. \tag{4.9}$$

This gives that the finiteness of $I_2$ relies on the integrability of the function $w(x)^{q(1/a'-1/2)}x^{-q\mu}u(x)$ near infinity. This establishes condition (4.2).

(B) Assume now that (4.3) holds. Choose $r > 0$, $d_0 \in \mathbb{R}$, and sufficiently small number $\varepsilon > 0$ such that

$$\left|\cos\left(d(x) - d_0 - \frac{\pi\mu}{2}\right)\right| \asymp 1, \quad u(x) \asymp 1, \qquad x \in [r, r+\varepsilon].$$

For $0 < \mu < 1$, consider the function

$$f_\mu(y) = s(y)^{-1/2}y^{\mu-1}\cos(ry-d_0)\chi_{[r,R)}(y),$$

where $R > r$ is such that $R_0 = R/\ln R > \varepsilon^{-1}$. Let us estimate its transform for $x \in [r, r+\varepsilon]$:

$$Ff_\mu(x) = \int_0^{\infty} f_\mu(y)K(x,y)s(y)\,dy = \int_r^R y^{\mu-1}\cos(ry-d_0)K(x,y)s(y)^{1/2}\,dy := I.$$

In virtue of (4.3),

$$I = \frac{D}{w(x)^{1/2}}\int_r^R y^{\mu-1}\cos(ry-d_0)\left[\cos(xy-d(x)) + O(y^{-1})\right]dy.$$

Note that $Dw(x)^{-1/2} \asymp 1$, $x \in [r, r+\varepsilon]$. Moreover, for arbitrary $R$,

$$\left|\int_r^R y^{\mu-1}\cos(ry-d_0)O(y^{-1})\,dy\right| \lesssim \int_r^{\infty} y^{\mu-2}\,dy = O(1),$$

and

$$\int_r^R y^{\mu-1}\cos(ry-d_0)\cos(xy-d(x))\,dy = \int_0^R - \int_0^r = \int_0^R + O(1) = \frac{1}{2}(I_1+I_2)+O(1),$$

where

$$I_1 := \int_0^R y^{\mu-1}\cos\big((x+r)y-d(x)-d_0\big)\,dy,$$

$$I_2 := \int_0^R y^{\mu-1}\cos\big((x-r)y-d(x)+d_0\big)\,dy.$$

Let $\xi = x+r$, $\delta(\xi)=d(x)+d_0$. Then

$$I_1 = \int_0^R y^{\mu-1}\cos\big(\xi y-\delta(\xi)\big)\,dy = \xi^{-\mu}\int_0^{R\xi} t^{\mu-1}\cos\big(t-\delta(\xi)\big)\,dt = \xi^{-\mu}G_\mu(R\xi,\delta(\xi)),$$

where $G_\mu$ is given by (4.5). Since $\xi \asymp 1$ for $x\in[r,r+\varepsilon]$, taking into account Lemma 4.1, we find out that $|I_1| = O(1)$. Therefore,

$$Ff_\mu(x) = (D/2)w(x)^{-1/2}I_2 + O(1) \quad \text{for} \quad x\in[r,r+\varepsilon].$$

Consider $I_2$. Let now $\xi = x-r$ $\delta(\xi)=d(x)-d_0$. Then, analogously to $I_1$, we have $I_2 = \xi^{-\mu}G_\mu(\xi R,\delta(\xi))$. For $x\in[r+1/R_0,r+\varepsilon]$, we have that $\xi R \ge R/R_0 = \ln R$. It follows from this and Lemma 4.1 that for sufficiently large $R$

$$\begin{aligned} |I_2| &= \xi^{-\mu}\left|\Gamma(\mu)\cos\left(\delta(\xi)-\frac{\pi\mu}{2}\right)+O((\ln R)^{\mu-1})\right| \\ &\gtrsim x^{-\mu}\left[\left|\cos\left(d(x)-d_0-\frac{\pi\mu}{2}\right)\right|+O((\ln R)^{\mu-1})\right] \gtrsim x^{-\mu} \end{aligned}$$

and

$$Ff_\mu(x) \gtrsim x^{-\mu}+O(1) \quad \text{for} \quad x\in[r+1/R_0,r+\varepsilon].$$

Applying Pitt's inequality (3.15) to $f_\mu$, we obtain for its right-hand side

$$\begin{aligned} \|s^{1/a}f_\mu\|_{p,v}^p &= \int_r^R s(y)^{p/a}\left[s(y)^{-1/2}y^{\mu-1}\cos(ry-d_0)\right]^p v(y)\,dy \\ &\le \int_r^\infty s(y)^{p(1/a-1/2)}y^{(\mu-1)p}v(y)\,dy, \end{aligned}$$

while for the left-hand side

$$\|w^{1/a'}Ff_\mu\|_{q,u}^q \gtrsim \int_{r+1/R_0}^{r+\varepsilon} w(x)^{q/a'}|Ff_\mu(x)|^q u(x)\,dx \asymp \int_{r+1/R_0}^{r+\varepsilon}|Ff_\mu(x)|^q\,dx \gtrsim J+O(1),$$

where

$$J := \int_{r+1/R_0}^{r+\varepsilon}(x-r)^{-q\mu}\,dx = \int_{1/R_0}^{\varepsilon}x^{-q\mu}\,dx \asymp \begin{cases} R_0^{\mu q-1}, & \mu\ne 1/q, \\ \ln R_0, & \mu = 1/q, \end{cases}.$$

If $\mu \ge 1/q$ and $R_0 = R/\ln R\to\infty$, then $J\to\infty$. This implies (4.4), which completes the proof. □

## 5. Pitt's inequality for piecewise power weights

In this section, Pitt's inequality is proved for the power type weights (see (1.9))

$$s(y)=y^{\overline{\nu}}:=\begin{cases} y^{\nu_1}, & y\le 1,\\ y^{\nu_2}, & y\ge 1,\end{cases}\qquad \nu_1,\nu_2\ge 0.$$

Set $a\overline{\nu}+b=(a\nu_1+b,a\nu_2+b)$ and $(1/y)^{\overline{\nu}}=y^{-\overline{\nu}^\circ}$, where

$$\overline{\nu}=(\nu_1,\nu_2)\quad\text{ and }\quad \overline{\nu}^\circ=(\nu_2,\nu_1).$$

If the operator generating weight $s$ is of the desired form $s(y)\asymp y^{\overline{\nu}}$ then, by (1.6),

$$w(x)\asymp x^{\overline{\nu}^\circ}. \tag{5.1}$$

Such weights with different $\nu_1$ and $\nu_2$ appear while considering the Jacobi transform. The case $\nu_1=\nu_2$ is in agreement with the Hankel transform.

**5.1. Interrelations of the parameters.** Similarly to the Fourier and Hankel transforms (see (1.11) and (1.16)), we obtain Pitt's inequality in the form

$$\|x^{-\overline{\gamma}^\circ}Ff\|_{q,x^{\overline{\nu}^\circ}}\lesssim \|y^{\overline{\beta}}f\|_{p,y^{\overline{\nu}}}, \tag{5.2}$$

where $\overline{\nu}\ge 0$ is as above, while $\overline{\gamma}=(\gamma_1,\gamma_2)$ and $\overline{\beta}=(\beta_1,\beta_2)$ are the varying parameters. We are going to determine the domains for the validity of Pitt's inequality (5.2) in terms of these parameters. We will also establish sharpness of the corresponding conditions in some cases.

Inequality (5.2) follows from the general results of Section 3 for monotone weights $u$ and $v$. Pitt's inequality there (see Corollary 3.2) is of the form $\|w^{1/a'}Ff\|_{q,u}\lesssim\|s^{1/a}f\|_{p,v}$, where $1<p\le q<\infty$, $1<a\le 2$. Observe that it can be equivalently written as

$$\|w^{-t}u^{1/q}Ff\|_{q,w}\lesssim \|s^r v^{1/p}f\|_{p,s}, \tag{5.3}$$

where

$$\begin{aligned} t&=\frac1q-\frac1{a'}\in[t_0,t_1), &\quad t_0&=\frac1q-\frac12,\quad t_1=\frac1q,\\ r&=\frac1{p'}-\frac1{a'}\in[r_0,r_1), &\quad r_0&=\frac1{p'}-\frac12,\quad r_1=\frac1{p'}.\end{aligned} \tag{5.4}$$

Comparing (5.2) and (5.3), we set

$$x^{-\overline{\gamma}^\circ}=w(x)^{-t}u(x)^{1/q},\qquad y^{\overline{\beta}}=s(y)^r v(y)^{1/p}, \tag{5.5}$$

which implies

$$u(x)=x^{q(t\overline{\nu}^\circ-\overline{\gamma}^\circ)},\qquad v(y)=y^{p(\overline{\beta}-r\overline{\nu})}. \tag{5.6}$$

The results of Section 3 that we are going to use are proven for the case where the weight $u$ is non-increasing, the weight $v$ is non-decreasing, and $u,v^{1-p'}\in L^1_{\mathrm{loc}}$. This is equivalent to

$$\overline{\gamma}\ge t\overline{\nu},\qquad \overline{\beta}\ge r\overline{\nu} \tag{5.7}$$

and

$$\gamma_2<t\nu_2+t_1,\qquad \beta_1<r\nu_1+r_1. \tag{5.8}$$

For each $a \in (1,2]$ (and corresponding ranges of $t$ and $r$) conditions (5.7) and (5.8) determine half-open strips

$$\mathfrak{T}_a := \{t\nu_1 \le \gamma_1,\ t\nu_2 \le \gamma_2 < t\nu_2 + t_1\} \quad \text{and} \quad \mathfrak{R}_a := \{r\nu_1 \le \beta_1 < r\nu_1 + r_1,\ r\nu_2 \le \beta_2\}$$

(see Figure 1). This justifies the following

DEFINITION 5.1. The parameters $\overline{\gamma}$ and $\overline{\beta}$ will be called **dual** if there is $a \in (1,2]$ such that simultaneously $\overline{\gamma} \in \mathfrak{T}_a$ and $\overline{\beta} \in \mathfrak{R}_a$.

Varying $a \in (1,2]$ (along with $t \in [t_0, t_1)$ and $r \in [r_0, r_1)$), we obtain the domains where the corresponding parameters $\overline{\gamma}$ and $\overline{\beta}$ live, which is illustrated in Figure 1 (see $A_1A_2A_3A_4A_5$ and $B_1B_2B_3B_4B_5$). Note that the left and right domains here are similarly assigned by quadruplets of parameters $(\overline{\gamma}, \overline{\nu}, q, t)$ and $\big(\overline{\beta}^\circ, \overline{\nu}^\circ, p', r\big)$.

