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REVERSE HÖLDER'S INEQUALITY FOR SPHERICAL HARMONICS

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Abstract. This paper determines the sharp asymptotic order of the following reverse Hölder inequality for spherical harmonics Y_n of degree n on the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$ as $n \to \infty$:

$$\|Y_n\|_{L^q(\mathbb{S}^{d-1})} \leq C n^{\alpha(p,q)} \|Y_n\|_{L^p(\mathbb{S}^{d-1})}, \quad 0 < p < q \leq \infty.$$

In many cases, these sharp estimates turn out to be significantly better than the corresponding estimates in the Nilkolskii inequality for spherical polynomials. Furthermore, they allow us to improve two recent results on the restriction conjecture and the sharp Pitt inequalities for the Fourier transform on $\mathbb{R}^d$.

1. Introduction

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ denote the unit sphere of $\mathbb{R}^d$ endowed with the usual Haar measure $d\sigma(x)$, where $\|\cdot\|$ denotes the Euclidean norm of $\mathbb{R}^d$. Given $0 < p \leq \infty$, we denote by $L^p(\mathbb{S}^{d-1})$ the usual Lebesgue L^p-space defined with respect to the measure $d\sigma(x)$ on $\mathbb{S}^{d-1}$, and by $\|\cdot\|_p$ the norm of $L^p(\mathbb{S}^{d-1})$. Throughout the paper, unless otherwise stated, all functions on $\mathbb{S}^{d-1}$ will be assumed to be real-valued and measurable, and the notation $A \sim B$ means that there exists an inessential constant $c > 0$, called the constant of equivalence, such that $c^{-1}A \leq B \leq cA$.

Let Π_n^d denote the space of all spherical polynomials of degree at most n on $\mathbb{S}^{d-1}$ (i.e., restrictions on $\mathbb{S}^{d-1}$ of polynomials in d variables of total degree at most n), and $\mathcal{H}_n^d$ the space of all spherical harmonics of degree n on $\mathbb{S}^{d-1}$. As is well known (see, for instance, [1, chapter 1]), $\mathcal{H}_n^d$ and Π_n^d are all finite dimensional spaces with $\dim\mathcal{H}_n^d \sim n^{d-2}$ and $\dim\Pi_n^d \sim n^{d-1}$ as $n \to \infty$. Furthermore, the spaces $\mathcal{H}_k^d$, $k = 0, 1, \cdots$ are mutually orthogonal with respect to the inner product of $L^2(\mathbb{S}^{d-1})$, and each space Π_n^d can be written as a direct sum $\Pi_n^d = \sum_{j=0}^n \mathcal{H}_j^d$. Since the space of spherical polynomials is dense in $L^2(\mathbb{S}^{d-1})$, each $f \in L^2(\mathbb{S}^{d-1})$ has a spherical harmonic expansion, $f = \sum_{k=0}^\infty \operatorname{proj}_k f$, where proj_k is the orthogonal projection of $L^2(\mathbb{S}^{d-1})$ onto the space $\mathcal{H}_k^d$ of spherical harmonics. The orthogonal projection proj_k has an integral representation:

$$\operatorname{proj}_k f(x) = C_{k,d} \int_{\mathbb{S}^{d-1}} f(y) P_k^{(\frac{d-3}{2}, \frac{d-3}{2})}(x \cdot y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{1.1}$$

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where

$$C_{k,d} := \frac{\Gamma(\frac{d}{2})\Gamma(\frac{d-1}{2})}{2\pi^{d/2}\Gamma(d-1)} \frac{(2k+d-2)\Gamma(k+d-2)}{\Gamma(k+\frac{d-1}{2})},$$

and $P_k^{(\alpha,\beta)}$ denotes the usual Jacobi polynomial of degree k and indices α, β, as defined in [10, Chapter IV].

Our goal in this paper is to find a sharp asymptotic order of the quantity $\sup_{Y_n\in\mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_p}$ for $0 < p < q \le \infty$ as $n \to \infty$. The background of this problem is as follows. In 1986, Sogge [7] proved that for $d \ge 3$, and $\lambda := \frac{d-2}{2}$,

$$\sup_{Y_n\in\mathcal{H}_n^d} \frac{\|Y_n\|_{L^q(\mathbb{S}^{d-1})}}{\|Y_n\|_{L^2(\mathbb{S}^{d-1})}} \sim \begin{cases} n^{\lambda(\frac{1}{2}-\frac{1}{q})}, & 2 \le q \le 2(1+\frac{1}{\lambda}), \\ n^{2\lambda(\frac{1}{2}-\frac{1}{q})-\frac{1}{q}}, & 2(1+\frac{1}{\lambda}) \le q \le \infty, \end{cases} \tag{1.2}$$

