



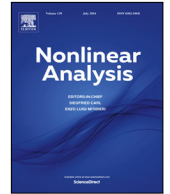
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# A quasiperiodically forced skew-product on the cylinder without fixed-curves



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## ABSTRACT

In Fabbri et al. (2005) the Sharkovskii Theorem was extended to periodic orbits of strips of quasiperiodic skew products in the cylinder.

In this paper we deal with the following natural question that arises in this setting: *Does Sharkovskii Theorem hold when restricted to curves instead of general strips?*

We answer this question in the negative by constructing a counterexample: We construct a map having a periodic orbit of period 2 of curves (which is, in fact, the upper and lower circles of the cylinder) and without any invariant curve.

In particular this shows that there exist quasiperiodic skew products in the cylinder without invariant curves.

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## 1. Introduction

We consider the coexistence and implications between periodic objects of maps on the cylinder  $\Omega = \mathbb{S}^1 \times \mathbb{I}$ , of the form:

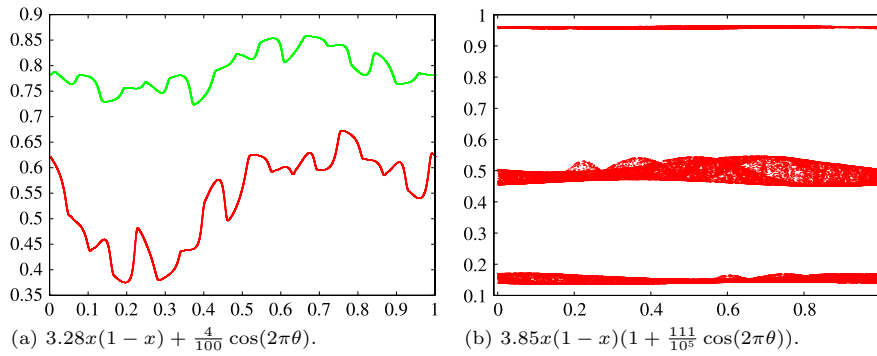
$$T: \begin{pmatrix} \theta \\ x \end{pmatrix} \longrightarrow \begin{pmatrix} R_\omega(\theta) \\ \zeta(\theta, x) \end{pmatrix},$$

where  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ ,  $\mathbb{I}$  is a compact interval of the real line,  $R_\omega(\theta) = \theta + \omega \pmod{1}$  with  $\omega \in \mathbb{R} \setminus \mathbb{Q}$  and  $\zeta(\theta, x) = \zeta_\theta(x)$  is continuous on both variables. The class of all maps of the above type will be denoted by  $\mathcal{S}(\Omega)$ .

In this setting a very basic and natural question is the following: *is it true that any map in the class  $\mathcal{S}(\Omega)$  has an invariant curve?*

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**Fig. 1.** In the left picture we show an example of two periodic orbit curves, and in the second we show a possible example of a three periodic orbit solid strips.

In [1], the authors created an appropriate topological framework that allowed them to obtain the following extension of the Sharkovskii Theorem to the class  $\mathcal{S}(\Omega)$ .<sup>1</sup>

Let  $X$  be a compact metric space. We recall that a subset  $G \subset X$  is *residual* if it contains the intersection of a countable family of open dense subsets in  $X$ .

In what follows,  $\pi: \Omega \rightarrow \mathbb{S}^1$  will denote the standard projection from  $\Omega$  to the circle. Given a set  $B \subset \mathbb{S}^1$ , for convenience we will use the following notation:

$$\uparrow\uparrow B := \pi^{-1}(B) = B \times \mathbb{I} \subset \Omega.$$

In the particular case when  $B = \{\theta\}$ , instead of  $\uparrow\uparrow\{\theta\}$  we will simply write  $\uparrow\uparrow\theta$ . Also, given  $A \subset \Omega$ , we will denote by  $A^{\uparrow\uparrow B}$  the set

$$A \cap \uparrow\uparrow B = \{(\theta, x) \in \Omega : \theta \in B \text{ and } (\theta, x) \in A\}.$$

In the particular case when  $B = \{\theta\}$ , instead of  $A^{\uparrow\uparrow\theta}$  we will simply write  $A^\theta$ .

Instead of periodic points we use objects that project over the whole  $\mathbb{S}^1$ , called *strips* in [1, Definition 3.9]. A set  $B \subset \Omega$  such that  $\pi(B) = \mathbb{S}^1$  (i.e.,  $B$  projects on the whole  $\mathbb{S}^1$ ) will be called a *circular set*.

**Definition 1.1.** A *strip* in  $\Omega$  is a compact circular set  $B \subset \Omega$  such that  $B^\theta$  is a closed interval (perhaps degenerate to a point) for every  $\theta$  in a residual set of  $\mathbb{S}^1$ . ■

Given two strips  $A$  and  $B$ , we will write  $A < B$  and  $A \leq B$  [1, Definition 3.13] if there exists a residual set  $G \subset \mathbb{S}^1$ , such that for every  $(\theta, x) \in A^{\uparrow\uparrow G}$  and  $(\theta, y) \in B^{\uparrow\uparrow G}$  it follows that  $x < y$  and, respectively,  $x \leq y$ . We say that the strips  $A$  and  $B$  are *ordered* (respectively *weakly ordered*) if either  $A < B$  or  $A > B$  (respectively  $A \leq B$  or  $A \geq B$ ).

**Definition 1.2** ([1, Definition 3.15]). A strip  $B \subset \Omega$  is called *n-periodic* for  $F \in \mathcal{S}(\Omega)$  if  $F^n(B) = B$  and the image sets  $B, F(B), F^2(B), \dots, F^{n-1}(B)$  are pairwise disjoint and pairwise ordered (see Fig. 1 for examples). ■

<sup>1</sup>As already remarked in [1], instead of  $\mathbb{S}^1$  we could take any compact metric space  $\Theta$  that admits a minimal homeomorphism  $R: \Theta \rightarrow \Theta$  such that  $R^\ell$  is minimal for every  $\ell > 1$ . However, for simplicity and clarity we will remain in the class  $\mathcal{S}(\Omega)$ .

To state the main theorem of [1] we need to recall the *Sharkovskii Ordering* [2,3]. The *Sharkovskii Ordering* is a linear ordering of  $\mathbb{N}$  defined as follows:

$$\begin{aligned}
 &3 \text{ Sh} > 5 \text{ Sh} > 7 \text{ Sh} > 9 \text{ Sh} > \cdots \text{ Sh} > \\
 &2 \cdot 3 \text{ Sh} > 2 \cdot 5 \text{ Sh} > 2 \cdot 7 \text{ Sh} > 2 \cdot 9 \text{ Sh} > \cdots \text{ Sh} > \\
 &4 \cdot 3 \text{ Sh} > 4 \cdot 5 \text{ Sh} > 4 \cdot 7 \text{ Sh} > 4 \cdot 9 \text{ Sh} > \cdots \text{ Sh} > \\
 &\quad \vdots \\
 &2^n \cdot 3 \text{ Sh} > 2^n \cdot 5 \text{ Sh} > 2^n \cdot 7 \text{ Sh} > 2^n \cdot 9 \text{ Sh} > \cdots \text{ Sh} > \\
 &\quad \vdots \\
 &\cdots \text{ Sh} > 2^n \text{ Sh} > \cdots \text{ Sh} > 16 \text{ Sh} > 8 \text{ Sh} > 4 \text{ Sh} > 2 \text{ Sh} > 1.
 \end{aligned}$$

In the ordering  $\text{Sh} \geq$  the least element is 1 and the largest one is 3. The supremum of the set  $\{1, 2, 4, \dots, 2^n, \dots\}$  does not exist.

**Sharkovskii Theorem for maps from  $\mathcal{S}(\Omega)$  ([1]).** Assume that the map  $F \in \mathcal{S}(\Omega)$  has a  $p$ -periodic strip. Then  $F$  has a  $q$ -periodic strip for every  $q <_{\text{Sh}} p$ .

In view of this result, the new following natural question (that is stronger than the previous one) arises: Does Sharkovskii Theorem for maps from  $\mathcal{S}(\Omega)$  hold when restricted to curves? where a curve is defined as the graph of a continuous map from  $\mathbb{S}^1$  to  $\mathbb{I}$ . More precisely, is it true that if  $F$  has a  $q$ -periodic curve and  $p \leq_{\text{Sh}} q$  then does there exist a  $p$ -periodic curve of  $F$ ?

The aim of this paper is to answer both of the above questions in the negative by constructing a counterexample. This is done by the following result which is the main result of the paper.

**Theorem A.** There exists a map  $T \in \mathcal{S}(\Omega)$  with  $f(\theta, \cdot)$  non-increasing for every  $\theta \in \mathbb{S}^1$ , such that  $T$  permutes the upper and lower circles of  $\Omega$  (thus having a periodic orbit of period two of curves), and  $T$  does not have any invariant curve.

The construction will be done in two steps. We fix  $\omega \in \mathbb{R} \setminus \mathbb{Q}$  and we consider the orbit of 0 by the rotation of angle  $\omega$ :

$$O^*(\omega) = \text{Orb}_\omega(0) = \{\ell\omega \pmod{1} : \ell \in \mathbb{Z}\}.$$

First, in Section 3, we construct a strip  $A$  that is the closure of the graph of a continuous function from  $\mathbb{S}^1 \setminus O^*(\omega)$  to  $\mathbb{I}$ . Moreover, the strip  $A$  is limit of strips  $A_n$ , being each of them the closure of the graph of a continuous function

$$\gamma_n : \mathbb{S}^1 \setminus \{\ell\omega \pmod{1} : \ell = -n, -n + 1, \dots, -1, 0, 1, \dots, n - 1, n\} \longrightarrow \mathbb{I}.$$

The map  $\gamma_n$  is obtained by perturbing the graph of the map  $\gamma_{n-1}$  in appropriate neighbourhoods (in  $\Omega$ ) of the points

$$(n\omega \pmod{1}, \gamma_{n-1}(n\omega \pmod{1})) \text{ and } (-n\omega \pmod{1}, \gamma_{n-1}(-n\omega \pmod{1})).$$

These neighbourhoods are defined to be *winged boxes*

$$\mathcal{R}^\sim(n\omega \pmod{1}), \mathcal{R}^\sim(-n\omega \pmod{1}) \subset \Omega.$$

The perturbation inserted in  $\mathcal{R}^\sim(n\omega \pmod{1})$  and  $\mathcal{R}^\sim(-n\omega \pmod{1})$  to obtain the map  $\gamma_n$  is a convenient rescaling of the function  $\sin(\frac{1}{x})$ . The set  $A$  obtained in this way can be described by putting a rescaled infinitely oscillating sine curve in a neighbourhood of every point  $\ell\omega \pmod{1}$  with  $\ell \in \mathbb{Z}$ .

Second, we construct a Cauchy sequence  $\{T_m\}_{m=0}^\infty$  that gives as a limit the function  $T$  from Theorem A having  $A$  as invariant set. To this end, in Section 4 we define a collection of auxiliary functions  $G_i$  defined

on the winged boxes  $\mathcal{R}^\sim(i^*)$ . Next, in Section 5 we introduce a notion of *depth* in the set of winged boxes  $\mathcal{R}^\sim(i^*)$  which defines a convenient stratification in the set of winged boxes  $\mathcal{R}^\sim(i^*)$ . In Section 6 we study the wings of box and its interaction with boxes of higher depth. In Section 7, by using the auxiliary functions from Section 4, the stratification from Section 5 and the technical results from Section 6 we construct the Cauchy sequence  $\{T_m\}_{m=0}^\infty \subset \mathcal{S}(\Omega)$ , we define the map  $T = \lim_{m \rightarrow \infty} T_m$ , and we prove [Theorem A](#).

For clarity, we omit the proofs of all results from Section 7. These proofs will be provided in the [Appendix \(Appendices A.2–A.4\)](#). Section 2 is devoted to introduce the necessary definitions and, in particular, to introduce the notion of pseudo-curve and some necessary results on the space of pseudo-curves.

## 2. Definitions and preliminary results

The main aim of this section is to introduce the definition and basic results about pseudo-curves.

Given  $G \subset \mathbb{S}^1$  and a map  $\varphi: G \rightarrow \mathbb{I}$ ,  $\text{Graph}(\varphi)$  denotes the *graph of  $\varphi$* . Also, given a set  $A$  we will denote the closure of  $A$  by  $\overline{A}$ .

**Definition 2.1** (*Pseudo-Curve*). Let  $G$  be a residual set of  $\mathbb{S}^1$  and let  $\varphi: G \rightarrow \mathbb{I}$  be a continuous map from  $G$  to  $\mathbb{I}$ . The set  $\overline{\text{Graph}(\varphi)}$ , denoted by  $A_{(\varphi, G)}$ , will be called a *pseudo-curve*. Notice that every pseudo-curve is a compact circular set.

Also,  $\mathcal{A}$  will denote the class of all pseudo-curves. ■

A set  $A \subset \Omega$  is *F-invariant* (respectively *strongly F-invariant*) if  $F(A) \subset A$  (respectively  $F(A) = A$ ). Observe that if  $F \in \mathcal{S}(\Omega)$ , every compact *F-invariant* set is circular. A closed invariant set is called *minimal* if it does not contain any proper closed invariant set.

An *arc of a curve* is the graph of a continuous function from an arc of  $\mathbb{S}^1$  to  $\mathbb{I}$ .

The pseudo-curves have the following properties which are easy to prove:

**Lemma 2.2.** *Given a pseudo-curve  $A_{(\varphi, G)} \in \mathcal{A}$  the following statements hold.*

(a)  $A_{(\varphi, G)}^\theta$  consists of a single point for every  $\theta \in G$ . Consequently,

$$A_{(\varphi, G)}^{\uparrow G} = \text{Graph}(\varphi).$$

(b) Every circular compact set contained in a pseudo-curve coincides with the pseudo-curve.

(c)  $A_{(\varphi, G)} = \overline{\text{Graph}(\varphi|_{\tilde{G}})}$  for every  $\tilde{G} \subset G$  dense in  $\mathbb{S}^1$ .

(d) If  $A_{(\varphi, G)}$  contains a curve then it is a curve.

**Proof.** We start by proving (a). By the definition of a pseudo-curve we have  $\text{Graph}(\varphi) \subset A_{(\varphi, G)}^{\uparrow G}$ . To prove the other inclusion fix  $\theta \in G$  and  $x \in \mathbb{I}$  such that  $(\theta, x) \in A_{(\varphi, G)}$ . Then, there exists a sequence  $\{(\theta_n, \varphi(\theta_n))\}_{n=1}^\infty \subset \text{Graph}(\varphi)$  such that  $\lim_{n \rightarrow \infty} (\theta_n, \varphi(\theta_n)) = (\theta, x)$ . The continuity of  $\varphi$  in  $G$  (and hence in  $\theta$ ) implies  $x = \varphi(\theta)$  and, therefore,  $(\theta, x) \in \text{Graph}(\varphi)$ .

Now we prove (b). Assume that  $B \subset A_{(\varphi, G)}$  is a circular compact set. From the assumptions and statement (a) we get  $A_{(\varphi, G)}^{\uparrow G} = B^{\uparrow G}$ . Hence,

$$A_{(\varphi, G)} = \overline{\text{Graph}(\varphi)} = \overline{A_{(\varphi, G)}^{\uparrow G}} = \overline{B^{\uparrow G}} \subset B.$$

Now (d) follows directly from (b) and the fact that a curve is compact since it is the graph of a continuous function. Statement (c) also follows from (b) because  $\overline{\text{Graph}(\varphi|_{\tilde{G}})} \subset A_{(\varphi, G)}$  and  $\overline{\text{Graph}(\varphi|_{\tilde{G}})}$  is a circular set (since  $\tilde{G}$  is dense in  $\mathbb{S}^1$ ). □

We also will be interested in the pseudo-curves as a possible invariant object of maps from  $\mathcal{S}(\Omega)$ . The next lemma studies their properties in this case.

**Lemma 2.3.** *Let  $F \in \mathcal{S}(\Omega)$  and assume that  $A_{(\varphi,G)} \in \mathcal{A}$  is an  $F$ -invariant pseudo-curve. Then,*

- (a)  $A_{(\varphi,G)}$  is strongly  $F$ -invariant and minimal.
- (b) If  $A_{(\varphi,G)}$  contains an arc of a curve then it is a curve.

**Proof.** We start by proving (a). Let  $B \subset A_{(\varphi,G)}$  be a closed invariant set. We have that  $B$  is circular and, by Lemma 2.2(b),  $B = A_{(\varphi,G)}$ . Hence,  $A_{(\varphi,G)}$  is minimal.

On the other hand,  $F(A_{(\varphi,G)}) \subset A_{(\varphi,G)}$  implies  $F^2(A_{(\varphi,G)}) \subset F(A_{(\varphi,G)})$  and, hence,  $F(A_{(\varphi,G)})$  is a compact  $F$ -invariant set. Therefore, by the part already proven,  $F(A_{(\varphi,G)}) = A_{(\varphi,G)}$ .

Now we prove (b). Let  $S$  be an (open) arc of  $\mathbb{S}^1$  and let  $\xi: S \rightarrow \mathbb{I}$  be a continuous map such that  $\text{Graph}(\xi) \subset A_{(\varphi,G)}$ . Clearly, there exists  $m \in \mathbb{N}$  such that  $\bigcup_{i=0}^m R_\omega^i(S) = \mathbb{S}^1$ . Now we set  $\xi_0 := \xi$  and, for  $i = 1, 2, \dots, m$ , we define  $\xi_i: R_\omega^i(S) \rightarrow \mathbb{I}$  by

$$\xi_i(\theta) := f(R_\omega^{-1}(\theta), \xi_{i-1}(R_\omega^{-1}(\theta))).$$

The continuity of  $f$  implies that every  $\xi_i$  is an arc of a curve and  $\text{Graph}(\xi_i) = F(\text{Graph}(\xi_{i-1}))$ . Hence,

$$\bigcup_{i=0}^m \text{Graph}(\xi_i) = \bigcup_{i=0}^m F^i(\text{Graph}(\xi)) \subset A_{(\varphi,G)}$$

because  $A_{(\varphi,G)}$  is  $F$ -invariant.

In view of Lemma 2.2(d) we only have to show that  $\bigcup_{i=0}^m \text{Graph}(\xi_i)$  is a curve. We will prove this by induction.

Assume that  $\emptyset \neq M \subsetneq \{0, 1, 2, \dots, m\}$  verifies that  $S_M := \bigcup_{i \in M} R_\omega^i(S)$  is an (open) arc of  $\mathbb{S}^1$  and  $\bigcup_{i \in M} \text{Graph}(\xi_i)$  is an arc of a curve (initially we can take  $M$  to be any unitary subset of  $\{0, 1, 2, \dots, m\}$ ). Then, there exists a continuous map  $\xi_M: S_M \rightarrow \mathbb{I}$  such that  $\text{Graph}(\xi_M) = \bigcup_{i \in M} \text{Graph}(\xi_i)$ .

Clearly, there exists  $j \in \{0, 1, 2, \dots, m\} \setminus M$  such that  $S_{M,j} := S_M \cap R_\omega^j(S) \neq \emptyset$ . The set  $S_{M,j}$  is an open arc of  $\mathbb{S}^1$  and, by Lemma 2.2(a),  $\xi_M|_{S_{M,j} \cap G} = \xi_j|_{S_{M,j} \cap G}$  because  $\text{Graph}(\xi_M), \text{Graph}(\xi_j) \subset A_{(\varphi,G)}$ . Since  $S_{M,j} \cap G$  is dense in  $S_{M,j}$ , given  $\theta \in S_{M,j} \setminus G$ , there exists a sequence  $\{\theta_n\}_{n=0}^\infty \subset S_{M,j} \cap G$  converging to  $\theta$ . The continuity of  $\xi_M$  and  $\xi_j$  on  $S_{M,j}$  implies that  $\xi_M(\theta) = \lim_{n \rightarrow \infty} \xi_M(\theta_n) = \lim_{n \rightarrow \infty} \xi_j(\theta_n) = \xi_j(\theta)$ . Consequently,  $\xi_M|_{S_{M,j}} = \xi_j|_{S_{M,j}}$  and  $\text{Graph}(\xi_M) \cup \text{Graph}(\xi_j)$  is an arc of a curve (defined on the open arc  $S_M \cup R_\omega^j(S)$ ). By redefining  $M$  as  $M \cup \{j\}$  and iterating this procedure until  $M \cup \{j\} = \{0, 1, 2, \dots, m\}$  we see that the whole  $\bigcup_{i=0}^m \text{Graph}(\xi_i)$  is a curve.  $\square$

Next we will introduce and study the space of pseudo-curves.

**Definition 2.4.** We define the *space of pseudo-curve generators* as

$$\mathcal{PCG} := \{(\varphi, G) : G \text{ is a residual set in } \mathbb{S}^1 \text{ and } \varphi: G \rightarrow \mathbb{I} \text{ is a continuous map}\}.$$

On  $\mathcal{PCG}$  we also define the *supremum pseudo-metric*  $d_\infty: \mathcal{PCG} \times \mathcal{PCG} \rightarrow \mathbb{R}^+$  by:

$$d_\infty((\varphi, G), (\varphi', G')) := \sup_{\theta \in G \cap G'} |\varphi(\theta) - \varphi'(\theta)|.$$

Clearly,  $d_\infty((\varphi, G), (\varphi', G')) = 0$  if and only if  $\varphi|_{G \cap G'} = \varphi'|_{G \cap G'}$  and, hence,  $d_\infty$  is a pseudo-metric.  $\blacksquare$

The next lemma will be useful in using the metric  $d_\infty$ .

**Lemma 2.5.** Let  $(\varphi, G), (\varphi', G') \in \mathcal{PCG}$ . Then,

$$d_\infty((\varphi, G), (\varphi', G')) = \sup_{\theta \in \tilde{G}} |\varphi(\theta) - \varphi'(\theta)|$$

for every  $\tilde{G} \subset G \cap G'$  dense in  $\mathbb{S}^1$ .

**Proof.** Set  $d_{\infty, \tilde{G}}((\varphi, G), (\varphi', G')) := \sup_{\theta \in \tilde{G}} |\varphi(\theta) - \varphi'(\theta)|$ . With this notation, we clearly have  $d_{\infty, \tilde{G}}((\varphi, G), (\varphi', G')) \leq d_\infty((\varphi, G), (\varphi', G'))$ .

To prove the reverse inequality take  $\theta \in (G \cap G') \setminus \tilde{G}$ . Since  $\tilde{G}$  is dense in  $\mathbb{S}^1$ , there exists a sequence  $\{\theta_n\}_{n=0}^\infty \subset \tilde{G}$  converging to  $\theta$ . On the other hand, by definition, the maps  $\varphi$  and  $\varphi'$ , are continuous in  $G \cap G'$  (and, hence, in  $\theta$ ). Consequently,  $|\varphi(\theta), \varphi'(\theta)| = \lim_{n \rightarrow \infty} |\varphi(\theta_n) - \varphi'(\theta_n)| \leq d_{\infty, \tilde{G}}((\varphi, G), (\varphi', G'))$ . This ends the proof of the lemma.  $\square$

As it is customary we will introduce an equivalent relation in the space of pseudo-curve generators so that the quotient space will be a metric space.

**Definition 2.6.** Two pseudo-curve generators  $(\varphi, G), (\varphi', G') \in \mathcal{PCG}$  are said to be equivalent, denoted by  $(\varphi, G) \sim (\varphi', G')$  if and only if  $A_{(\varphi, G)} = A_{(\varphi', G')}$ . Clearly  $\sim$  is an equivalence relation in  $\mathcal{PCG}$ . The  $\sim$ -equivalence class of  $(\varphi, G) \in \mathcal{PCG}$  will be denoted by  $[\varphi, G]$ .  $\blacksquare$

**Remark 2.7.** From Lemma 2.2(a), (c) it follows that  $(\varphi, G) \sim (\varphi', G')$  if and only if  $\varphi|_{\tilde{G}} = \varphi'|_{\tilde{G}}$  for every  $\tilde{G} \subset G \cap G'$  dense in  $\mathbb{S}^1$ . In particular, by taking  $\tilde{G} = G \cap G'$ , we get that  $d_\infty((\varphi, G), (\varphi', G')) = 0$  if and only if  $(\varphi, G) \sim (\varphi', G')$ .  $\blacksquare$

**Definition 2.8.** The space  $\mathcal{PCG}/\sim$  will be called the *space of pseudo-curves generator classes* and denoted by  $\mathcal{PC}$ . Also, on  $\mathcal{PC}$  we define the *supremum metric*, also denoted  $d_\infty : \mathcal{PC} \times \mathcal{PC} \rightarrow \mathbb{R}^+$  by abuse of notation, in the following way. Given  $A = [\varphi_A, G_A], B = [\varphi_B, G_B] \in \mathcal{PC}$  we set

$$d_\infty(A, B) := d_\infty((\varphi_A, G_A), (\varphi_B, G_B)).$$

Note that  $d_\infty$  is well defined. To see this take  $[\varphi_A, G_A] = [\varphi^A, G_{A'}], [\varphi_B, G_B] \in \mathcal{PCG}$ . Then, by Lemma 2.5 and Remark 2.7 applied to  $\tilde{G} = G_A \cap G_{A'} \cap G_B$  we get  $d_\infty((\varphi_A, G_A), (\varphi_B, G_B)) = d_\infty((\varphi^A, G_{A'}), (\varphi_B, G_B))$ .  $\blacksquare$

The next result establishes the basic properties of the space of pseudo-curves generator classes  $(\mathcal{PC}, d_\infty)$ .

**Proposition 2.9.** The space of pseudo-curves generator classes  $\mathcal{PC}$  is a complete metric space.

**Proof.** The fact that  $d_\infty$  is a metric in  $\mathcal{PC}$  follows from Remark 2.7.

Now we prove that  $\mathcal{PC}$  is complete. Assume that  $\{[\varphi_n, G_n]\}_{n=1}^\infty$  is a Cauchy sequence in  $\mathcal{PC}$ . We have to see that  $\lim_{n \rightarrow \infty} [\varphi_n, G_n] \in \mathcal{PC}$ .

Set,  $G := \bigcap_{i=1}^\infty G_n$ . Since this intersection is countable,  $G$  is still a residual set. The definition of  $d_\infty$  implies that the sequence  $\{\varphi_n(\theta)\}_{n=1}^\infty \subset \mathbb{I}$  is a Cauchy sequence in  $\mathbb{I}$  for every  $\theta \in G$ . So, it is convergent and we can define a map  $\varphi: G \rightarrow \mathbb{I}$  by  $\varphi(\theta) := \lim_{n \rightarrow \infty} \varphi_n(\theta)$ .

If  $(\varphi, G) \in \mathcal{PCG}$  we have  $[\varphi, G] \in \mathcal{PC}$  and, from the definition of  $\varphi$  it follows that

$$\lim_{n \rightarrow \infty} d_\infty([\varphi, G], [\varphi_n, G_n]) = \sup_{\theta \in G \cap G_n} \lim_{n \rightarrow \infty} |\varphi(\theta) - \varphi_n(\theta)| = 0.$$

Consequently,  $[\varphi, G] = \lim_{n \rightarrow \infty} [\varphi_n, G_n]$ . Since  $\varphi$  is the uniform limit of a sequence of continuous functions on  $G$ , it is continuous on  $G$ . That is,  $(\varphi, G) \in \mathcal{PCG}$ .  $\square$

In what follows we want to look at the space  $\mathcal{A}$  as a metric space and relate this metric space with  $(\mathcal{PC}, d_\infty)$ .