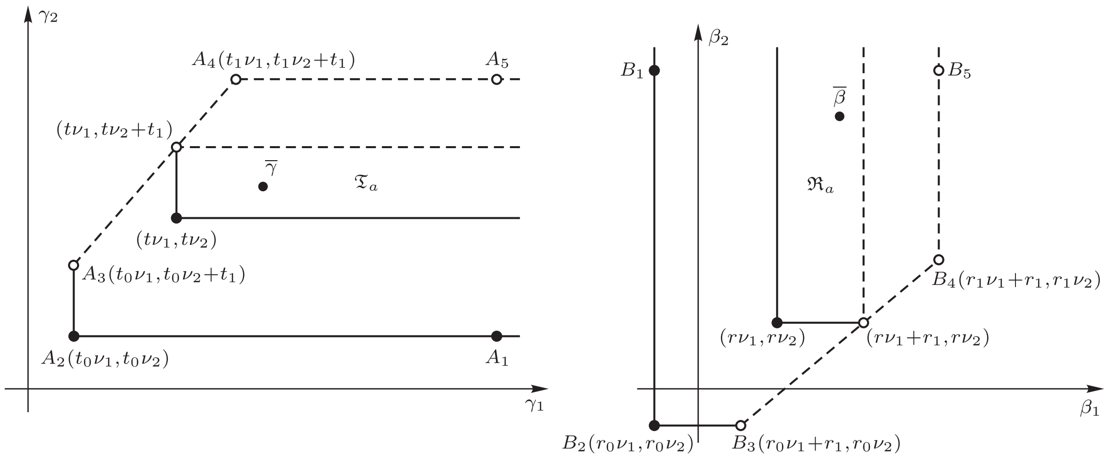


FIGURE 1. Ranges for $\overline{\gamma}$ and $\overline{\beta}$ (corresponding to conditions (5.7) and (5.8))

In order to formulate the basic results, let us sketch the domain for the validity of Pitt's inequality, written

$$D_{p,q} := D_{p,q}(\nu_1, \nu_2), \qquad 1 < p \le q < \infty.$$

First, let $(p,q) \neq (2,2)$. With (5.7) and (5.8) in hand, we additionally assume that $\overline{\gamma}$ and $\overline{\beta}$ satisfy

$$\beta_1 - \gamma_1 \le \left(\frac{1}{p'} - \frac{1}{q}\right)(\nu_1 + 1), \qquad \beta_2 - \gamma_2 \ge \left(\frac{1}{p'} - \frac{1}{q}\right)(\nu_2 + 1), \tag{5.9}$$

$$\begin{cases} \gamma_1 \ge t\nu_1 + t_*, & t \in [t_0, t_*), \\ \gamma_1 > t\nu_1 + t, & t \in [t_*, t_1), \end{cases} \tag{5.10}$$

and

$$\begin{cases} \beta_2 \ge r\nu_2 + r_*, & r \in [r_0, r_*), \\ \beta_2 > r\nu_2 + r, & r \in [r_*, r_1), \end{cases} \tag{5.11}$$

where

$$t_* = \max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\}, \qquad r_* = \max\left\{0, \frac{1}{p'} - \frac{1}{q}\right\}.$$

For $p = q = 2$, we have $t_0 = t_* = 0$, $r_0 = r_* = 0$ in conditions (5.10) and (5.11). Let us define the domain $D_{2,2}$ as above with (5.10) and (5.11) replaced by

$$\begin{cases} \gamma_1 \geq 0, & t = 0, \\ \gamma_1 > t\nu_1 + t, & t \in (0, t_1), \end{cases} \quad \text{and} \quad \begin{cases} \beta_2 \geq 0, & r = 0, \\ \beta_2 > r\nu_2 + r, & r \in (0, r_1), \end{cases} \tag{5.12}$$

In Figure 2, the projections of the domain $D_{p,q}$ on the planes $\overline{\gamma}$ and $\overline{\beta}$ are drawn (see $A_1A_1^*A_2^*A_3^*A_4^*A_5$ and $B_1B_1^*B_2^*B_3^*B_4^*B_5$). They are subdomains of the corresponding domains in Figure 1.

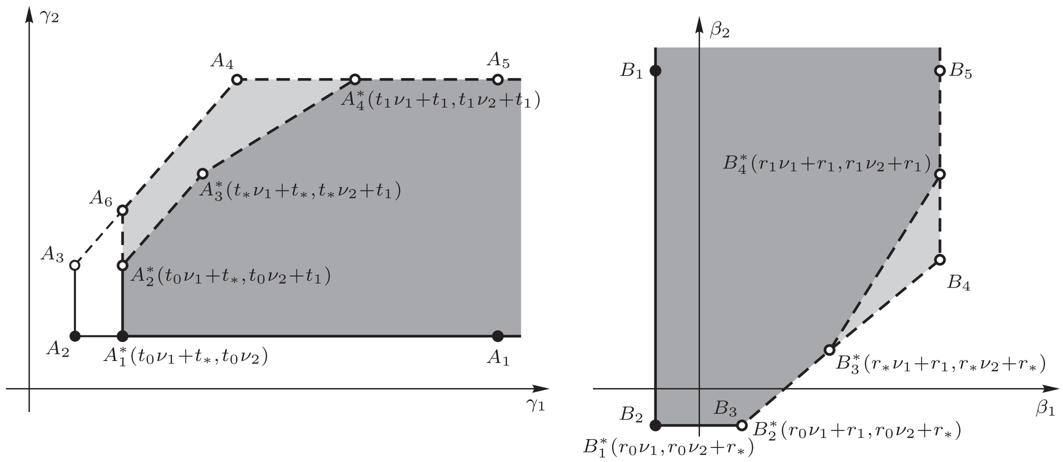


FIGURE 2. Projections of the domain $D_{p,q}$

The geometry of $D_{p,q}$ may vary in accordance with the values of parameters. Figures 1 and 2 are drawn for $q < 2 < p'$. In this case the point $A_2$ lies in the 1st quadrant, while its dual $B_2$ in the 3rd one. Note also that the domains may become simpler. For instance, if $q = p'$, then it follows from $t_* = r_* = 0$ that $A_1^* \to A_2$, $A_2^* \to A_3$, and $A_3^* \in [A_3, A_4)$.

**5.2. The main result.** We are now in a position to formulate the main result of this section. Our goal is to establish sufficient conditions for the validity of Pitt's inequality and similarly the necessary ones. These conditions not always coincide.

THEOREM 5.1. *Suppose* $1 < p \leq q < \infty$, $\overline{\nu} \geq 0$.

(A) *Let* $\overline{\gamma}$ *and* $\overline{\beta}$ *be dual and such that* $(\overline{\gamma}, \overline{\beta}) \in D_{p,q}$. *Then Pitt's inequality* (5.2) *holds.*

(B1) *Let the kernel* $K$ *satisfy* (3.17). *Then condition* (5.9) *is necessary for Pitt's inequality* (5.2) *to be valid.*

(B2) *If the kernel* $K$ *satisfies* (4.1) *and* (4.3), *then the conditions*

$$\gamma_1 \geq \left(\frac{1}{q} - \frac{1}{2}\right)\nu_1 + \frac{1}{q} - \frac{1}{p'}, \qquad \beta_2 \geq \left(\frac{1}{p'} - \frac{1}{2}\right)\nu_2 + \frac{1}{p'} - \frac{1}{q}$$

*are necessary for Pitt's inequality* (5.2) *to be valid.*

REMARK 5.1. Figure 1 presents the conditions of monotonicity and local integrability of the weights (5.7) and (5.8), respectively. Moreover, Theorem 5.1 shows that the parameters determining the dark-grey domain in Figure 2 ($A_1A_1^*A_2^*A_3^*A_4^*A_5$ and $B_1B_1^*B_2^*B_3^*B_4^*B_5$)

correspond to the sufficient conditions which ensure the validity of Pitt's inequality. Similarly, the parameters for the dark-grey domain along with those on the plain grey domain correspond to the necessary ones provided conditions (3.17), (4.1) and (4.3) take place (see $A_1A_1^*A_2^*A_6A_4A_4^*A_5$ and $B_1B_1^*B_2^*B_3^*B_4B_4^*B_5$). In particular, it follows from this that the border lines $\gamma_1 \geq t_0\nu_1 + t_*$ and $\beta_2 \geq r_0\nu_2 + r_*$ ($A_2A_3$ and $B_1B_2$ in Figure 2) and $\gamma_2 < t_1\nu_2 + t_1$ and $\beta_1 < r_1\nu_1 + r_1$ (the upper ones in Figure 2) of $D_{p,q}$ cannot be shifted.

Proof. We derive (A) from Corollary 3.2 with monotone weights (5.5). More precisely, we verify that conditions (3.13) and (3.14) applied to our weights lead to (5.8)–(5.11).

Condition (3.13) is of the form

$$\left(P_{1/r}x^{q(t\overline{\nu}^\circ-\overline{\gamma}^\circ)}\right)^{1/q}\left(P_r y^{p'(r\overline{\nu}-\overline{\beta})}\right)^{1/p'} \lesssim 1, \qquad r>0, \tag{5.13}$$

where

$$x^{q(t\overline{\nu}^\circ-\overline{\gamma}^\circ)} = \begin{cases} x^{q(t\nu_2-\gamma_2)}, & x\leq 1,\\ x^{q(t\nu_1-\gamma_1)}, & x\geq 1,\end{cases} \qquad y^{p'(r\overline{\nu}-\overline{\beta})} = \begin{cases} y^{p'(r\nu_1-\beta_1)}, & y\leq 1,\\ y^{p'(r\nu_2-\beta_2)}, & y\geq 1.\end{cases}$$

For $r \leq 1$, we have

(5.14)

$$P_{1/r}x^{q(t\overline{\nu}^\circ-\overline{\gamma}^\circ)} = \int_0^1 x^{q(t\nu_2-\gamma_2)}dx + \int_1^{1/r} x^{q(t\nu_1-\gamma_1)}dx, \quad P_r y^{p'(r\overline{\nu}-\overline{\beta})} = \int_0^r y^{p'(r\nu_1-\beta_1)}dy.$$

These values exist provided the corresponding functions are integrable near the origin. This leads to conditions (5.8):

$$q(t\nu_2-\gamma_2) > -1, \quad p'(r\nu_1-\beta_1) > -1$$

or, equivalently,

$$\gamma_2 < t\nu_2 + \frac{1}{q}, \quad \beta_1 < r\nu_1 + \frac{1}{p'}.$$

This and (5.14) yield, for $0<r<1$,

$$\left(P_{1/r}u\right)^{1/q} \asymp \begin{cases} (1/r)^{(q(t\nu_1-\gamma_1)+1)/q}, & q(t\nu_1-\gamma_1)+1>0,\\ (\ln(1/r))^{1/q}, & q(t\nu_1-\gamma_1)+1=0,\\ 1, & q(t\nu_1-\gamma_1)+1<0,\end{cases}$$

$$\left(P_r v^{1-p'}\right)^{1/p'} \asymp r^{(p'(r\nu_1-\beta_1)+1)/p'}.$$

Since by (5.8) there holds $p'(r\nu_1-\beta_1)+1>0$, we have, for $0<r<1$ and $q(t\nu_1-\gamma_1)+1\leq 0$,

$$\left(P_{1/r}u\right)^{1/q}\left(P_r v^{1-p'}\right)^{1/p'} \lesssim 1,$$

and condition (3.13) (and, equivalently (5.13)) is satisfied. In the case of $q(t\nu_1-\gamma_1)+1>0$, we get

$$\left(P_{1/r}u\right)^{1/q}\left(P_r v^{1-p'}\right)^{1/p'} \asymp r^{(p'(r\nu_1-\beta_1)+1)/p'-(q(t\nu_1-\gamma_1)+1)/q}.$$

For small $r$, this value is bounded provided

$$\frac{p'(r\nu_1-\beta_1)+1}{p'} - \frac{q(t\nu_1-\gamma_1)+1}{q} \geq 0,$$

which is equivalent to

$$\gamma_1-\beta_1\ge (t-r)(\nu_1+1)=\left(\frac{1}{q}-\frac{1}{p'}\right)(\nu_1+1). \tag{5.15}$$

Thus, (3.13) is valid for $r\le 1$ if and only if (5.8) and (5.15) are.