which confirms a conjecture of Stanton–Weinstein [8] in the case of $d = 3$ and $q = 4$. Here and throughout the paper, it is agreed that $0/0 = 0$. Recently, De Carli and Grafakos [4] proved that if $1 \le p \le q \le 2$ and $Y_n \in \mathcal{H}_n^d$ can be written in the form

$$Y_n(x) = e^{im_{d-2}x_{d-1}} \prod_{k=0}^{d-2} (\sin x_{k+1})^{m_{k+1}} P_{m_k - m_{k+1}}^{(m_{k+1}+\frac{d-2-k}{2}, m_{k+1}+\frac{d-2-k}{2})}(\cos x_{k+1}), \tag{1.3}$$

with $n = m_0 \ge m_1 \ge \cdots m_{d-2} \ge 0$ being integers, then

$$\frac{\|Y_n\|_{L^q(\mathbb{S}^{d-1})}}{\|Y_n\|_{L^p(\mathbb{S}^{d-1})}} \leqslant C n^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{q})}, \quad 1 \le p < q \le 2, \tag{1.4}$$

which was further applied in [4] to prove the restriction conjecture for the class of functions consisting of products of radial functions and spherical harmonics that are in the form (1.3). Note that the set of functions Y_n in (1.3) with $n = m_0 \ge m_1 \ge \cdots m_{d-2} \ge 0$ forms a linear basis of the space $\mathcal{H}_n^d$. It is therefore natural to ask whether or not (1.4) holds for all spherical harmonics Y_n of degree n. A related work in this direction was done recently by De Carli, Gorbachev and Tikhonov in [3], where the following weaker estimate was obtained for all spherical harmonics and applied to study a sharp Pitt inequality for the Fourier transform on $\mathbb{R}^d$:

$$\sup_{Y_n\in\mathcal{H}_n^d} \frac{\|Y_n\|_{p'}}{\|Y_n\|_p} \le C n^{(d-1)(\frac{1}{p}-\frac{1}{2})}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 1 \le p \le 2, \tag{1.5}$$

Finally, let us recall the following well-known result of Kamzolov [6] on the Nikolskii inequality for spherical polynomials:

$$\|P_n\|_q \leqslant C n^{(d-1)(\frac{1}{p}-\frac{1}{q})} \|P_n\|_p, \qquad \forall P_n \in \Pi_n^d, \quad 0 < p < q \le \infty. \tag{1.6}$$

Since $\mathcal{H}_n^d \subset \Pi_n^d$, the Nikolskii inequality (1.6) is applicable to every spherical harmonics $Y_n \in \mathcal{H}_n^d$. It turns out, however, that the resulting estimates are not sharp for spherical harmonics in many cases (see, for instance, (1.2), (1.5) and (1.4)).

In this paper, we will prove the following result, which, in particular, shows that (1.4) holds for all spherical harmonics $Y_n \in \mathcal{H}_n^d$, and the upper bound on the right hand side of (1.5) can be improved to be $Cn^{(d-2)(\frac{1}{p}-\frac{1}{2})}$.

Theorem 1.1. *Assume that* $d \geq 3$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$ *if* $p \geq 1$. *Set* $\lambda := \frac{d-2}{2}$.

(i) *If either* $0 < p \leq 1$ *and* $p < q \leq \infty$, *or* $1 \leq p \leq 2$ *and* $p < q \leq (1 + \frac{1}{\lambda})p'$, *then*

$$\sup_{Y_n \in \mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_p} \sim n^{\lambda(\frac{1}{p} - \frac{1}{q})}. \tag{1.7}$$

(ii) *If either* $1 \leq p \leq 2$ *and* $q \geq (1 + \frac{1}{\lambda})p'$, *or* $2 \leq p < 2 + \frac{1}{\lambda}$ *and* $q > 2 + \frac{2}{\lambda}$, *then*

$$\sup_{Y_n \in \mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_p} \sim n^{2\lambda(\frac{1}{2} - \frac{1}{q}) - \frac{1}{q}}.$$

(iii) *If* $2 + \frac{1}{\lambda} < p < q \leq \infty$, *then*

$$\sup_{Y_n \in \mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_p} \sim n^{(2\lambda+1)(\frac{1}{p} - \frac{1}{q})}.$$

(iv) *If* $d = 3$ *and* $2 \leq p < 4 = 2 + \frac{1}{\lambda}$, *then for* $q \geq 3p' = (1 + \frac{1}{\lambda})p'$,

$$\sup_{Y_n \in \mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_p} \sim n^{\frac{1}{2} - \frac{2}{q}},$$

whereas for $p < q \leq 3p'$,

$$\sup_{Y_n \in \mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_p} \sim n^{\frac{1}{2}(\frac{1}{p} - \frac{1}{q})}.$$

Of particular interest is the case when $1 \leq p \leq 2$ and $q = p'$, where our result can be stated as follows:

Corollary 1.2. *If* $Y_n \in \mathcal{H}_n^d$ *and* $1 \leq p \leq 2$, *then*

$$\|Y_n\|_{p'} \leq C n^{(d-2)(\frac{1}{p} - \frac{1}{2})} \|Y_n\|_p, \quad 1 \leq p \leq 2. \tag{1.8}$$

Furthermore, this estimate is sharp.