Let  $\rho$  denote the euclidean metric in  $\Omega$ . Then, the space  $(\Omega, \rho)$  is a compact metric space. We recall that the Hausdorff metric is defined in the space of compact subsets of  $(\Omega, \rho)$ , by

$$H_\rho(A, B) = \max \left\{ \max_{(\theta, x) \in A} \rho((\theta, x), B), \max_{(\theta, x) \in B} \rho((\theta, x), A) \right\}.$$

Then,  $(\mathcal{A}, H_\rho)$  is a metric space. To study the relation between  $(\mathcal{PC}, d_\infty)$  and  $(\mathcal{A}, H_\rho)$  we need a couple of simple technical results.

**Lemma 2.10.** *Let  $A, B \subset \Omega$  be compact circular sets. Then,*

$$H_\rho(A, B) \leq \max_{\theta \in \mathbb{S}^1} H_\rho(A^\theta, B^\theta).$$

**Proof.** It follows directly from the definitions:

$$\begin{aligned} H_\rho(A, B) &\leq \max \left\{ \sup_{(\theta, x) \in A} \rho((\theta, x), B^\theta), \sup_{(\theta, x) \in B} \rho((\theta, x), A^\theta) \right\} \\ &= \max \left\{ \sup_{\theta \in \mathbb{S}^1} \max_{\{x \in \mathbb{I} : (\theta, x) \in A\}} \rho((\theta, x), B^\theta), \sup_{\theta \in \mathbb{S}^1} \max_{\{x \in \mathbb{I} : (\theta, x) \in B\}} \rho((\theta, x), A^\theta) \right\} \\ &= \sup_{\theta \in \mathbb{S}^1} \max \left\{ \max_{\{x \in \mathbb{I} : (\theta, x) \in A\}} \rho((\theta, x), B^\theta), \max_{\{x \in \mathbb{I} : (\theta, x) \in B\}} \rho((\theta, x), A^\theta) \right\} \\ &= \sup_{\theta \in \mathbb{S}^1} H_\rho(A^\theta, B^\theta). \quad \square \end{aligned}$$

**Proposition 2.11.** *Let  $(\varphi, G), (\tilde{\varphi}, \tilde{G}) \in \mathcal{PCG}$ . Then,*

$$H_\rho(A_{(\varphi, G)}, A_{(\tilde{\varphi}, \tilde{G})}) \leq \sup_{\theta \in \mathbb{S}^1} H_\rho(A_{(\varphi, G)}^\theta, A_{(\tilde{\varphi}, \tilde{G})}^\theta) = d_\infty((\varphi, G), (\tilde{\varphi}, \tilde{G})).$$

**Proof.** The first inequality follows from Lemma 2.10.

Now we prove the second equality. By Lemma 2.2(a),

$$d_\infty((\varphi, G), (\tilde{\varphi}, \tilde{G})) = \sup_{\theta \in G \cap \tilde{G}} |\varphi(\theta) - \tilde{\varphi}(\theta)| = \sup_{\theta \in G \cap \tilde{G}} H_\rho(A_{(\varphi, G)}^\theta, A_{(\tilde{\varphi}, \tilde{G})}^\theta).$$

So, to end the proof of the lemma, we have to see that

$$H_\rho(A_{(\varphi, G)}^\theta, A_{(\tilde{\varphi}, \tilde{G})}^\theta) \leq d_\infty((\varphi, G), (\tilde{\varphi}, \tilde{G})) \quad \text{for every } \theta \in \mathbb{S}^1 \setminus (G \cap \tilde{G}).$$

Fix  $\theta \in \mathbb{S}^1 \setminus (G \cap \tilde{G})$ . From the definition of the Hausdorff metric it follows that there exist  $x, y \in \mathbb{I}$  such that  $H_\rho(A_{(\varphi, G)}^\theta, A_{(\tilde{\varphi}, \tilde{G})}^\theta) = |x - y|$ ,  $(\theta, x) \in A_{(\varphi, G)}^\theta$ , and  $(\theta, y) \in A_{(\tilde{\varphi}, \tilde{G})}^\theta$ .

Since  $G \cap \tilde{G}$  is residual (and thus dense) in  $\mathbb{S}^1$ , from Lemma 2.2(a), (c) it follows that there exists sequences  $\{(\theta_n, \varphi(\theta_n))\}_{n=0}^\infty, \{(\theta_n, \tilde{\varphi}(\theta_n))\}_{n=0}^\infty \subset \uparrow(G \cap \tilde{G})$  such that  $\lim_{n \rightarrow \infty} (\theta_n, \varphi(\theta_n)) = (\theta, x)$  and  $\lim_{n \rightarrow \infty} (\theta_n, \tilde{\varphi}(\theta_n)) = (\theta, y)$ . Hence,

$$H_\rho(A_{(\varphi, G)}^\theta, A_{(\tilde{\varphi}, \tilde{G})}^\theta) = |x - y| = \lim_{n \rightarrow \infty} |\varphi(\theta_n) - \tilde{\varphi}(\theta_n)| \leq d_\infty((\varphi, G), (\tilde{\varphi}, \tilde{G})). \quad \square$$



Proposition 2.11 tells us that if  $\{[\varphi_n, G_n]\}_{n=1}^\infty$  is a Cauchy sequence in  $\mathcal{PC}$  then  $A_{(\varphi_n, G_n)}$  is a Cauchy sequence in  $(\mathcal{A}, H_\rho)$ , and if  $[\varphi, G] = \lim_{n \rightarrow \infty} [\varphi_n, G_n]$  then  $A_{(\varphi, G)} = \lim_{n \rightarrow \infty} A_{(\varphi_n, G_n)}$ . Unfortunately the space  $(\mathcal{A}, H_\rho)$  is not complete as the following simple example shows.

**Example 2.12** (*The Space  $(\mathcal{A}, H_\rho)$  is not Complete*). Consider continuous maps  $\xi_n : \mathbb{S}^1 \rightarrow \mathbb{I}$  with  $n \in \mathbb{N}$ ,  $n \geq 2$ , defined by

$$\xi_n(\theta) = \begin{cases} 2n\theta & \text{if } \theta \in [0, \frac{1}{2n}], \\ 2(1 - n\theta) & \text{if } \theta \in [\frac{1}{2n}, \frac{1}{n}], \\ 0 & \text{if } \theta \geq \frac{1}{n}. \end{cases}$$

Clearly,  $(\xi_n, \mathbb{S}^1) \in \mathcal{PCG}$  and  $H_\rho(A_{(\xi_n, \mathbb{S}^1)}, A_{(\xi_m, \mathbb{S}^1)}) \leq \frac{1}{\min\{n, m\}}$ . Hence,  $\{A_{(\xi_n, \mathbb{S}^1)}\}$  is a Cauchy sequence in  $\mathcal{A}$ . However, the sequence  $\{A_{(\xi_n, \mathbb{S}^1)}\}$  has no limit in  $\mathcal{A}$ . Indeed,  $\lim_{n \rightarrow \infty} A_{(\xi_n, \mathbb{S}^1)} = L = (\mathbb{S}^1 \times \{0\}) \cup (\{0\} \times [0, 1])$ , which is not the closure of the graph of a continuous map on a residual set of  $\mathbb{S}^1$  (in other words,  $L \notin \mathcal{A}$ ). This is consistent with the fact that, clearly,  $\{[\xi_n, \mathbb{S}^1]\}$  is not a Cauchy sequence in  $(\mathcal{PC}, d_\infty)$ . ■

### 3. Construction of a connected pseudo-curve

The aim of this subsection is to construct a strip  $A = A_{(\gamma, G)}$  as a connected pseudo-curve with certain topological properties that will allow us to define the map  $T \in \mathcal{S}(\Omega)$  having this pseudo-curve as the only proper invariant object. The pseudo-curve  $A_{(\gamma, G)}$  will be obtained as a limit in  $\mathcal{PC}$  of a sequence of pseudo-curves that will be constructed recursively.

We will start by introducing the necessary notation.

In what follows, for simplicity, we will take the interval  $\mathbb{I}$  as the interval  $[-2, 2]$ . Also, fix  $\omega \in [0, 1] \setminus \mathbb{Q}$ . For any  $\ell \in \mathbb{Z}$  set  $\ell^* = \ell\omega \pmod{1}$  and  $O^*(\omega) = \{\ell^* : \ell \in \mathbb{Z}\}$ . That is,  $O^*(\omega)$  is the orbit of 0 by the rotation of angle  $\omega$ .

We will denote by  $d_{\mathbb{S}^1}$  the arc distance on  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ . That is, for  $\theta_1, \theta_2 \in \mathbb{S}^1$ , we set

$$d_{\mathbb{S}^1}(\theta_1, \theta_2) := \begin{cases} \theta_2 - \theta_1 & \text{when } \theta_1 \leq \theta_2, \text{ and} \\ (\theta_2 + 1) - \theta_1 & \text{when } \theta_1 > \theta_2. \end{cases}$$

The closed arc of  $\mathbb{S}^1$  joining  $\theta_1$  and  $\theta_2$  in the natural direction will be denoted by  $[\theta_1, \theta_2]$ . That is,

$$[\theta_1, \theta_2] = \begin{cases} \{t \pmod{1} : \theta_1 \leq t \leq \theta_2\} & \text{when } \theta_1 \leq \theta_2, \text{ and} \\ \{t \pmod{1} : \theta_1 \leq t \leq \theta_2 + 1\} & \text{when } \theta_1 > \theta_2. \end{cases}$$

The open arc of  $\mathbb{S}^1$  joining  $\theta_1$  and  $\theta_2$  will be denoted by  $(\theta_1, \theta_2) = [\theta_1, \theta_2] \setminus \{\theta_1, \theta_2\}$ , and is defined analogously with strict inequalities. Given an arc  $B \subset \mathbb{S}^1$ ,  $\text{Bd}(B)$  will denote the set of endpoints of  $B$ .

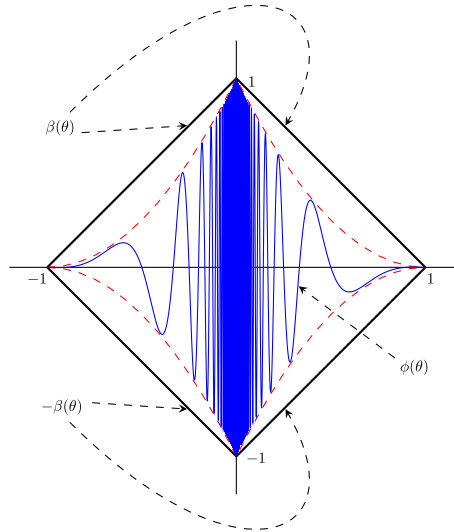
We will denote the open (respectively closed) ball (in  $\mathbb{S}^1$ ) of radius  $\delta$  centred at  $\theta \in \mathbb{S}^1$  by  $B_\delta(\theta)$  (respectively  $\overline{B}_\delta[\theta]$ ):

$$B_\delta(\theta) = \{\tilde{\theta} \in \mathbb{S}^1 : d_{\mathbb{S}^1}(\theta, \tilde{\theta}) < \delta\} = (\theta - \delta \pmod{1}, \theta + \delta \pmod{1}), \text{ and} \\ \overline{B}_\delta[\theta] = \{\tilde{\theta} \in \mathbb{S}^1 : d_{\mathbb{S}^1}(\theta, \tilde{\theta}) \leq \delta\} = [\theta - \delta \pmod{1}, \theta + \delta \pmod{1}].$$

We consider the space  $\Omega$  endowed the metric induced by the maximum of  $d_{\mathbb{S}^1}$  and the absolute value on  $\mathbb{I}$ . That is, given  $(\theta, x), (\nu, y) \in \Omega$  we set

$$d_\Omega((\theta, x), (\nu, y)) := \max\{d_{\mathbb{S}^1}(\theta, \nu), |x - y|\}.$$

Then, given  $A \subset \Omega$  we will denote the interior of  $A$  by  $\text{Int}(A)$  and  $\text{diam}(A)$  will denote the diameter of  $A$  whenever  $A$  is compact.



**Fig. 2.** The graphs of the functions  $\phi$  (in blue) and  $\pm\beta$  in thick black. The red dashed curve is  $(1 - |x|)^2$ . (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

To define the sequence of pseudo-curves that will converge to  $A_{(\gamma,G)}$  we first need to construct an auxiliary family  $\{\mathcal{R}(\ell^*)\}_{\ell \in \mathbb{Z}}$  of compact regions in  $\Omega$  and a family of compact sets  $\{\Gamma\varphi_{\ell^*}\}_{\ell \in \mathbb{Z}}$  such that, for every  $\ell \in \mathbb{Z}$ ,  $\Gamma\varphi_{\ell^*} \subset \mathcal{R}(\ell^*)$  and it is the restriction of a pseudo-curve generator to  $\pi(\mathcal{R}(\ell^*))$ . To do this we define the auxiliary functions  $\beta: [-1, 1]$  and  $\phi: [-1, 1] \setminus \{0\} \rightarrow [-1, 1]$  by (see Fig. 2):

$$\beta(x) := 1 - |x| \quad \text{and} \quad \phi(x) := (1 - |x|)^2 \sin\left(\frac{\pi}{x}\right).$$

Note that  $-\beta(x) < \phi(x) < \beta(x)$ , for all  $x \in [-1, 1] \setminus \{0\}$  and the graphs of  $-\beta$  and  $\beta$  intersect the closure of the graph of  $\phi$  only at the points  $(0, -1)$ ,  $(0, 1)$ ,  $(-1, 0)$  and  $(1, 0)$ . The function  $\phi$  is an infinitely oscillating sine curve contained in the region delimited by the graphs of the functions  $\beta$  and  $-\beta$ .

To define the families  $\{\mathcal{R}(\ell^*)\}_{\ell \in \mathbb{Z}}$  and  $\{\Gamma\varphi_{\ell^*}\}_{\ell \in \mathbb{Z}}$  we use the following *generic boxes*.

For every  $\theta \in \mathbb{S}^1$  and  $\delta < \frac{1}{2}$ ,  $\vartheta_\theta: [-\delta, \delta] \rightarrow \mathbb{S}^1$  denotes the map defined by  $\vartheta_\theta(x) = x + \theta \pmod{1}$ . Clearly  $\vartheta_\theta$  is a homeomorphism between  $[-\delta, \delta]$  and  $B_\delta[\theta]$ . Finally  $\vartheta_\theta^{-1}: B_\delta[\theta] \rightarrow [-\delta, \delta]$  denotes the inverse homeomorphism of  $\vartheta_\theta$ .

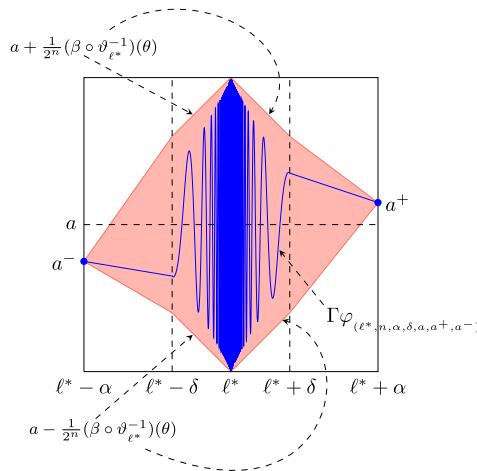
**Definition 3.1 (Generic Boxes).** Fix  $\ell, n \in \mathbb{Z}$ ,  $n \geq |\ell|$ ,  $\alpha \in (0, 2^{-n})$ ,  $\delta \in (0, \alpha)$ ,  $a \in [-1, 1]$  and  $a^+, a^- \in B_\alpha(2^{-n}\beta(\delta))$  (see Fig. 3). Now we consider the Jordan closed curve in  $\Omega$ , formed by the graphs of the functions

$$a + 2^{-n}(\beta \circ \vartheta_\theta^{-1})|_{B_\delta[\ell^*]} \quad \text{and} \quad a - 2^{-n}(\beta \circ \vartheta_\theta^{-1})|_{B_\delta[\ell^*]},$$

together with the four segments that join the points:

$$\begin{aligned} &(\ell^* - \alpha, a^-) \quad \text{with} \quad (\ell^* - \delta, a - 2^{-n}\beta(-\delta)), \\ &(\ell^* - \alpha, a^-) \quad \text{with} \quad (\ell^* - \delta, a + 2^{-n}\beta(-\delta)), \\ &(\ell^* + \alpha, a^+) \quad \text{with} \quad (\ell^* + \delta, a - 2^{-n}\beta(\delta)), \quad \text{and} \\ &(\ell^* + \alpha, a^+) \quad \text{with} \quad (\ell^* + \delta, a + 2^{-n}\beta(\delta)). \end{aligned}$$

We denote the closure of the connected component of the complement of the above Jordan curve in  $\Omega$  that contains the point  $(\ell^*, a)$  by  $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$  (the coloured region in Fig. 3). Observe that  $\pi(\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-))$ , the projection of  $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$  to  $\mathbb{S}^1$ , is  $B_\alpha[\ell^*] = [\ell^* - \alpha, \ell^* + \alpha]$ .



**Fig. 3.** The region  $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$  is the coloured filled area, delimited in the rectangle  $\uparrow\uparrow B_\delta[\ell^*]$  by the graphs of the functions  $a \pm \frac{1}{2^n}(\beta \circ \vartheta_{\ell^*}^{-1})(\theta)$ . In blue the set  $\Gamma\varphi_{(\ell^*, n, \alpha, \delta, a, a^+, a^-)}$  inductively defining the pseudo-curve. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

We denote by

$$\varphi_{\ell^*} = \varphi_{(\ell^*, n, \alpha, \delta, a, a^+, a^-)} : B_\alpha[\ell^*] \setminus \{\ell^*\} \longrightarrow \mathbb{I}$$

the continuous map defined as follows:

- (i)  $\varphi_{\ell^*} \Big|_{B_\delta[\ell^*] \setminus \{\ell^*\}} = a + (-1)^\ell 2^{-n}(\phi \circ \vartheta_{\ell^*}^{-1})$ .
- (ii)  $\varphi_{\ell^*}(\ell^* - \alpha) = a^-$  and  $\varphi_{\ell^*}(\ell^* + \alpha) = a^+$ .
- (iii)  $\varphi_{\ell^*} \Big|_{[\ell^* - \alpha, \ell^* - \delta]}$  and  $\varphi_{\ell^*} \Big|_{[\ell^* + \delta, \ell^* + \alpha]}$  are affine.

We also denote by  $\Gamma\varphi_{(\ell^*, n, \alpha, \delta, a, a^+, a^-)} \subset \mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$  the closure in  $\Omega$  of the graph of  $\varphi_{(\ell^*, n, \alpha, \delta, a, a^+, a^-)}$ . ■

**Remark 3.2.** The region  $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$  and the set  $\Gamma\varphi_{(\ell^*, n, \alpha, \delta, a, a^+, a^-)}$  satisfy the following properties:

- (1)  $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-) \subset B_\alpha[\ell^*] \times [a - 2^{-n}, a + 2^{-n}]$ .
- (2)  $\text{diam}(\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)) = \text{diam}(\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)^{\ell^*}) = 2 \cdot 2^{-n}$ .
- (3) The sets  $\Gamma\varphi_{(\ell^*, n, \alpha, \delta, a, a^+, a^-)}$  and  $\partial\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$  only intersect at the points  $(\ell^*, a - 2^{-n}), (\ell^*, a + 2^{-n}), (\ell^* - \alpha, a^-)$  and  $(\ell^* + \alpha, a^+)$ .
- (4)  $\left(\Gamma\varphi_{(\ell^*, n, \alpha, \delta, a, a^+, a^-)}\right)^{\ell^*} = \mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)^{\ell^*}$  is an interval.
- (5) Let  $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$  and  $\mathcal{R}(k^*, \tilde{n}, \tilde{\alpha}, \tilde{\delta}, \tilde{a}, \tilde{a}^+, \tilde{a}^-)$  be two regions, then  $B_\alpha[\ell^*] \cap B_{\tilde{\alpha}}[k^*] = \emptyset$  implies

$$\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-) \cap \mathcal{R}(k^*, \tilde{n}, \tilde{\alpha}, \tilde{\delta}, \tilde{a}, \tilde{a}^+, \tilde{a}^-) = \emptyset. \quad \blacksquare$$

For every  $j \in \mathbb{Z}^+$ , we set

$$Z_j := \{i \in \mathbb{Z} : |i| \leq j\} = \{-j, -j + 1, \dots, -1, 0, 1, \dots, j - 1, j\} \quad \text{and} \\ Z_j^* := \{i^* : i \in Z_j\}.$$

With the help of the sets  $\mathcal{R}(\ell^*, n, \alpha, \delta, a, a^+, a^-)$  and  $\Gamma\varphi_{(\ell^*, n, \alpha, \delta, a, a^+, a^-)}$ , which are the “bricks” of our construction we are ready to define the sequence of pseudo-curve generators  $\{(\gamma_j, \mathbb{S}^1 \setminus Z_j^*)\}_{j=0}^\infty$  that we are looking for.

To do this, for every  $j \geq 0$  we define

- a strictly increasing sequence  $\{n_j\}_{j=0}^\infty \subset \mathbb{N}$ ,
- a strictly decreasing sequence  $\{\alpha_j\}_{j=0}^\infty$  such that  $2^{-n_{j+1}} < \alpha_j < 2^{-n_j}$
- and a sequence  $\{\delta_j\}_{j=0}^\infty$  with  $2^{-n_{j+1}} < \delta_j < \alpha_j$

verifying some technical properties that we will make explicit below, and we define a sequence of boxes  $\mathcal{R}(j^*) := \mathcal{R}(j^*, n_j, \alpha_j, \delta_j, a_j, a_j^+, a_j^-)$  and  $\mathcal{R}((-j)^*) := \mathcal{R}((-j)^*, n_j, \alpha_j, \delta_j, a_{-j}, a_{-j}^+, a_{-j}^-)$  (for  $j = 0$  both sets coincide) with projections

$$\pi(\mathcal{R}(j^*)) = B_{\alpha_j}[j^*] \quad \text{and} \quad \pi(\mathcal{R}((-j)^*)) = B_{\alpha_j}[(-j)^*].$$

Finally, with the use of all these sequences and objects we can define our functions  $\gamma_j|_{\mathbb{S}^1 \setminus Z_j^*}$ .

Observe that we are using the intervals of the form  $B_{\alpha_{|\ell|}}[\ell^*]$ ,  $B_{\delta_{|\ell|}}[\ell^*]$  and also  $B_{\alpha_{|\ell|-1}}[\ell^*]$  when  $\ell$  is negative. To ease the use of these intervals we introduce the following notation:

$$B_\ell^\sim[\ell^*] := \begin{cases} B_{\alpha_\ell}[\ell^*] & \text{if } \ell \geq 0, \text{ or} \\ B_{\alpha_{|\ell+1}}[\ell^*] & \text{if } \ell < 0, \end{cases} \quad \text{and} \quad B_\ell^\sim(\ell^*) := \begin{cases} B_{\alpha_\ell}(\ell^*) & \text{if } \ell \geq 0, \text{ or} \\ B_{\alpha_{|\ell+1}}(\ell^*) & \text{if } \ell < 0. \end{cases}$$

Notice that the ball  $B_\ell^\sim[\ell^*]$  has diameter  $\alpha_j$  for  $\ell \in \{j, -(j+1)\}$ .

**Remark 3.3.** With the above notation  $B_{\alpha_{|\ell|}}[\ell^*] \subsetneq B_\ell^\sim(\ell^*)$  for every  $\ell < 0$ . Moreover, for  $\ell \in \mathbb{Z}$  and  $j \in \mathbb{Z}^+$ ,

$$R_\omega(B_{\alpha_j}[\ell^*]) = B_{\alpha_j}[(\ell+1)^*], \quad \text{and} \\ R_\omega(B_\ell^\sim[\ell^*]) = \begin{cases} B_{\alpha_\ell}[(\ell+1)^*] & \text{if } \ell \geq 0, \text{ or} \\ B_{\alpha_{|\ell+1}}[(\ell+1)^*] & \text{if } \ell < 0. \end{cases}$$

Also, the same formulae holds with  $\alpha$  replaced by  $\delta$  and for open balls. ■

The next crucial definition fixes in detail all quantities and objects mentioned above.

**Definition 3.4.** We start by defining  $\mathcal{R}(0^*) := \mathcal{R}(0^*, n_0, \alpha_0, \delta_0, 0, 0, 0)$  and  $\varphi_{0^*} := \varphi_{(0^*, n_0, \alpha_0, \delta_0, 0, 0, 0)}$  by choosing (Definition 3.1)  $n_0 = 1$ ,  $\alpha_0 < \frac{1}{2} = 2^{-n_0}$  and  $\delta_0 < \alpha_0$  small enough so that the intervals  $B_0^\sim[0^*] = B_{\alpha_0}[0^*]$ ,  $B_{\alpha_0}[1^*]$  and  $B_{-1}^\sim[(-1)^*] = B_{\alpha_0}[(-1)^*]$  are pairwise disjoint; and  $(-2)^*, 2^* \notin B_{-1}^\sim[(-1)^*]$  and, additionally,  $\text{Bd}(B_{\alpha_0}[0^*]) \cap O^*(\omega) = \emptyset$ .

We also set  $a_0^+ = a_0^- = a_0 = 0$ , and we define the map  $\gamma_0 : \mathbb{S}^1 \setminus \{0\} \rightarrow \mathbb{I}$  by

$$\gamma_0(\theta) = \begin{cases} \varphi_{0^*}(\theta) & \text{if } \theta \in B_{\alpha_0}[0^*] \setminus \{0\}, \\ 0 & \text{if } \theta \notin B_{\alpha_0}[0^*]. \end{cases}$$

For consistency with the definition of  $\gamma_j$  in the case  $j \geq 1$ , we define the map  $\gamma_{-1} : \mathbb{S}^1 \setminus \{0\} \rightarrow \mathbb{I}$  by  $\gamma_{-1}(\theta) = 0$  for every  $\theta \in \mathbb{S}^1$ . Then, notice that,  $a_0 = \gamma_{-1}(0^*), a_0^\pm = \varphi_{0^*}(0^* \pm \alpha_0) = \gamma_{-1}(0^* \pm \alpha_0)$ , and  $\gamma_0(\theta) = \gamma_{-1}(\theta)$  for every  $\theta \notin B_{\alpha_0}[0^*]$ .