By duality, or, as above, by straightforward calculations, we obtain, for $r\ge 1$,

$$\gamma_2-\beta_2\le (t-r)(\nu_2+1)=\left(\frac{1}{q}-\frac{1}{p'}\right)(\nu_2+1). \tag{5.16}$$

Now we deal with condition (3.14). Note that in case $p=q=a=2$, this condition can be skipped by Corollary 3.2. Changing variables $x\to 1/x$, $y\to 1/y$, and $r\to 1/r$ in (3.14), we arrive at

$$\Big[P_{1/r}(x^{q/a'-2}u(1/x))\Big]^{1/q}\Big[P_r(y^{p'/a'-2}v(1/y)^{1-p'})\Big]^{1/p'}\lesssim 1,$$

which is equivalent to (3.14) if $a'\ge\max\{q,p'\}$. Inserting then our weights

$$u(1/x)=(1/x)^{q(t\overline{\nu}^\circ-\overline{\gamma}^\circ)}=x^{q(\overline{\gamma}-t\overline{\nu})},\qquad v(1/y)=(1/y)^{p(\overline{\beta}-r\overline{\nu})}=y^{p(r\overline{\nu}^\circ-\overline{\beta}^\circ)},$$

we get

$$\Big(P_{1/r}x^{q/a'-2+q(\overline{\gamma}-t\overline{\nu})}\Big)^{1/q}\Big(P_ry^{p'/a'-2-p'(r\overline{\nu}^\circ-\overline{\beta}^\circ)}\Big)^{1/p'}\lesssim 1.$$

The inequality above is of the same form as (5.13), with obvious alterations. Hence, the same operations that we used for deriving (5.15) and (5.16) imply

$$\frac{q}{a'}-2-q(t\nu_1-\gamma_1)>-1,\quad \frac{p'}{a'}-2-p'(r\nu_2-\beta_2)>-1.$$

Using

$$\frac{1}{a'}=\frac{1}{q}-t\quad\text{and}\quad\frac{1}{a'}=\frac{1}{p'}-r,$$

we derive from this

$$\gamma_1>t(\nu_1+1),\qquad \beta_2>r(\nu_2+1). \tag{5.17}$$

Finally, if $r\le 1$, then

$$\frac{p'/a'-2-p'(r\nu_2-\beta_2)+1}{p'}-\frac{q/a'-2+q(\gamma_2-t\nu_2)+1}{q}\ge 0$$

or, equivalently,

$$\gamma_2-\beta_2\le\left(\frac{1}{q}-\frac{1}{p'}\right)(\nu_2+1),$$

which is just (5.16). For $r\ge 1$, we get (5.15) in a similar manner.

Thus, conditions (5.7), (5.8), (5.9) and (5.17), with $a'\ge\max\{q,p'\}$, imply Pitt's inequality (5.2). Let us show that these conditions are equivalent to (5.7)–(5.11). In fact, what is to be proved are (5.10) and (5.11). We restrict ourselves to inequalities for $\overline{\gamma}$, since, by Definition 5.1, those for $\overline{\beta}$ are similar.

Conditions (5.9) and (5.7) along with $1/q-1/p'=t-r$ yield

$$\gamma_1\ge (t-r)(\nu_1+1)+\beta_1\ge (t-r)(\nu_1+1)+r\nu_1=t\nu_1+\frac{1}{q}-\frac{1}{p'}.$$

Together with $\gamma_1 \geq t\nu_1$ this gives

$$\gamma_1 \geq t\nu_1 + t_*, \qquad t \in [t_0, t_1), \tag{5.18}$$

which ensures the first inequality in (5.10). We then observe that $a' \geq \max\{q, p'\}$ is equivalent to

$$\frac{1}{a'} \leq \min\left\{\frac{1}{q}, \frac{1}{p'}\right\},$$

and, in turn, to

$$t \geq \max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\} = t_*.$$

Taking into account this and (5.17), we derive the second inequality in (5.10):

$$t\nu_1 + t < \gamma_1, \qquad t \in [t_*, t_1). \tag{5.19}$$

The latter is more restrictive than (5.18) for $t \geq t_*$, which completes the proof of (A) in Theorem 5.1 for all cases except $p = q = a = 2$.

In the case $p = q = a = 2$, which corresponds to $t = r = t_* = r_*$, the conditions in (5.17) are not needed, as mentioned above. Therefore, one should set $t \in (t_*, t_1)$ in (5.19). This explains why (5.12) fits this case.

Let us proceed to (B).

(B1) If the kernel $K$ satisfies (3.17), then (3.13) (and, correspondingly, (5.13)) holds. It is proved in (A) that (5.15) and (5.16) follow from (5.13), and hence we have (5.9).

(B2) Let the kernel $K$ satisfy (4.1) and (4.3). Rewriting (4.2) and (4.4) in present notation, we have

$$\int_0^1 x^{\nu_2 q/a' + q(t\nu_2 - \gamma_2)}\, dx + \int_1^\infty x^{\nu_1 q(1/a' - 1/2) - q\mu + q(t\nu_1 - \gamma_1)}\, dx < \infty, \qquad \mu > \frac{1}{p'},$$

$$\int_1^\infty y^{\nu_2 p(1/a - 1/2) + (\mu - 1)p + p(\beta_2 - r\nu_2)}\, dy = \infty, \qquad \mu \geq \frac{1}{q}.$$

What follows from this is $\frac{\nu_2 q}{a'} + q(t\nu_2 - \gamma_2) > -1$, which is equivalent to

$$\gamma_2 < \frac{1}{q} + \nu_2\left(t + \frac{1}{a'}\right) = \frac{\nu_2 + 1}{q},$$

and $\nu_1 q\left(\frac{1}{a'} - \frac{1}{2}\right) - q\mu + q(t\nu_1 - \gamma_1) < -1$, which is equivalent to

$$\gamma_1 > \frac{1}{q} - \mu + \nu_1\left(\frac{1}{q} - \frac{1}{2}\right).$$

Letting $\mu \to \frac{1}{p'}$, we derive

$$\gamma_1 \geq \left(\frac{1}{q} - \frac{1}{2}\right)\nu_1 + \frac{1}{q} - \frac{1}{p'},$$

and

$$\nu_2 p\left(\frac{1}{a} - \frac{1}{2}\right) + (\mu - 1)p + p(\beta_2 - r\nu_2) \geq -1,$$

which is equivalent to

$$\beta_2 \geq -\frac{1}{p} + r\nu_2 + \nu_2\left(\frac{1}{a'} - \frac{1}{2}\right) + 1 - \mu.$$

Letting now $\mu \to \frac{1}{q}$, we obtain

$$\beta_2 \ge \left(\frac{1}{p'} - \frac{1}{2}\right)\nu_2 + \frac{1}{p'} - \frac{1}{q},$$

which completes the proof of the theorem. □

**5.3. The special case: Hausdorff–Young-type inequalities.** There is one case of special interest, the one where $q = p'$. In this case, we have $t = r$, $t_0 = r_0 \le 0$, $t_1 = r_1$, and $t_* = r_* = 0$. Then the condition (5.7)-(5.9) as well as (5.10)–(5.11) for $(p, q) \ne (2, 2)$ or (5.12) for $(p, q) = (2, 2)$, that define the domain $D_{p,q}$, reduce to

$$t\nu_2 \le \gamma_2 < t\nu_2 + t_1, \qquad t\nu_1 \le \beta_1 < t\nu_1 + t_1, \qquad t \in [t_0, t_1), \tag{5.20}$$

$$\gamma_1 \ge \beta_1, \qquad \beta_2 \ge \gamma_2, \tag{5.21}$$

$$\gamma_1 > t\nu_1 + t, \qquad \beta_2 > t\nu_2 + t, \qquad t \in \begin{cases} (0, t_1), & p = 2, \\ [0, t_1), & p \ne 2. \end{cases} \tag{5.22}$$

Observe that (5.7) and (5.8) in this situation reduce to (5.20).

Corollary 5.1. *Suppose $q = p' \ge 2$. Let $\overline{\gamma}$ and $\overline{\beta}$ be dual and such that $(\overline{\gamma}, \overline{\beta}) \in D_{p,q}$ described by* (5.20)–(5.22)*. Then Pitt's inequality* (5.2) *holds.*

The case $t = 0$ is a good example of the application of Corollary 5.1. In this case, the conditions

$$0 \le \gamma_2, \quad \beta_1 < \frac{1}{p'}, \quad \gamma_1 \ge \beta_1, \quad \beta_2 \ge \gamma_2$$

guarantee the validity of Pitt's inequality

$$\|x^{-(\gamma_2,\gamma_1)} Ff\|_{p',x^{(\nu_2,\nu_1)}} \lesssim \|y^{(\beta_1,\beta_2)} f\|_{p,y^{(\nu_1,\nu_2)}}, \qquad 1 < p \le 2.$$

It is of great interest to figure out what conditions are necessary and sufficient for Pitt's inequality (3.15) to hold in the case of usual power weights $u(x) = x^{-\gamma}$ and $v(y) = y^{\beta}$. We have $\gamma_1 = \gamma_2$ and $\beta_1 = \beta_2$ there, which results in drawing in Figure 2 the lines crossing the origin on the angle 45°. Analyzing the coordinates of the points $A_1^*, A_4^*, B_1^*$, and $B_4^*$ in Figure 2, one can see that necessary and sufficient conditions coincide only if $\nu_1 = \nu_2$. And this is what next section focus on.