Several remarks are in order.

Remark 1.1. Estimate (1.8) for $p = p_\lambda := 1 + \frac{\lambda}{\lambda+2}$ follows directly from the well-known result of Sogge [7] on the orthogonal projection $\operatorname{proj}_n : L^2(\mathbb{S}^{d-1}) \to \mathcal{H}_n^d$. However, for $1 \leq p < 2$ and $p \neq p_\lambda$, the sharp estimate (1.8) in Corollary 1.2 is nontrivial and cannot be deduced from the result of Sogge [7]. Indeed, it was shown in [7] that for $1 \leq p \leq p_\lambda := 1 + \frac{\lambda}{\lambda+2}$,

$$\|\operatorname{proj}_n f\|_2 \leq C n^{\lambda(\frac{1}{p} - \frac{1}{2}) + \frac{1}{2p(\lambda+2)}(p_\lambda - p)} \|f\|_p, \quad \forall f \in L^p(\mathbb{S}^{d-1}), \tag{1.9}$$

and this estimate is sharp. Since $\operatorname{proj}_n f = f$ for $f \in \mathcal{H}_n^d$, this leads to the inequality

$$\|Y_n\|_2 \leq C n^{\lambda(\frac{1}{p} - \frac{1}{2}) + \frac{1}{2p(\lambda+2)}(p_\lambda - p)} \|Y_n\|_p, \quad \forall\, Y_n \in \mathcal{H}_n^d, \ \ 1 \leq p \leq p_\lambda,$$

which, according to Corollary 1.2, is not sharp unless $p = p_\lambda$.

Remark 1.2. Interesting reverse Hölder inequalities for spherical harmonics,

$$\sup_{Y_n\in\mathcal{H}_n^d}\frac{\|Y_n\|_q}{\|Y_n\|_p}\leq C(n,q)$$

with the constant $C(n,q)$ being independent of the dimension d but dependent on the degree n of spherical harmonics, were obtained in [5] for some pairs of (p,q), $0<p<q<\infty$. The general constants C in our paper are dependent on the dimension d, but independent of the degree n.

Remark 1.3. For $d\geq 4$, it remains open to find the asymptotic estimate of the supremum on the left hand side of (1.7) for $2<p<1+\frac{1}{\lambda}$ and $p<q<2+\frac{2}{\lambda}$.

This paper is organized as follows. In Section 2, we construct a sequence of convolution operators $\{T_n\}_{n=0}^\infty$ on $L^1(\mathbb{S}^{d-1})$ with the properties that $T_nf=f$ for $f\in\mathcal{H}_n^d$, $|T_nf|\leq C\sup_{0\leq j\leq d}|\operatorname{proj}_{n+2j}f|$ and $\|T_nf\|_\infty\leq Cn^\lambda\|f\|_1$ for all $f\in L^1(\mathbb{S}^{d-1})$. These operators play an indispensable role in the proof of Theorem 1.1, which is given in the third section. Finally, in Section 4, we give two applications of our main result, improving a recent result of [4] on restriction conjecture and a result of [3] on sharp Pitt's inequality.

2. A sequence of convolution operators

We start with the following well-known result of Sogge [7] on the operator norms of the orthogonal projections $\operatorname{proj}_n: L^2(\mathbb{S}^{d-1})\to\mathcal{H}_n^d$.

Lemma 2.1. [7] *Let $n\in\mathbb{N}$, $d\geq 3$ and $\lambda=\frac{d-2}{2}$. Then the following statements hold:*

(i) *If $1\leq p\leq p_\lambda:=1+\frac{\lambda}{\lambda+2}$, then*

$$\|\operatorname{proj}_n f\|_2\leq Cn^{(2\lambda+1)(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}}\|f\|_p.$$

(ii) *If $p_\lambda\leq p\leq 2$, then*

$$\|\operatorname{proj}_n f\|_2\leq Cn^{\lambda(\frac{1}{p}-\frac{1}{2})}\|f\|_p.$$

(iii) *If $2+\frac{2}{\lambda}\leq q\leq\infty$, then*

$$\|\operatorname{proj}_n f\|_q\leq Cn^{(2\lambda+1)(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}}\|f\|_2.$$

(iv) *If $2\leq q\leq 2+\frac{2}{\lambda}$, then*

$$\|\operatorname{proj}_n f\|_q\leq Cn^{\lambda(\frac{1}{2}-\frac{1}{q})}\|f\|_2.$$

Here, the letter C denotes a general positive constant independent of n and f.