Next, for every  $j \in \mathbb{N}$  we define  $\mathcal{R}(j^*)$ ,  $\mathcal{R}((-j)^*)$  and  $(\gamma_j, \mathbb{S}^1 \setminus Z_j^*)$  from the corresponding boxes  $\mathcal{R}(i^*)$  and  $B_{\alpha_{|i|}}[i^*] \subset B_i^\sim[i^*]$  for  $i \in Z_{j-1}$ , and  $(\gamma_{j-1}, \mathbb{S}^1 \setminus Z_{j-1}^*)$  as follows. We take  $n_j$ ,  $\delta_j$  and  $\alpha_j$  such that (see Fig. 4 to fix ideas):

(R.1)  $n_j > n_{j-1}$ ,  $\delta_j < \alpha_j < 2^{-n_j} < \delta_{j-1} < \alpha_{j-1}$  and

$$\left( \text{Bd} (B_{\alpha_j} [(-j)^*]) \cup \text{Bd} (B_{\alpha_j} [j^*]) \right) \cap O^*(\omega) = \emptyset.$$

(R.2) The intervals

$$\begin{aligned} B_j^\sim [j^*] &= B_{\alpha_j} [j^*], \\ R_\omega (B_{\alpha_j} [j^*]) &= B_{\alpha_j} [(j+1)^*], \\ B_{-j}^\sim [(-j)^*] &= B_{\alpha_{j-1}} [(-j)^*] \quad \text{and} \\ B_{-(j+1)}^\sim [(-(j+1))^*] &= B_{\alpha_j} [(-(j+1))^*] \end{aligned}$$

are pairwise disjoint,

$$\gamma_{j-1} (B_{\alpha_j} [\ell^*]) \subset [\gamma_{j-1}(\ell^*) - 2^{-n_j}, \gamma_{j-1}(\ell^*) + 2^{-n_j}]$$

for every  $\ell \in \{j+1, -(j+1)\}$ ,

$$\begin{aligned} B_\ell^\sim [\ell^*] \cap Z_{j+1}^* &= \{\ell^*\} \quad \text{for } \ell \in \{j, -(j+1)\} \text{ and} \\ B_{\alpha_j} [(j+1)^*] \cap Z_{j+1}^* &= \{(j+1)^*\}, \end{aligned}$$

and  $(-(j+2))^*, (j+2)^* \notin B_{-(j+1)}^\sim [(-(j+1))^*] = B_{\alpha_j} [(-(j+1))^*]$ .

(R.3)  $\text{Bd} (B_{\alpha_{|k|}} [(k+1)^*]) \cap (B_{\alpha_j} [j^*] \cup B_{\alpha_j} [(-j)^*]) = \emptyset$  for every  $k \in Z_{j-1}$ .

(R.4) Assume that there exists  $k \in Z_{j-1}$  such that  $B_{\alpha_j} [(j+1)^*] \cap B_k^\sim [k^*] \neq \emptyset$  and  $|k|$  is maximal verifying these conditions. Then,  $B_{\alpha_j} [(j+1)^*]$  is contained in one of the two connected components of  $B_{\alpha_{|k|}} (k^*) \setminus \{k^*\}$  when  $B_{\alpha_j} [(j+1)^*] \cap B_{\alpha_{|k|}} [k^*] \neq \emptyset$ , and  $B_{\alpha_j} [(j+1)^*]$  is contained in one of the two connected components of  $B_k^\sim (k^*) \setminus B_{\alpha_{|k|}} [k^*]$  if  $B_{\alpha_j} [(j+1)^*] \cap B_{\alpha_{|k|}} [k^*] = \emptyset$  (note that, in this case,  $k$  must be negative).

(R.5) Let  $\ell \in \{j, -(j+1)\}$  (recall that the ball  $B_\ell^\sim [\ell^*]$  has diameter  $\alpha_j$  for these two values of  $\ell$  and only for them).

(R.5.i) If  $\ell^* \notin \bigcup_{i \in Z_{j-1}} B_i^\sim [i^*]$  then,  $B_\ell^\sim [\ell^*] \cap B_i^\sim [i^*] = \emptyset$  for every  $i \in Z_{j-1}$ .

(R.5.ii) If  $\ell^* \in B_m^\sim [m^*]$  for some  $m \in Z_{j-1}$  such that  $|m|$  is maximal with these properties, then

(R.5.ii.1)  $B_\ell^\sim [\ell^*] \cap B_i^\sim [i^*] = \emptyset$  for every  $i \in Z_{j-1}$  such that  $|i| \geq |m|$ ,  $i \neq m$ , and

(R.5.ii.2)  $B_\ell^\sim [\ell^*]$  is contained in (a connected component of)

$$\begin{aligned} B_m^\sim (m^*) \setminus (\text{Bd} (B_{\alpha_{|m|}} [m^*]) \cup \{m^*\}) &= (m^* - \alpha_{|m|-1}, m^* - \alpha_{|m|}) \cup (m^* - \alpha_{|m|}, m^*) \\ &\quad \cup (m^*, m^* + \alpha_{|m|}) \cup (m^* + \alpha_{|m|}, m^* + \alpha_{|m|-1}) \end{aligned}$$

(observe that  $B_\ell^\sim [\ell^*] \subset B_m^\sim (m^*) \setminus B_{\alpha_{|m|}} [m^*]$  can only happen when  $m < 0$  since  $B_m^\sim [m^*] = B_{\alpha_{|m|}} [m^*]$  for  $m \geq 0$ ).

(R.6) Let  $\ell \in \{j, -j\}$ . If  $B_\ell^\sim [\ell^*] \cap B_m^\sim [m^*] = \emptyset$  for every  $m \in Z_j$ ,  $m \neq \ell$  then, to define  $\mathcal{R}(\ell^*)$  and the map  $\varphi_{\ell^*}$ , we set

$$a_\ell = \gamma_{j-1}(\ell^*) = a_\ell^\pm = \gamma_{j-1}(\ell^* \pm \alpha_j) = 0.$$

Otherwise, there exists  $m \in Z_{j-1}$  such that  $B_\ell^\sim [\ell^*]$  is contained in a connected component of  $B_m^\sim (m^*) \setminus (\text{Bd} (B_{\alpha_{|m|}} [m^*]) \cup \{m^*\})$  and  $|m|$  is maximal with these properties. Then, to define  $\mathcal{R}(\ell^*)$  and the map  $\varphi_{\ell^*}$ , we set

(R.6.i)  $a_\ell := \gamma_{|m|}(\ell^*)$ ,  $a_\ell^\pm := \gamma_{|m|}(\ell^* \pm \alpha_j)$  and  $\text{Graph}(\gamma_{|m|}|_{B_{\alpha_j}[\ell^*]}) \subset \mathcal{R}(\ell^*)$ .

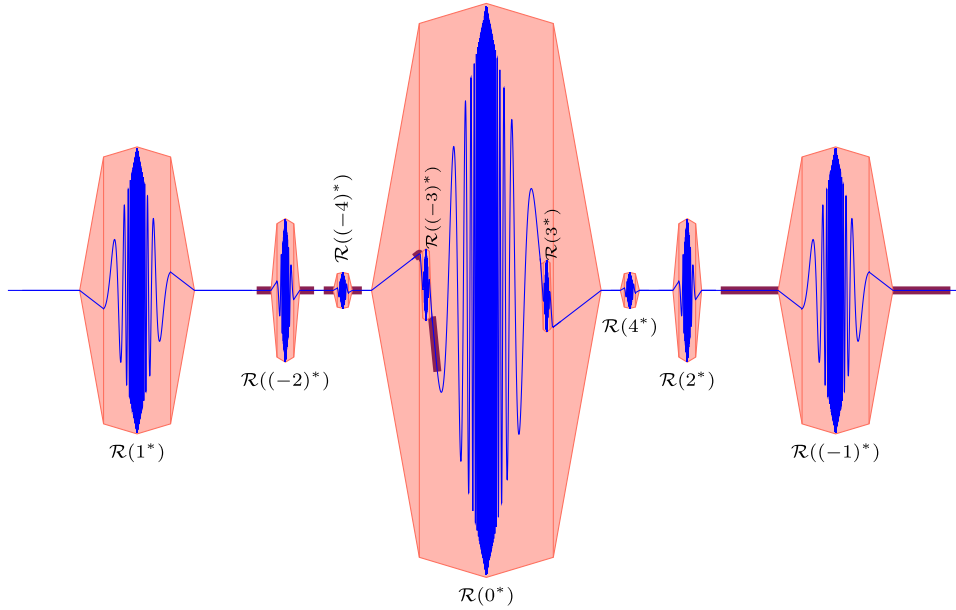


Fig. 4. The boxes  $\mathcal{R}(\ell^*)$  for  $\ell \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$  and the graph of  $\gamma_4$ . The wings are represented as a thick garnet curve surrounding the graph of  $\gamma_4$ . For clarity the scale and separation between boxes are not preserved. The circle  $\mathbb{S}^1$  is parametrized as  $[-\frac{1}{2}, \frac{1}{2})$ .

(R.6.ii) Assume that there exists  $k \in Z_{|m|} \subset Z_{j-1}$  such that  $B_\ell^{\sim}[\ell^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ . Then,  $\mathcal{R}(\ell^*)$  is contained in one of the two connected components of  $\text{Int}(\mathcal{R}(k^*) \setminus \uparrow\uparrow k^*)$ .

Finally we define  $\gamma_j : \mathbb{S}^1 \setminus Z_j^* \rightarrow \mathbb{I}$  by

$$\gamma_j(\theta) = \begin{cases} \varphi_{j^*}(\theta) & \text{if } \theta \in B_{\alpha_j}[j^*] \setminus \{j^*\}, \\ \varphi_{(-j)^*}(\theta) & \text{if } \theta \in B_{\alpha_j}[(-j)^*] \setminus \{(-j)^*\}, \\ \gamma_{j-1}(\theta) & \text{if } \theta \notin (B_{\alpha_j}[j^*] \cup B_{\alpha_j}[(-j)^*] \cup Z_{j-1}^*). \end{cases}$$

(notice that  $Z_j^* = Z_{j-1}^* \cup \{j^*, (-j)^*\}$ ). ■

For every  $\ell \in \mathbb{Z}$  we define the *winged region associated to  $\ell$*  as

$$\mathcal{R}^{\sim}(\ell^*) := \begin{cases} \mathcal{R}(\ell^*) & \text{if } \ell \geq 0, \text{ or} \\ \mathcal{R}(\ell^*) \cup \text{Graph}\left(\gamma_{|\ell|} \Big|_{B_\ell^{\sim}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right) & \text{if } \ell < 0. \end{cases}$$

**Remark 3.5** (*Explicit Consequences of Definition 3.4*). The following statements are easy consequences of Definition 3.4. They are stated explicitly for easiness of usage.

(R.1)  $n_j > j$ . This follows from Definition 3.4(R.1) and the fact that we have set  $n_0 = 1$  and  $n_j > n_{j-1}$  for  $j \in \mathbb{N}$ .

(R.2) For every  $j \in \mathbb{N}$ ,

$$B_{-j}^{\sim}[(-j)^*] \cap Z_{j+1}^* = \{(-j)^*\}.$$

This follows from Definition 3.4(R.2) for  $j - 1$ . We get

$$B_{-j}^{\sim} [(-j)^*] \cap Z_j^* = \{(-j)^*\} \quad \text{and} \quad (-(j+1))^*, (j+1)^* \notin B_{-j}^{\sim} [(-j)^*]$$

which shows the statement.

(R.3) Let  $j \in \mathbb{N}$  and  $\ell \in \{j, -j\}$ , and assume that  $B_{\ell}^{\sim} [\ell^*] \cap B_m^{\sim} [m^*] = \emptyset$  for every  $m \in Z_j$ ,  $m \neq \ell$ . Then,  $\gamma_r|_{B_{\ell}^{\sim} [\ell^*]} = \gamma_0|_{B_{\ell}^{\sim} [\ell^*]} \equiv 0$  for  $r = 1, 2, \dots, j - 1$ .

(R.6.i) Assume that there exists  $m \in Z_{j-1}$  such that  $B_{\ell}^{\sim} [\ell^*]$  is contained in a connected component of  $B_m^{\sim} (m^*) \setminus (\text{Bd}(B_{\alpha_{|m|}} [m^*]) \cup \{m^*\})$  and  $|m|$  is maximal with these properties. Then,  $\gamma_r|_{B_{\ell}^{\sim} [\ell^*]} = \gamma_{|m|}|_{B_{\ell}^{\sim} [\ell^*]}$  for  $r = |m| + 1, |m| + 2, \dots, j - 1$ .

(R.6.ii) Assume that there exists  $k \in Z_{|m|} \subset Z_{j-1}$  such that  $B_{\ell}^{\sim} [\ell^*] \subset B_{\alpha_{|k|}} (k^*) \setminus \{k^*\}$  and  $|k|$  is maximal with these properties. Then,  $\gamma_r|_{B_{\ell}^{\sim} [\ell^*]} = \gamma_{|k|}|_{B_{\ell}^{\sim} [\ell^*]}$  for  $r = |k| + 1, |k| + 2, \dots, |m|$ .

To prove (R.6) notice that when  $B_{\ell}^{\sim} [\ell^*] \cap B_{\alpha_{|m|}} [m^*] \subset B_{\ell}^{\sim} [\ell^*] \cap B_m^{\sim} [m^*] = \emptyset$  for every  $m \in Z_j$ ,  $m \neq \ell$ , from the definition of  $\gamma_r$  for  $0 \leq r < j$  we get that  $\gamma_r|_{B_{\ell}^{\sim} [\ell^*]} = \gamma_0|_{B_{\ell}^{\sim} [\ell^*]} \equiv 0$  for  $r = 1, 2, \dots, j - 1$ .

(R.6.i) The maximality of  $|m|$ , together with Definition 3.4(R.2), imply that  $B_{\ell}^{\sim} [\ell^*] \cap B_{\alpha_{|i|}} [i^*] \subset B_{\ell}^{\sim} [\ell^*] \cap B_i^{\sim} [i^*] = \emptyset$  for every  $i \in Z_{j-1}$ ,  $|i| \geq |m|$ ,  $i \neq m$ . So, by the definition of the functions  $\gamma_r$ ,

$$\gamma_r|_{B_{\alpha_j} [\ell^*]} = \gamma_{|m|}|_{B_{\alpha_j} [\ell^*]} \quad \text{for } r = |m| + 1, |m| + 2, \dots, j - 1.$$

(R.6.ii) When  $|k| = |m|$  (R.6.ii) holds trivially. So, assume that  $|k| < |m|$ . As in the case (R.6.i), the maximality of  $|k|$  and Definition 3.4(R.2) imply that  $B_{\ell}^{\sim} [\ell^*] \cap B_{\alpha_{|r|}} [r^*] = \emptyset$  for every  $r \in Z_{j-1}$ ,  $|r| \geq |k|$ ,  $r \neq k$ . So, (R.6.ii) follows from the definition of the functions  $\gamma_r$ . ■

The next technical lemma shows that the objects from Definition 3.4 exist (that is, they are well defined), and studies some of the basic properties of the family of pseudo-curve generators  $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^{\infty}$ . For easiness of reading, its long and complicate proof is displaced to the Appendix (Appendix A.1).

**Lemma 3.6.** For every  $j \in \mathbb{Z}^+$  the regions  $\mathcal{R}(j^*)$  and  $\mathcal{R}((-j)^*)$  (and hence  $\mathcal{R}^{\sim}(j^*)$  and  $\mathcal{R}^{\sim}((-j)^*)$ ), and the maps  $(\gamma_j, \mathbb{S}^1 \setminus Z_j^*)$  are well defined. Moreover, the following statements hold:

(a)  $(\gamma_j, \mathbb{S}^1 \setminus Z_j^*) \in \text{PCG}$ . Furthermore, for every  $\ell \in \{j + 1, -(j + 1)\}$ ,

$$\gamma_j (B_{\alpha_j} [\ell^*]) \subset [\gamma_j(\ell^*) - 2^{-n_j}, \gamma_j(\ell^*) + 2^{-n_j}].$$

(b)  $\bigcup_{\ell \in \mathbb{Z}} \mathcal{R}^{\sim}(\ell^*) \subset \mathbb{S}^1 \times [-1, 1]$  and  $\text{Graph}(\gamma_j|_{\mathbb{S}^1 \setminus Z_j^*}) \subset \mathbb{S}^1 \times [-1, 1]$ .

(c) For  $\ell \in \{j, -j\}$  we have  $\text{Graph}(\gamma_{j-1}|_{B_{\alpha_j} [\ell^*]}) \subset \mathcal{R}(\ell^*)$ ,  $a_{\ell} = \gamma_{j-1}(\ell^*)$ , and  $a_{\ell}^{\pm} = \varphi_{\ell^*}(\ell^* \pm \alpha_j) = \gamma_{j-1}(\ell^* \pm \alpha_j)$ .

(d)  $\text{Graph}(\gamma_n|_{B_{\alpha_j} [\ell^*] \setminus Z_n^*}) \subset \mathcal{R}(\ell^*)$  for every  $n \geq j$  and  $\ell \in \{j, -j\}$ .

(e) For every  $\ell \in \{j, -j\}$ ,

$$\gamma_j|_{(B_{\ell}^{\sim} [\ell^*] \setminus B_{\alpha_j} (\ell^*)) \cup R_{\omega} (B_{\ell}^{\sim} [\ell^*] \setminus B_{\alpha_j} (\ell^*))} = \gamma_{j-1}|_{(B_{\ell}^{\sim} [\ell^*] \setminus B_{\alpha_j} (\ell^*)) \cup R_{\omega} (B_{\ell}^{\sim} [\ell^*] \setminus B_{\alpha_j} (\ell^*))}.$$

Moreover, for every  $\theta \in \text{Bd}(B_{\ell}^{\sim} [\ell^*] \setminus B_{\alpha_j} (\ell^*)) = \text{Bd}(B_{\alpha_j} [\ell^*]) \cup \text{Bd}(B_{\ell}^{\sim} [\ell^*])$ , we have  $\theta \notin B_n^{\sim} [n^*] \cup B_{-n}^{\sim} [(-n)^*]$  and  $\gamma_n(\theta) = \gamma_j(\theta) = \gamma_{j-1}(\theta)$  for every  $n > j$ , and  $R_{\omega}(\theta) \notin B_{\alpha_n} [n^*] \cup B_{\alpha_n} [(-n)^*]$  and  $\gamma_n(R_{\omega}(\theta)) = \gamma_{j-1}(R_{\omega}(\theta))$  for every  $n \geq j$ .

(f) For every  $\ell \in \mathbb{Z}$ ,  $\mathcal{R}^\rceil(\ell^*)$  is a compact connected set such that  $\pi(\mathcal{R}^\rceil(\ell^*)) = B_\ell^\rceil[\ell^*]$ ,  $\gamma_{|\ell}|_{B_\ell^\rceil[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}$  is continuous and

$$\text{diam}(\mathcal{R}^\rceil(\ell^*)) = \begin{cases} \text{diam}(\mathcal{R}(\ell^*)) = \text{diam}(\mathcal{R}((- \ell)^*)) = 2 \cdot 2^{-n_\ell} \leq 2^{-\ell} & \text{if } \ell \geq 0, \\ 2 \cdot 2^{-n_{|\ell+1|}} \leq 2 \cdot 2^{-|\ell|} & \text{if } \ell < 0. \end{cases}$$

(g) Given  $\ell, m \in \mathbb{Z}$  such that  $|\ell| \geq |m|$ ,  $\ell \neq m$  and  $B_\ell^\rceil[\ell^*] \cap B_m^\rceil[m^*] \neq \emptyset$ , it follows that  $|\ell| > |m|$ , and either  $B_\ell^\rceil[\ell^*] \subset B_{\alpha_{|m|}}(m^*) \setminus \{m^*\}$  and the region  $\mathcal{R}^\rceil(\ell^*)$  is contained in one of the two connected components of  $\text{Int}(\mathcal{R}(m^*) \setminus \uparrow\uparrow m^*)$ , or  $m < 0$  and  $B_\ell^\rceil[\ell^*]$  is contained in one of the two connected components of  $B_m^\rceil(m^*) \setminus B_{\alpha_{|m|}}[m^*]$ .

The next results allow us to define the limit pseudo-curve generated by the sequence  $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty$ .

**Lemma 3.7.** *The sequence  $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty \subset \mathcal{PCG}$  is convergent in  $\mathcal{PCG}$ .*

**Proof.** By Proposition 2.9 it suffices to show that  $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty$  is a Cauchy sequence in  $\mathcal{PCG}$ . By the definition of  $\gamma_i$  (Definition 3.4) we have

$$\begin{aligned} d_\infty(\gamma_{i-1}, \gamma_i) &= \sup_{\theta \in \mathbb{S}^1 \setminus Z_i^*} |\gamma_{i-1}(\theta) - \gamma_i(\theta)| \\ &= \sup_{\theta \in (B_{\alpha_i}[i^*] \setminus \{i^*\}) \cup (B_{\alpha_i}[(-i)^*] \setminus \{(-i)^*\})} |\gamma_{i-1}(\theta) - \gamma_i(\theta)|. \end{aligned}$$

By Lemma 3.6(c), (d), and Definition 3.4(R.2) and Remark 3.5(R.2),

$$(\theta, \gamma_{i-1}(\theta)), (\theta, \gamma_i(\theta)) \in \mathcal{R}(\ell^*) \quad \text{for } \theta \in B_{\alpha_i}[\ell^*] \setminus \{\ell^*\} \text{ and } \ell \in \{i, -i\}.$$

Hence, by Lemma 3.6(f),

$$d_\infty(\gamma_{i-1}, \gamma_i) \leq \text{diam}(\mathcal{R}(i^*)) = \text{diam}(\mathcal{R}((-i)^*)) \leq 2^{-i}.$$

Since  $n_i$  is a strictly increasing sequence, for every  $m \geq 0$ ,

$$d_\infty(\gamma_{i+m}, \gamma_i) \leq \sum_{k=i+1}^{i+m} 2^{-k} < 2^{-(i+1)} \sum_{k=0}^\infty \frac{1}{2^k} = 2 \cdot 2^{-(i+1)},$$

and consequently  $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty$  is a Cauchy sequence in  $\mathcal{PCG}$ .  $\square$

Lemma 3.7 allows us to define the following limit pseudo-curve generator of the sequence  $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty$ .

**Definition 3.8.** There exists  $(\gamma, \mathbb{S}^1 \setminus O^*(\omega)) \in \mathcal{PCG}$  such that

$$(\gamma, \mathbb{S}^1 \setminus O^*(\omega)) = \lim_{i \rightarrow \infty} (\gamma_i, \mathbb{S}^1 \setminus Z_i^*)$$

(that is,  $\gamma(\theta) = \lim_{i \rightarrow \infty} \gamma_i(\theta)$  for every  $\theta \in \mathbb{S}^1 \setminus O^*(\omega)$ ). Observe that

$$\mathbb{S}^1 \setminus O^*(\omega) = \bigcap_{i=1}^\infty (\mathbb{S}^1 \setminus Z_i^*)$$

is a residual set in  $\mathbb{S}^1$ .  $\blacksquare$

Now, we are ready to define the sequence of pseudo-curves associated to the sequence  $\{(\gamma_i, \mathbb{S}^1 \setminus Z_i^*)\}_{i=0}^\infty$ , and to the limit pseudo-curve generator  $(\gamma, \mathbb{S}^1 \setminus O^*(\omega))$ . This will finally define the pseudo-curve **A** that we want to construct.



**Definition 3.9.** We denote by

$$A_j := A_{(\gamma_j, \mathbb{S}^1 \setminus Z_j^*)} = \overline{\text{Graph}(\gamma_j, \mathbb{S}^1 \setminus Z_j^*)}$$

the pseudo-curve defined by  $(\gamma_j, \mathbb{S}^1 \setminus Z_j^*) \in \mathcal{PCG}$ , and

$$A = A_{(\gamma, \mathbb{S}^1 \setminus O^*(\omega))} := \overline{\text{Graph}(\gamma, \mathbb{S}^1 \setminus O^*(\omega))}.$$

By [Definition 3.8](#) and [Proposition 2.11](#),  $A = \lim_{j \rightarrow \infty} A_{(\gamma_j, \mathbb{S}^1 \setminus Z_j^*)}$ . ■

The next lemmas study the properties the pseudo-curves  $A_j$  and  $A$ .

**Lemma 3.10.** *The following statements hold for every  $\ell \in \mathbb{Z}$ :*

- (a)  $A_n^\theta \subset \mathcal{R}(\ell^*)^\theta$  for every  $n \geq |\ell| - 1$  and  $\theta \in B_{\alpha_{|\ell|}}[\ell^*]$ .
- (b)  $A_n^{\ell^*} = A_{|\ell|}^{\ell^*} \subset \mathcal{R}(\ell^*)^{\ell^*}$  for every  $n \geq |\ell|$ . Moreover,  $A_{|\ell|}^{\ell^*} = \mathcal{R}(\ell^*)^{\ell^*}$  is a non-degenerate interval.
- (c)  $A_\ell^\theta = \{(\theta, \gamma_\ell(\theta))\}$  for every  $\theta \in \mathbb{S}^1 \setminus Z_\ell^*$ .
- (d)  $A_{|\ell|} \subset \mathbb{S}^1 \times [-1, 1]$ .

**Proof.** (a) By [Lemma 3.6](#)(c), (d),  $\text{Graph}(\gamma_n|_{B_{\alpha_{|\ell|}}[\ell^*] \setminus Z_n^*}) \subset \mathcal{R}(\ell^*)$ . Then, the statement follows from the compacty of  $\mathcal{R}(\ell^*)$ .

(b) From the definition of  $\gamma_i$  ([Definition 3.4](#)) and [Definition 3.4](#)(R.2), for every  $n > |\ell|$  there exists an  $\varepsilon(n) > 0$  such that  $\gamma_n(\theta) = \gamma_{|\ell|}(\theta)$  for every  $\theta \in B_{\varepsilon(n)}(\ell^*) \setminus \{\ell^*\}$ . Hence  $A_n^{\ell^*} = A_{|\ell|}^{\ell^*}$ . Moreover,  $\gamma_{|\ell|}$  coincides with  $\varphi_{\ell^*}$  in a neighbourhood of  $\ell^*$ . Thus,  $A_{|\ell|}^{\ell^*} = \mathcal{R}(\ell^*)^{\ell^*}$  and it is an interval by [Definition 3.1](#) and [Remark 3.2](#)(4).

Finally statement (c) follows from [Lemma 2.2](#)(a) and [Definition 3.9](#), and (d) from [Lemma 3.6](#)(b). □

**Lemma 3.11.** *The following statements hold.*

- (a)  $A^\theta \subset \mathcal{R}(\ell^*)^\theta$  for every  $\ell \in \mathbb{Z}$  and  $\theta \in B_{\alpha_{|\ell|}}[\ell^*]$ .
- (b)  $A^{\ell^*} = A_{|\ell|}^{\ell^*}$  for every  $\ell \in \mathbb{Z}$ . In particular  $A^{\ell^*}$  is a non-degenerate interval.
- (c) If  $\theta \notin O^*(\omega)$ , then  $A^\theta = \{(\theta, \gamma(\theta))\}$ .
- (d)  $A \subset \mathbb{S}^1 \times [-1, 1]$ .

**Proof.** Statement (c) follows directly from [Lemma 2.2](#)(a).

Now we prove (a). From [Lemma 3.10](#)(a),  $A_n^\theta \subset \mathcal{R}(\ell^*)^\theta$  for every  $\ell \in \mathbb{Z}$  and  $n \geq |\ell|$ . On the other hand, by [Definition 3.8](#) and [Proposition 2.11](#),  $A^\theta = \lim_{n \rightarrow \infty} A_n^\theta$ . Hence the result follows from the compacty of  $\mathcal{R}(\ell^*)^\theta$ .