## 6. Weighted inequalities for Hankel transforms

Let $\alpha \ge -1/2$. Recall (see (1.15)) that the normalized Bessel function is defined by $j_\alpha(t) = \Gamma(\alpha+1)(t/2)^{-\alpha} J_\alpha(t)$, where $J_\alpha$ is the classical Bessel function. In terms of this normalized Bessel function, the direct and inverse Hankel transform are defined by

$$Hf(\lambda) := H_\nu f(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) t^\nu \, dt, \qquad \nu = 2\alpha + 1 \ge 0,$$

$$H^{-1}f(t) = b_\alpha \int_0^\infty f(\lambda) j_\alpha(\lambda t) \lambda^\nu \, d\lambda, \qquad b_\alpha^{-1} = 2^{2\alpha}\Gamma(\alpha+1)^2;$$

for a convenient presentation, see [**36**, Ch. 5]. By this, Parseval's identity holds true

$$\int_0^\infty |f(t)|^2 t^\nu \, dt = b_\alpha \int_0^\infty |Hf(\lambda)|^2 \lambda^\nu \, d\lambda.$$

In particular cases, the Hankel transform reduces to the cosine Fourier transform ($\alpha = -1/2$) and the sine Fourier transform ($\alpha = 1/2$). For $\alpha = -1/2$, we have $j_{-1/2}(t) = \cos t$, $\nu = 0$, and

$$\widehat{f_c}(\lambda) := F_c f(\lambda) = \int_0^\infty f(t) \cos(\lambda t) \, dt, \qquad F_c^{-1} f(\lambda) = \frac{2}{\pi} \int_0^\infty f(\lambda) \cos(\lambda t) \, d\lambda. \tag{6.1}$$

If $\alpha = 1/2$, then $j_{1/2}(t) = t^{-1} \sin t$, $\nu = 2$,

$$Hf(\lambda) = \int_0^\infty f(t)(\lambda t)^{-1} \sin(\lambda t) t^2 \, dt = \lambda^{-1} F_s[tf(t)](\lambda), \qquad H^{-1} f(t) = t^{-1} F_s^{-1}[\lambda f(\lambda)](t),$$

where

$$\widehat{f_s}(\lambda) := F_s f(\lambda) = \int_0^\infty f(t) \sin(\lambda t) \, dt, \qquad F_s^{-1} f(t) = \frac{2}{\pi} \int_0^\infty f(\lambda) \sin(\lambda t) \, d\lambda \tag{6.2}$$

denote the sine Fourier transform and its inverse, respectively.

**6.1. Properties of the normalized Bessel function.** The kernel $K(xy) = j_\alpha(xy)$ satisfies all the properties from Subsection 1.1. This follows from the corresponding properties of the classical Bessel function $J_\alpha$ [**2**, Ch. 7]. Let us list the main ones.

1) The functions $j_\alpha(\lambda t)$ are the eigenfunctions of the Sturm–Liouville problem

$$\frac{d}{dt}\left(t^\nu \frac{d}{dt} j_\alpha(\lambda t)\right) + \lambda^2 t^\nu j_\alpha(\lambda t) = 0, \qquad j_\alpha(0) = 1, \quad \frac{d}{dt} j_\alpha(0) = 0;$$

2) For $\alpha > -1/2$, the Poisson integral representation holds

$$j_\alpha(t) = \frac{2\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+1/2)} \int_0^1 (1-u^2)^{\alpha-1/2} \cos(tu) \, du;$$

3) If $t \to +\infty$,

$$j_\alpha(t) = \frac{2^\alpha \Gamma(\alpha+1)(2/\pi)^{1/2}}{t^{\alpha+1/2}} \left[\cos(t - c_\alpha) + O(t^{-1})\right], \qquad c_\alpha = \frac{\pi(\alpha+1/2)}{2}.$$

**6.2. Sharp weighted inequalities.** Since the Hankel transform coincides with its inverse up to a constant, we will present weighted inequalities only for the direct transform.

Setting for $\nu = 2\alpha + 1 \geq 0$, $x, y \geq 0$

$$Ff(x) = \int_0^\infty f(y) K(x,y) s(y) \, dy, \qquad K(x,y) = j_\alpha(xy), \quad s(y) = y^\nu, \quad w(x) = b_\alpha x^\nu,$$

we have

$$w(x)s(1/x) = b_\alpha, \qquad \|f\|_{2,s}^2 = b_\alpha \|Ff\|_{2,w}^2.$$

It follows from Subsection 6.1 that

$$|K(x,y)| \asymp 1, \quad xy \lesssim 1, \qquad |K(x,y)| \lesssim [w(x)s(y)]^{-1/2}, \quad xy \gtrsim 1.$$

By this, the kernel $K$ satisfies (1.8).

The main result of this section is the sharp Pitt's inequality for the Hankel transform. It comes as a consequence of Theorem 5.1 for the case where $\nu_1 = \nu_2 = \nu \geq 0$ (for $\nu = 0$, we have the case of the cosine Fourier transform).

In the sequel, let $1 < p \le q < \infty$, $\nu \ge 0$, and, as usual, $\overline{\gamma} = (\gamma_1, \gamma_2)$, and $\overline{\beta} = (\beta_1, \beta_2)$.

COROLLARY 6.1. *Let $\beta_1 = \beta_2 = \beta$ and $\gamma_1 = \gamma_2 = \gamma$. Pitt's inequality*

$$\|x^{-\gamma} Hf\|_{q,x^\nu} \lesssim \|y^\beta f\|_{p,y^\nu} \tag{6.3}$$

*holds true if and only if*

$$\gamma - \beta = \left(\frac{1}{q} - \frac{1}{p'}\right)(\nu + 1), \tag{6.4}$$

*and*

$$\left(\frac{1}{q} - \frac{1}{2}\right)\nu + \max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\} \le \gamma < \frac{\nu+1}{q}. \tag{6.5}$$

In [**16**], this assertion has been proved by different means.

PROOF. This result may be considered as a reformulation of Theorem 5.1, whereas (6.4) follows from (5.9), while (6.5) follows from (5.10) and (5.11). □

Let us investigate one more interesting case. Consider Figure 3, where

$$\begin{aligned} &a_1(t_0\nu + t_*, t_0\nu + t_*), \qquad a_2(t_0\nu + t_*, t_0\nu + t_1), \\ &a_3(t_*(\nu+1), t_*\nu + t_1), \qquad a_4(t_1(\nu+1), t_1(\nu+1)), \end{aligned} \tag{6.6}$$

and the points $\overline{a}_i$ are symmetric to $a_i$ with respect to the line $a_1a_4$, $i = 1, 2$.

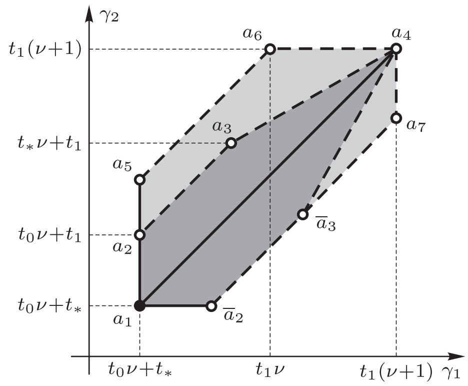


FIGURE 3. The case of the Hankel transform for $\gamma_1 - \beta_1 = \gamma_2 - \beta_2$

COROLLARY 6.2. *Let $\gamma_1 - \beta_1 = \gamma_2 - \beta_2$. If*

$$\overline{\beta} = \overline{\gamma} + \left(\frac{1}{p'} - \frac{1}{q}\right)(\nu + 1)$$

*and parameter $\overline{\gamma}$ lies in the domain $a_1a_2a_3a_4\overline{a}_3\overline{a}_2a_1$ symmetric with respect to the line $a_1a_4$ (see Figure 3), then Pitt's inequality holds:*

$$\|x^{-\overline{\gamma}^\circ} Hf\|_{q,x^\nu} \lesssim \|y^{\overline{\beta}} f\|_{p,y^\nu}. \tag{6.7}$$

*In addition, for the points which are off the domain $a_1a_2a_5a_6a_4a_7\overline{a}_3\overline{a}_2a_1$, outlined in Figure 3, Pitt's inequality does not hold.*

PROOF. By (5.9), we have

$$\gamma_i - \beta_i = \left(\frac{1}{q} - \frac{1}{p'}\right)(\nu+1), \quad i = 1, 2,$$

that implies

$$\overline{\beta} = \overline{\gamma} + \left(\frac{1}{p'} - \frac{1}{q}\right)(\nu+1). \tag{6.8}$$

To complete the proof, it remains to use the fact that Figure 3 is the intersection of the two domains in Figure 2: one is in the $\overline{\gamma}$ coordinates, while the other is in the $\overline{\beta}$ coordinates but is drawn shifted in accordance with (6.8).

Let us give a more detailed explanation. It is routine to show that (5.10) and (5.11) (for $(p, q) \neq 2$) or (5.12) (for $p = q = 2$) imply, for $\gamma_i$, $i = 1, 2$,

$$t\nu + t_* \le \gamma_i < t\nu + t_1, \quad t \in [t_0, t_*), \qquad t\nu + t < \gamma_i < t\nu + t_1, \quad t \in [t_*, t_1), \tag{6.9}$$

and $t_*\nu + t_* \le \gamma_i$ for $p = q = 2$.

Figure 3 presents two coordinates $\gamma_1$ and $\gamma_2$. However, in the inequalities (6.9) there is also the parameter $t$. It expresses how the results and conditions depend on $a$. Generally, inequalities (6.9) define a polytope in that spatial coordinate system $(\gamma_1, \gamma_2, t)$. What we see in Figure 3 is the projection of that polytope on the plane $(\gamma_1, \gamma_2)$ (i.e., $t = 0$). Observe that for each $t$, the sets of $(\gamma_1, \gamma_2)$ that satisfy (6.9) form a half-open square with the sides parallel to the coordinate axes $\gamma_1$ and $\gamma_2$, two vertices on the bisectrix $\gamma_1 = \gamma_2$, and two other vertices symmetric with respect to this bisectrix.

If $t = t_0$, the points $a_1$, $a_2$, $\overline{a}_2$ in Figure 3 are three vertices of the initial square. If $t = t_*$, two symmetric vertices $a_3$ and $\overline{a}_3$ of the corresponding square are denoted in Figure 3. Finally, if $t = t_1$, the corresponding square degenerates into one point, whose projection in Figure 3 is denoted by $a_4(t_1(\nu+1), t_1(\nu+1))$. By this, the projection of the whole polytope $D_{p,q}$ is exactly $a_1a_2a_3a_4\overline{a}_3\overline{a}_2a_1$, as claimed.

The optimality condition follows from Theorem 5.1 and relation (6.8) between the parameters $\gamma_i$ and $\beta_i$. □

Note that Corollary 6.1 is illustrated in Figure 3 by the interval $a_1a_4$.