As was pointed out in the introduction, Lemma 2.1 will not be enough for the proof of our main result. The crucial step in the proof of Theorem 1.1 is to construct a sequence of linear operators $\{T_n\}_{n=0}^\infty$ with the properties that $T_nf=f$ for $f\in\mathcal{H}_n^d$, $|T_nf|\leq C\sup_{0\leq j\leq d}|\operatorname{proj}_{n+2j}f|$ and $\|T_nf\|_\infty\leq Cn^\lambda\|f\|_1$ for all $f\in L^1(\mathbb{S}^{d-1})$.

To define the operators T_n, we need to recall several notations. First, given $h\in\mathbb{N}$, and a sequence $\{a_n\}_{n=0}^\infty$ of real numbers, define

$$\triangle_h a_n=a_n-a_{n+h},\quad \triangle_h^{\ell+1}=\triangle_h\triangle_h^\ell,\quad \ell=1,2,\dots.$$

Next, let

$$R_n^\lambda(\cos\theta) := \frac{P_n^{(\lambda-\frac12,\lambda-\frac12)}(\cos\theta)}{P_n^{(\lambda-\frac12,\lambda-\frac12)}(1)}, \quad \theta\in[0,\pi]$$

denote the normalized Jacobi polynomial, and for a step $h\in\mathbb{N}$, define

$$\triangle_h^\ell R_n^\lambda(\cos\theta) = \triangle_h^\ell a_n, \quad \ell = 1,2,\dots, \quad n = 0,1,\cdots,$$

with $a_n := R_n^\lambda(\cos\theta)$. Here and throughout, the difference operator in $\triangle_h^\ell R_n^\lambda(\cos\theta)$ is always acting on the integer n. In the case when the step $h = 1$, we have the following estimate ([1, Lemma B.5.1], [2]):

$$\left|\triangle_1^\ell R_n^\lambda(\cos\theta)\right| \le C\theta^\ell(1+n\theta)^{-\lambda}, \quad \theta\in[0,\pi/2], \quad \ell\in\mathbb{N}. \tag{2.1}$$

On the other hand, however, the ℓ-th order difference $\triangle_1^\ell R_n^\lambda(\cos\theta)$ with step $h = 1$ does not provide a desirable upper estimate when θ is close to π, and as will be seen in our later proof, estimate (2.1) itself will not be enough for our purpose.

To overcome this difficulty, instead of the difference with step 1, we consider the ℓ-th order difference $\triangle_2^\ell R_n^\lambda(\cos\theta)$ with step $h = 2$. Since $\triangle_2^\ell a_n = \sum_{j=0}^\ell \binom{\ell}{j}\triangle_1^\ell a_{n+j}$, on one hand, (2.1) implies that

$$\left|\triangle_2^\ell R_n^\lambda(\cos\theta)\right| \le C\theta^\ell(1+n\theta)^{-\lambda}, \quad \theta\in[0,\pi/2].$$

On the other hand, however, since

$$\triangle_2^\ell R_n^\lambda(\cos\theta) = \sum_{j=0}^\ell (-1)^j\binom{\ell}{j} R_{n+2j}^\lambda(\cos\theta),$$

and since $R_{n+2j}^\lambda(-z) = (-1)^n R_{n+2j}^\lambda(z)$, we have $\triangle_2^\ell R_n^\lambda(\cos(\pi-\theta)) = (-1)^n\triangle_2^\ell R_n^\lambda(\cos\theta)$. It follows that

$$\left|\triangle_2^\ell R_n^\lambda(\cos\theta)\right| \le C\begin{cases}\theta^\ell(1+n\theta)^{-\lambda}, & \theta\in[0,\pi/2],\\ (\pi-\theta)^\ell(1+n(\pi-\theta))^{-\lambda}, & \theta\in[\pi/2,\pi].\end{cases} \tag{2.2}$$