By [Lemma 3.10](#)(b) and the part of the lemma already proved we have

$$A^{\ell^*} = \lim_{n \rightarrow \infty} A_n^{\ell^*} = A_{|\ell|}^{\ell^*}.$$

Statement (d) follows from [Lemma 3.10](#)(d), the compacty of  $\mathbb{S}^1 \times [-1, 1]$  and the fact that  $A = \lim_{j \rightarrow \infty} A_j$ . □

The next proposition, summarizes the main properties of the set  $A$ .

**Proposition 3.12.** *The set  $A$  is a connected, does not contain any arc of curve and  $\Omega \setminus A$  has two connected components.*

**Proof.** From statements (b) and (c) of the previous lemma, we know that  $A^\theta$  is connected for every  $\theta \in \mathbb{S}^1$ .

If  $A$  is not connected there exist closed (in  $A$ ) sets  $U$  and  $V$  such that  $U \cap V = \emptyset$  and  $U \cup V = A$ . Observe that  $\pi(U) \cup \pi(V) = \pi(A) = \mathbb{S}^1$  because every pseudo-curve is a circular set. Moreover, since  $A$  is compact,  $U$  and  $V$  are also compact sets of  $\Omega$ . Hence,  $\pi(U)$  and  $\pi(V)$  compact in  $\mathbb{S}^1$ . Since  $\mathbb{S}^1$  is connected,  $\pi(U) \cap \pi(V) \neq \emptyset$ . For every  $\theta \in \pi(U) \cap \pi(V)$  we have,

$$A^\theta = (U \cup V)^\theta = U^\theta \cup V^\theta.$$

The sets  $U^\theta$  and  $V^\theta$  are closed, non-empty and disjoint. Consequently,  $A^\theta$  is not connected; a contradiction. This proves that  $A$  is connected.

By Lemma 3.11(b),  $A^{\ell^*}$  is a non-degenerate interval for every  $\ell \in O^*(\omega)$ . Then, since  $O^*(\omega)$  is dense in  $\mathbb{S}^1$ ,  $A$  does not contain any arc of curve by Lemma 2.3(b).

To prove that  $\Omega \setminus A$  has two connected components we define

$$\begin{aligned} \Omega_- &:= \{(\theta, y) \in \Omega : y < \min\{x \in \mathbb{I} : (\theta, x) \in A\}\}, \quad \text{and} \\ \Omega_+ &:= \{(\theta, y) \in \Omega : y > \max\{x \in \mathbb{I} : (\theta, x) \in A\}\}. \end{aligned}$$

By Lemma 3.11(d) we know that

$$-1 \leq \min\{x \in \mathbb{I} : (\theta, x) \in A\} \leq \max\{x \in \mathbb{I} : (\theta, x) \in A\} \leq 1.$$

Hence,  $\Omega \setminus A = \Omega_- \cup \Omega_+$ ,  $\Omega_+$  and  $\Omega_-$  are disjoint open circular subsets of  $\Omega$  and  $\Omega_- \supset \mathbb{S}^1 \times [-2, -1]$  and  $\Omega_+ \supset \mathbb{S}^1 \times [1, 2]$  (in particular, for every  $\theta \in \mathbb{S}^1$ ,  $\Omega_+^\theta$  and  $\Omega_-^\theta$  are non-degenerate intervals). Thus,  $\Omega_+$  and  $\Omega_-$  are arc-wise connected and, hence, connected.  $\square$

#### 4. A collection of auxiliary functions $G_i$ defined on the boxes $\mathcal{R}^-(i^*)$

In this section we define a family of auxiliary functions

$$G_i : \mathcal{R}^-(i^*) \longrightarrow \mathcal{R}^-((i+1)^*)$$

with  $i \in \mathbb{Z}$  and study their properties. These functions will play an essential role in the construction of the map which leaves the set  $A$  invariant.

In what follows we consider the supremum metric  $d_\infty$  on the class of all functions  $F : A \longrightarrow \Omega$  with  $A \subset \Omega$ . That is, given  $F, G : A \longrightarrow \Omega$  we set

$$d_\infty(F, G) := \sup_{(\theta, x) \in A} d_\Omega(F(\theta, x), G(\theta, x)).$$

In the special case when  $F$  and  $G$  are skew products with the same base, that is when  $F(\theta, x) = (R(\theta), f(\theta, x))$  and  $G(\theta, x) = (R(\theta), g(\theta, x))$ , then

$$d_\infty(F, G) := \sup_{(\theta, x) \in A} |f(\theta, x) - g(\theta, x)|.$$

Observe that  $(\mathcal{S}(\Omega), d_\infty)$  is a complete metric space.

Before defining the maps  $G_i$  we need to introduce the necessary notation, and recall and collect some basic facts that we will use in this definition and to study their properties.

For every  $i \in \mathbb{Z}$ , we define

$$\begin{aligned} M_i : B_i^-(i^*) &\longrightarrow \mathbb{I} \quad \text{by } M_i(\theta) := \max\{x \in \mathbb{I} : (\theta, x) \in \mathcal{R}^-(i^*)\}, \quad \text{and} \\ m_i : B_i^-(i^*) &\longrightarrow \mathbb{I} \quad \text{by } m_i(\theta) := \min\{x \in \mathbb{I} : (\theta, x) \in \mathcal{R}^-(i^*)\}. \end{aligned}$$

The next simple lemma states the basic properties of the maps  $m_i$  and  $M_i$ .

**Lemma 4.1.** *The following statements hold for every  $i \in \mathbb{Z}$*

- (a)  $-1 \leq m_i(\theta) \leq M_i(\theta) \leq 1$  for every  $\theta \in B_i^\curvearrowright [i^*]$ .
- (b)  $m_i$  and  $M_i$  are continuous.
- (c)  $m_i|_{B_{\alpha_{|i|}} [i^*]}$  and  $M_i|_{B_{\alpha_{|i|}} [i^]}$  are piecewise linear.
- (d)  $m_i(\theta) = M_i(\theta) = \gamma_{|i|}(\theta)$  if and only if  $\theta \in B_i^\curvearrowright [i^*] \setminus B_{\alpha_{|i|}} (i^*)$ .

**Proof.** It follows easily from Definition 3.1, the definition of a winged region and Lemma 3.6(b), (f).  $\square$

Notice that, for every  $i \in \mathbb{Z}$ ,

$$\mathcal{R}^\curvearrowright (i^*) = \bigcup_{\theta \in B_i^\curvearrowright [i^*]} \mathcal{R}^\curvearrowright (i^*)^\theta = \bigcup_{\theta \in B_i^\curvearrowright [i^*]} \{\theta\} \times [m_i(\theta), M_i(\theta)].$$

In what follows the interval  $[m_i(\theta), M_i(\theta)] \subset \mathbb{I}$ , defined for every  $\theta \in B_i^\curvearrowright [i^*]$ , will be denoted by  $\mathbb{I}_{i,\theta}$ . Clearly, for every  $\theta \in B_i^\curvearrowright [i^*]$ ,  $\mathcal{R}^\curvearrowright (i^*)^\theta = \{\theta\} \times \mathbb{I}_{i,\theta}$ .

By Definition 3.4(R.2) and Remark 3.5(R.2),

$$B_i^\curvearrowright [i^*] \setminus \{i^*\} \text{ is disjoint from } Z_{|i|}^*.$$

Hence, Lemmas 3.6(a), (d) and 3.10(c) can be summarized as:

$$\begin{cases} \gamma_{|\ell|}|_{B_\ell^\curvearrowright [\ell^*] \setminus \{\ell^*\}} \text{ is continuous,} \\ \gamma_{|\ell|}(\theta) \in \mathbb{I}_{\ell,\theta} \text{ for every } \theta \in B_\ell^\curvearrowright [\ell^*] \setminus \{\ell^*\}, \text{ and} \\ \mathbf{A}_{|\ell|}^\theta = \{(\theta, \gamma_{|\ell|}(\theta))\} \text{ for every } \theta \in B_\ell^\curvearrowright [\ell^*] \setminus \{\ell^*\} \end{cases} \tag{1}$$

for  $\ell \in \{i, i + 1\}$ .

Now we define a family of continuous maps  $G_i: \mathcal{R}^\curvearrowright (i^*) \rightarrow \mathcal{R}^\curvearrowright ((i + 1)^*)$  with  $i \in \mathbb{Z}$ , by

$$G_i(\theta, x) = (R_\omega(\theta), g_i(\theta, x)).$$

Also, for every  $\theta \in B_i^\curvearrowright [i^*]$ , we will denote the map  $g_i(\theta, \cdot): \mathbb{I}_{i,\theta} \rightarrow \mathbb{I}$  by  $g_{i,\theta}$ .

To define the functions  $g_{i,\theta}$ , for clarity, we will consider separately two different situations:

- $i \geq 0$ , when  $\mathcal{R}^\curvearrowright (i^*) = \mathcal{R}(i^*)$ ,  $B_i^\curvearrowright [i^*] = B_{\alpha_{|i|}} [i^*]$  and  $G_i(\mathcal{R}(i^*))$  strictly contains the smaller box  $\mathcal{R}((i + 1)^*)$ , and
- $i \leq -1$ , when  $G_i(\mathcal{R}^\curvearrowright (i^*))$  is strictly contained in the bigger box  $\mathcal{R}((i + 1)^*)$ .

We start by defining  $g_{i,\theta}$  for  $i \geq 0$  in three different ways, depending on the base point  $\theta \in B_{\alpha_i} [i^*]$ . In this definition, for simplicity we will use  $\mathcal{R}(i^*)$  instead of  $\mathcal{R}^\curvearrowright (i^*)$  and  $B_{\alpha_{|i|}} [i^*]$  instead of  $B_i^\curvearrowright [i^*]$ .

Notice that, by Definition 3.4(R.1) and Lemma 3.6(c),

$$\begin{aligned} &\text{for every } i \geq 0 \\ &B_{\delta_{i+1}} [i^*] \subset B_{\alpha_{i+1}} (i^*) \text{ and } B_{\alpha_{i+1}} [i^*] \subset B_{\delta_i} (i^*) \subset B_{\alpha_i} (i^*), \text{ and} \\ &\gamma_{i-1}(i^*) = a_i \text{ and } \gamma_i((i + 1)^*) = a_{i+1}. \end{aligned} \tag{2}$$

**Definition 4.2** (Definition of  $g_i$  for  $i \geq 0$ ).

$$\theta \in B_{\delta_{i+1}} [i^*]: g_{i,\theta}(x) := \gamma_i((i + 1)^*) + \frac{2^{n_i}}{2^{n_{i+1}}} (\gamma_{i-1}(i^*) - x).$$

$\theta \in B_{\alpha_{i+1}} [i^*] \setminus B_{\delta_{i+1}} (i^*)$ : we define  $g_{i,\theta}$  to be the unique piecewise affine map with two affine pieces, defined on  $\mathbb{I}_{i,\theta}$ , whose graph joins  $(m_i(\theta), M_{i+1}(R_\omega(\theta)))$  with  $(\gamma_i(\theta), \gamma_{i+1}(R_\omega(\theta)))$ , and this with the point  $(M_i(\theta), m_{i+1}(R_\omega(\theta)))$  (in particular,  $g_{i,\theta}(\gamma_i(\theta)) = \gamma_{i+1}(R_\omega(\theta))$ ),  
 $\theta \in B_{\alpha_i} [i^*] \setminus B_{\alpha_{i+1}} (i^*)$ :  $g_{i,\theta}(x) := \gamma_{i+1}(R_\omega(\theta))$  (that is,  $g_{i,\theta}$  is constant). ■

The next lemma states the basic properties of the functions  $G_i$  for  $i \geq 0$ .

**Lemma 4.3.** *The following statements hold for every  $i \geq 0$ :*

- (a) *The map  $g_{i,\theta}$  is well defined and non-increasing for every  $\theta \in B_{\alpha_i} [i^*]$ . Moreover,  $-1 \leq g_{i,\theta}(x) \leq 1$  for every  $\theta \in B_{\alpha_i} [i^*]$  and  $x \in \mathbb{I}_{i,\theta}$ . Furthermore, the function  $G_i$  is continuous.*
- (b)  *$G_i|_{\mathcal{R}(i^*)^\theta}$  is affine and  $G_i(\mathcal{R}(i^*)^\theta) = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$  for every  $\theta \in B_{\delta_{i+1}} [i^*]$ ;  $G_i|_{\mathcal{R}(i^*)^\theta}$  is piecewise affine with two pieces and  $G_i(\mathcal{R}(i^*)^\theta) = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$  for every  $\theta \in B_{\alpha_{i+1}} [i^*] \setminus B_{\delta_{i+1}} (i^*)$ ; and  $G_i(\mathcal{R}(i^*)^\theta) = A_{i+1}^{R_\omega(\theta)}$  for every  $\theta \in B_{\alpha_i} [i^*] \setminus B_{\alpha_{i+1}} (i^*)$ .*
- (c)  *$G_i(A_i^\theta) = A_{i+1}^{R_\omega(\theta)}$  for every  $\theta \in B_{\alpha_i} [i^*]$ .*

**Proof.** We will prove all statements of the lemma simultaneously and according to the regions in the definition of the map  $g_i$ .

- We start with the region  $\mathcal{R}(i^*) \cap B_{\delta_{i+1}} [i^*]$ .

Let  $z \in [-\delta_i, \delta_i] \subset \mathbb{R}$  and let  $\theta = i^* + z \in B_{\delta_i} [i^*]$ . From Definition 3.1 and (2) we get

$$\begin{aligned} m_i(\theta) &= a_i - 2^{-n_i}(1-z) = \gamma_{i-1}(i^*) - 2^{-n_i}(1-z), \quad \text{and} \\ M_i(\theta) &= a_i + 2^{-n_i}(1-z) = \gamma_{i-1}(i^*) + 2^{-n_i}(1-z). \end{aligned} \tag{3}$$

In a similar way, for every  $\theta \in B_{\delta_{i+1}} [i^*]$  (that is,  $z \in [-\delta_{i+1}, \delta_{i+1}]$ ), we have  $R_\omega(\theta) = (i+1)^* + z \in B_{\delta_{i+1}} [(i+1)^*]$ , and

$$\begin{aligned} m_{i+1}(R_\omega(\theta)) &= a_{i+1} - 2^{-n_{i+1}}(1-z) = \gamma_i((i+1)^*) - 2^{-n_{i+1}}(1-z), \text{ and} \\ M_{i+1}(R_\omega(\theta)) &= a_{i+1} + 2^{-n_{i+1}}(1-z) = \gamma_i((i+1)^*) + 2^{-n_{i+1}}(1-z). \end{aligned} \tag{4}$$

Hence, for every  $\theta \in B_{\delta_{i+1}} [i^*]$ ,

$$\begin{aligned} g_{i,\theta}(m_i(\theta)) &= \gamma_i((i+1)^*) + \frac{2^{n_i}}{2^{n_{i+1}}} 2^{-n_i}(1-z) = \gamma_i((i+1)^*) + 2^{-n_{i+1}}(1-z) \\ &= M_{i+1}(R_\omega(\theta)), \\ g_{i,\theta}(M_i(\theta)) &= \gamma_i((i+1)^*) - \frac{2^{n_i}}{2^{n_{i+1}}} 2^{-n_i}(1-z) = \gamma_i((i+1)^*) - 2^{-n_{i+1}}(1-z) \\ &= m_{i+1}(R_\omega(\theta)). \end{aligned} \tag{5}$$

So,  $g_{i,\theta}|_{\mathbb{I}_{i,\theta}}$  is the affine map whose graph joins the point  $(m_i(\theta), M_{i+1}(R_\omega(\theta)))$  with  $(M_i(\theta), m_{i+1}(R_\omega(\theta)))$ . In particular,  $g_{i,\theta}$  sends the interval  $\mathbb{I}_{i,\theta}$  affinely onto  $\mathbb{I}_{i+1, R_\omega(\theta)}$  or, equivalently,  $G_i$  sends the interval  $\mathcal{R}(i^*)^\theta$  affinely onto  $\mathcal{R}((i+1)^*)^{R_\omega(\theta)}$ . Then, by Lemma 3.6(b), this implies that  $-1 \leq g_{i,\theta}(x) \leq 1$  for every  $x \in \mathbb{I}_{i,\theta}$ . Moreover, the continuity of the maps  $m_i$ ,  $M_i$ ,  $m_{i+1} \circ R_\omega$  and  $M_{i+1} \circ R_\omega$  implies that  $g_i$  is well defined and continuous on  $\mathcal{R}(i^*) \cap B_{\delta_{i+1}} [i^*]$ .

Next we will prove that  $G_i(A_i^\theta) = A_{i+1}^{R_\omega(\theta)}$  for every  $\theta \in B_{\delta_{i+1}} [i^*]$ . We take  $\theta = i^* + z \in B_{\delta_{i+1}} [i^*] \setminus \{i^*\}$ . Then, clearly,  $z \in [-\delta_{i+1}, \delta_{i+1}] \setminus \{0\} \subset \mathbb{R}$ . By Definitions 3.4 and 3.1 and statement (2),

$$\begin{aligned} \gamma_i(\theta) &= \varphi_{i^*}(\theta) = a_i + 2^{-n_i}d = \gamma_{i-1}(i^*) + 2^{-n_i}d \in \mathbb{I}_{i,\theta}, \quad \text{and} \\ \gamma_{i+1}(R_\omega(\theta)) &= \varphi_{(i+1)^*}(\theta) = a_{i+1} - 2^{-n_{i+1}}d = \gamma_{i-1}(i^*) - 2^{-n_{i+1}}d \in \mathbb{I}_{i+1, R_\omega(\theta)}, \end{aligned}$$

where  $d = (-1)^i \phi(z)$ . So, for every  $\theta \in B_{\delta_{i+1}} [i^*] \setminus \{i^*\}$ ,

$$g_{i,\theta}(\gamma_i(\theta)) = \gamma_i((i+1)^*) - \frac{2^{n_i}}{2^{n_{i+1}}} 2^{-n_i} d = \gamma_{i+1}(R_\omega(\theta)). \tag{6}$$

Thus, from (2) and (1) we get

$$\begin{aligned} G_i(A_i^\theta) &= G_i(\{(\theta, \gamma_i(\theta))\}) = \{(R_\omega(\theta), g_{i,\theta}(\gamma_i(\theta)))\} \\ &= \{(R_\omega(\theta), \gamma_{i+1}(R_\omega(\theta)))\} = A_{i+1}^{R_\omega(\theta)} \end{aligned}$$

for every  $\theta \in B_{\delta_{i+1}} [i^*] \setminus \{i^*\}$ . On the other hand, by the part already proven,  $g_{i,i^*}$  sends the interval  $\mathbb{I}_{i,i^*}$  affinely to  $\mathbb{I}_{i+1,(i+1)^*}$  or, equivalently,  $G_i$  sends the interval  $\mathcal{R}(i^*)^{i^*} = \{i^*\} \times \mathbb{I}_{i,i^*}$  affinely onto  $\mathcal{R}((i+1)^*)^{(i+1)^*} = \{(i+1)^*\} \times \mathbb{I}_{i,(i+1)^*}$ . This implies that  $G_i(A_i^{i^*}) = A_{i+1}^{(i+1)^*}$  by Lemma 3.10(b). Hence,  $G_i(A_i^\theta) = A_{i+1}^{R_\omega(\theta)}$  for every  $\theta \in B_{\delta_{i+1}} [i^*]$ .

- Now we study  $\mathcal{R}(i^*) \cap (B_{\alpha_{i+1}} [i^*] \setminus B_{\delta_{i+1}} (i^*))$ .

Observe that  $R_\omega(B_\alpha [i^*] \setminus \{i^*\}) = B_\alpha [(i+1)^*] \setminus \{(i+1)^*\}$  for  $\alpha \in \{\alpha_i, \alpha_{i+1}\}$ . Then, by (1)

$$\begin{aligned} \gamma_{i+1} \circ R_\omega|_{B_{\alpha_i} [i^*] \setminus \{i^*\}} &\text{ is continuous, and} \\ \gamma_{i+1}(R_\omega(\theta)) &\in \mathbb{I}_{i+1,R_\omega(\theta)} \quad \text{for every } \theta \in B_{\alpha_{i+1}} [i^*] \setminus \{i^*\}. \end{aligned} \tag{7}$$

So, the continuity of the maps  $m_i$ ,  $M_i$ ,  $m_{i+1} \circ R_\omega$  and  $M_{i+1} \circ R_\omega$  implies that  $g_i$  is well defined and continuous on  $\mathcal{R}(i^*) \cap (B_{\alpha_{i+1}} [i^*] \setminus B_{\delta_{i+1}} (i^*))$ , and

$$(\gamma_i(\theta), \gamma_{i+1}(R_\omega(\theta))) \in \mathbb{I}_{i,\theta} \times \mathbb{I}_{i+1,R_\omega(\theta)}$$

for every  $\theta \in B_{\alpha_{i+1}} [i^*] \setminus B_{\delta_{i+1}} (i^*)$ . Consequently,  $g_{i,\theta}$  maps  $\mathbb{I}_{i,\theta}$  piecewise affinely with two pieces onto  $\mathbb{I}_{i+1,R_\omega(\theta)}$  or, equivalently,  $G_i$  sends the interval  $\mathcal{R}(i^*)^\theta$  piecewise affinely with two pieces onto  $\mathcal{R}((i+1)^*)^{R_\omega(\theta)}$ . Again, by Lemma 3.6(b), this implies that  $-1 \leq g_{i,\theta}(x) \leq 1$  for every  $x \in \mathbb{I}_{i,\theta}$ . On the other hand, from (2) and (1) we have

$$\begin{aligned} G_i(A_i^\theta) &= G_i(\{(\theta, \gamma_i(\theta))\}) = \{(R_\omega(\theta), g_{i,\theta}(\gamma_i(\theta)))\} \\ &= \{(R_\omega(\theta), \gamma_{i+1}(R_\omega(\theta)))\} = A_{i+1}^{R_\omega(\theta)} \end{aligned}$$

for every  $\theta \in B_{\alpha_{i+1}} [i^*] \setminus B_{\delta_{i+1}} (i^*)$ .

- Finally, we study the region  $\mathcal{R}(i^*) \cap (B_{\alpha_i} [i^*] \setminus B_{\alpha_{i+1}} (i^*))$ .

In this case, by definition and Lemma 3.6(b) we have  $-1 \leq g_{i,\theta}(x) \leq 1$  for every  $x \in \mathbb{I}_{i,\theta}$ . By (7),  $g_i(\cdot, x) = \gamma_{i+1} \circ R_\omega$  is well defined and continuous in both variables on  $\mathcal{R}(i^*) \cap (B_{\alpha_i} [i^*] \setminus B_{\alpha_{i+1}} (i^*))$  because  $m_i$  and  $M_i$  are continuous. Moreover, for every  $\theta \in B_{\alpha_i} [i^*] \setminus B_{\alpha_{i+1}} (i^*)$  and  $x$  such that  $(\theta, x) \in \mathcal{R}(i^*)^\theta$ , we have

$$\{G_i(\theta, x)\} = \{(R_\omega(\theta), g_i(\theta, x))\} = \{(R_\omega(\theta), \gamma_{i+1}(R_\omega(\theta)))\} = A_{i+1}^{R_\omega(\theta)}$$

by Definition 3.9 and Lemma 2.2(a). Thus, by Lemma 3.10(a),

$$G_i(A_i^\theta) = G_i(\mathcal{R}(i^*)^\theta) = A_{i+1}^{R_\omega(\theta)}.$$

From all the previous arguments (b) and (c) follow. To end the proof of (a) we have to see that  $G_i$  is well defined and globally continuous. This amounts to show that it is well defined on the fibres

$$\begin{aligned} \mathcal{R}(i^*)^{(i^* \pm \delta_{i+1})} &= \{i^* \pm \delta_{i+1}\} \times \mathbb{I}_{i,i^* \pm \delta_{i+1}} \quad \text{and} \\ \mathcal{R}(i^*)^{(i^* \pm \alpha_{i+1})} &= \{i^* \pm \alpha_{i+1}\} \times \mathbb{I}_{i,i^* \pm \alpha_{i+1}}. \end{aligned}$$

We will only show that the two definitions of  $g_i$  coincide on  $\{\theta\} \times \mathbb{I}_{i,\theta}$  with  $\theta \in \{i^* + \delta_{i+1}, i^* + \alpha_{i+1}\}$ . The case  $\theta \in \{i^* - \delta_{i+1}, i^* - \alpha_{i+1}\}$  follows analogously.

We start with  $\theta = i^* + \alpha_{i+1} \in B_{\delta_i}(i^*)$ . In this case,  $R_\omega(\theta) = (i + 1)^* + \alpha_{i+1} \in \text{Bd}(B_{\alpha_{i+1}}[(i + 1)^*])$  and, by Definition 3.1 and Lemma 3.6(c),

$$M_{i+1}(R_\omega(\theta)) = m_{i+1}(R_\omega(\theta)) = a_{i+1}^+ = \gamma_{i+1}(R_\omega(\theta)).$$

Thus, the piecewise affine map whose graph joins the points  $(m_i(\theta), M_{i+1}(R_\omega(\theta)))$ ,  $(\gamma_i(\theta), \gamma_{i+1}(R_\omega(\theta)))$ , and  $(M_i(\theta), m_{i+1}(R_\omega(\theta)))$  is the constant map  $\gamma_{i+1}(R_\omega(\theta))$ . Hence,  $g_{i,\theta}$  is well defined for  $\theta = i^* + \alpha_{i+1}$ .

Now we deal with the case  $\theta = i^* + \delta_{i+1} \in B_{\delta_i}[i^*]$ . By (5) and (6) we know that the points  $(m_i(\theta), M_{i+1}(R_\omega(\theta)))$ ,  $(\gamma_i(\theta), \gamma_{i+1}(R_\omega(\theta)))$  and  $(M_i(\theta), m_{i+1}(R_\omega(\theta)))$  belong to  $\text{Graph}(x \mapsto \gamma_i((i + 1)^* + \frac{2^{n_i}}{2^{n_{i+1}}}(\gamma_{i-1}(i^*) - x)))$ . Consequently, the map  $\gamma_i((i + 1)^* + \frac{2^{n_i}}{2^{n_{i+1}}}(\gamma_{i-1}(i^*) - x))$  coincides with the piecewise affine map whose graph joins  $(m_i(\theta), M_{i+1}(R_\omega(\theta)))$ ,  $(\gamma_i(\theta), \gamma_{i+1}(R_\omega(\theta)))$  and  $(M_i(\theta), m_{i+1}(R_\omega(\theta)))$ . This ends the proof of (a).  $\square$

Now we define  $g_{i,\theta}$  for  $i < 0$ . In this case, since we are going from a smaller box  $\mathcal{R}^\sim(i^*)$  to a bigger one, we only need to define  $g_{i,\theta}$  in two different ways, depending on the base point  $\theta \in B_i^\sim[i^*]$ .

As in the previous case we need to fix some facts about the elements that we will use in the definition.