**6.3. The Hankel transform of the Bochner–Riesz kernel.** The goal of this subsection is to show that necessity in Corollary 6.1, i.e., the sharpness of condition (6.5) can easily be observed by means of the Bochner–Riesz kernel

$$f_\sigma(t) = (1 - t^2)_+^\sigma, \qquad \sigma > -1. \tag{6.10}$$

It follows from the formula (see, e.g., [**50**, Ch. IV, Lemma 4.13])

$$\int_0^1 (xy)^{1/2} J_\alpha(xy) x^{\alpha+1/2} (1 - x^2)^\sigma \, dx = 2^\sigma \Gamma(\sigma+1) y^{-\sigma-1/2} J_{\alpha+\sigma+1}(y), \quad \sigma > -1,$$

that

$$\int_0^1 (1 - t^2)^\sigma j_\alpha(\lambda t) t^\nu \, dt = c_{\alpha\sigma} j_{\alpha+\sigma+1}(\lambda), \qquad c_{\alpha\sigma} = \frac{\Gamma(\alpha+1)\Gamma(\sigma+1)}{2\Gamma(\alpha+\sigma+2)}.$$

This implies

$$H f_\sigma(\lambda) = c_{\alpha\sigma} j_{\alpha+\sigma+1}(\lambda). \tag{6.11}$$

Note that $\sigma = 0$ gives the Hankel transform of the indicator function $\chi_{[0,1]}(t)$.

To prove the sharpness of the range in (6.5), we take the Bochner–Riesz kernel (6.10). We have

$$\|y^\beta f_\sigma\|_{p,y^\nu} = \left(\int_0^1 y^{\beta p}(1-y^2)^{\sigma p} y^\nu\, dy\right)^{1/p} \asymp \left(\int_0^{1/2} y^{\beta p+\nu}\, dy + \int_{1/2}^1 (1-y)^{\sigma p}\, dy\right)^{1/p}.$$

It follows from (6.11), properties of the normalized Bessel function (see Subsection 6.1 and also [**17**]), and arguments similar to (4.9) that

$$\begin{aligned}\|x^{-\gamma} F f_\sigma\|_{q,x^\nu} &\asymp \left(\int_0^\infty |j_{\alpha+\sigma+1}(x)|^q x^{-\gamma q+\nu}\, dx\right)^{1/q}\\ &\asymp \left(\int_0^1 x^{-\gamma q+\nu}\, dx + \int_1^\infty x^{-q(\alpha+\sigma+3/2)-\gamma q+\nu}\, dx\right)^{1/q}.\end{aligned}$$

Therefore, if $\beta p + \nu > -1$ and $\sigma p > -1$, that is, $\|y^\beta f_\sigma\|_{p,y^\nu} < \infty$, then the inequality

$$\|x^{-\gamma} F f_\sigma\|_{q,x^\nu} \lesssim \|y^\beta f_\sigma\|_{p,y^\nu}$$

does not hold if either $-q\gamma + \nu \le -1$ or $-q(\alpha+\sigma+3/2) - \gamma q + \nu \ge -1$. Taking into account how $\beta$ and $\gamma$ are related as well as the equality $\alpha + 3/2 = (\nu+2)/2$, we derive that the inequality (6.3) in Corollary 6.1 does not hold if

$$\gamma \ge \frac{\nu+1}{q}, \qquad \frac{\nu+1}{q} - (\nu+1) < \gamma \le \frac{\nu+1}{q} - \frac{\nu+2}{2} - \sigma < \frac{\nu+1}{q} - \frac{\nu+2}{2} + \frac{1}{p}.$$

Interchanging $f$ and $Ff$ and using that the Hankel transform coincides with its inverse, we get that the inequality (6.3) in Corollary 6.1 is invalid also when

$$-p\left(\frac{\nu+2}{2} + \sigma\right) + \beta p + \nu < -1, \quad \sigma q > -1 \implies \gamma < \frac{\nu+1}{q} - \frac{\nu+2}{2} + \frac{1}{q'}.$$

Summarizing, we see that (6.3) does not hold for

$$\gamma < \frac{\nu+1}{q} - \frac{\nu+2}{2} + \max\left\{\frac{1}{p}, \frac{1}{q'}\right\}, \qquad \gamma \ge \frac{\nu+1}{q},$$

which proves the sharpness of condition (6.5).

## 7. Weighted inequalities for Fourier transforms

In this section, we discuss well-known special cases of the Hankel transform each of which corresponds to the particular case of the Fourier transform: cosine and sine in dimension one and the radial case in several dimensions.

**7.1. The Fourier transform of a radial function.** We start with the case $\nu = n-1$. This is the very important case of the Fourier transform of a radial function in the $n$-dimensional Euclidean space.

Corollary 7.1. *For $\nu = n-1$ and $1 < p \le q < \infty$, Pitt's inequality*

$$\|x^{-\gamma} \widehat{f}\|_{q,x^\nu} \lesssim \|y^\beta f\|_{p,y^\nu} \tag{7.1}$$

*holds if and only if*

$$\beta = \gamma + n\left(1 - \frac{1}{p} - \frac{1}{q}\right), \qquad \frac{n}{q} - \frac{n+1}{2} + \max\left\{\frac{1}{p}, \frac{1}{q'}\right\} \le \gamma < \frac{n}{q}.$$

This is the result in [**16**]. We remark that it does not follow from the known techniques dealing with the Fourier transforms, like in [**5, 26**]. Indeed, taking in Theorem 4 (i) and (iii) in [**5**] the weights to be the powers of $|x|$ and $|y|$, we obtain the classical Pitt inequality (1.11), which corresponds to (7.1) in the case of radial functions, under the more restrictive condition (1.12). This is also shown in Examples 4 and 5 in [**5**].

Let us assume $1 < p \le q < \infty$, and, as above, $\overline{\gamma} = (\gamma_1, \gamma_2)$ and $\overline{\beta} = (\beta_1, \beta_2)$.

**7.2. The cosine Fourier transform.** It is worth getting a closer look at the case $\nu = 0$, which corresponds to the cosine Fourier transform.

THEOREM 7.1. *Let* $\gamma_1 - \beta_1 = \gamma_2 - \beta_2$. *Pitt's inequality*

$$\|x^{-\overline{\gamma}^\circ} \widehat{f}_c\|_q \lesssim \|y^{\overline{\beta}} f\|_p$$

*holds if and only if*

$$\overline{\beta} = \overline{\gamma} + \frac{1}{p'} - \frac{1}{q} \tag{7.2}$$

*and*

$$\max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\} \le \gamma_i < \frac{1}{q}, \quad i = 1, 2.$$

*These inequalities correspond to the half-open square in Figure 4.*

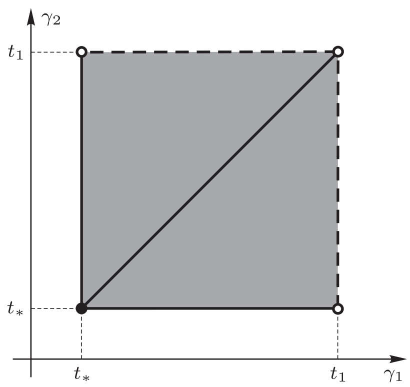


FIGURE 4. The case of the cosine Fourier transform

*In particular, for* $\gamma_1 = \gamma_2 = \gamma$, *we get the bisectrix*

$$\max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\} \le \gamma < \frac{1}{q}, \tag{7.3}$$

*which gives the necessary and sufficient condition.*

PROOF. Indeed, it follows from (6.9) that for $i = 1, 2$,

$$\text{(i)} \quad t_* \le \gamma_i < t_1, \quad t \in [t_0, t_*), \qquad \text{(ii)} \quad t < \gamma_i < t_1, \quad t \in [t_*, t_1).$$

We again have a polytope (cf. Remark 5.1) for the whole range of $a$ (or $t$). However, its form is different. Inequalities (i) define the same square with the sides of length $t_1 - t_*$.

For $t$ varying from $t_*$ to $t_1$, the squares defined by (ii) become smaller and smaller, with just a point for $t = t_1$. This polytope has no slope, contrary to that in Figure 3, and all the projections fall down into the large square in Figure 4. Equivalently, we take the widest range of inequalities corresponding to (7.2). In the case $\gamma_1 = \gamma_2 = \gamma$, the bisectrix (7.3) is the intersection of the necessary and sufficient conditions. □

It is important to note that (7.3) coincides with (1.12) and (1.13) for $n = 1$. We also remark that Theorem 7.1 can be derived from results in [**26**]. Indeed, let $w = C_1$ and $s = C_2$, where $C_1$ and $C_2$ are some constants. This is the case of the cosine Fourier transform. Condition (3.13) in Corollary 3.2 is the same as condition (2.7) in [**26**]. Then Theorem 3.1 from [**26**] implies Pitt's inequality.

**7.3. The sine Fourier transform.** The other interesting case is $\nu = 2$, which corresponds to the sine Fourier transform. Using the relation between the Hankel transform for $\nu = 2$ and the sine Fourier transform, we can prove the following assertion.

THEOREM 7.2. *Let $\gamma_1 - \beta_1 = \gamma_2 - \beta_2$. Pitt's inequality*

$$\|x^{-\overline{\gamma}^\circ} \widehat{f}_s\|_q \lesssim \|y^{\overline{\beta}} f\|_p \tag{7.4}$$

*holds for*

$$\overline{\beta} = \overline{\gamma} + \frac{1}{p'} - \frac{1}{q}, \tag{7.5}$$

*where $\overline{\gamma}$ belongs to the domain $a_1a_2a_3a_4\overline{a}_3\overline{a}_2a_1$ (see Figure 5) with*

$$a_1(t_*, t_*), \quad a_2(t_*, t_1), \quad a_3(3t_* - 2t_1 + 1, 2t_* - t_1 + 1), \quad a_4(t_1 + 1, t_1 + 1),$$

$$t_* = \max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\}, \quad t_1 = \frac{1}{q}. \tag{7.6}$$

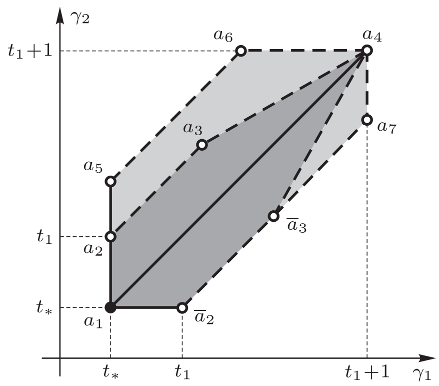


FIGURE 5. The case of the sine Fourier transform

REMARK 7.1. This assertion is similar to Corollary 6.2 with respect to assumptions on $\overline{\beta}$ and $\overline{\gamma}$. Of course, necessary conditions for (7.4) are the conditions provided by (B1) and (B2) in Theorem 5.1 (if $\overline{\gamma}$ belongs to the domain $a_1a_5a_6a_4a_7\overline{a}_2a_1$).