By (1.1), we obtain that for every $P\in\mathcal{H}_n^d$,

$$P(x) = c_n\int_{\mathbb{S}^{d-1}} P(y)R_n^\lambda(x\cdot y)\,d\sigma(y), \quad x\in\mathbb{S}^{d-1},$$

where

$$c_n := \frac{\Gamma(\frac d2)}{2\pi^{d/2}}\frac{d+2n-2}{d+n-2}\frac{\Gamma(d+n-1)}{\Gamma(n+1)\Gamma(d-1)} \sim n^{d-2},$$

and $x\cdot y$ denotes the dot product of $x,y\in\mathbb{R}^d$. Since $R_j^\lambda(x\cdot)\in\mathcal{H}_j^d$ for any fixed $x\in\mathbb{S}^{d-1}$, it follows by the orthogonality of spherical harmonics that for any $P\in\mathcal{H}_n^d$, and any $\ell\in\mathbb{N}$,

$$\begin{aligned} P(x) &= c_n\sum_{j=0}^\ell(-1)^j\binom{\ell}{j}\int_{\mathbb{S}^{d-1}} P(y)R_{n+2j}^\lambda(x\cdot y)\,d\sigma(y)\\ &= c_n\int_{\mathbb{S}^{d-1}} P(y)\triangle_2^\ell R_n^\lambda(x\cdot y)\,d\sigma(y). \end{aligned} \tag{2.3}$$

For the rest of the paper, we will choose ℓ to be an integer bigger than λ (for instance, we may set $\ell = d-2$), so that by (2.2), we have

$$\Big|\triangle_2^\ell R_n^\lambda(\cos\theta)\Big| \le Cn^{-\lambda}. \tag{2.4}$$

Now we are in a position to define the operators T_n.

Definition 2.2. *For $f \in L(\mathbb{S}^{d-1})$, we define*

$$T_n f(x) := \int_{\mathbb{S}^{d-1}} f(y)\Phi_n(x\cdot y)\,d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{2.5}$$

where

$$\Phi_n(\cos\theta) := c_n \sum_{j=0}^{d-2} (-1)^j \binom{d-2}{j} R_{n+2j}^\lambda(\cos\theta).$$

By (2.4), we have

$$|\Phi_n(\cos\theta)| \le Cn^\lambda, \quad \theta \in [0,\pi], \tag{2.6}$$

whereas by (2.3)

$$T_n P(x) = P(x), \quad \forall P \in \mathcal{H}_n^d, \quad \forall x \in \mathbb{S}^{d-1}. \tag{2.7}$$

The main result of this section can now be stated as follows.

Theorem 2.3. (i) *If $1 \le p \le 2$ and $p' \le q \le (1+\frac{1}{\lambda})p'$, then*

$$\|T_n f\|_q \le Cn^{\lambda(\frac{1}{p}-\frac{1}{q})}\|f\|_p, \quad \forall f \in L^p(\mathbb{S}^{d-1}). \tag{2.8}$$

(ii) *If $1 \le p \le 2$ and $q \ge (1+\frac{1}{\lambda})p'$, then*

$$\|T_n f\|_q \le Cn^{\lambda - \frac{2\lambda+1}{q}}\|f\|_p, \quad \forall f \in L^p(\mathbb{S}^{d-1}).$$

Proof. First, we prove the assertion (i). Note that by definition, for each $f \in L^2(\mathbb{S}^{d-1})$,

$$T_n f = \sum_{j=0}^{d-2} (-1)^j \binom{d-2}{j} \frac{c_n}{c_{n+2j}} \operatorname{proj}_{n+2j} f, \tag{2.9}$$

which implies that

$$\|T_n f\|_2 \le C\|f\|_2, \quad \forall f \in L^2(\mathbb{S}^{d-1}). \tag{2.10}$$

On the other hand, however, using (2.6), we have

$$\|T_n f\|_\infty \le Cn^\lambda \|f\|_1, \quad \forall f \in L^1(\mathbb{S}^{d-1}). \tag{2.11}$$

Thus, applying the Riesz-Thorin interpolation theorem, and using (2.10) and (2.11), we deduce that for $1 \le p \le 2$,

$$\|T_n f\|_{p'} \le Cn^{(d-2)(\frac{1}{p}-\frac{1}{2})}\|f\|_p, \quad \forall f \in L^p(\mathbb{S}^{d-1}). \tag{2.12}$$

Next, by (iv) of Lemma 2.1, and using (2.9), we obtain that for $2 \le r \le 2(1+\frac{1}{\lambda})$,

$$\|T_n f\|_r \le Cn^{\lambda(\frac{1}{2}-\frac{1}{r})}\|f\|_2, \quad \forall f \in L^2(\mathbb{S}^{d-1}). \tag{2.13}$$

Assume that $1 \le p \le 2$ and $p' \le q \le (1+\frac{1}{\lambda})p'$. Let $\theta = \frac{2}{p'} \in [0,1]$, and let $r = \theta q = \frac{2}{p'}q$. Then $2 \le r \le 2(1+\frac{1}{\lambda})$, and

$$\frac{1}{p} = 1-\theta+\frac{\theta}{2}, \quad \frac{1}{q} = \frac{1-\theta}{\infty}+\frac{\theta}{r}.$$

Thus, by (2.12), (2.13) and applying the Riesz-Thorin interpolation theorem, we obtain that

$$\|T_n f\|_q \le C n^{\lambda(1-\theta)} n^{\lambda(\frac{1}{2}-\frac{1}{r})\theta} \|f\|_p = C n^{\lambda(\frac{1}{p}-\frac{1}{q})}\|f\|_p.$$

This completes the proof of the assertion (i).