By Definition 3.4(R.1) and Lemma 3.6(c),

$$\begin{aligned} &\text{for every } i < 0 \\ &B_{\delta_{|i|}}[(i + 1)^*] \subset B_{\alpha_{|i|}}[(i + 1)^*] \subset B_{\delta_{|i+1|}}((i + 1)^*) \subset B_{\alpha_{|i+1|}}((i + 1)^*), \\ &R_\omega(B_i^\sim[i^*]) = B_{\alpha_{|i+1|}}[(i + 1)^*], \quad B_{\delta_{|i|}}[i^*] \subset B_{\alpha_{|i|}}(i^*), \quad \text{and} \\ &\gamma_{|i+1|}(i^*) = a_i \quad \text{and} \quad \gamma_{|i+2|}((i + 1)^*) = a_{i+1}. \end{aligned} \tag{8}$$

Consequently, from (1) and Definitions 3.1 and 3.4 we get

$$\begin{aligned} m_i(\theta) &< \gamma_{|i|}(\theta) < M_i(\theta) \quad \text{and} \\ m_{i+1}(R_\omega(\theta)) &< \gamma_{|i+1|}(R_\omega(\theta)) < M_{i+1}(R_\omega(\theta)) \end{aligned}$$

for every  $\theta \in B_{\alpha_{|i|}}(i^*) \setminus \{i^*\}$  (and  $R_\omega(\theta) \in B_{\alpha_{|i|}}((i + 1)^*) \setminus \{(i + 1)^*\}$ ). Then,

$$\tilde{\kappa}_i(\theta) = \min \left\{ 1, \frac{m_{i+1}(R_\omega(\theta)) - \gamma_{|i+1|}(R_\omega(\theta))}{\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}(\gamma_{|i|}(\theta) - M_i(\theta))}, \frac{M_{i+1}(R_\omega(\theta)) - \gamma_{|i+1|}(R_\omega(\theta))}{\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}(\gamma_{|i|}(\theta) - m_i(\theta))} \right\} > 0$$

defines a continuous function  $\tilde{\kappa}_i: B_{\alpha_{|i|}}(i^*) \setminus B_{\delta_{|i|}}(i^*) \rightarrow (0, 1]$ . To define the map  $g_i$  we need an auxiliary function

$$\kappa_i: B_{\alpha_{|i|}}[i^*] \setminus B_{\delta_{|i|}}(i^*) \rightarrow [0, 1]$$

such that  $\kappa_i$  is non-decreasing and continuous,  $\kappa_i(i^* \pm \delta_{|i|}) = \tilde{\kappa}_i(i^* \pm \delta_{|i|})$ , and  $\kappa_i(\theta) \leq \tilde{\kappa}_i(\theta)$  for every  $\theta \in B_{\alpha_{|i|}}(i^*) \setminus B_{\delta_{|i|}}(i^*)$ . In principle any such function would do, but for definiteness, and to show that such function exists, we note that we can take, for instance,

$$\kappa_i(\theta) = \begin{cases} \inf_{t \in [\theta, i^* - \delta_{|i|}] \cap B_{\alpha_{|i|}}(i^*)} \tilde{\kappa}_i(t) & \text{if } \theta \leq i^* - \delta_{|i|}, \\ \inf_{t \in [i^* + \delta_{|i|}, \theta] \cap B_{\alpha_{|i|}}(i^*)} \tilde{\kappa}_i(t) & \text{if } \theta \geq i^* + \delta_{|i|}. \end{cases}$$

It is easy to check that this map verifies the desired properties.

**Definition 4.4** (Definition of  $g_i$  for  $i < 0$ ). For every  $(\theta, x) \in \mathcal{R}^\sim(i^*)$  we set

$$g_{i,\theta}(x) := \begin{cases} \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}} (\gamma_{|i+1|}(i^*) - x) + \gamma_{|i+2|}((i+1)^*) & \text{if } \theta \in B_{\delta_{|i|}}[i^*], \\ \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}} \kappa_i(\theta) (\gamma_{|i|}(\theta) - x) + \gamma_{|i+1|}(R_\omega(\theta)) & \text{if } \theta \in B_{\alpha_{|i|}}[i^*] \setminus B_{\delta_{|i|}}(i^*), \\ \gamma_{|i+1|}(R_\omega(\theta)) & \text{if } \theta \in B_i^\sim[i^*] \setminus B_{\alpha_{|i|}}(i^*). \blacksquare \end{cases}$$

The next lemma states the basic properties of the functions  $G_i$  for  $i < 0$ .

**Lemma 4.5.** *The following statements hold for every  $i < 0$ :*

- (a) *The map  $g_{i,\theta}$  is well defined and non-increasing for every  $\theta \in B_{\alpha_i}[i^*]$ . Moreover,  $-1 \leq g_{i,\theta}(x) \leq 1$  for every  $\theta \in B_{\alpha_i}[i^*]$  and  $x \in \mathbb{I}_{i,\theta}$ . Furthermore, the function  $G_i$  is continuous.*
- (b)  *$G_i|_{\mathcal{R}^\sim(i^*)^\theta}$  is affine,  $G_i(\mathcal{R}^\sim(i^*)^\theta) \subset \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$  for every  $\theta \in B_i^\sim[i^*]$  and  $G_i(\mathcal{R}^\sim(i^*)^\theta) = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$  for every  $\theta \in B_{\delta_{|i|}}[i^*]$ .*
- (c)  *$G_i(A_{|i|}^\theta) = A_{|i+1|}^{R_\omega(\theta)}$  for every  $\theta \in B_i^\sim[i^*]$ .*

**Proof.** First we will prove that the map  $G_i$  is continuous and that  $G_i|_{\mathcal{R}^\sim(i^*)^\theta}$  is affine, according to the three regions in the definition.

- As in the previous lemma we start with  $\mathcal{R}^\sim(i^*)^{\uparrow B_{\delta_{|i|}}[i^*]} = \mathcal{R}(i^*)^{\uparrow B_{\delta_{|i|}}[i^*]}$ .

As in the same case of Lemma 4.3, by using (8) instead of (2), it follows that  $g_{i,\theta}|_{\mathbb{I}_{i,\theta}}$  is the affine map whose graph joins the points  $(m_i(\theta), M_{i+1}(R_\omega(\theta)))$  and  $(M_i(\theta), m_{i+1}(R_\omega(\theta)))$ ,  $g_i$  is well defined and continuous on  $\mathcal{R}(i^*)^{\uparrow B_{\delta_{|i|}}[i^*]}$ ,

$$\begin{aligned} g_{i,\theta}(\gamma_{|i|}(\theta)) &= \gamma_{|i+1|}(R_\omega(\theta)) \quad \text{for every } \theta \in B_{\delta_{|i|}}[i^*] \setminus \{i^*\}, \\ G_i &\text{ sends the interval } \mathcal{R}(i^*)^\theta \text{ affinely onto } \mathcal{R}((i+1)^*)^\theta, \quad \text{and} \\ G_i(A_{|i|}^\theta) &= A_{|i+1|}^{R_\omega(\theta)} \quad \text{for every } \theta \in B_{\delta_{|i|}}[i^*]. \end{aligned}$$

- $\mathcal{R}^\sim(i^*)^{\uparrow (B_{\alpha_{|i|}}[i^*] \setminus B_{\delta_{|i|}}(i^*))} = \mathcal{R}(i^*)^{\uparrow (B_{\alpha_{|i|}}[i^*] \setminus B_{\delta_{|i|}}(i^*))}$ .

From (1) we know that the maps  $\gamma_{|i|}$  and  $\gamma_{|i+1|} \circ R_\omega$  are continuous on the domain  $B_{\alpha_{|i|}}[i^*] \setminus B_{\delta_{|i|}}(i^*)$ . Hence, the continuity of  $g_i$  follows from the continuity of the maps  $\kappa_i$ ,  $m_i$ ,  $M_i$ ,  $m_{i+1} \circ R_\omega$  and  $M_{i+1} \circ R_\omega$ .

Notice that, from the definition of  $g_i$  in this region we clearly have that

$$g_{i,\theta}(\gamma_{|i|}(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)), \quad \text{and}$$

$G_i|_{\mathcal{R}^\sim(i^*)^\theta} = g_i(\theta, \cdot)$  is affine.

- $\mathcal{R}^\sim(i^*)^{\uparrow (B_i^\sim[i^*] \setminus B_{\alpha_{|i|}}(i^*))}$ .

In this case we have  $m_i(\theta) = \gamma_{|i|}(\theta) = M_i(\theta)$  by definition. Then, the map  $G_i|_{\mathcal{R}^\sim(i^*)^\theta} = g_i(\theta, \cdot)$  is affine because it is constant, and  $g_i$  is continuous because  $\gamma_{|i|}$  and  $\gamma_{|i+1|} \circ R_\omega$  are continuous on the domain  $B_i^\sim[i^*] \setminus \{i^*\}$  by (1).

To end the proof of (a) we have to see that  $G_i$  is well defined and globally continuous. This amounts to show that it is well defined on the fibres

$$\mathcal{R}(i^*)^{(i^* \pm \delta_{|i|})} \quad \text{and} \quad \mathcal{R}(i^*)^{(i^* \pm \alpha_{|i|})}.$$

We start by showing that the two definitions of  $g_i$  coincide on the fibres  $\mathcal{R}(i^*)^\theta$  for  $\theta \in \{i^* \pm \alpha_{|i|}\}$ . In this case we have  $m_i(\theta) = \gamma_{|i|}(\theta) = M_i(\theta)$ . Consequently,  $\mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\}$  and

$$\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}} \kappa_i(\theta) (\gamma_{|i|}(\theta) - x) + \gamma_{|i+1|}(R_\omega(\theta)) = \gamma_{|i+1|}(R_\omega(\theta))$$

for  $x \in \mathbb{I}_{i,\theta}$ .

Next we consider  $\mathcal{R}(i^*)^\theta = \{\theta\} \times \mathbb{I}_{i,\theta}$  with  $\theta = i^* + \delta_{|i|}$ . We will show that the two definitions of  $g_i$  coincide on this set. The case  $\theta = i^* - \delta_{|i|}$  follows analogously.

For simplicity we will denote

$$g_{i,\theta}^{\delta_{|i|}}(x) := \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}} (\gamma_{|i+1|}(i^*) - x) + \gamma_{|i+2|}((i+1)^*), \quad \text{and}$$

$$\xi_{i,\theta}(x) := \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}} (\gamma_{|i|}(\theta) - x) + \gamma_{|i+1|}(R_\omega(\theta)).$$

Notice that  $g_{i,\theta}^{\delta_{|i|}}$  is the map  $g_{i,\theta}$  as defined in the first region while

$$\kappa_i(\theta) (\xi_{i,\theta} - \gamma_{|i+1|}(R_\omega(\theta))) + \gamma_{|i+1|}(R_\omega(\theta))$$

is the map  $g_{i,\theta}$  as defined in the second region. In a similar way to the previous lemma we have that  $(\gamma_{|i|}(\theta), \gamma_{|i+1|}(R_\omega(\theta))) \in \text{Graph}(g_{i,\theta}^{\delta_{|i|}})$ . Hence, since  $g_{i,\theta}^{\delta_{|i|}}$  is affine with slope  $-\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}}$ , it follows that  $g_{i,\theta}^{\delta_{|i|}} = \xi_{i,\theta}$ . So, to end the proof of the lemma, we only have to see that  $\kappa_i(i^* + \delta_{|i|}) = \tilde{\kappa}_i(i^* + \delta_{|i|}) = 1$ .

Since the points  $(m_i(\theta), M_{i+1}(R_\omega(\theta)))$  and  $(M_i(\theta), m_{i+1}(R_\omega(\theta)))$  also belong to  $\text{Graph}(g_{i,\theta}^{\delta_{|i|}}) = \text{Graph}(\xi_{i,\theta})$ , it follows that

$$m_{i+1}(R_\omega(\theta)) = \xi_{i,\theta}(M_i(\theta)) = \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}} (\gamma_{|i|}(\theta) - M_i(\theta)) + \gamma_{|i+1|}(R_\omega(\theta)), \quad \text{and}$$

$$M_{i+1}(R_\omega(\theta)) = \xi_{i,\theta}(m_i(\theta)) = \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}} (\gamma_{|i|}(\theta) - m_i(\theta)) + \gamma_{|i+1|}(R_\omega(\theta)).$$

This shows that  $\tilde{\kappa}_i(i^* + \delta_{|i|}) = \tilde{\kappa}_i(\theta) = 1$  and ends the proof of (a).

Now we prove (b) according to the three regions in the definition. From the part of the lemma already proven we already know that  $G_i|_{\mathcal{R}^-(i^*)^\theta}$  is affine, and  $G_i(\mathcal{R}^-(i^*)^\theta) = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$  for every  $\theta \in B_{\delta_{|i|}}[i^*]$ . So, to end the proof of (b) we have to see that

$$g_{i,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{i+1,R_\omega(\theta)} \tag{9}$$

for every  $\theta \in B_i^-[i^*] \setminus B_{\delta_{|i|}}[i^*]$  (by definition, since  $i < 0$ ,  $B_i^-[i^*] = B_{\alpha_{|i+1|}}[i^*]$ ; therefore,  $R_\omega(\theta) \in B_{\alpha_{|i+1|}}[(i+1)^*]$  and  $\mathbb{I}_{i+1,R_\omega(\theta)} = \mathcal{R}((i+1)^*)^{R_\omega(\theta)}$ ).

For  $\theta \in B_i^-[i^*] \setminus B_{\alpha_{|i|}}(i^*)$ , by (1), we have

$$g_{i,\theta}(\mathbb{I}_{i,\theta}) = \{\gamma_{|i+1|}(R_\omega(\theta))\} \subset \mathbb{I}_{i+1,R_\omega(\theta)}.$$

Now we consider  $\theta \in B_{\alpha_{|i|}}(i^*) \setminus B_{\delta_{|i|}}[i^*]$ . Since

$$\kappa_i(\theta) \leq \tilde{\kappa}_i(\theta) \leq \frac{M_{i+1}(R_\omega(\theta)) - \gamma_{|i+1|}(R_\omega(\theta))}{\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}} (\gamma_{|i|}(\theta) - m_i(\theta))},$$

we have

$$g_{i,\theta}(m_i(\theta)) \leq \frac{2^{n_{|i|}}}{2^{n_{|i+1|}}} \frac{M_{i+1}(R_\omega(\theta)) - \gamma_{|i+1|}(R_\omega(\theta))}{\frac{2^{n_{|i|}}}{2^{n_{|i+1|}}} (\gamma_{|i|}(\theta) - m_i(\theta))} (\gamma_{|i|}(\theta) - m_i(\theta)) + \gamma_{|i+1|}(R_\omega(\theta))$$

$$= M_{i+1}(R_\omega(\theta)).$$



An analogous computation shows that  $g_{i,\theta}(M_i(\theta)) \geq m_{i+1}(R_\omega(\theta))$ . Hence, (9) holds because  $g_{i,\theta}$  is affine. This ends the proof of (b).

Then, by Lemma 3.6(b), Statement (b) of the lemma implies that  $-1 \leq g_{i,\theta}(x) \leq 1$  for every  $x \in \mathbb{I}_{i,\theta}$ .

By the part of the lemma already proved we know that  $G_i(A_{|i|}^\theta) = A_{|i+1|}^{R_\omega(\theta)}$  for every  $\theta \in B_{\delta_{|i|}}[i^*]$ . On the other hand, as in the previous lemma, from (8) and (1) we get

$$\begin{aligned} G_i(A_{|i|}^\theta) &= G_i(\{(\theta, \gamma_{|i|}(\theta))\}) = \{(R_\omega(\theta), g_{i,\theta}(\gamma_{|i|}(\theta)))\} \\ &= \{(R_\omega(\theta), \gamma_{|i+1|}(R_\omega(\theta)))\} = A_{|i+1|}^{R_\omega(\theta)} \end{aligned}$$

for every  $\theta \in B_i^- [i^*] \setminus B_{\delta_{|i|}} [i^*]$ . So, (c) holds.  $\square$

Up to now we have defined the family of auxiliary functions  $G_i: \mathcal{R}^-(i^*) \rightarrow \Omega$  with  $i \in \mathbb{Z}$ . The next step before being able to define the family  $\{T_m\} \subset \mathcal{S}(\Omega)$  is to fix some stratification in the set of boxes  $\mathcal{R}^-(i^*)$ .

### 5. A stratification in the set of boxes $\mathcal{R}^-(i^*)$

In this section we introduce a notion of *depth* in the set of arcs  $B_i^- [i^*]$  defined earlier. This notion introduces a stratification in the set of boxes  $\mathcal{R}^-(i^*)$  that we study below.

**Definition 5.1.** For every  $\ell \in \mathbb{Z}$  we define the *depth* of  $\ell$ , which will be denoted by  $\text{depth}(\ell)$ , as the cardinality of the set (see Lemma 3.6(g))

$$\begin{aligned} \{i \in \mathbb{Z}: B_\ell^- [\ell^*] \subsetneq B_i^- [i^*]\} &= \{i \in \mathbb{Z}: B_\ell^- [\ell^*] \cap B_i^- [i^*] \neq \emptyset\} \\ &= \{i \in \mathbb{Z}: \mathcal{R}^-(\ell^*) \subsetneq \mathcal{R}^-(i^*)\} = \{i \in \mathbb{Z}: \mathcal{R}^-(\ell^*) \cap \mathcal{R}^-(i^*) \neq \emptyset\}. \end{aligned}$$

Also, for every  $m \in \mathbb{Z}^+$ , we denote

$$\begin{aligned} \mathfrak{D}_m &:= \{\ell \in \mathbb{Z}: \text{depth}(\ell) = m\}, \\ \mathfrak{D}_m^* &:= \{i^*: i \in \mathfrak{D}_m\}, \quad \text{and} \\ \mu_m &:= \min\{|i|: i \in \mathfrak{D}_m\}. \quad \blacksquare \end{aligned}$$

The next lemma studies the stratification on  $\mathbb{Z}$  created by the notion of *depth*.

**Lemma 5.2.** *The following statements hold:*

- (a)  $\mathfrak{D}_{m+1} \subset \{\ell \in \mathbb{Z}: \exists i \in \mathfrak{D}_m \text{ such that } B_\ell^- [\ell^*] \subsetneq B_i^- [i^*]\}$ .
- (b) For every  $\ell, k \in \mathfrak{D}_m$  it follows that  $B_\ell^- [\ell^*] \cap B_k^- [k^*] = \emptyset$ .

**Proof.** Observe that if  $B_\ell^- [\ell^*] \subsetneq B_i^- [i^*]$  then  $\text{depth}(\ell) \geq \text{depth}(i) + 1$ . Hence, (a) holds.

Statement (b) follows from Lemma 3.6(g).  $\square$

In what follows, for every  $m \in \mathbb{Z}^+$  we set

$$\mathbb{B}_m^- := \bigcup_{i \in \mathfrak{D}_m} B_i^- [i^*] \supset \mathfrak{D}_m^*.$$

Note that, by Lemma 5.2(b),  $\mathbb{B}_m^-$  is a disjoint union of closed arcs. Therefore, for every  $\theta \in \mathbb{B}_m^-$ , there exists a unique  $i \in \mathfrak{D}_m$  such that  $\theta \in B_i^- [i^*]$ . We will denote such integer  $i$  by  $\mathbf{b}^-(\theta, m) \in \mathfrak{D}_m$ .

The next two lemmas study the properties of the winged boxes  $B_i^- [i^*]$  and  $\mathcal{R}^-(i^*)$  according to the depth stratification. Lemma 5.4 is the real motivation to introduce the winged boxes.

**Lemma 5.3.** *The following statements hold:*

- (a) *The sequence  $\{\mu_m\}_{m=0}^\infty$  is strictly increasing. In particular  $\lim_{m \rightarrow \infty} \mu_m = \infty$ .*
- (b) *For every  $m \in \mathbb{Z}^+$ ,  $\mathbb{B}_m^\rceil$  is dense in  $\mathbb{S}^1$ ,  $\mathbb{B}_{m+1}^\rceil \subset \mathbb{B}_m^\rceil$  and  $\mathfrak{D}_m^* \cap \mathbb{B}_{m+1}^\rceil = \emptyset$ .*
- (c)  *$O^*(\omega) \subset \mathbb{B}_0^\rceil$ , and  $A^\theta = \{(\theta, 0)\}$  for every  $\theta \in \mathbb{S}^1 \setminus \mathbb{B}_0^\rceil$ .*
- (d) *Let  $i \in \mathbb{Z}$  and  $\theta \in B_i^\rceil[i^*] \setminus \mathbb{B}_{\text{depth}(i)+1}^\rceil$ . Then,  $\theta \notin O^*(\omega)$  unless  $\theta = i^*$ , and  $A_n^\theta = A_{|i|}^\theta$  for every  $n \geq |i|$ . In particular  $A^\theta = A_{|i|}^\theta$ .*

**Proof.** By Lemmas 5.2(a) and 3.6(g) it follows that for every  $m \in \mathbb{Z}^+$  and  $\ell \in \mathfrak{D}_{m+1}$  there exists  $i \in \mathfrak{D}_m$  such that  $B_\ell^\rceil[k^*] \not\subset B_i^\rceil[i^*]$  and  $|i| < |\ell|$ . Thus,  $\mathbb{B}_{m+1}^\rceil \subset \mathbb{B}_m^\rceil$  and  $\mu_m < \mu_{m+1}$ . This proves (a) and the second statement of (b).

Next we will show that  $i^* \notin \mathbb{B}_{m+1}^\rceil$  for every  $i \in \mathfrak{D}_m$ . Assume by way of contradiction that there exists  $i \in \mathfrak{D}_m$  such that  $i^* \in \mathbb{B}_{m+1}^\rceil$ . Let  $k = \mathbf{b}^\rceil(i^*, m+1) \in \mathfrak{D}_{m+1}$ . Clearly,  $i \neq k$  and  $i^* \in B_k^\rceil[k^*]$ . Then, by Lemma 3.6(g),  $|k| < |i|$  and  $B_i^\rceil[i^*] \not\subset B_k^\rceil[k^*]$ . Thus,

$$m = \text{depth}(i) \geq \text{depth}(k) + 1 = m + 2;$$

a contradiction.

Now we prove the first statement of (c). From the definitions and the part of (b) already proven we have

$$O^*(\omega) \subset \bigcup_{i \in \mathbb{Z}} B_i^\rceil[i^*] \subset \bigcup_{m=0}^\infty \mathbb{B}_m^\rceil = \mathbb{B}_0^\rceil.$$

To end the proof of (b) it remains to show the density of  $\mathbb{B}_m^\rceil$ . We will do it by induction on  $m$ . Clearly  $\mathbb{B}_0^\rceil \supset O^*(\omega)$  is dense in  $\mathbb{S}^1$  because so is  $O^*(\omega)$ . Suppose that (b) holds for  $\mathbb{B}_m^\rceil$ . We will show that (b) also holds for  $\mathbb{B}_{m+1}^\rceil$ . Choose  $\theta \in \mathbb{B}_m^\rceil$  and set  $i = \mathbf{b}^\rceil(\theta, m)$ . Since  $O^*(\omega)$  is dense in  $\mathbb{S}^1$ , there exists a sequence  $\{s_n\}_{n=0}^\infty \subset \mathbb{Z}$  such that  $s_n^* \in B_i^\rceil[i^*]$  and  $\lim_{n \rightarrow \infty} s_n^* = \theta$ . As above, we get that  $\text{depth}(s_n) \geq \text{depth}(i) + 1 = m + 1$ . Moreover,  $s_n^* \in \mathbb{B}_{\text{depth}(s_n)}^\rceil \subset \mathbb{B}_{m+1}^\rceil$  for every  $n$ . Consequently,  $\mathbb{B}_m^\rceil \subset \overline{\mathbb{B}_{m+1}^\rceil}$ , and the density of  $\mathbb{B}_{m+1}^\rceil$  follows from the density of  $\mathbb{B}_m^\rceil$ .

Next we prove the second statement of (c). From above it follows that

$$\bigcup_{i \in \mathbb{Z}} B_{\alpha_{|i|}}[i^*] \subset \bigcup_{i \in \mathbb{Z}} B_i^\rceil[i^*] \subset \mathbb{B}_0^\rceil.$$

Hence, by the definition of the maps  $\gamma_m$  (Definition 3.4) it follows that  $\gamma_m(\theta) = \gamma_0(\theta) = 0$  for every  $\theta \notin \mathbb{B}_0^\rceil$  and  $m \in \mathbb{Z}^+$ . So,  $\gamma(\theta) = \lim_{m \rightarrow \infty} \gamma_m(\theta) = 0$ , and  $A^\theta = \{(\theta, \gamma(\theta))\} = \{(\theta, 0)\}$  by Lemma 3.11(c). This ends the proof of (c).

(d) If  $\theta = i^*$  then the statement follows from Lemmas 3.10(b) and 3.11(b). So, we assume that  $\theta \neq i^*$ .

By Definition 3.4(R.2) and Remark 3.5(R.2) we get that  $\theta \notin Z_{|i|+1}^*$ . Hence, if  $\theta \in O^*(\omega)$ , it follows that  $\theta = k^* \in \mathbb{B}_{\text{depth}(k)}^\rceil$  with  $|k| > |i| + 1$  and  $B_k^\rceil[k^*] \cap B_i^\rceil[i^*] \neq \emptyset$ . Thus, by Lemma 3.6(g),  $\text{depth}(k) \geq \text{depth}(i) + 1$ . By (b), this implies that  $\theta = k^* \in \mathbb{B}_{\text{depth}(i)+1}^\rceil$ ; a contradiction. Therefore,  $\theta \notin O^*(\omega)$ . On the other hand,  $\theta \notin B_{-i}^\rceil[(-i)^*]$  by Definition 3.4(R.2).

If  $\theta \notin B_{\alpha_{|k|}}[k^*]$  for every  $k \in \mathbb{Z}$  such that  $|k| > |i|$ , then  $\gamma_n(\theta) = \gamma_{|i|}(\theta)$  and  $A_n^\theta = A_{|i|}^\theta$  for every  $n \geq |i|$ , by Definition 3.4 and Lemma 3.10(c).