Note that for $\gamma_1 = \gamma_2 = \gamma$, we get the bisectrix $a_1a_4$

$$\max\left\{0, \frac{1}{q} - \frac{1}{p'}\right\} \le \gamma < \frac{1}{q} + 1, \tag{7.7}$$

which gives the necessary and sufficient condition.

Proof. The sine transform can be represented via the Hankel transform of the function $H_2$. Indeed, we have

$$\|x^{-\overline{\gamma}^\circ}\widehat{f}_s\|_q = \|x^{-(\overline{\gamma}^\circ-1+2/q)}H_2 g\|_{q,x^2},$$

where $g(y) = y^{-1}f(y)$. Applying now (6.7) in Corollary 6.2 for $\nu = 2$, $\overline{\gamma}^\circ - 1 + 2/q$ in place of $\overline{\gamma}^\circ$ and $\overline{\beta} + 1 - 2/p$ in place of $\overline{\beta}$, and taking into account (7.6), we obtain (7.5). The rest of the conditions follow from (6.9) taken with $\nu = 2$. □

In [**37**], the right-hand bound in (7.7) has been obtained only for functions satisfying special type monotonicity conditions. This result is also new as compared with that in [**45**] for functions with mean zero.

## 8. Jacobi transforms: basic properties and Pitt's inequalities

Let us begin with the needed prerequisites; they can be found in [**19**], [**20**], [**33**], [**34**] (see also [**8**], [**13**], [**14**]). The Jacobi functions are defined by

$$\varphi_\lambda^{(\alpha,\beta)}(t) = F\left(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \alpha+1; -(\sinh t)^2\right), \qquad t \ge 0, \quad \alpha,\beta,\lambda \in \mathbb{C},$$

where $\rho = \alpha + \beta + 1$ and $F(a, b; c; z)$ is the hypergeometric Gauss function (1.17). We consider the case

$$\alpha \ge \beta \ge -\frac{1}{2}.$$

In particular cases, we have more transparent representations for the Jacobi functions:

$$\varphi_\lambda^{(-1/2,-1/2)}(t) = \cos(\lambda t), \qquad \varphi_\lambda^{(1/2,1/2)}(t) = \frac{2\sin(\lambda t)}{\lambda \sinh 2t}, \tag{8.1}$$

$$\varphi_\lambda^{(\alpha,\alpha)}(t) = \varphi_{\lambda/2}^{(\alpha,-1/2)}(2t) = \frac{2^\alpha\Gamma(\alpha+1)P^{-\alpha}_{-1/2+i\lambda/2}(\cosh 2t)}{(\sinh 2t)^\alpha},$$

where $P_\nu^\mu$ is the Legendre function.

The direct and inverse Jacobi transforms are defined by the identities

$$Jf(\lambda) = \int_0^\infty f(t)\varphi_\lambda^{(\alpha,\beta)}(t)m(t)\,dt, \qquad J^{-1}f(t) = \int_0^\infty f(\lambda)\varphi_\lambda^{(\alpha,\beta)}(t)n(\lambda)\,d\lambda,$$

respectively. Here

$$m(t) = (2\pi)^{-1/2}\Delta(t), \qquad \Delta(t) = 2^{2\rho}(\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}, \tag{8.2}$$

$$n(\lambda) = (2\pi)^{-1/2}|c(\lambda)|^{-2}, \qquad c(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma\left(\frac{\rho+i\lambda}{2}\right)\Gamma\left(\frac{\rho+i\lambda}{2}-\beta\right)}. \tag{8.3}$$

Note that the Jacobi transform is a bijection between the space of even infinitely differentiable functions with compact support and the space of even entire functions of exponential type with rapid decay. It can be extended to the isomorphism between the two weighted spaces $L^2_m$ and $L^2_n$, the norm in which is defined by (1.1). Parseval's identities are true as well:

$$\|Jf\|_{L^2_n} = \|f\|_{L^2_m}, \qquad \|f\|_{L^2_n} = \|J^{-1}f\|_{L^2_m}. \tag{8.4}$$

When $\alpha=\beta$, the Jacobi transform is also known as the Mehler–Fock transform. If $\alpha=\beta=-1/2$, then (8.1), (8.2), and (8.3) reduce the Jacobi transform to the cosine Fourier transform. Since this case has been studied in detail in the previous section, in what follows we will be interested in the case $\alpha>-1/2$.

The Jacobi transform does not immediately fit the general outline given in Sections 2–5, since the weight $m$ does not satisfy the $\Delta_2$-condition. However, it can easily be modified to become such. We postpone this to Subsection 8.2.

**8.1. Properties of Jacobi functions.** Let us list several useful properties of the Jacobi functions

$$\varphi_\lambda(t):=\varphi_\lambda^{(\alpha,\beta)}(t),\qquad \alpha\ge\beta\ge-1/2,\quad \alpha>-1/2.$$

There is a substantial similarity of their properties with those of the normalized Bessel functions $j_\alpha(\lambda t)$:

1. For $t\ge 0$, the functions $\varphi_\lambda(t)$ are eigenfunctions of the Sturm–Liouville problem

$$\frac{d}{dt}\,\Delta(t)\,\frac{d}{dt}\,\varphi_\lambda(t)+(\rho^2+\lambda^2)\Delta(t)\varphi_\lambda(t)=0,\qquad \varphi_\lambda(0)=1,\qquad \varphi_\lambda'(0)=0. \tag{8.5}$$

2. Mehler's integral representation is true:

$$\varphi_\lambda(t)=\frac{2}{\Delta(t)}\int_0^t A(s,t)\cos{(\lambda s)}\,ds,\qquad A(s,t)\ge 0 \tag{8.6}$$

(see [**34**], where the explicit expression for $A(s,t)$ is given).

3. Let us establish the asymptotic behavior of the weight $n(\lambda)$, $\lambda>0$. It follows from the identity $\Gamma(z+1)=z\Gamma(z)$ that

$$|c(\lambda)|=\frac{2^\rho\Gamma(\alpha+1)}{\Gamma(\rho/2)\Gamma(\rho/2-\beta)}\,\lambda^{-1}\,(1+O(\lambda)),\qquad \text{as}\quad \lambda\to 0.$$

For $|z|\to\infty$, $|\arg z|<\pi$, we have (see, e.g., [**1**, Ch. 1])

$$\ln\Gamma(z+a)=\left(z+a-\frac12\right)\ln z-z+\frac12\ln{(2\pi)}+O(z^{-1}).$$

This yields, as $\lambda\to\infty$,

$$\ln c(\lambda)=(\rho+\alpha)\ln 2+\ln\Gamma(\alpha+1)-\left(\alpha+\frac12\right)\ln i\lambda-\frac12\ln{(2\pi)}+O(\lambda^{-1})$$

and

$$|c(\lambda)|=\frac{2^{\rho+\alpha}\Gamma(\alpha+1)(2\pi)^{-1/2}}{\lambda^{\alpha+1/2}}\left[1+O(\lambda^{-1})\right].$$

Hence, using (8.3), we obtain

$$n(\lambda)=(2\pi)^{-1/2}\left(\frac{\Gamma(\rho/2)\Gamma(\rho/2-\beta)}{2^\rho\Gamma(\alpha+1)}\right)^2\lambda^2\left[1+O(\lambda)\right],\quad \lambda\lesssim 1, \tag{8.7}$$

$$n(\lambda)=\frac{(2\pi)^{1/2}}{\left(2^{\rho+\alpha}\Gamma(\alpha+1)\right)^2}\,\lambda^{2\alpha+1}\left[1+O(\lambda^{-1})\right],\quad \lambda\gtrsim 1. \tag{8.8}$$

4. We now give some asymptotic properties of Jacobi functions. They result from the following asymptotic formulas for hypergeometric functions:

$$F(a,b;c;z)=1+O(|z|),\qquad \text{as}\quad |z|\to 0; \tag{8.9}$$

while for $|\arg(-z)| < \pi$ and $|z| \to \infty$, we have the following two relations:

$$F(a,a;c;z) = B_0(-z)^{-a}\ln(-z)\left[1+O(|\ln(-z)|^{-1})\right], \qquad c-a\notin\mathbb{Z}, \quad a\neq 0, \tag{8.10}$$

$$F(a,b;c;z) = B_1(-z)^{-a}\left[1+O(|z|^{-1})\right] + B_2(-z)^{-b}\left[1+O(|z|^{-1})\right], \qquad a-b\notin\mathbb{Z}, \tag{8.11}$$

$$B_0 = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}, \quad B_1 = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}, \quad B_2 = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}$$

(see [**1**, Ch. 2]);

$$F\left(a+\mu, b-\mu; c; \frac{1-z}{2}\right) = \frac{\Gamma(c)2^{(a+b-1)/2}\left(\theta\sinh\theta\right)^{1/2}}{\gamma^{c-1}\left(z-1\right)^{c/2}\left(z+1\right)^{(a+b+1-c)/2}}\left[I_{c-1}(\gamma\theta)+\varepsilon_1\right], \tag{8.12}$$

where

$$z=\cosh\theta, \quad \gamma = \frac{a-b}{2}+\mu, \quad |\varepsilon_1| \le \frac{A_1 e^{A_2/|\gamma|}}{\theta|\gamma|^2|K_c(\gamma\theta)|}, \qquad \theta>0, \quad |\arg\mu|<\pi,$$

with $A_1$ and $A_2$ being positive constants independent of $\theta$ and $\gamma$. Here $I_\nu$ and $K_\nu$ are modified Bessel functions (see [**30**]).