Assertion (ii) can be proved similarly. Indeed, using (2.9) and (iii) of Lemma 2.1, we have that for $r \ge 2(1+\frac{1}{\lambda})$,

$$\|T_n f\|_r \le C n^{2\lambda(\frac{1}{2}-\frac{1}{r})-\frac{1}{r}}\|f\|_2, \quad \forall f \in L^2(\mathbb{S}^{d-1}). \tag{2.14}$$

Assume that $1 \le p \le 2$ and $q \ge (1+\frac{1}{\lambda})p'$. Let $\theta = \frac{2}{p'}$ and $r = \theta q = \frac{2}{p'}q$. Then $r \ge 2(1+\frac{1}{\lambda})$. Using (2.14), (2.12) and applying the Riesz-Thorin interpolation theorem, we deduce that

$$\begin{aligned}\|T_n f\|_q &\le C n^{\lambda(1-\theta)} n^{(d-2)\theta(\frac{1}{2}-\frac{1}{r})-\frac{\theta}{r}}\|f\|_p = C n^{\lambda-\frac{2\lambda+1}{q}}\|f\|_p \\ &= C n^{(d-2)(\frac{1}{2}-\frac{1}{q})-\frac{1}{q}}\|f\|_p.\end{aligned}$$

This completes the proof of (ii).

□

3. Proof of Theorem 1.1

The stated lower estimates of Theorem 1.1 follow directly from the following two known lemmas.

Lemma 3.1. [7] *Let*

$$f_n(x) = (x_1 + i x_2)^n, \quad x \in \mathbb{S}^{d-1}.$$

Then $f \in \mathcal{H}_n^d$ *and*

$$\|f_n\|_p \sim n^{-\lambda/p}, \quad 0 < p < \infty.$$

Lemma 3.2. [10, p.391] *Let*

$$g_n(x) = P_n^{(\frac{d-3}{2},\frac{d-3}{2})}(x \cdot e)$$

for a fixed point $e \in \mathbb{S}^{d-1}$. *Then* $g_n \in \mathcal{H}_n^d$, *and*

$$\|g_n\|_p \sim \begin{cases} n^{\frac{d-3}{2}} n^{-\frac{d-1}{p}}, & p > \frac{2(d-1)}{d-2}, \\ n^{-\frac{1}{2}}(\log n)^{\frac{1}{p}}, & p = \frac{2(d-1)}{d-2}, \\ n^{-\frac{1}{2}}, & p < \frac{2(d-1)}{d-2}. \end{cases}$$

For the proof of the upper estimates, we let $P \in \mathcal{H}_n^d$. The crucial tool in our proof is Theorem 2.3, where we recall that $T_n P = P$ for all $P \in \mathcal{H}_n^d$. We consider the following cases:

Case 1. $1 \le p \le q \le p'$.

In this case, $1 \le p \le 2 \le p'$, and the stated upper estimate for $q = p'$ follows directly from Theorem 2.3. In general, for $p \le q \le p'$, let $\theta \in [0,1]$ be such that $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p'}$. Then by the log-convexity of the L^p-norm, we have

$$\|P\|_q \le \|P\|_p^\theta \|P\|_{p'}^{1-\theta} \le Cn^{\lambda(\frac{1}{p}-\frac{1}{p'})(1-\theta)}\|P\|_p \le Cn^{\lambda(\frac{1}{p}-\frac{1}{q})}\|P\|_p,$$

which is as desired in this case.

Case 2. $0 < p \le 1$ and $p < q$.

In this case, note that

$$\|P\|_1 \le \|P\|_p^p \|P\|_\infty^{1-p} \le Cn^{\lambda(1-p)}\|P\|_p^p \|P\|_1^{1-p}.$$

It follows that

$$\|P\|_1 \le Cn^{\lambda(\frac{1}{p}-1)}\|P\|_p, \quad 0 < p \le 1,$$

which, in turn, implies that for $p < q$ and $\frac{1}{q} = \frac{1-\theta}{p}$,

$$\|P\|_q \le \|P\|_\infty^\theta \|P\|_p^{1-\theta} \le Cn^{\lambda\theta}\|P\|_1^\theta \|P\|_p^{1-\theta} \le Cn^{\lambda(\frac{1}{p}-\frac{1}{q})}\|P\|_p.$$

Case 3. $1 \le p \le 2$ and $q \ge p'$.