Now assume that  $\theta \in B_{\alpha_{|k|}}[k^*]$  for some  $k \in \mathbb{Z}$  such that  $|k| > |i|$  and  $|k|$  is minimal with these properties. If  $\theta \in B_k^\rceil(k^*)$ , as above we get that  $\text{depth}(k) \geq \text{depth}(i) + 1$  and  $\theta \in \mathbb{B}_{\text{depth}(k)}^\rceil \subset \mathbb{B}_{\text{depth}(i)+1}^\rceil$ . Thus,  $\theta \in \text{Bd}(B_k^\rceil[k^*]) = \text{Bd}(B_{\alpha_{|k|}}[k^*])$  and  $k \geq 0$ . So, by Lemma 3.6(c) and the definition of the maps  $\gamma_j$

(Definition 3.4),  $\gamma_{|k|}(\theta) = \gamma_{|k|-1}(\theta)$ . Moreover, by Lemma 3.6(e),  $\gamma_j(\theta) = \gamma_{|k|}(\theta)$  for every  $j > |k|$ . On the other hand, the minimality of  $|k|$  implies that  $\theta \notin B_{\alpha_{|\ell|}}[\ell^*]$  for every  $\ell \in \mathbb{Z}$  such that  $|k| > |\ell| > |i|$ . Hence, by the definition of the maps  $\gamma_j$  (Definition 3.4),  $\gamma_j(\theta) = \gamma_{|i|}(\theta)$  for every  $|k| > j > |i|$ . In short, we have proved that  $\gamma_j(\theta) = \gamma_{|i|}(\theta)$  for every  $j \geq |i|$ . Thus, as above,  $A_n^\theta = A_{|i|}^\theta$  for every  $n \geq |i|$ . This ends the proof of the lemma.  $\square$

**Lemma 5.4.** *Assume that  $B_i^\sim[i^*] \subset B_k^\sim[k^*]$  for some  $i \in \mathfrak{D}_m$ ,  $k \in \mathfrak{D}_{m-1}$  and  $m \in \mathbb{N}$ . Then,  $|k| < |i|$  and  $|k + 1| < |i + 1|$  unless  $k \geq 0$  and  $i = -(k + 2)$  (whence  $|k + 1| = |i + 1|$ ). Moreover, the following statements hold:*

(a) *For every  $\theta \in B_i^\sim[i^*]$ ,*

$$\gamma_{|k|}(\theta) = \gamma_{|k|+1}(\theta) = \dots = \gamma_{|i|-1}(\theta) \in \mathbb{I}_{i,\theta}$$

*and, when  $|k + 1| < |i + 1|$ ,*

$$\gamma_{|k+1|}(R_\omega(\theta)) = \gamma_{|k+1|+1}(R_\omega(\theta)) = \dots = \gamma_{|i+1|-1}(R_\omega(\theta)).$$

(b) *For every  $\theta \in B_i^\sim[i^*] \setminus B_{\alpha_{|i|}}(i^*)$ ,*

$$\gamma_{|i|}(\theta) = \gamma_{|i|-1}(\theta) \quad \text{and} \quad \mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta}.$$

**Proof.** The fact that  $|k| < |i|$  follows from Lemma 3.6(g). Therefore, either  $|k + 1| < |i + 1|$  or  $k \geq 0$ ,  $i = -(k + 2)$  and  $|k + 1| = |i + 1|$  or  $k \geq 0$ ,  $i = -(k + 1)$  and  $|k + 1| > |i + 1|$ . In the last case,  $B_i^\sim[i^*] = B_{-(k+1)}^\sim[(-(k+1))^*]$  and  $B_k^\sim[k^*]$  must be disjoint by Definition 3.4(R.2) (with  $j = k$ ); which is a contradiction. Thus  $|k + 1| < |i + 1|$  unless  $k \geq 0$  and  $i = -(k + 2)$  ( $|k + 1| = |i + 1|$ ).

By Definition 3.4(R.2) and Remark 3.5(R.2),  $B_i^\sim[i^*] \cap Z_{|i|-1}^* = \emptyset$ . Hence, from the definition of the maps  $\gamma_j$  (Definition 3.4), to prove that

$$\gamma_{|k|}|_{B_i^\sim[i^*]} = \gamma_{|k|+1}|_{B_i^\sim[i^*]} = \dots = \gamma_{|i|-2}|_{B_i^\sim[i^*]} = \gamma_{|i|-1}|_{B_i^\sim[i^*]},$$

it is enough to show that  $B_{\alpha_{|\ell|}}[\ell^*] \cap B_i^\sim[i^*] = \emptyset$  for every  $\ell$  such that  $|k| < |\ell| < |i|$ . Assume that  $B_{\alpha_{|\ell|}}[\ell^*] \cap B_i^\sim[i^*] \neq \emptyset$  for some  $\ell$  such that  $|k| < |\ell| < |i|$ . Then,

$$\emptyset \neq B_{\alpha_{|\ell|}}[\ell^*] \cap B_i^\sim[i^*] \subset B_\ell^\sim[\ell^*] \cap B_i^\sim[i^*] \subset B_\ell^\sim[\ell^*] \cap B_k^\sim[k^*]$$

and, by Lemma 3.6(g),

$$B_i^\sim[i^*] \subsetneq B_\ell^\sim[\ell^*] \subsetneq B_k^\sim[k^*].$$

So, in a similar way as before,

$$m = \text{depth}(i) \geq \text{depth}(\ell) + 1 \geq \text{depth}(k) + 2 = m + 1;$$

a contradiction. This ends the proof of the first statement of (a).

Now we show that if  $|k + 1| < |i + 1| - 1$ , then

$$\gamma_{|k+1|}(R_\omega(\theta)) = \gamma_{|k+1|+1}(R_\omega(\theta)) = \dots = \gamma_{|i+1|-1}(R_\omega(\theta)),$$

and are well defined.

First we prove that  $\gamma_\ell(R_\omega(\theta))$  is well defined for every  $\ell = 0, 1, \dots, |i + 1| - 1$ . For every  $\theta \in B_i^\rceil[i^*]$  we have

$$R_\omega(\theta) \in R_\omega(B_i^\rceil[i^*]) = \begin{cases} B_{\alpha_i}[(i + 1)^*] & \text{when } i \geq 0, \text{ and} \\ B_{\alpha_{|i+1|}}[(i + 1)^*] \subset B_{i+1}^\rceil[(i + 1)^*] & \text{when } i < 0. \end{cases}$$

In any case, by Definition 3.4(R.2) and Remark 3.5(R.2) with  $j = i$  when  $i \geq 0$  and  $\ell = -(j + 1) = i + 1$  when  $i < 0$ , and Lemma 3.6(a),

$$R_\omega(\theta) \notin \begin{cases} Z_i^* & \text{when } i \geq 0, \text{ and} \\ Z_{|i+1|-1}^* & \text{when } i < 0, \end{cases}$$

and  $\gamma_\ell(R_\omega(\theta))$  is well defined for  $\ell = 0, 1, \dots, |i + 1| - 1$  (recall that  $Z_m^* \subset Z_{m+1}^*$  for every  $m \geq 0$ ).

Now, assume by way of contradiction that

$$\gamma_\ell(R_\omega(\theta)) \neq \gamma_{\ell-1}(R_\omega(\theta)) \quad \text{for some } \ell \in \{|k + 1| + 1, |k + 1| + 2, \dots, |i + 1| - 1\},$$

and  $\ell$  is minimal with this property (observe that  $\ell \geq 1$ ). By the definition of the map  $\gamma_\ell$  (Definition 3.4),

$$R_\omega(\theta) \in B_{\alpha_\ell}((q + 1)^*) \quad \text{with } q \in \{\ell - 1, -(\ell + 1)\}$$

and, hence,  $\theta \in B_{\alpha_\ell}(q^*)$ .

Since  $|k + 1| + 1 \leq \ell < |i + 1|$ , when  $q = -(\ell + 1) \leq -2$ ,

$$|k + 1| + 2 \leq -q \leq |i + 1| \quad \text{and} \quad B_{\alpha_\ell}(q^*) = B_{-(\ell+1)}^\rceil((-(\ell + 1))^*) = B_q^\rceil(q^*).$$

Otherwise, when  $q = \ell - 1 \geq 0$ ,  $|k + 1| \leq q \leq |i + 1| - 2$  and

$$B_{\alpha_\ell}(q^*) \subset B_{\alpha_{\ell-1}}((\ell - 1)^*) = B_{\ell-1}^\rceil((\ell - 1)^*) = B_q^\rceil(q^*),$$

by Definition 3.4(R.1).

Next we want to use Lemma 3.6(g) to show that  $B_i^\rceil[i^*] \subsetneq B_q^\rceil[q^*] \subsetneq B_k^\rceil[k^*]$ . To this end we have to compare  $|q|$  with  $|i|$  and  $|k|$ .

Notice  $B_q^\rceil[q^*] \cap B_k^\rceil[k^*] \neq \emptyset$  because

$$\theta \in B_q^\rceil(q^*) \cap B_i^\rceil[i^*] \subset B_q^\rceil(q^*) \cap B_k^\rceil[k^*].$$

If  $k \geq 0$ ,  $|q| \geq |k + 1| > |k|$ . When  $k, q < 0$ ,  $|q| \geq |k + 1| + 2 = |k| + 1 > |k|$ . If  $k < 0$  and  $q \geq 0$ ,  $|q| = q \geq |k + 1| = |k| - 1$ . If  $q = |k| - 1$  (that is,  $k = -(q + 1)$ ), as above, by Definition 3.4(R.2) with  $j = q$  we get  $B_k^\rceil[k^*] \cap B_q^\rceil[q^*] = \emptyset$ ; a contradiction. So,  $|q| > |k|$  unless  $|q| = |k|$  and  $k < 0 \leq q$ . Summarizing, we have shown that  $|q| \geq |k|$  and  $q \neq k$ . Then, from Lemma 3.6(g) we get that  $|q| > |k|$  and  $B_q^\rceil[q^*] \subsetneq B_k^\rceil[k^*]$ .

Now we will study the relation of  $B_q^\rceil[q^*]$  with the box  $B_i^\rceil[i^*]$ . From above we get that  $B_q^\rceil[q^*] \cap B_i^\rceil[i^*] \neq \emptyset$ . If  $i < 0$ ,  $|q| \leq |i + 1| = |i| - 1$ . When  $q, i \geq 0$ , we have  $|q| = q \leq |i + 1| - 2 = |i| - 1$ . If  $i \geq 0$  and  $q < 0$ ,  $|q| \leq |i + 1| = |i| + 1$ .

Assume that  $i \geq 0$  and  $q = -(i + 1) < 0$ . In this case, additionally,  $q = -(\ell + 1)$  and, thus,  $i = \ell \geq 1$ . Then,

$$\begin{aligned} R_\omega(\theta) \in R_\omega(B_i^\rceil[i^*]) &= R_\omega(B_{\alpha_i}[i^*]) = B_{\alpha_i}[(i + 1)^*], \quad \text{and} \\ R_\omega(\theta) \in B_{\alpha_\ell}((q + 1)^*) &= B_{\alpha_i}((-i)^*) \subset B_{-i}^\rceil((-i)^*), \end{aligned}$$

which is a contradiction by Definition 3.4(R.2). Summarizing,  $|q| < |i|$  unless  $|q| = |i|$  and  $q < 0 \leq i$  (that is,  $|q| \leq |i|$  and  $q \neq i$ ). Then, again by Lemma 3.6(g),  $|q| < |i|$  and  $B_i^- [i^*] \subsetneq B_q^- [q^*] \subsetneq B_k^- [k^*]$ . So, as before,

$$m = \text{depth}(i) \geq \text{depth}(q) + 1 \geq \text{depth}(k) + 2 = m + 1;$$

a contradiction. This ends the proof of (a).

Now we assume that  $\theta \in B_i^- [i^*] \setminus B_{\alpha_{|i|}} (i^*)$ . By Lemmas 3.6(e) and 4.1(d),

$$\gamma_{|i|}(\theta) = \gamma_{|i|-1}(\theta) \quad \text{and} \quad \mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|i|-1}(\theta)\} = \{\gamma_{|k|}(\theta)\}.$$

On the other hand, by Lemma 5.3(b),  $\mathfrak{D}_{m-1}^* \cap \mathbb{B}_m^- = \emptyset$  which implies that  $\theta \neq k^*$  because  $k^* \in \mathfrak{D}_{m-1}^*$  and  $\theta \in B_i^- [i^*] \setminus B_{\alpha_{|i|}} (i^*) \subset \mathbb{B}_m^-$ . So, by (1),

$$\mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta}.$$

Now we prove that  $\gamma_{|i|-1}(\theta) \in \mathbb{I}_{i,\theta}$  for every  $\theta \in B_i^- [i^*]$ . From above, we have  $\mathbb{I}_{i,\theta} = \{\gamma_{|i|-1}(\theta)\}$  for every  $\theta \in B_i^- [i^*] \setminus B_{\alpha_{|i|}} (i^*)$ . Moreover, when  $\theta \in B_{\alpha_{|i|}} (i^*)$  the statement follows directly from Lemma 3.6(c). Thus, (b) is proved.  $\square$

### 6. Boxes in the wings

To prove Theorem A we will inductively construct a Cauchy sequence  $\{T_m\}_{m=0}^\infty \subset \mathcal{S}(\Omega)$  that gives the function  $T$  from Theorem A as a limit.

This section is devoted to study the points in the wings of boxes in the circle and its interaction with boxes of higher depth. The resulting technology is necessary to be able to construct the sequence  $\{T_m\}_{m=0}^\infty$  so that it is Cauchy sequence. Unfortunately this will complicate even more the definition of the functions  $T_m$  and the proof of its continuity.

We start by introducing some more notation. For every  $m \in \mathbb{Z}^+$  we set

$$\mathbb{B}_m := \bigcup_{i \in \mathfrak{D}_m} B_{\alpha_{|i|}} [i^*] \subset \mathbb{B}_m^-, \quad \text{and}$$

$$\text{WIDB}_m := \left\{ \theta \in \mathbb{B}_m^- \setminus \mathbb{B}_m : \theta \in \mathbb{B}_j \text{ for some } j > m \right\}.$$

On the other hand, the smallest number  $j$  from the above definition will be called the *least essential depth* of  $\theta$  below  $m$ , and will be denoted by  $\text{led}(\theta, m)$ . That is,  $\text{led}(\theta, m)$  denotes the positive integer larger than  $m$  such that

$$\theta \in \mathbb{B}_j^- \setminus \mathbb{B}_j \quad \text{for } j = m, m + 1, \dots, \text{led}(\theta, m) - 1 \quad \text{and} \quad \theta \in \mathbb{B}_{\text{led}(\theta, m)}.$$

The following simple lemmas are useful to better understand and use the above definitions. The next lemma establishes the relation between boxes in the wings of increasing depth.

**Lemma 6.1.** *Assume that  $\theta \in \text{WIDB}_m$  for some  $m \in \mathbb{Z}^+$  and set  $\ell = \text{led}(\theta, m)$ . Then, the following statements hold.*

(a) *For every  $j = m, m + 1, \dots, \ell$  the numbers  $i_j = \mathbf{b}^-(\theta, j) \in \mathfrak{D}_j$  are well defined and are all of them negative except, perhaps,  $i_\ell = \mathbf{b}^-(\theta, \text{led}(\theta, m))$ .*

(b)

$$\begin{aligned} & |i_m| < |i_{m+1}| < \dots < |i_{\ell-1}| < |i_\ell|, \quad \text{and} \\ & \theta \in B_{\alpha_{|i_\ell|}} [(i_\ell)^*] \subset B_{i_{\ell-1}}^- ((i_{\ell-1})^*) \setminus B_{\alpha_{|i_{\ell-1}|}} [(i_{\ell-1})^*] \\ & \subset B_{i_{\ell-2}}^- ((i_{\ell-2})^*) \setminus B_{\alpha_{|i_{\ell-2}|}} [(i_{\ell-2})^*] \subset \dots \subset B_{i_m}^- ((i_m)^*) \setminus B_{\alpha_{|i_m|}} [(i_m)^*]. \end{aligned}$$

- (c) For every  $j = m, m + 1, \dots, \ell - 1$ ,  $B_{\alpha_{|i_\ell|}} [(i_\ell)^*] \subset \text{WDB}_j$ ,  $\text{led}(\nu, j) = \text{led}(\theta, m)$  and  $\mathbf{b}^\sim(\nu, \text{led}(\nu, j)) = \mathbf{b}^\sim(\theta, \text{led}(\theta, m)) = i_\ell$  for every  $\nu \in B_{\alpha_{|i_\ell|}} [(i_\ell)^*]$ .
- (d)  $\mathbb{I}_{i_m, \nu} = \{\gamma_{|i_m|}(\nu)\} \subset \mathbb{I}_{i_\ell, \nu}$  for every  $\nu \in B_{\alpha_{|i_\ell|}} ((i_\ell)^*)$  and

$$\mathbb{I}_{i_m, \nu} = \{\gamma_{|i_m|}(\nu)\} = \{m_{i_\ell}(\nu)\} = \{M_{i_\ell}(\nu)\} = \{\gamma_{|i_\ell|}(\nu)\} = \mathbb{I}_{i_\ell, \nu}$$

for every  $\nu \in \text{Bd} \left( B_{\alpha_{|i_\ell|}} [(i_\ell)^*] \right)$ .

**Proof.** Since  $B_i^\sim [i^*] = B_{\alpha_i} [i^*]$  for every  $i \geq 0$ ,

$$\mathbb{B}_m^\sim \setminus \mathbb{B}_m = \bigcup_{\substack{i \in \mathcal{D}_m \\ i < 0}} (B_i^\sim [i^*] \setminus B_{\alpha_{|i|}} [i^*]) \tag{10}$$

for every  $m \in \mathbb{Z}^+$ .

Statement (a) follows from Lemma 5.3(b) and (10). Then, (b) follows from Lemma 3.6(g). Statement (c) is an easy consequence of (b) and the definitions.

Now we prove (d) iteratively. Fix  $\nu \in B_{\alpha_{|i_\ell|}} ((i_\ell)^*)$ . By (b)

$$\nu \in B_{i_{m+1}}^\sim ((i_{m+1})^*) \setminus B_{\alpha_{|i_{m+1}|}} [(i_{m+1})^*] \subset B_{i_m}^\sim ((i_m)^*) \setminus B_{\alpha_{|i_m|}} [(i_m)^*]$$

provided that  $\ell = \text{led}(\theta, m) > m + 1$ . Hence, by Lemmas 4.1(d) and 5.4,

$$\begin{aligned} \gamma_{|i_m|}(\nu) &= \gamma_{|i_{m+1}|}(\nu) = \dots = \gamma_{|i_{m+1}|}(\nu), \quad \text{and} \\ \mathbb{I}_{i_m, \nu} &= \{\gamma_{|i_m|}(\nu)\} = \{\gamma_{|i_{m+1}|}(\nu)\} = \mathbb{I}_{i_{m+1}, \nu}. \end{aligned}$$

By iterating this argument we get,

$$\gamma_{|i_m|}(\nu) = \gamma_{|i_{m+1}|}(\nu) = \dots = \gamma_{|i_{\ell-1}|}(\nu) \quad \text{and} \quad \mathbb{I}_{i_m, \nu} = \mathbb{I}_{i_{\ell-1}, \nu}.$$

Again by (b) and Lemmas 4.1(d) and 5.4,

$$\gamma_{|i_m|}(\nu) = \gamma_{|i_{m+1}|}(\nu) = \dots = \gamma_{|i_\ell|}(\nu) \quad \text{and} \quad \mathbb{I}_{i_m, \nu} = \mathbb{I}_{i_\ell, \nu}$$

when  $\nu \in \text{Bd} \left( B_{\alpha_{|i_\ell|}} [(i_\ell)^*] \right)$  and, otherwise,

$$\gamma_{|i_m|}(\nu) = \gamma_{|i_{m+1}|}(\nu) = \dots = \gamma_{|i_{\ell-1}|}(\nu) \quad \text{and} \quad \mathbb{I}_{i_m, \nu} \subset \mathbb{I}_{i_\ell, \nu}. \quad \square$$

Equipped with above results and definition we are going to define two maps, analogous to the maps  $m_i$  and  $M_i$ , on the wings of the negative boxes.

**Definition 6.2.** For every  $m \in \mathbb{Z}^+$  we define

$$\begin{aligned} \mathfrak{W}\mathfrak{F}\mathfrak{D}_m &:= \{\mathbf{b}^\sim(\theta, \text{led}(\theta, m)) : \theta \in \text{WDB}_m\} \subset \mathbb{Z}, \\ \text{WIB}_m &:= \text{Int}(\text{WDB}_m) = \bigcup_{i \in \mathfrak{W}\mathfrak{F}\mathfrak{D}_m} B_{\alpha_{|i|}} (i^*), \\ \text{WB}_m^\sim &:= \bigcup_{\substack{i \in \mathcal{D}_m \\ i < 0}} (B_i^\sim [i^*] \setminus B_{\alpha_{|i|}} (i^*)), \quad \text{and} \\ \text{EB}_m^\sim &:= \bigcup_{i \in \mathcal{D}_m} \text{Bd} (B_i^\sim [i^*]) \subset \mathbb{B}_m^\sim. \end{aligned}$$

By Lemma 6.1(a), (c),  $\mathfrak{W}\mathfrak{F}\mathfrak{D}_m$  is well defined and

$$\text{WIB}_m \subset \text{WDB}_m \subset \mathbb{B}_m^\sim \setminus \mathbb{B}_m \subset \text{WB}_m^\sim.$$

Consequently,

$$\mathbb{B}_m^\sim = \mathbb{B}_m \cup \text{WB}_m^\sim.$$

Then, we can define functions  $\tau_m : \text{WB}_m^\sim \rightarrow \mathbb{I}$  and  $\lambda_m : \text{WB}_m^\sim \rightarrow \mathbb{I}$  as follows:

$$\begin{aligned} \tau_m(\theta) &:= \begin{cases} M_{\mathbf{b}^\sim(\theta, \text{led}(\theta, m))}(\theta) & \text{if } \theta \in \text{WIB}_m, \\ \gamma_{|\mathbf{b}^\sim(\theta, m)|}(\theta) & \text{otherwise,} \end{cases} \\ \lambda_m(\theta) &:= \begin{cases} m_{\mathbf{b}^\sim(\theta, \text{led}(\theta, m))}(\theta) & \text{if } \theta \in \text{WIB}_m, \\ \gamma_{|\mathbf{b}^\sim(\theta, m)|}(\theta) & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly, by Lemmas 4.1(a) and 3.6(b),

$$-1 \leq \lambda_m(\theta) \leq \tau_m(\theta) \leq 1$$

for every  $\theta \in \text{WB}_m^\sim$ . So, we can define

$$\text{IW}_{m,\theta} := [\lambda_m(\theta), \tau_m(\theta)] \subset [0, 1]. \quad \blacksquare$$

The next lemmas will help us in the definition and study of the maps  $T_m$ .

**Lemma 6.3.** *The following statements hold for every  $m \in \mathbb{Z}^+$ .*

(a)  $\text{WIB}_m \cap \mathbb{B}_m = \text{WIB}_m \cap \mathbb{EB}_m^\sim = \emptyset$ .

(b) Let  $\theta \in \text{WB}_m^\sim$ . Then,  $\mathbb{I}_{\mathbf{b}^\sim(\theta, m), \theta} = \left\{ \gamma_{|\mathbf{b}^\sim(\theta, m)|}(\theta) \right\}$ ,

$$\mathbb{I}_{\mathbf{b}^\sim(\theta, m), \theta} = \text{IW}_{m,\theta} \quad \text{when } \theta \notin \text{WIB}_m, \text{ and}$$

$$\mathbb{I}_{\mathbf{b}^\sim(\theta, m), \theta} \subset \text{IW}_{m,\theta} \quad \text{when } \theta \in \text{WIB}_m.$$

(c) Assume that  $m \in \mathbb{N}$  and let  $U$  be a connected component of  $\text{WB}_m^\sim$  such that  $U \subset \text{WB}_{m-1}^\sim$ . Then,  $\text{WDB}_m \cap U \subset \text{WDB}_{m-1}$ ,  $\text{WIB}_m \cap U = \text{WIB}_{m-1} \cap U$  and  $\text{IW}_{m,\theta} = \text{IW}_{m-1,\theta}$  for every  $\theta \in U$ .

**Proof.** (a) By Lemma 6.1(b),

$$\theta \in B_{\mathbf{b}^\sim(\theta, m)}^\sim \left( (\mathbf{b}^\sim(\theta, m))^* \right) \setminus B_{\alpha_{|\mathbf{b}^\sim(\theta, m)|}} \left[ (\mathbf{b}^\sim(\theta, m))^* \right]$$

and  $\mathbf{b}^\sim(\theta, m) < 0$  for every  $\theta \in \text{WIB}_m \subset \text{WDB}_m$ . So, by Lemma 5.2(b), we get  $\theta \notin \mathbb{B}_m \cup \mathbb{EB}_m^\sim$ .

(b) The fact that  $\mathbb{I}_{\mathbf{b}^\sim(\theta, m), \theta} = \left\{ \gamma_{|\mathbf{b}^\sim(\theta, m)|}(\theta) \right\}$  follows from Lemma 4.1(d). The other two statements follow from Definition 6.2 and Lemma 6.1(d).

(c) The assumption that  $U$  is a connected component of  $\text{WB}_m^\sim$  and  $U \subset \text{WB}_{m-1}^\sim$  implies by Lemmas 5.2(b) and 3.6(g) that there exist  $i \in \mathfrak{D}_m$  and  $k \in \mathfrak{D}_{m-1}$ ,  $i, k < 0$ , such that  $U$  is a connected component of

$$B_i^\sim [i^*] \setminus B_{\alpha_{|i|}} (i^*) \subset B_k^\sim (k^*) \setminus B_{\alpha_{|k|}} [k^*] \subset \text{WB}_{m-1}^\sim.$$

Again by Lemma 5.2(b) this implies that  $U \subset \mathbb{B}_{m-1}^\sim \setminus \mathbb{B}_{m-1}$ . Moreover, by definition,  $\text{WDB}_m \subset \mathbb{B}_m^\sim \setminus \mathbb{B}_m$ . Consequently,  $\text{WDB}_m \cap U \subset \text{WDB}_{m-1}$ .

Let  $\theta \in \text{WIB}_m \cap U \subset \text{WDB}_m \cap U \subset \text{WDB}_{m-1} \cap U$ . By Definition 6.2 and Lemma 6.1(a), (b),  $i = \mathbf{b}^\sim(\theta, m)$  and there exists  $\ell = \mathbf{b}^\sim(\theta, \text{led}(\theta, m)) \in \mathfrak{W}\mathfrak{F}\mathfrak{D}_m$  such that

$$\theta \in B_{\alpha_{|\ell|}} (\ell^*) \subset B_i^\sim [i^*] \setminus B_{\alpha_{|i|}} (i^*) \subset B_k^\sim (k^*) \setminus B_{\alpha_{|k|}} [k^*].$$

Therefore, again by Lemma 6.1(a)–(c) and Definition 6.2,  $\text{led}(\theta, m - 1) = \text{led}(\theta, m)$ ,

$$\ell = \mathbf{b}^\rceil(\theta, \text{led}(\theta, m)) = \mathbf{b}^\rceil(\theta, \text{led}(\theta, m - 1)) \in \mathfrak{W}\mathfrak{F}\mathfrak{D}_{m-1}$$

and  $\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset \text{WIB}_{m-1}$ . Hence,  $\text{WIB}_m \cap U \subset \text{WIB}_{m-1}$ .

Now assume that  $\theta \in \text{WIB}_{m-1} \cap U$ . As above, there exist  $r = \mathbf{b}^\rceil(\theta, m) \in \mathfrak{D}_m$  and  $\ell = \mathbf{b}^\rceil(\theta, \text{led}(\theta, m - 1)) \in \mathfrak{W}\mathfrak{F}\mathfrak{D}_{m-1}$  such that

$$\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset B_r^\rceil(r^*) \setminus B_{\alpha_{|r|}}[r^*] \subset B_k^\rceil(k^*) \setminus B_{\alpha_{|k|}}[k^*].$$

Since  $\theta \in U \subset B_i^\rceil[i^*]$ , Lemma 5.2(b) gives  $i = r$  and  $\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset U$ . Moreover, by Lemma 6.1(c),  $\ell = \mathbf{b}^\rceil(\theta, \text{led}(\theta, m - 1)) = \mathbf{b}^\rceil(\theta, \text{led}(\theta, m)) \in \mathfrak{W}\mathfrak{F}\mathfrak{D}_m$  and, so,  $\theta \in B_{\alpha_{|\ell|}}(\ell^*) \subset \text{WIB}_m$ . Thus,  $\text{WIB}_m \cap U = \text{WIB}_{m-1} \cap U$ .