Proposition 8.1. The following properties of the Jacobi functions hold true:

$$\varphi_0(t) = 1+O(t^2), \quad 0\le t\lesssim 1,$$

$$\varphi_0(t) = \frac{2^{\rho+1}\Gamma(\alpha+1)}{(2\pi)^{1/4}\Gamma(\rho/2)\Gamma(\rho/2-\beta)}\,\frac{t}{m(t)^{1/2}}\left[1+O(t^{-1})\right], \quad t\gtrsim 1; \tag{8.13}$$

and for $\lambda t\gtrsim 1$,

$$\varphi_\lambda(t) = \frac{(2/\pi)^{1/2}}{[m(t)n(\lambda)]^{1/2}}\left[\cos\left(\lambda t+\arg c(\lambda)\right)+O(e^{-2t})\right], \qquad 0<\lambda\lesssim 1, \tag{8.14}$$

and

$$\varphi_\lambda(t) = \frac{(2/\pi)^{1/2}}{[m(t)n(\lambda)]^{1/2}}\left[\cos\left(\lambda t-c_\alpha\right)+O(\lambda^{-1})\right], \qquad \lambda\gtrsim 1, \quad t>0. \tag{8.15}$$

Proof. 1. Set in (8.9) and (8.10)

$$a=b=\frac{\rho}{2}, \quad c=\alpha+1, \quad z=-(\sinh t)^2.$$

In virtue of (8.9), we have $\varphi_0(t) = 1+O(t^2)$, $t\lesssim 1$. Using (8.10), asymptotic equalities

$$\sinh t = \cosh t\left(1+O(e^{-2t})\right), \quad -z = (\sinh t)^2 = \frac{e^{2t}}{4}\left(1+O(e^{-2t})\right), \qquad t\to\infty, \tag{8.16}$$

and (8.2), we obtain for $t\gtrsim 1$

$$\begin{aligned}\varphi_0(t) &= \frac{\Gamma(\alpha+1)2t}{\Gamma(\rho/2)\Gamma(\rho/2-\beta)(\sinh t)^\rho}\left[1+O(t^{-1})\right]\\ &= \frac{2^{\rho+1}\Gamma(\alpha+1)}{(2\pi)^{1/4}\Gamma(\rho/2)\Gamma(\rho/2-\beta)}\,\frac{t}{m(t)^{1/2}}\left[1+O(t^{-1})\right].\end{aligned}$$

By this, (8.13) is established.

2. Let $\lambda t\gtrsim 1$, $0<\lambda\lesssim 1$. Then $t\gtrsim 1$. Set in (8.11)

$$a=\frac{\rho+i\lambda}{2}, \quad b=\frac{\rho-i\lambda}{2}, \quad c=\alpha+1, \quad z=-(\sinh t)^2.$$

It follows from (8.11), the properties of the Gamma function and (8.3) that

$$\varphi_\lambda(t) = 2\operatorname{Re}\left(B_2(\sinh t)^{-\rho+i\lambda}\right)\left[1+O\left((\sinh t)^{-2}\right)\right],$$

where

$$B_2 = \overline{B_1} = \frac{\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma\left(\frac{\rho+i\lambda}{2}\right)\Gamma\left(\frac{\rho+i\lambda}{2}-\beta\right)} = 2^{-\rho+i\lambda}c(\lambda).$$

This yields

$$\begin{aligned}\varphi_\lambda(t) &= 2\operatorname{Re}\left(2^{-\rho+i\lambda}c(\lambda)(\sinh t)^{-\rho+i\lambda}\right)\left[1+O(e^{-2t}\right]\\ &= 2(2\sinh t)^{-\rho}|c(\lambda)|\operatorname{Re}\left(e^{i[\lambda\ln(2\sinh t))+\arg c(\lambda)]}\right)\left[1+O(e^{-2t})\right].\end{aligned}$$

Here $\arg c(\lambda)$ is a continuous function when $0<\lambda\lesssim 1$ because of the analyticity of the function $c(\lambda)$. Making use of (8.2), (8.3), and (8.16), we get

$$\varphi_\lambda(t) = \frac{(2/\pi)^{1/2}}{[m(t)n(\lambda)]^{1/2}}\left[\cos\left(\lambda\ln(2\sinh t)+\arg c(\lambda)\right)\right]\left[1+O(e^{-2t})\right],$$

where $\ln(2\sinh t) = t+O(e^{-2t})$ and

$$\cos\left(\lambda\ln(2\sinh t)+\arg c(\lambda)\right) = \cos\left(\lambda t+\arg c(\lambda)\right)+O(e^{-2t}), \qquad t\gtrsim 1.$$

Thus, the proof of (8.14) is complete.

3. Let $\lambda t\gtrsim 1$, $\lambda\gtrsim 1$, $t>0$. Putting in (8.12)

$$a=b=\frac{\rho}{2},\quad \mu=\frac{i\lambda}{2},\quad c=\alpha+1,\quad \theta=2t,$$

we have

$$z=\cosh 2t,\quad \frac{1-z}{2}=-(\sinh t)^2,\quad \gamma=\frac{i\lambda}{2},\quad \gamma\theta=i\lambda t,\quad I_{c-1}(\gamma\theta)=i^\alpha J_\alpha(\lambda t).$$

It follows from this and from the expression for the normalized Bessel function (1.15) that

$$\varphi_\lambda(t) = \frac{t^{\alpha+1/2}}{(\sinh t)^{\alpha+1/2}(\cosh t)^{\beta+1/2}}\left[j_\alpha(\lambda t)+\widetilde{\varepsilon}_1\right],$$

where

$$\widetilde{\varepsilon}_1 = 2^\alpha\Gamma(\alpha+1)(i\lambda t)^{-\alpha}\varepsilon_1,\qquad |\widetilde{\varepsilon}_1|\le\frac{A_1 2^{\alpha+1}\Gamma(\alpha+1)e^{2A_2/\lambda}}{\lambda^{\alpha+2}t^{\alpha+1}|K_{\alpha+1}(i\lambda t)|}. \tag{8.17}$$

For $z\gtrsim 1$, $\alpha\ge -1/2$, we have $|K_{\alpha+1}(iz)|\asymp z^{-1/2}$. This, (8.17) and $\lambda t\gtrsim 1$, $\lambda\gtrsim 1$ yield

$$|\widetilde{\varepsilon}_1|\lesssim\frac{1}{\lambda(\lambda t)^{\alpha+1/2}}.$$

Finally, using the asymptotics of the normalized Bessel function and (8.2), we obtain

$$\varphi_\lambda(t) = \frac{(2/\pi)^{1/2}}{[m(t)n(\lambda)]^{1/2}}\left[\cos(\lambda t-c_\alpha)+O(\lambda^{-1})\right],\qquad \lambda\gtrsim 1.$$

We now have (8.15), which completes the proof of the proposition. □

**8.2. The modified Jacobi transform.** The Jacobi transform can be reduced to a transform with weights of power type (1.9). For this, set

$$\widetilde{\varphi}_\lambda(t) := \widetilde{\varphi}_\lambda^{(\alpha,\beta)}(t) = \frac{\varphi_\lambda(t)}{\varphi_0(t)}, \qquad \widetilde{m}(t) = [\varphi_0(t)]^2 m(t), \qquad \widetilde{n}(\lambda) = n(\lambda). \tag{8.18}$$

The modified Jacobi transforms are thus defined by the relations

$$\widetilde{J}f(\lambda) = \int_0^\infty f(t)\widetilde{\varphi}_\lambda(t)\widetilde{m}(t)\,dt, \quad \widetilde{J}^{-1}f(t) = \int_0^\infty f(\lambda)\widetilde{\varphi}_\lambda(t)\widetilde{n}(\lambda)\,d\lambda.$$

From (8.18), (8.2), (8.13) and (8.7), (8.8), we find out that

$$\widetilde{m}(t) \asymp m(t) \asymp t^{2\alpha+1}, \quad t \lesssim 1, \qquad \widetilde{m}(t) \asymp (tm(t)^{-1/2})^2 m(t) \asymp t^2, \quad t \gtrsim 1. \tag{8.19}$$

This and definition (1.9) yield

$$\widetilde{m}(t) \asymp t^{(2\alpha+1,2)}, \quad t > 0. \tag{8.20}$$

It follows from (8.7), (8.8) and (8.18), (8.3) that

$$\widetilde{n}(\lambda) \asymp \lambda^{(2,2\alpha+1)}, \quad \lambda > 0. \tag{8.21}$$

The appropriate relation between the weights

$$\widetilde{m}(t)\widetilde{n}(1/t) \asymp 1, \quad t > 0 \tag{8.22}$$

follows now from (8.20) and (8.21). Therefore, the modified weights satisfy all the required properties.

We have the following simple connection between the transforms:

$$\widetilde{J}f(\lambda) = J\left[f(t)\varphi_0(t)\right](\lambda), \qquad \widetilde{J}^{-1}f(t) = \varphi_0(t)^{-1}J^{-1}f(t). \tag{8.23}$$

It follows from (8.4) that

$$\|\widetilde{J}f\|_{L^2_{\widetilde{n}}} = \|f\|_{L^2_{\widetilde{m}}}, \quad \|f\|_{L^2_{\widetilde{n}}} = \|\widetilde{J}^{-1}f\|_{L^2_{\widetilde{m}}}.$$

For $\alpha = \beta = -1/2$, using the duplication formula for the Gamma function, we obtain

$$\widetilde{\varphi}_\lambda(t) = \cos(\lambda t), \quad \widetilde{m}(t) \equiv (2\pi)^{-1/2}, \quad \widetilde{n}(\lambda) \equiv 2^2(2\pi)^{-1/2}.$$

As mentioned, this is the case of the cosine Fourier transform, or, which is the same, the Hankel transform with $\alpha = -1/2$. For $\alpha = \beta = 1/2$, we have

$$\widetilde{\varphi}_\lambda(t) = \frac{\sin(\lambda t)}{\lambda t}, \quad \widetilde{m}(t) = 2^4(2\pi)^{-1/2}t^2, \quad \widetilde{n}(\lambda) = 2^{-2}(2\pi)^{-1/2}\lambda^2.$$

This case reduces to the Hankel transform with $\alpha = 1/2$, or the sine Fourier transform.

**8.3. Properties of the modified Jacobi functions.** Let us give basic properties of the modified Jacobi functions when $\alpha \ge \beta \ge -1/2$, $\alpha > -1/2$. They simply follow from the corresponding properties of the (standard) Jacobi functions. By this, we will see that the kernel $K(xy) = \widetilde{\varphi}_x(y)$ satisfies all the properties from Subsection 1.1.

1. It follows from (8.5) for Jacobi functions that the modified Jacobi functions are eigenfunctions of the Sturm–Liouville problem

$$\frac{d}{dt}\left(\widetilde{m}(t)\,\frac{d}{dt}\,\widetilde{\varphi}_\lambda(t)\right) + \lambda^2\widetilde{m}(t)\widetilde{\varphi}_\lambda(t) = 0, \qquad \widetilde{\varphi}_\lambda(0) = 1, \quad \frac{d}{dt}\,\widetilde{\varphi}_\lambda(0) = 0. \tag{8.24}$$

2. Mehler's representation for the modified Jacobi function

$$\widetilde{\varphi}_\lambda(t)=\int_0^t \widetilde{A}(s,t)\cos\lambda s\,ds,\qquad \widetilde{A}(s,t)=\frac{A(s,t)}{\int_0^t A(u,t)\,du}\ge 0$$

follows from (8.6). This and (8.24) yield

$$|\widetilde{\varphi}_\lambda(t)|\le\widetilde{\varphi}_0(t)=\widetilde{\varphi}_\lambda(0)=1,\qquad \widetilde{\varphi}_\lambda(t)\asymp 1,\quad \lambda t\lesssim 1$$

(see also Subsection 1.1).