The desired estimate in this case follows directly from the first and the second parts of Theorem 2.3 since $T_n P = P$ for all $P \in \mathcal{H}_n^d$.

Case 4. $2 \le p \le 2 + \frac{1}{\lambda}$ and $q \ge 2 + \frac{2}{\lambda}$.

For $P \in \mathcal{H}_n^d$, by the already proven cases it follows that

$$\|P\|_q \le Cn^{(d-2)(\frac{1}{2}-\frac{1}{q})-\frac{1}{q}}\|P\|_2 \le Cn^{(d-2)(\frac{1}{2}-\frac{1}{q})-\frac{1}{q}}\|P\|_p.$$

Case 5. $2 + \frac{1}{\lambda} < p < q \le \infty$.

The reverse Hölder inequality in this case follows directly from the corresponding Nikolskii inequality for spherical polynomials given by (1.6).

Case 6. $d = 3$ and $2 \le p < 4 = 2 + \frac{1}{\lambda}$.

The proof in this case relies on the following result of Sogge [7]:

Lemma 3.3. *If* $d = 3$, $\frac{4}{3} < p < 4$ *and* $q = 3p' = p'(1 + \frac{1}{\lambda})$, *then*

$$\|\operatorname{proj}_n f\|_q \le Cn^{\frac{1}{2}-\frac{2}{q}}\|f\|_p.$$

Now we return to the proof in Case 6. Again, in view of Lemmas 3.1 and 3.2, it is enough to prove the upper estimates. Assume first that $q \geq 3p'$. Let $2 \leq p < p_1 < 4$ and let $\theta \in [0,1]$ be such that

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{2}.$$

Set $q_1 = 3p_1'$. Then by Lemma 3.3,

$$\|Tf\|_{q_1} \leq Cn^{\frac{1}{2}-\frac{2}{q_1}}\|f\|_{p_1}. \tag{3.1}$$

For $q \geq 3p' > 3p_1' = q_1$, let $q_2 \geq q$ be such that

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

Then

$$\frac{1}{3} \geq \frac{1}{3p} + \frac{1}{q} = \theta(\frac{1}{6} + \frac{1}{q_2}) + \frac{1}{3}(1-\theta) = \theta(\frac{1}{q_2} - \frac{1}{6}) + \frac{1}{3}.$$

This implies that $q_2 \geq 6 = 2 + \frac{2}{\lambda}$, hence by (ii) of Theorem 2.3,

$$\|T_nf\|_{q_2} \leq Cn^{\frac{1}{2}-\frac{2}{q_2}}\|f\|_2. \tag{3.2}$$

Thus, using (3.1), (3.2), and the Riesz-Thorin theorem, we obtain

$$\|T_nf\|_q \leq Cn^{\frac{1}{2}-\frac{2}{q}}\|f\|_p,$$

which implies the desired estimate for the case of $q \geq 3p'$.

The case of $p < q < 3p'$ can be treated similarly. In fact, let p_1, q_1 and θ be as above. Observing that $\frac{1}{2} - \frac{2}{q_1} = \frac{1}{2}(\frac{1}{p_1} - \frac{1}{q_1})$, we may rewrite (3.1) as

$$\|Tf\|_{q_1} \leq Cn^{\frac{1}{2}(\frac{1}{p_1}-\frac{1}{q_1})}\|f\|_{p_1}.$$

Furthermore, we may choose $p_1 > p$ to be very close to p so that $q < q_1 = 3p_1' < 3p'$. Let $q_3 \leq q$ be such that

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_3}.$$

Then

$$\frac{1}{3} < \frac{1}{3p} + \frac{1}{q} = \theta(\frac{1}{6} + \frac{1}{q_3}) + \frac{1}{3}(1-\theta) = \theta(\frac{1}{q_3} - \frac{1}{6}) + \frac{1}{3}.$$

Hence $2 < q_3 < 6$, and using (i) of Theorem 2.3, we deduce

$$\|T_nf\|_{q_3} \leq Cn^{\frac{1}{2}(\frac{1}{2}-\frac{1}{q_3})}\|f\|_2.$$

The stated estimate for $p < q < 3p'$ then follows by the Riesz-Thorin interpolation theorem. □

4. Applications: Fourier inequalities

4.1. **The restriction conjecture.** One of the most challenging problems in classical Fourier analysis is the restriction conjecture, which states that if $1 \leq p < \frac{2d}{d+1}$ and $q \leq \frac{d-1}{d+1}p'$, then there exists a constant C depending only on p, q, d such that

$$\frac{\|\widehat{F}\|_{L^q(\mathbb{S}^{d-1})}}{\|F\|_{L^p(\mathbb{R}^d)}} \leq C, \quad \forall F \in C_0^\infty(\mathbb{R}^d), \tag{4.1}$$

where $\hat{F}(\xi) := \int_{\mathbb{R}^d} F(x) e^{-2\pi i x\cdot\xi}\, dx$, $\xi \in \mathbb{R}^d$. This conjecture has been completely proved only in the case of $d = 2$. We refer to the book [9, Chapter IX] for more background information of this problem.