To end the proof of the lemma we have to show that  $\text{IW}_{m,\theta} = \text{IW}_{m-1,\theta}$  for every  $\theta \in U$ . Assume first that  $\theta \in U \setminus \text{WIB}_m \subset \text{WIB}_m^\rceil \setminus \text{WIB}_m$ . Then,

$$\theta \in U \setminus \text{WIB}_m = U \setminus \text{WIB}_{m-1} \subset \text{WIB}_{m-1}^\rceil \setminus \text{WIB}_{m-1}$$

and, by (b) and Lemmas 4.1(d) and 5.4,

$$\text{IW}_{m,\theta} = \mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} = \mathbb{I}_{k,\theta} = \text{IW}_{m-1,\theta}.$$

If  $\theta \in U \cap \text{WIB}_m = U \cap \text{WIB}_{m-1}$  then we get

$$\begin{aligned} \text{IW}_{m,\theta} &= [m_{\mathbf{b}^\rceil(\theta, \text{led}(\theta, m))}(\theta), M_{\mathbf{b}^\rceil(\theta, \text{led}(\theta, m))}(\theta)] \\ &= [m_{\mathbf{b}^\rceil(\theta, \text{led}(\theta, m-1))}(\theta), M_{\mathbf{b}^\rceil(\theta, \text{led}(\theta, m-1))}(\theta)] = \text{IW}_{m-1,\theta} \end{aligned}$$

from Definition 6.2 and Lemma 6.1(c).  $\square$

**Lemma 6.4.** *Let  $m \in \mathbb{Z}^+$  and let  $U$  be a connected component of  $\text{WIB}_m^\rceil$ . Then, the functions  $\lambda_m|_U$  and  $\tau_m|_U$  are continuous.*

**Proof.** We will prove only the continuity of  $\lambda_m|_U$ . The proof of the continuity of  $\tau_m|_U$  is analogous.

By Lemmas 6.1(c) and 4.1(b) we get

$$\begin{aligned} &\text{for every } \ell \in \mathfrak{W}\mathfrak{F}\mathfrak{D}_m, \ell = \mathbf{b}^\rceil(\nu, \text{led}(\nu, m)) \text{ for every } \nu \in B_{\alpha_{|\ell|}}[\ell^*], \text{ and the} \\ &\text{function } m_\ell \text{ is continuous on } B_{\alpha_{|\ell|}}[\ell^*]. \end{aligned} \tag{11}$$

Let  $\ell \in \mathfrak{W}\mathfrak{F}\mathfrak{D}_m$  be such that  $B_{\alpha_{|\ell|}}(\ell^*) \subset \text{WIB}_m \cap U$ . Thus, by (11), the function  $\lambda_m = m_\ell$  is continuous on  $B_{\alpha_{|\ell|}}(\ell^*)$ .

So, we have to show that  $\lambda_m$  is continuous at every  $\theta \in U \setminus \text{WIB}_m$ . To show this we will use a simple usual  $\varepsilon$ - $\delta$  game. Fix  $\varepsilon > 0$ .

By Lemma 5.2(b) it follows that  $U$  is a connected component of  $B_i^\rceil[i^*] \setminus B_{\alpha_{|i|}}(i^*)$  for some  $i \in \mathfrak{D}_m$ ,  $i < 0$ , and

$$\mathbf{b}^\rceil(\nu, m) = i \quad \text{for every } \nu \in U. \tag{12}$$

By Lemma 3.6(a) and Definition 3.4(R.2) and Remark 3.5(R.2), the function  $\gamma_{|i|}|_U$  is continuous. So,

$$\text{there exists } \bar{\delta}_{|i|} = \bar{\delta}_{|i|}(\theta) > 0 \text{ such that } |\gamma_{|i|}(\theta), \gamma_{|i|}(\nu)| < \varepsilon/2 \text{ provided that } \mathbf{d}_{s_1}(\theta, \nu) < \bar{\delta}_{|i|}. \tag{13}$$



On the other hand, by (11),

$$\begin{aligned} &\text{for every } \ell \in \mathfrak{WF}\mathfrak{D}_m, \text{ there exists } \delta_\ell > 0 \text{ such that } |m_\ell(\tilde{\theta}), m_\ell(\nu)| < \varepsilon/2 \\ &\text{for every } \tilde{\theta} \in \text{Bd}(B_{\alpha_{|\ell|}}[\ell^*]) \text{ and } \nu \in \text{Bd}B_{\alpha_{|\ell|}}[\ell^*] \text{ such that } d_{s^1}(\theta, \nu) < \delta_\ell. \end{aligned} \tag{14}$$

Now we will define  $\delta$ . Note that there exists  $N \in \mathbb{N}$  such that  $2^{-N} < \varepsilon/2$ . Then we set:

$$\delta = \delta(\theta) := \min\left\{\bar{\delta}_{|i|}(\theta), \min\{\delta_\ell : \ell \in \mathfrak{WF}\mathfrak{D}_m \text{ and } |\ell| < N\}\right\}.$$

Clearly,  $\delta > 0$  because the set  $\{\ell \in \mathfrak{WF}\mathfrak{D}_m : |\ell| < N\}$  is finite.

To end the proof of the lemma we have to show that

$$|\lambda_m(\theta) - \lambda_m(\nu)| < \varepsilon$$

whenever  $\nu \in U$  and  $d_{s^1}(\theta, \nu) < \delta$ .

Assume that  $\nu \in U$  and  $d_{s^1}(\theta, \nu) < \delta$  (recall that we have the assumption that  $\theta \notin \text{WIB}_m$ ). If  $\nu \notin \text{WIB}_m$ , then  $d_{s^1}(\theta, \nu) < \delta \leq \bar{\delta}_{|i|}(\theta)$  and, by (12) and (13),

$$|\lambda_m(\theta) - \lambda_m(\nu)| = |\gamma_{|i|}(\theta) - \gamma_{|i|}(\nu)| < \varepsilon/2 < \varepsilon.$$

Now assume that there exists  $\ell \in \mathfrak{WF}\mathfrak{D}_m$  such that  $\nu \in B_{\alpha_{|\ell|}}(\ell^*) \subset \text{WIB}_m$ . Clearly, there exists  $\tilde{\theta} \in \text{Bd}(B_{\alpha_{|\ell|}}[\ell^*])$  such that

$$\begin{aligned} d_{s^1}(\theta, \tilde{\theta}) &< d_{s^1}(\theta, \nu) < \delta \leq \bar{\delta}_{|i|}(\theta) \quad \text{and} \\ d_{s^1}(\tilde{\theta}, \nu) &< d_{s^1}(\theta, \nu) < \delta. \end{aligned}$$

Observe that, by Lemma 5.2(b),  $\tilde{\theta} \notin \text{WIB}_m$ . Hence, by (12) and Lemma 6.1(c), (d),

$$\lambda_m(\tilde{\theta}) = \gamma_{|i|}(\tilde{\theta}) = m_\ell(\tilde{\theta}).$$

If  $|\ell| < N$ , then  $d_{s^1}(\tilde{\theta}, \nu) < \delta \leq \delta_\ell$  and, by (14),  $|m_\ell(\tilde{\theta}) - m_\ell(\nu)| < \varepsilon/2$ . Otherwise, by Lemma 3.6(f),

$$|m_\ell(\tilde{\theta}) - m_\ell(\nu)| < \text{diam}(\mathcal{R}(\ell^*)) \leq 2^{-|\ell|} \leq 2^{-N} < \varepsilon/2.$$

In any case,  $|m_\ell(\tilde{\theta}) - m_\ell(\nu)| < \varepsilon/2$ . Thus, again by (12) and (13),

$$\begin{aligned} |\lambda_m(\theta) - \lambda_m(\nu)| &\leq |\lambda_m(\theta) - \lambda_m(\tilde{\theta})| + |\lambda_m(\tilde{\theta}) - \lambda_m(\nu)| \\ &= |\gamma_{|i|}(\theta) - \gamma_{|i|}(\tilde{\theta})| + |m_\ell(\tilde{\theta}) - m_\ell(\nu)| < \varepsilon. \quad \square \end{aligned}$$

### 7. A Cauchy sequence of skew products. Proof of Theorem A

In this section prove Theorem A. To do this we inductively construct a Cauchy sequence  $\{T_m\}_{m=0}^\infty \subset \mathcal{S}(\Omega)$  that gives the function  $T$  from Theorem A as a limit.

The sequence  $\{T_m\}_{m=0}^\infty \subset \mathcal{S}(\Omega)$  is defined so that

$$T_m(\theta, x) = (R_\omega(\theta), f_m(\theta, x))$$

and  $f_m : \Omega \rightarrow \mathbb{I}$  is continuous. The maps  $f_m(\theta, \cdot)$  will also be denoted as  $f_{m,\theta}$ , and will be defined non-increasing, and such that  $f_{m,\theta}(2) = -2$  and  $f_{m,\theta}(-2) = 2$  for every  $\theta \in \mathbb{S}^1$ .

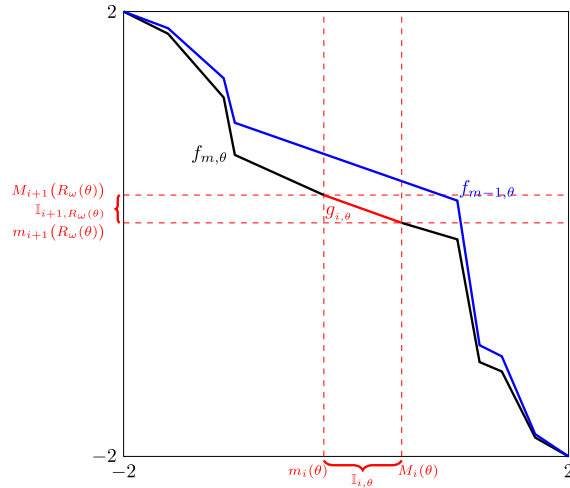


Fig. 5. A symbolic representation of the map  $f_{m,\theta}$  in a generic case (red and black) together with the map  $f_{m-1,\theta}$  in blue. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

To build these functions we will use the auxiliary functions

$$G_i: \mathcal{R}^{\sim}(i^*) \longrightarrow \mathcal{R}^{\sim}((i+1)^*)$$

with  $i \in \mathbb{Z}$  from Section 4. The basic property of the maps  $T_m$  is that for every  $i$  of depth  $m$  (that is,  $i \in \mathfrak{D}_m$ ),

$$T_m|_{\mathcal{R}^{\sim}(i^*)} = G_i,$$

and  $f_{m,\theta}$  send  $\mathbb{I}_{i,\theta}$  affinely into either  $\mathbb{I}_{i+1,R_\omega(\theta)}$  or  $A_{|i+1|}^{R_\omega(\theta)}$  (the red part in Fig. 5). Moreover, for those integers  $i$ ,

$$T_m|_{\uparrow\uparrow B_i^{\sim}[i^*] \setminus \mathcal{R}^{\sim}(i^*)}$$

is obtained by an affine perturbation of  $T_{m-1}$  that preserves the continuity of  $f_{m,\theta}$  (the black part in Fig. 5 which is obtained from the blue graph).

To make more evident the strategy of the construction of this sequence of maps we will separate several cases, and we will state without proofs the results that study these maps. After establishing all the definitions and results related to the construction of the sequence  $\{T_m\}_{m=0}^\infty$  without having been distracted by the technicalities involving the proofs, we will proceed to provide the missing proofs. More precisely, we will start by defining the map  $T_0$  and stating without proof the proposition that summarizes the necessary properties of this map. Next we will inductively define the maps  $\{T_m\}_{m=1}^\infty \subset \mathcal{S}(\Omega)$  and state without proof the proposition that establishes the properties of the whole sequence  $\{T_m\}_{m=0}^\infty$ .

Then, as we have said, we prove Theorem A and in the Appendix (Appendices A.2–A.4). we will provide all pending proofs.

In what follows  $\mathcal{C}(\mathbb{I}, \mathbb{I})$  will denote the class of all continuous maps from  $\mathbb{I}$  to itself. We endow  $\mathcal{C}(\mathbb{I}, \mathbb{I})$  with the supremum metric denoted by  $\|\cdot\|$  so that  $(\mathcal{C}(\mathbb{I}, \mathbb{I}), \|\cdot\|)$  is a complete metric space.

Next we define the map  $T_0$ .

**Definition 7.1** (*The Map  $T_0$* ). Assume first that  $\theta \in \mathbb{B}_0^\rceil$  and let  $i = \mathbf{b}^\rceil(\theta, 0)$  (that is  $\theta \in B_i^\rceil [i^*]$ ). In this case we set:

$$f_{0,\theta}(x) = \begin{cases} g_{i,\theta}(x) & \text{if } x \in \mathbb{I}_{i,\theta}, \\ \frac{g_{i,\theta}(m_i(\theta)) - 2}{m_i(\theta) + 2}(x + 2) + 2 & \text{if } x \in [-2, m_i(\theta)], \\ \frac{g_{i,\theta}(M_i(\theta)) + 2}{M_i(\theta) - 2}(x - 2) - 2 & \text{if } x \in [M_i(\theta), 2]. \end{cases}$$

If  $\theta \in \mathbb{S}^1 \setminus \mathbb{B}_0^\rceil$  then we define  $f_{0,\theta}$  to be the unique piecewise affine map with two affine pieces whose graph joins the point  $(-2, 2)$  with  $(0, \gamma(R_\omega(\theta)))$ , and this with the point  $(2, -2)$ . ■

Next we introduce some more notation to be able to define the maps  $\{T_m\}_{m=1}^\infty$ . For every  $k \in \mathbb{Z}$  we set

$$V_{k^*}^\rceil := \prod B_k^\rceil [k^*] = B_k^\rceil [k^*] \times \mathbb{I}$$

and, for every  $m \in \mathbb{Z}^+$ ,

$$V_m^\rceil := \prod B_m^\rceil = B_m^\rceil \times \mathbb{I} = \bigcup_{i \in \mathcal{D}_m} V_{i^*}^\rceil.$$

**Definition 7.2** (*The Maps  $T_m$  with  $m > 0$* ). Now we assume that we have defined the function  $T_{m-1}$  for some  $m \geq 1$  and we define

$$T_m(\theta, x) = (R_\omega(\theta), f_m(\theta, x))$$

as follows. By Lemma 5.2(b), for every  $(\theta, x) \in V_m^\rceil$ , we have

$$\theta \in B_i^\rceil [i^*] \subset B_m^\rceil \quad \text{with } i = \mathbf{b}^\rceil(\theta, m) \in \mathcal{D}_m$$

(and, of course,  $x \in \mathbb{I}$ ). Then we define:

$$f_{m,\theta}(x) = \begin{cases} f_{m-1,\theta}(x) & \text{if } \theta \in \mathbb{S}^1 \setminus \mathbb{B}_m^\rceil; x \in \mathbb{I}, \\ g_{i,\theta}(x) & \text{if } \theta \in \mathbb{B}_m; x \in \mathbb{I}_{i,\theta}, \\ \frac{2 - g_{i,\theta}(m_i(\theta))}{2 - f_{m-1,\theta}(m_i(\theta))}(f_{m-1,\theta}(x) - 2) + 2 & \text{if } \theta \in \mathbb{B}_m; x \in [-2, m_i(\theta)], \\ \frac{2 + g_{i,\theta}(M_i(\theta))}{2 + f_{m-1,\theta}(M_i(\theta))}(f_{m-1,\theta}(x) + 2) - 2 & \text{if } \theta \in \mathbb{B}_m; x \in [M_i(\theta), 2], \\ \gamma_{|i+1|}(R_\omega(\theta)) & \text{if } \theta \in \mathbb{WB}_m^\rceil; x \in \mathbb{IW}_{m,\theta}, \\ \frac{2 - \gamma_{|i+1|}(R_\omega(\theta))}{2 - f_{m-1,\theta}(\lambda_m(\theta))}(f_{m-1,\theta}(x) - 2) + 2 & \text{if } \theta \in \mathbb{WB}_m^\rceil; x \in [-2, \lambda_m(\theta)], \\ \frac{2 + \gamma_{|i+1|}(R_\omega(\theta))}{2 + f_{m-1,\theta}(\tau_m(\theta))}(f_{m-1,\theta}(x) + 2) - 2 & \text{if } \theta \in \mathbb{WB}_m^\rceil; x \in [\tau_m(\theta), 2]. \end{cases}$$

Since  $V_m^\rceil \subset V_{m-1}^\rceil$ ,  $f_{m-1,\theta}$  is defined on  $V_m^\rceil$ . Moreover, the above formula defines  $f_{m,\theta}$  for every  $\theta \in B_m^\rceil$  since, by Definition 6.2,  $B_m^\rceil = B_m \cup \mathbb{WB}_m^\rceil$ . We also remark that  $f_{m,\theta}$  formally is defined in two different ways when  $\theta \in \mathbb{WB}_m^\rceil \cap B_m$ . Later on we will show that  $f_{m,\theta}$  is well defined. ■

The next proposition studies the maps  $\{T_m\}_{m=0}^\infty$  and describes their properties.

**Proposition 7.3.** *The following statements hold for every  $m \in \mathbb{Z}^+$ .*

- (a) The map  $T_m$  is well defined, continuous and belongs to  $\mathcal{S}(\Omega)$ .
- (b) For every  $\theta \in \mathbb{S}^1$ ,  $f_{m,\theta}$  is non-increasing, and  $f_{m,\theta}(2) = -2$ ,  $f_{m,\theta}(-2) = 2$ . Moreover,  $-1 \leq f_{0,\theta}(M_{\mathbb{b}^{\leftarrow}(\theta,m)}(\theta)) \leq f_{0,\theta}(m_{\mathbb{b}^{\leftarrow}(\theta,m)}(\theta)) \leq 1$  for every  $\theta \in \mathbb{B}_m^{\leftarrow}$ .
- (c) For every  $i \in \mathfrak{D}_m$ ,  $T_m|_{\mathcal{R}^{\leftarrow}(i^*)} = G_i$ ,  $T_m(A_{|i|}^{i^*}) = A_{|i+1|}^{(i+1)^*}$ , and  $T_k|_{\{i^*\} \times \mathbb{I}} = T_m|_{\{i^*\} \times \mathbb{I}}$  (that is,  $f_{k,i^*} = f_{m,i^*}$ ) for every  $k > m$ .

The next result shows that the sequence  $\{T_m\}_{m=0}^\infty$  has a limit in  $\mathcal{S}(\Omega)$ .

**Proposition 7.4.** For every  $m \geq 2$  and  $\theta \in \mathbb{S}^1$ ,

$$\|f_{m,\theta} - f_{m-1,\theta}\| \leq 2 \cdot 2^{-|b^{\leftarrow}(\theta,m-1)|}. \tag{15}$$

Moreover, the sequence  $\{T_m\}_{k=0}^\infty$  is a Cauchy sequence.

Finally we are ready to prove the main result of the paper. It follows from the next result which gives a more concrete version of [Theorem A](#).

**Theorem 7.5.** There exists a map  $T \in \mathcal{S}(\Omega)$  with  $f(\theta, \cdot)$  non-increasing for every  $\theta \in \mathbb{S}^1$ , such that  $T$  permutes the upper and lower circles of  $\Omega$  (thus having a periodic orbit of period two of curves), and there exists a connected pseudo-curve  $A \subset \Omega$  which does not contain any arc of a curve such that  $T(A) = A$  and there does not exist any  $T$ -invariant curve.

**Proof.** By [Propositions 7.3](#) and [7.4](#), there exists a map

$$T(\theta, x) = (R_\omega(\theta), f(\theta, x)) = (R_\omega(\theta), \lim_{m \rightarrow \infty} f_m(\theta, x)) \in \mathcal{S}(\Omega)$$

with  $f(\theta, \cdot)$  non-increasing for every  $\theta \in \mathbb{S}^1$  such that  $T$  permutes the upper and lower circles of  $\Omega$  (that is,  $f(\theta, 2) = -2$  and  $f(\theta, -2) = 2$ ). As the connected set  $A$  we take the one given by [Proposition 3.12](#) (and [Definition 3.9](#)).

To end the proof of the theorem we need to show that  $T(A) = A$ , since this already implies that there does not exist any  $T$ -invariant curve. To see it, assume by way of contradiction that there exists an invariant curve and denote its graph by  $B$ . Since  $B$  is the graph of a (continuous) curve, it is compact and connected. On the other hand, let  $\Omega_+$  and  $\Omega_-$  be the two connected components of  $\Omega \setminus A$  from the proof of [Proposition 3.12](#). The facts that  $T(A) = A$ ,  $f(\theta, \cdot)$  is decreasing for every  $\theta \in \mathbb{S}^1$ , and  $T$  permutes the upper and lower circles of  $\Omega$  imply that  $T(\Omega_+) = \Omega_-$  and  $T(\Omega_-) = \Omega_+$ . Hence, by the invariance of  $B$ ,  $B \not\subseteq \Omega_+$  and  $B \not\subseteq \Omega_-$ . The connectivity of  $A$  and  $B$  implies that there exists  $(\theta, x) \in A \cap B$ . Consequently,

$$B = \overline{\{T^n(\theta, x) : n \in \mathbb{Z}^+\}} \subset A;$$

a contradiction because  $A$  does not contain any arc of a curve.

So, only it remains to prove that  $T(A) = A$ . By using [Proposition 7.3\(c\)](#) and [Lemma 3.11\(b\)](#) we get that  $T_m(A^{i^*}) = A^{(i+1)^*}$ , and  $T_k|_{A^{i^*}} = T_m|_{A^{i^*}}$  for every  $k, m \in \mathbb{Z}^+$ ,  $k \geq m$  and  $i \in \mathfrak{D}_m$ . Consequently, by the definition of the map  $T$  we have,  $T(A^{i^*}) = A^{(i+1)^*}$  for every  $i \in \mathbb{Z}$  or, equivalently,  $T(A^{\uparrow O^*(\omega)}) = A^{\uparrow O^*(\omega)}$ .

Now we consider  $A^\theta$  with  $\theta \in \mathbb{S}^1 \setminus O^*(\omega)$ . Since  $O^*(\omega)$  is dense in  $\mathbb{S}^1$ , there exists a sequence  $\{(\theta_n, x_n)\}_{n=0}^\infty \subset A^{\uparrow O^*(\omega)}$  such that  $\lim_{n \rightarrow \infty} \theta_n = \theta$ . By the compacity of  $A$  we can assume (by taking a convergent subsequence, if necessary) that  $\{(\theta_n, x_n)\}_{n=0}^\infty$  is convergent to a point  $(\theta, x) \in A$ . By [Lemma 3.11\(c\)](#),  $A^\theta = (\theta, x)$  (and  $x = \gamma(\theta)$ ). On the other hand, by the part of the statement already proven,  $T(\theta_n, x_n) \in A$  for every  $n$ . Hence, by the continuity of  $T$  and the compacity of  $A$ ,

$$T(\theta, x) = (R_\omega(\theta), f(\theta, x)) = \lim_{n \rightarrow \infty} T(\theta_n, x_n) \in A^{R_\omega(\theta)}.$$

Since  $\theta \notin O^*(\omega)$  we have that  $R_\omega(\theta) \notin O^*(\omega)$  and, again by Lemma 3.11(c),  $A^{R_\omega(\theta)}$  consists of a unique point. Hence,  $T(A^\theta) = A^{R_\omega(\theta)}$  for every  $\theta \in \mathbb{S}^1 \setminus O^*(\omega)$ . Equivalently,  $T\left(A^\uparrow(\mathbb{S}^1 \setminus O^*(\omega))\right) = A^\uparrow(\mathbb{S}^1 \setminus O^*(\omega))$ . This ends the proof of the theorem.  $\square$

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## Appendix. The most technical proofs

### A.1. Proof of Lemma 3.6

We start by proving the first statement of the lemma and (a) by induction.

Observe that  $n_0 = 1$ ,  $\alpha_0$ ,  $\delta_0$  and  $\gamma_0$  are defined so that Definition 3.4(R.1–2) for  $j = 0$  and  $(\gamma_0, \mathbb{S}^1 \setminus Z_0^*) \in \mathcal{PCG}$  are verified except for the obvious fact that  $B_{-j}^\uparrow [(-j)^*] = B_j^\uparrow [j^*]$ . On the other hand, by construction,  $B_{\alpha_0} [0^*]$  is disjoint from  $B_{\alpha_0} [1^*]$  and  $B_{\alpha_0} [(-1)^*]$ . Then, by the definition of  $\gamma_0$ ,

$$\gamma_0(B_{\alpha_0}[\ell^*]) = \{0\} \subset \left[-\frac{1}{2}, \frac{1}{2}\right] = [\gamma_0(\ell^*) - 2^{-n_0}, \gamma_0(\ell^*) + 2^{-n_0}]$$

for  $\ell \in \{1, -1\}$ . Hence, (a) holds.

Fix  $j > 0$  and assume that we have defined  $n_\ell$ ,  $\alpha_\ell$ ,  $\delta_\ell$  and  $\gamma_\ell$  such that all Definition 3.4(R.1–6) and (a) hold for  $\ell = 0, 1, \dots, j-1$ .

Since the elements of  $Z_{j+2}^*$  are pairwise different, we can choose an integer  $n_j > n_{j-1}$  and  $\delta_j$  and  $\alpha_j$  small enough so that

- $0 < \delta_j < \alpha_j < 2^{-n_j} < \delta_{j-1}$ ,
- $(-(j+2))^*, (j+2)^* \notin B_{-(j+1)}^\uparrow [(-(j+1))^*] = B_{\alpha_j} [(-(j+1))^*]$ ,
- the three intervals  $B_j^\uparrow [j^*] = B_{\alpha_j} [j^*]$ ,  $R_\omega(B_{\alpha_j} [j^*]) = B_{\alpha_j} [(j+1)^*]$  and  $B_{-(j+1)}^\uparrow [(-(j+1))^*]$  are pairwise disjoint,
- $B_\ell^\uparrow [\ell^*] \cap Z_{j+1}^* = \{\ell^*\}$  for  $\ell \in \{j, -(j+1)\}$ ,  
 $B_{\alpha_j} [(j+1)^*] \cap Z_{j+1}^* = \{(j+1)^*\}$  and, additionally,
- $\left(\text{Bd}(B_{\alpha_j} [(-j)^*]) \cup \text{Bd}(B_{\alpha_j} [j^*])\right) \cap O^*(\omega) = \emptyset$ .

Then, Definition 3.4(R.1) is verified. Moreover, from the above conditions it follows that  $B_{\alpha_j} [\ell^*] \cap Z_{j+1}^* = \{\ell^*\}$  for every  $\ell \in \{j+1, -(j+1)\}$ . Thus, by statement (a) for  $j-1$ ,  $\gamma_{j-1}$  is defined and continuous on  $\ell^* \in B_{\alpha_j} [\ell^*]$  because this interval is disjoint from  $Z_{j-1}^*$ . Hence, we can decrease the value of  $\alpha_j$  (and, accordingly, the value of  $0 < \delta_j < \alpha_j$ ), if necessary, to get

- $\gamma_{j-1}(B_{\alpha_j}[\ell^*]) \subset [\gamma_{j-1}(\ell^*) - 2^{-n_j}, \gamma_{j-1}(\ell^*) + 2^{-n_j}]$   
for every  $\ell \in \{j+1, -(j+1)\}$ .