3. It follows from Proposition 8.1 (properties (8.14) and (8.15)) and (8.18) that for $\lambda t\gtrsim 1$

$$\widetilde{\varphi}_\lambda(t)=\frac{(2/\pi)^{1/2}}{[\widetilde{m}(t)\widetilde{n}(\lambda)]^{1/2}}\begin{cases}\cos(\lambda t-c_\alpha)+O(\lambda^{-1}), & \lambda\gtrsim 1,\quad t>0,\\ \cos(\lambda t+\arg c(\lambda))+O(t^{-1}), & t\gtrsim 1,\quad 0<\lambda\lesssim 1,\end{cases}$$

cf. Subsection 1.1. Here we also used $e^{2t}\gtrsim t$ for $t\gtrsim 1$.

**8.4. Pitt's inequalities for the Jacobi transforms.** We first write the modified Jacobi transforms in a different notation. Let $\nu=2\alpha+1>0$.

*Direct transform.* In this case

$$K(x,y)=\widetilde{\varphi}_x^{(\alpha,\beta)}(y),\qquad s(y)=\widetilde{m}(y)\asymp y^{\overline{\nu}},\quad \overline{\nu}=(\nu,2),\qquad w(x)=\widetilde{n}(x)\asymp x^{\overline{\nu}'}. \tag{8.25}$$

*Inverse transform.* In this case

$$K(x,y)=\widetilde{\varphi}_y^{(\alpha,\beta)}(x),\qquad s(y)=\widetilde{n}(y)\asymp y^{\overline{\nu}},\quad \overline{\nu}=(2,\nu),\qquad w(x)=\widetilde{m}(x)\asymp x^{\overline{\nu}'}.$$

All the properties of the kernel and weights of the integral transform given in the introduction needed for deriving Pitt's inequality are valid in both cases. In particular, the assumptions of Theorems 3.2 and 4.1 take place, which follows from the above results of this section.

Applying the results for general power weights from Section 5, we derive from Theorem 5.1 the following Pitt's inequalities for the modified Jacobi transforms.

THEOREM 8.1. *Suppose $1<p\le q<\infty$, $\nu>0$.*

(A) *Let $\overline{\gamma}$ and $\overline{\beta}$ be dual and $(\overline{\gamma},\overline{\beta})\in D_{p,q}(\nu,2)$, then*

$$\|x^{-\overline{\gamma}^\circ}\widetilde{J}f\|_{q,x^{(2,\nu)}}\lesssim\|y^{\overline{\beta}}f\|_{p,y^{(\nu,2)}} \tag{8.26}$$

*for the direct transform. If $(\overline{\gamma},\overline{\beta})\in D_{p,q}(2,\nu)$, then*

$$\|x^{-\overline{\gamma}^\circ}\widetilde{J}^{-1}f\|_{q,x^{(\nu,2)}}\lesssim\|y^{\overline{\beta}}f\|_{p,y^{(2,\nu)}} \tag{8.27}$$

*for the inverse one.*

(B) *Let $(\nu_1,\nu_2)=(\nu,2)$ or $(\nu_1,\nu_2)=(2,\nu)$ for the direct Jacobi transform or the inverse one, respectively. Then the conditions*

$$\beta_1-\gamma_1\le\left(\frac{1}{p'}-\frac{1}{q}\right)(\nu_1+1),\qquad \beta_2-\gamma_2\ge\left(\frac{1}{p'}-\frac{1}{q}\right)(\nu_2+1),$$

*and*

$$\gamma_1\ge\left(\frac{1}{q}-\frac{1}{2}\right)\nu_1+\frac{1}{q}-\frac{1}{p'},\qquad \beta_2\ge\left(\frac{1}{p'}-\frac{1}{2}\right)\nu_2+\frac{1}{p'}-\frac{1}{q}$$

*are necessary for Pitt's inequalities* (8.26) *and* (8.27) *to be valid.*

It is easy to rewrite Theorem 8.1 for the standard Jacobi transforms $J$ and $J^{-1}$. We omit this.

## Appendix. Observations concerning conditions on weights

In order to prove Pitt's inequality $\|w^{1/a'}Ff\|_{q,u} \lesssim \|s^{1/a}f\|_{p,v}$ in Theorem 3.1, we have used Hardy's inequalities (2.1) applied to the Calderón type estimate (2.5) in Theorem 2.2. However, the same result follows from the more general inequality (cf. (2.6))

$$(Fg)^*(x) \lesssim x^{-1/a'} \left( \int_0^x \left( \int_0^{1/t} g^*(\xi)\, d\xi \right)^a t^{a-2}\, dt \right)^{1/a}. \tag{8.28}$$

Using (8.28) instead of the inequality (2.5) in Theorem 2.2, we wish to obtain the conditions on weights such that the following inequalities hold:

$$\begin{aligned}\left( \int_0^\infty (Fg)^*(x)^q u^*(x)\, dx \right)^{1/q} &\lesssim \left( \int_0^\infty \left( \int_0^x \left( \int_0^{1/t} g^*(\xi)\, d\xi \right)^a t^{a-2}\, dt \right)^{q/a} x^{-q/a'} u^*(x)\, dx \right)^{1/q} \\ &\lesssim \left( \int_0^\infty \left( t \int_0^{1/t} g^*(\xi)\, d\xi \right)^p v_*(1/t) t^{-2}\, dt \right)^{1/p} \\ &\lesssim \left( \int_0^\infty g^*(t)^p v_*(t)\, dt \right)^{1/p}. \end{aligned} \tag{8.29}$$

Indeed, the left-hand side controls $\|Fg\|_{q,u}$, by the first inequality in (3.5), while the second inequality in (3.5) ensures the veracity of Theorem 3.1.

On the other hand, since all the functions in (8.29) are rearrangements (that is, only monotone functions are involved in (8.29)), we can apply a special version of Hardy's inequality for monotone functions instead of (2.1). Such a version is given in [**47**, Th. 2] (see also [**5**, Th. D]; recent developments may be found in [**22**, Th. 1.2], [**18**], [**7**], and some other works; all these are surveyed in [**21**], where large bibliography is given). To this end, we rewrite the second inequality in (8.29) as

$$\left( \int_0^\infty \left( \int_0^x G(t)\, dt \right)^{q/a} x^{-q/a'} u^*(x)\, dx \right)^{a/q} \lesssim \left( \int_0^\infty G(t)^{p/a}\, t^{2\frac{p}{a}-2} v_*(1/t)\, dt \right)^{a/p}, \tag{8.30}$$

where

$$G(t) := \left( \int_0^{1/t} g^*(\xi)\, d\xi \right)^a t^{a-2}.$$

To obtain necessary and sufficient conditions for this, we employ the following

LEMMA 8.1. *For*

$$\left( \int_0^\infty \left( \int_0^x G(t)h(t)\, dt \right)^q w(x)\, dx \right)^{1/q} \lesssim \left( \int_0^\infty G(x)^p z(x)\, dx \right)^{1/p} \tag{8.31}$$

*to be valid for a monotone $G$, it is necessary and sufficient that for all $r > 0$,*

(i) *if $1 < p \le q < \infty$ and $h(t) \equiv 1$, then*

$$\frac{\left( \int_0^r x^q w(x)\, dx \right)^{1/q}}{\left( \int_0^r z(x)\, dx \right)^{1/p}} \lesssim 1$$

*and*

$$\left(\int_r^\infty w(x)\,dx\right)^{1/q}\left(\int_0^r\left(\frac{1}{x}\int_0^x z(t)\,dt\right)^{-p'} z(x)\,dx\right)^{1/p'}\lesssim 1;$$

(ii) *if* $0<p\le q<\infty$, $0<p\le 1$, $H(t)=\int_0^t h(x)\,dx$ *and* $Z(t)=\int_0^t z(x)\,dx$, *then*

$$Z^{-1/p}(r)\left(\int_0^\infty H^q(\min(x,r))w(x)\,dx\right)^{1/q}\lesssim 1.$$

Part (a) of this lemma is Theorem 2 in [**47**], while part (b) is (d) in [**21**, Th. 2.5]. In particular, if $1<p=q<\infty$, $h(t)\equiv 1$ and $w(x):=x^{-q}z(x)$, then (8.31) holds for a monotone $G$ if and only if $z\in B_p$, i.e.,

$$\int\limits_r^\infty \frac{z(t)}{t^p}dt\le \frac{C}{x^p}\int\limits_0^r z(t)dt,\qquad r>0.$$

Now, using Lemma 8.1 with $h(t)\equiv 1$, $w(x)=x^{\frac{q}{a}-q}u^*(x)$ and $z(x)=x^{\frac{2p}{a}-2}v_*(1/x)$ gives the following result.

**Theorem 3.1′.** *Let* $1<p\le q<\infty$, $1<a\le 2$, $(p,q,a)\ne(2,2,2)$. *Let also* $u$ *and* $v$ *be the weights such that* $v_*\in B_p$ *and, for all* $t$,

(i) *for* $a<p$,

$$\frac{\left(\int_0^t \xi^{q/a}u^*(\xi)\,d\xi\right)^{1/q}}{\left(\int_0^t \xi^{2p/a-2}v_*(1/\xi)\,d\xi\right)^{1/p}}\lesssim 1 \tag{8.32}$$

*and*

$$\left(\int_0^t\left[\tau^{-1}\int_0^\tau \xi^{\frac{2p}{a}-2}v_*(1/\xi)\,d\xi\right]^{\frac{p}{a-p}}\tau^{\frac{2p}{a}-2}v_*(1/\tau)\,d\tau\right)^{\frac{p-a}{p}}\left(\int_t^\infty \xi^{\frac{q}{a}-q}u^*(\xi)\,d\xi\right)^{\frac{a}{q}}\lesssim 1; \tag{8.33}$$

(ii) *for* $a\ge p$,

$$\left(\int_0^t x^{2p/a-2}v_*(1/x)\,dx\right)^{\frac{a-p}{p}}\left(\int_0^\infty \min(x,t)^{q/a}x^{-q/a'}u^*(x)\,dx\right)^{a/q}\lesssim 1. \tag{8.34}$$

*Then*

$$\|w^{1/a'}Ff\|_{q,u}\lesssim \|s^{1/a}f\|_{p,v}$$

*holds.*

Let us figure out what these conditions yield for the power weights. Taking $u(x)=x^{q/a-q-q\gamma+(1/q-1/a')q\nu}$ and $v(x)=x^{p\beta-(1/p'-1/a')p\nu}$ and checking (8.32), (8.33) and (8.34) for them, in each of them we arrive at the familiar

$$\gamma-\beta=\left(\frac{1}{q}-\frac{1}{p'}\right)(\nu+1),$$

as expected.

There are other delicate tests, but they present much more technical difficulties. For example, one may try to apply [**22**, Th. 1.2] to the left-hand side of (8.30). For some cases (not for the considered power weights, where the results are presumably the same for every method), one may expect less restrictive conditions than (8.32)–(8.34) but verifying them seems to be very difficult.

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