De Carli and Grafakos [4] recently proved that the restriction conjecture is valid for all functions F that can be expressed in the form

$$F(x) = f(\|x\|)\|x\|^n g_n\big(\frac{x}{\|x\|}\big), \quad n = 0, 1, \cdots$$

with $f(\|\cdot\|) \in C_0^\infty(\mathbb{R}^d)$ and $g_n \in \mathcal{H}_n^d$ being given in (1.3) . Using Theorem 1.1 (i), and following the argument of [4], we may conclude here that the restriction conjecture holds for a wider class of functions

$$F \in \bigcup_{n=0}^\infty \Big\{ f(\|x\|)\|x\|^n Y_n\big(\frac{x}{\|x\|}\big) : \quad f(\|\cdot\|) \in C_0^\infty(\mathbb{R}^d), \quad Y_n \in \mathcal{H}_n^d \Big\}.$$

Indeed, it was shown in [4] that for $F(x) = f(\|x\|)\|x\|^n Y_n(x/\|x\|)$ with $f \in C_0^\infty(\mathbb{R}^d)$ and $Y_n \in \mathcal{H}_n^d$,

$$\begin{aligned} \frac{\|\widehat{F}\|_{L^q(\mathbb{S}^{d-1})}}{\|F\|_{L^p(\mathbb{R}^d)}} &= \frac{\Big| \int_0^\infty f(r) J_{\frac{d}{2}-1+n}(r) r^{\frac{d}{2}+n} dr \Big|}{\Big(\int_0^\infty |f(r)|^p r^{d-1+np} dr \Big)^{1/p}} \frac{\|Y_n\|_{L^q(\mathbb{S}^{d-1})}}{\|Y_n\|_{L^p(\mathbb{S}^{d-1})}} \\ &\leq C n^{(d-1)(\frac{1}{2}-\frac{1}{p})+\frac{1}{p'}} \frac{\|Y_n\|_{L^q(\mathbb{S}^{d-1})}}{\|Y_n\|_{L^p(\mathbb{S}^{d-1})}}, \end{aligned} \tag{4.2}$$

where $J_n(r)$ is the Bessel function of the first kind. However, according to (i) of Theorem 1.1 , we obtain that for $1 \leq p < \frac{2d}{d+1}$ and $q \leq \frac{d-1}{d+1}p'$,

$$\text{RHS of } (4.2) \leq C \sup_{m\geq 1} m^{(d-1)(\frac{1}{2}-\frac{1}{p})+\frac{1}{p'}+\frac{d-2}{2}(\frac{1}{p}-\frac{1}{q})} \leq C.$$

4.2. **The sharp Pitt inequality.** The following sharp Pitt inequality has been recently proved in [3]:

Theorem 4.1. *If $1 \leq p \leq 2$ and $s = (d-1)\left(\frac{1}{2}-\frac{1}{p}\right)$, then for every $Y_k \in \mathcal{H}_k^d$ and every radial $f \in \mathcal{S}(\mathbb{R}^d)$, the Pitt inequality*

$$\|\, |y|^{-s} \widehat{fY_k}\|_{L^{p'}(\mathbb{R}^d)} \leq C \|\, |x|^s f Y_k\|_{L^p(\mathbb{R}^d)} \tag{4.3}$$

holds with the best constant

$$C = (2\pi)^{\frac{d}{2}} 2^{\frac{1}{2}-\frac{1}{p'}} \frac{p^{\frac{(2k+d-1)p+2}{4p}} \Gamma\left(\frac{(2k+d-1)p'+2}{4}\right)^{\frac{1}{p'}}}{(p')^{\frac{(2k+d-1)p'+2}{4p'}} \Gamma\left(\frac{(2k+d-1)p+2}{4}\right)^{\frac{1}{p}}} \sup_{Y_k\in\mathcal{H}_k^d} \frac{\|Y_k\|_{L^{p'}(\mathbb{S}^{n-1})}}{\|Y_k\|_{L^{p}(\mathbb{S}^{n-1})}}. \tag{4.4}$$

According to Theorem 1.1, we have

$$\sup_{Y_k\in\mathcal{H}_k^d} \frac{\|Y_k\|_{L^{p'}(\mathbb{S}^{n-1})}}{\|Y_k\|_{L^{p}(\mathbb{S}^{n-1})}} \sim k^{(d-2)(\frac{1}{p}-\frac{1}{2})},$$

whereas only the weaker estimate (1.5) was obtained in [3].

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