To see that Definition 3.4(R.2) is verified it remains to show that the intervals  $B_j^\uparrow [j^*]$ ,  $B_{\alpha_j} [(j+1)^*]$  and  $B_{-(j+1)}^\uparrow [(-(j+1))^*]$  are disjoint from  $B_{-j}^\uparrow [(-j)^*]$ . By induction, Definition 3.4(R.2) holds for  $j-1$ . Thus we see, that  $(-(j+1))^*, (j+1)^* \notin B_{-j}^\uparrow [(-j)^*]$ , and  $R_\omega(B_{\alpha_{j-1}} [(j-1)^*]) = B_{\alpha_{j-1}} [j^*]$  is disjoint from  $B_{-j}^\uparrow [(-j)^*]$ . Hence, we can decrease the value of  $\alpha_j$  (and, accordingly, the value of  $0 < \delta_j < \alpha_j$ ), if necessary, until  $B_{\alpha_j} [(j+1)^*]$  and  $B_{-(j+1)}^\uparrow [(-(j+1))^*] = B_{\alpha_j} [(-(j+1))^*]$  are disjoint from  $B_{-j}^\uparrow [(-j)^*]$ . On the other hand we have that  $\alpha_j < 2^{-n_j} < \delta_{j-1} < \alpha_{j-1}$ . So,  $B_j^\uparrow [j^*] = B_{\alpha_j} [j^*] \subset B_{\alpha_{j-1}} [j^*]$  is disjoint from  $B_{-j}^\uparrow [(-j)^*]$ .

Up to now we have seen that we can choose  $n_j$ ,  $\delta_j$  and  $\alpha_j$  so that Definition 3.4(R.1–2) hold for  $j$ . Let us see that we can choose  $\alpha_j$  such that Definition 3.4(R.3) also holds. Observe that for every  $\ell, i \in \mathbb{Z}$  and every  $m \geq 0$  it follows that  $\text{Bd}(B_{\alpha_m}[\ell^*]) \cap O^*(\omega) \neq \emptyset$  if and only if  $\text{Bd}(R_\omega^i(B_{\alpha_m}[\ell^*])) \cap O^*(\omega) = \text{Bd}(B_{\alpha_m}[(\ell+i)^*]) \cap O^*(\omega) \neq \emptyset$ . Therefore, by using Definition 3.4(R.1) inductively, we obtain

$$\bigcup_{k \in Z_{j-1}} \text{Bd}(B_{\alpha_{|k|}}[(k+1)^*]) \cap \{(-j)^*, j^*\} \subset \bigcup_{k \in Z_{j-1}} \text{Bd}(B_{\alpha_{|k|}}[(k+1)^*]) \cap O^*(\omega) = \emptyset.$$

Consequently, since  $\bigcup_{k \in Z_{j-1}} \text{Bd}(B_{\alpha_{|k|}}[(k+1)^*])$  is a finite set, by decreasing again the value of  $\alpha_j$ , if necessary, we can achieve that Definition 3.4(R.3) holds for  $j$  and Definition 3.4(R.1–2) are still verified.

Next we will take care of Definition 3.4(R.4). If  $(j+1)^* \notin \bigcup_{i \in Z_{j-1}} B_i^- [i^*]$ , by decreasing again the value of  $\alpha_j$  (and  $\delta_j$ ), if necessary, we can achieve that  $B_{\alpha_j}[(j+1)^*] \cap \left(\bigcup_{i \in Z_{j-1}} B_i^- [i^*]\right) = \emptyset$  while preserving that Definition 3.4(R.1–3) is verified for  $j$ . In this case Definition 3.4(R.4) holds trivially.

Conversely, assume that there exists  $k \in Z_{j-1}$  such that  $(j+1)^* \in B_k^- [k^*]$  and  $|k|$  is maximal verifying these conditions. By Definition 3.4(R.2),  $k$  is unique (that is, the condition cannot be verified by  $k$  and  $-k$  simultaneously). On the other hand, by the Definition 3.4(R.1) for  $|k|$  and  $|k| - 1$  and the comment above,  $(j+1)^* \notin \text{Bd}(B_k^- [k^*]) \cup \text{Bd}(B_{\alpha_{|k|}} [k^*])$ . Since  $k \in Z_{j-1}$ ,  $|k| \leq j - 1$  and, hence,  $(j+1)^* \notin Z_{|k|}^*$  (in particular  $j^* \neq k^*$ ). Consequently,  $(j+1)^*$  is contained in one of the connected components of  $B_k^- (k^*) \setminus \left(\text{Bd}(B_{\alpha_{|k|}} [k^*]) \cup Z_{|k|}^*\right)$ . Then, by decreasing again the value of  $\alpha_j$ , if necessary, we can get that  $B_{\alpha_j}[(j+1)^*]$  is contained in the connected component of  $B_k^- (k^*) \setminus \left(\text{Bd}(B_{\alpha_{|k|}} [k^*]) \cup Z_{|k|}^*\right)$  where  $(j+1)^*$  lies, while preserving that Definition 3.4(R.1–3) are verified for  $j$ . Consequently, Definition 3.4(R.1–4) hold for  $j$ .

Now we will deal with Definition 3.4(R.5). If  $\ell^* \notin \bigcup_{i \in Z_{j-1}} B_i^- [i^*]$ , by decreasing again the value of  $\alpha_j$ , if necessary, we can get Definition 3.4(R.5.i) while preserving that Definition 3.4(R.1–4) are verified for  $j$ .

Assume that there exists  $m \in Z_{j-1}$  such that  $\ell^* \in B_m^- [m^*]$  and  $|m|$  is maximal with these properties. As in the above construction, by Definition 3.4(R.1–2),  $\ell^* \in B_m^- (m^*) \setminus \left(\text{Bd}(B_{\alpha_{|m|}} [m^*]) \cup \{m^*\}\right)$  and  $m$  is unique (that is, the condition cannot be verified simultaneously by  $m$  and  $-m$ ). Consequently,  $\ell^* \notin B_i^- [i^*]$  for every  $i \in Z_{j-1}$  such that  $|i| \geq |m|$ ,  $i \neq m$ . Thus, by decreasing again the value of  $\alpha_j$ , if necessary, we can get that Definition 3.4(R.1–4) still holds, Definition 3.4(R.5.ii.1) is verified and the interval  $B_\ell^- [\ell^*]$  is contained in the connected component of  $B_m^- (m^*) \setminus \left(\text{Bd}(B_{\alpha_{|m|}} [m^*]) \cup \{m^*\}\right)$  where  $\ell^*$  lies. So, Definition 3.4(R.5.ii.2) also holds.

We claim that

for every  $\ell, m \in \mathbb{Z}$  such that  $|m| \leq |\ell| \leq j$ ,  $\ell \neq m$ , either  $B_\ell^- [\ell^*] \cap B_m^- [m^*] = \emptyset$  or  $|m| < |\ell|$  and  $B_\ell^- [\ell^*]$  is contained in a connected component of

$$B_m^- (m^*) \setminus \left(\text{Bd}(B_{\alpha_{|m|}} [m^*]) \cup \{m^*\}\right).$$

We prove the claim by induction. Observe that the claim holds trivially for  $|m| \leq |\ell| \leq 1$  because  $B_0^- [0^*]$ ,  $B_1^- [1^*] = B_{\alpha_1} [1^*] \subset B_{\alpha_0} [1^*]$  and  $B_{-1}^- [(-1)^*]$  are pairwise disjoint by construction.

Assume that the claim holds for every  $|m| \leq |\ell| < j$ . So, to prove the claim, we may assume that  $\ell \in \{j, -j\}$ ,  $m \in Z_{j-1} \cup \{-\ell\}$  and  $B_\ell^- [\ell^*] \cap B_m^- [m^*] \neq \emptyset$ . By Definition 3.4(R.2),  $B_j^- [j^*] \cap B_{-j}^- [(-j)^*] = \emptyset$ . Consequently,  $m \neq -\ell$  (that is,  $m \in Z_{j-1}$  and  $|\ell| = j > |m|$ ). On the other hand, if  $\ell = -j$ , Definition 3.4(R.2) for  $j - 1$  shows that  $B_{j-1}^- [(j-1)^*]$ ,  $B_{-(j-1)}^- [(-(j-1))^*]$  and  $B_{-j}^- [(-j)^*]$  are pairwise disjoint. Thus,  $m \in Z_{j-2}$  in this case.

Hence, by Definition 3.4(R.5) for  $j$  when  $\ell = j$  and for  $j - 1$  when  $\ell = -j$ , there exists  $k \in Z_{j-1}$  (in fact when  $\ell = -j$ ,  $k \in Z_{j-2}$ ) such that  $B_\ell^- [\ell^*]$  is contained in a connected component of  $B_k^- (k^*) \setminus \left(\text{Bd}(B_{\alpha_{|k|}} [k^*]) \cup \{k^*\}\right)$  and  $|\ell| = j > |k| \geq |m|$ .

If  $m = k$  then the claim holds. Otherwise,  $m \neq k$  and since  $j = |\ell| > |k| \geq |m|$ , by the induction hypotheses,  $|k| > |m|$ , and  $B_k^{\sim}[k^*]$  is contained in a connected component of  $B_m^{\sim}(m^*) \setminus (\text{Bd}(B_{\alpha_{|m|}}[m^*]) \cup \{m^*\})$ . So, the claim holds also in this case. This ends the proof of the claim.

Finally, we consider Definition 3.4(R.6). The fact that either  $B_\ell^{\sim}[\ell^*] \cap B_m^{\sim}[m^*] = \emptyset$  for every  $m \in Z_j$ ,  $m \neq \ell$  or there exists  $m \in Z_{j-1}$  such that  $B_\ell^{\sim}[\ell^*]$  is contained in a connected component of  $B_m^{\sim}(m^*) \setminus (\text{Bd}(B_{\alpha_{|m|}}[m^*]) \cup \{m^*\})$  follows from the claim.

To show that Definition 3.4(R.6.i) can be guaranteed, it is enough to decrease again the value of  $\alpha_j$ , if necessary, until  $B_{\alpha_j}[\ell^*]$  is disjoint from  $Z_{|m|}^*$  and Definition 3.4(R.1–5) are still verified. Thus by (a) for  $|m|$ ,  $\gamma_{|m|}$  is well defined and continuous on  $B_{\alpha_j}[\ell^*]$ . So, we can set  $a_\ell := \gamma_{|m|}(\ell^*)$  and, by decreasing again  $\alpha_j$  (if necessary), we get  $\text{Graph}(\gamma_{|m|}|_{B_{\alpha_j}[\ell^*]}) \subset \mathcal{R}(j^*)$ .

To show that Definition 3.4(R.6.ii) can be guaranteed we first assume that  $k = m$ . As before, if necessary, we can increase the value of  $n_j$  and, accordingly, decrease the values of  $\alpha_j < 2^{-n_j}$  and  $0 < \delta_j < \alpha_j$  so that Definition 3.4(R.1–5) and (R.6.i) are still verified for  $j$  and in addition,

$$(\ell^*, a_\ell + 2^{-n_j}), (\ell^*, a_\ell - 2^{-n_j}) \in \text{Int}(\mathcal{R}(k^*))$$

and the region  $\mathcal{R}(\ell^*)$  is contained in one of the two connected components of  $\text{Int}(\mathcal{R}(k^*) \setminus \uparrow\uparrow k^*)$ .

Assume now that  $k \neq m$  (recall that  $|k| \leq |m| < j$ ). In this case we have  $B_\ell^{\sim}[\ell^*] \subset B_m^{\sim}(m^*) \cap B_{\alpha_{|k|}}(k^*)$ . In particular,  $B_m^{\sim}(m^*) \cap B_{\alpha_{|k|}}(k^*) \neq \emptyset$  and, by the above claim,  $|k| < |m|$  and  $B_\ell^{\sim}[\ell^*] \subset B_m^{\sim}[m^*]$  is contained in a connected component of  $B_k^{\sim}(k^*) \setminus (\text{Bd}(B_{\alpha_{|k|}}[k^*]) \cup \{k^*\})$ . The fact that  $B_\ell^{\sim}[\ell^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$  implies that  $B_\ell^{\sim}[\ell^*] \subset B_m^{\sim}[m^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ . Then, as above we can increase the value of  $n_j$  and, accordingly, decrease the values of  $\alpha_j < 2^{-n_j}$  and  $0 < \delta_j < \alpha_j$  so that Definition 3.4(R.1–5) and (R.6.i) are still verified,

$$(\ell^*, a_\ell + 2^{-n_j}), (\ell^*, a_\ell - 2^{-n_j}) \in \text{Int}(\mathcal{R}(k^*))$$

and the region  $\mathcal{R}(\ell^*)$  is contained in one of the two connected components of  $\text{Int}(\mathcal{R}(k^*) \setminus \uparrow\uparrow k^*)$ .

Now assume that  $|k|$  is not maximal verifying the assumptions. Then, there exists  $r \in Z_{|m|} \subset Z_{j-1}$  such that  $B_\ell^{\sim}[\ell^*] \subset B_{\alpha_{|r|}}(r^*) \setminus \{r^*\}$  and  $|r|$  is maximal with these properties.

We have  $|k| \leq |r| \leq |m| < j$  and

$$B_r^{\sim}[r^*] \cap B_k^{\sim}[k^*] \supset B_{\alpha_{|r|}}(r^*) \cap B_{\alpha_{|k|}}(k^*) \neq \emptyset$$

because  $B_\ell^{\sim}[\ell^*] \subset B_{\alpha_{|r|}}(r^*) \cap B_{\alpha_{|k|}}(k^*)$ . Then, by the claim,  $|k| < |r|$  and  $B_r^{\sim}[r^*]$  are contained in a connected component of  $B_k^{\sim}(k^*) \setminus (\text{Bd}(B_{\alpha_{|k|}}[k^*]) \cup \{k^*\})$ . The fact that  $B_\ell^{\sim}[\ell^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$  implies that  $B_r^{\sim}[r^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ . By the part already proven and Definition 3.4(R.6.ii) for  $|r| < j$  we get that  $\mathcal{R}(\ell^*)$  is contained in one of the two connected components of  $\text{Int}(\mathcal{R}(r^*) \setminus \uparrow\uparrow r^*)$  and  $\mathcal{R}(r^*)$  is contained in one of the two connected components of  $\text{Int}(\mathcal{R}(k^*) \setminus \uparrow\uparrow k^*)$ . This shows that Definition 3.4(R.6.ii) can be guaranteed.

Let us prove that (a) holds for  $j$ . Since the set  $\mathbb{S}^1 \setminus Z_j^*$  is residual, to prove that  $(\gamma_j, \mathbb{S}^1 \setminus Z_j^*) \in \mathcal{PCG}$  we have to show that  $\gamma_j|_{\mathbb{S}^1 \setminus Z_j^*}$  is continuous. Note that, from Definition 3.4(R.6.ii),  $a_\ell^\pm = \varphi_{\ell^*}(\ell^* \pm \alpha_j) = \gamma_{j-1}(\ell^* \pm \alpha_j)$ . Hence, the continuity of  $\gamma_j|_{\mathbb{S}^1 \setminus Z_j^*}$  follows from the fact that  $\gamma_{j-1}$  is continuous on  $\mathbb{S}^1 \setminus Z_{j-1}^* \supset \mathbb{S}^1 \setminus Z_j^*$  and the continuity of  $\varphi_{j^*}$  and  $\varphi_{(-j)^*}$  (Definition 3.1).

This ends the proof of the first statement of the lemma and the first statement of (a). For every  $\ell \in \{j+1, -(j+1)\}$ , from By Definition 3.4(R.1,2) we get:

$$\begin{aligned} \gamma_{j-1}(B_{\alpha_j}[\ell^*]) &\subset [\gamma_{j-1}(\ell^*) - 2^{-n_j}, \gamma_{j-1}(\ell^*) + 2^{-n_j}] \\ B_{\alpha_j}[\ell^*] &\text{ is disjoint from } B_{\alpha_j}[j^*] \text{ and } B_{\alpha_{j-1}}[(-j)^*] \supset B_{\alpha_j}[(-j)^*], \text{ and} \\ \{\ell^*\} &\notin B_{\alpha_j}[\ell^*] \cap Z_{j-1}^* \subset B_{\alpha_j}[\ell^*] \cap Z_{j+1}^* = \{\ell^*\}. \end{aligned}$$

So, from the definition of  $\gamma_j$  it follows that

$$\gamma_j|_{B_{\alpha_j}[\ell^*]} = \gamma_{j-1}|_{B_{\alpha_j}[\ell^*]}$$

and, thus, (a) holds.

Statement (c) follows immediately from Definition 3.4(R.6) and Remark 3.5(R.6).

Next we prove (b), (d), (e), (f), (g).

(d) When  $n = j$ , we get  $B_{\alpha_j}[\ell^*] \setminus Z_j^* = B_{\alpha_j}[\ell^*] \setminus \{\ell^*\}$  from Definition 3.4(R.2). Hence,  $\text{Graph}(\gamma_j|_{B_{\alpha_j}[\ell^*] \setminus Z_j^*}) \subset \mathcal{R}(\ell^*)$  by the definition of  $\gamma_j$  (Definition 3.4) and the definition of  $\varphi_{\ell^*}$  (Definition 3.1).

Now assume that  $n > j$  and fix  $\theta \in B_{\alpha_j}[\ell^*] \setminus Z_n^*$ . We have to show that the point  $(\theta, \gamma_n(\theta)) \in \mathcal{R}(\ell^*)$ . If  $\theta \notin B_{\alpha_{|m|}}[m^*]$  for every  $m$  such that  $j < |m| \leq n$  then, by the iterative use of the definition of  $\gamma_i$  for  $i = j + 1, j + 2, \dots, n$  (Definition 3.4) and Definition 3.1,

$$(\theta, \gamma_n(\theta)) = (\theta, \gamma_{n-1}(\theta)) = \dots = (\theta, \gamma_{j+1}(\theta)) = (\theta, \gamma_j(\theta)) = (\theta, \varphi_{\ell^*}(\theta)) \in \mathcal{R}(\ell^*).$$

Otherwise, by Definition 3.4(R.2), there exists  $m \in \mathbb{Z}$  such that  $|\ell| < |m| \leq n, \theta \in B_{\alpha_{|m|}}[m^*] \setminus Z_n^*$ , and  $\theta \notin B_{\alpha_{|s|}}[s^*]$  for every  $s$  such that  $|m| < |s| \leq n$ . This implies that  $B_\ell^{\sim}[\ell^*] \cap B_m^{\sim}[m^*] \supset B_{\alpha_j}[\ell^*] \cap B_{\alpha_{|m|}}[m^*] \neq \emptyset$  and  $|m|$  is maximal with these properties. So, by the claim for  $j = |m|$ ,  $B_m^{\sim}[m^*]$  is contained in a connected component of  $B_\ell^{\sim}(\ell^*) \setminus (\text{Bd}(B_{\alpha_{|m|}}[\ell^*]) \cup \{\ell^*\})$ . Moreover, since  $\theta \in B_m^{\sim}(m^*) \cap B_{\alpha_j}[\ell^*] \neq \emptyset$ ,  $B_m^{\sim}[m^*] \subset B_{\alpha_{|m|}}[\ell^*] \setminus \{\ell^*\}$ . Thus, by Definition 3.4(R.6.ii) and Remark 3.5(R.6.ii) for  $j = |m|$ ,  $\ell$  replaced by  $m$  and  $k$  replaced by  $\ell$ ,  $\mathcal{R}(m^*) \subset \mathcal{R}(\ell^*)$  and (d) follows from the part already proven by replacing  $\ell$  by  $m$  and  $j$  by  $|m|$ .

(g) By the claim we have that for every  $\ell, m \in \mathbb{Z}$  such that  $|\ell| \geq |m|$ ,  $\ell \neq m$  and  $B_\ell^{\sim}[\ell^*] \cap B_m^{\sim}[m^*] \neq \emptyset$ , it follows that  $|\ell| > |m|$ , and  $B_\ell^{\sim}[\ell^*]$  is contained in a connected component of  $B_m^{\sim}(m^*) \setminus (\text{Bd}(B_{\alpha_{|m|}}[m^*]) \cup \{m^*\})$ . Only it remains to show that if  $B_\ell^{\sim}[\ell^*] \subset B_{\alpha_{|m|}}(m^*) \setminus \{m^*\}$ , then the region  $\mathcal{R}^{\sim}(\ell^*)$  is contained in one of the two connected components of  $\text{Int}(\mathcal{R}(m^*) \setminus \uparrow\uparrow m^*)$ . By Definition 3.4(R.6.ii) we know that this holds for  $\mathcal{R}(\ell^*)$  instead of  $\mathcal{R}^{\sim}(\ell^*)$ . Hence, if  $\ell \geq 0$ , (g) holds because  $\mathcal{R}^{\sim}(\ell^*) = \mathcal{R}(\ell^*)$ . Assume now that  $\ell < 0$ . Since  $\mathcal{R}^{\sim}(\ell^*) = \mathcal{R}(\ell^*) \cup \text{Graph}(\gamma_{|\ell|}|_{B_\ell^{\sim}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)})$  is connected,  $\mathcal{R}(\ell^*) \subset \mathcal{R}(m^*)$ , and  $\text{Int}(\mathcal{R}(m^*) \setminus \uparrow\uparrow m^*)$  has two connected components, it is enough to show that

$$\text{Graph}(\gamma_{|\ell|}|_{B_\ell^{\sim}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}) \subset \mathcal{R}(m^*).$$

Since  $B_\ell^{\sim}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*) \subset B_\ell^{\sim}[\ell^*] \subset B_{\alpha_{|m|}}(m^*) \setminus \{m^*\}$ , statement (g) follows from (d) with  $\ell$  replaced by  $m$ ,  $j$  by  $|m|$  and  $n$  replaced by  $|\ell|$ .

(b) With (g) in mind we set

$$D := \{\ell \in \mathbb{Z} : \mathcal{R}^{\sim}(\ell^*) \not\subset \mathcal{R}(i^*) \text{ for every } i \in \mathbb{Z} \setminus \{\ell\}\}.$$

Clearly,

$$\begin{aligned} \bigcup_{\ell \in \mathbb{Z}} \mathcal{R}^{\sim}(\ell^*) &= \left( \bigcup_{i \in \mathbb{Z} \setminus D} \mathcal{R}^{\sim}(i^*) \right) \cup \left( \bigcup_{\ell \in D} \mathcal{R}^{\sim}(\ell^*) \right) \\ &\subset \left( \bigcup_{i \in D} \mathcal{R}(i^*) \right) \cup \left( \bigcup_{\ell \in D} \mathcal{R}^{\sim}(\ell^*) \right) = \bigcup_{\ell \in D} \mathcal{R}^{\sim}(\ell^*). \end{aligned}$$

**Claim.** For every  $\ell \in D$ ,  $\gamma_{|\ell-1}|_{B_\ell^{\sim}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)} \equiv 0$ .



First we prove statement (b) from the above claim and then we will prove the claim. To this end we start by pointing out few elementary facts.

From the definition of  $\mathcal{R}^{\leftarrow}(\ell^*)$  we see that  $\mathcal{R}^{\leftarrow}(\ell^*) \setminus \mathcal{R}(\ell^*) = \emptyset$  for every  $\ell \geq 0$  and  $\mathcal{R}^{\leftarrow}(\ell^*) \setminus \mathcal{R}(\ell^*) \subset \text{Graph}\left(\gamma_{|\ell|} \Big|_{B_{\ell}^{\leftarrow}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right)$  for every  $\ell < 0$ . So, in any case,

$$\mathcal{R}^{\leftarrow}(\ell^*) \setminus \mathcal{R}(\ell^*) \subset \text{Graph}\left(\gamma_{|\ell|} \Big|_{B_{\ell}^{\leftarrow}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right) \quad \text{for every } \ell \in \mathbb{Z}.$$

On the other hand, the arc  $B_{\ell}^{\leftarrow}[\ell^*] \supset B_{\ell}^{\leftarrow}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)$  is disjoint from the arc  $B_{-\ell}^{\leftarrow}[(\ell^*)^*] \supset B_{\alpha_{|\ell|}}[(\ell^*)^*]$  by Definition 3.4(R.2). Thus, by Definition 3.4 and (a),

$$\gamma_{|\ell-1|} \Big|_{B_{\ell}^{\leftarrow}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)} = \gamma_{|\ell|} \Big|_{B_{\ell}^{\leftarrow}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}.$$

Furthermore, by the Claim and Definition 3.4(R.6),  $a_{\ell}^+ = a_{\ell}^- = a_{\ell} = 0$  for every  $\ell \in D$ . So, by Remark 3.2(1),

$$\mathcal{R}(\ell^*) \subset B_{\alpha_{|\ell|}}[\ell^*] \times [-2^{-n_{|\ell|}}, 2^{-n_{|\ell|}}] \subset B_{\alpha_{|\ell|}}[\ell^*] \times [-2^{-|\ell|}, 2^{-|\ell|}] \subset B_{\alpha_{|\ell|}}[\ell^*] \times [-1, 1].$$

Therefore, summarizing and using again by the Claim,

$$\begin{aligned} \bigcup_{\ell \in \mathbb{Z}} \mathcal{R}^{\leftarrow}(\ell^*) &\subset \bigcup_{\ell \in \mathbb{D}} \mathcal{R}^{\leftarrow}(\ell^*) \subset \bigcup_{\ell \in \mathbb{D}} \left( \mathcal{R}(\ell^*) \cup \text{Graph}\left(\gamma_{|\ell|} \Big|_{B_{\ell}^{\leftarrow}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right) \right) \\ &= \left( \bigcup_{\ell \in \mathbb{D}} \mathcal{R}(\ell^*) \right) \cup \left( \bigcup_{\ell \in \mathbb{D}} \text{Graph}\left(\gamma_{|\ell-1|} \Big|_{B_{\ell}^{\leftarrow}[\ell^*] \setminus B_{\alpha_{|\ell|}}(\ell^*)}\right) \right) \\ &\subset \left( \bigcup_{\ell \in \mathbb{D}} B_{\alpha_{|\ell|}}[\ell^*] \right) \times [-1, 1] \cup \mathbb{S}^1 \times \{0\} \subset \mathbb{S}^1 \times [-1, 1]. \end{aligned}$$

So, the first part of (b) is proved, provided that the claim holds. Let us prove the second statement of (b). Observe that, since

$$\left( \bigcup_{\ell \in \mathbb{Z}} \mathcal{R}(\ell^*) \right) \cup \mathbb{S}^1 \times \{0\} \subset \left( \bigcup_{\ell \in \mathbb{Z}} \mathcal{R}^{\leftarrow}(\ell^*) \right) \cup \mathbb{S}^1 \times \{0\} \subset \mathbb{S}^1 \times [-1, 1],$$

it is enough to show that

$$\text{Graph}\left(\gamma_j \Big|_{\mathbb{S}^1 \setminus Z_j^*}\right) \subset \left( \bigcup_{\ell \in \mathbb{Z}} \mathcal{R}(\ell^*) \right) \cup \mathbb{S}^1 \times \{0\}$$

for every  $j \in \mathbb{Z}^+$ . We will prove this statement by induction on  $j$ .

By construction we have

$$\text{Graph}\left(\gamma_0 \Big|_{\mathbb{S}^1 \setminus \{0^*\}}\right) \subset \mathcal{R}(0^*) \cup \mathbb{S}^1 \times \{0\} \subset \left( \bigcup_{\ell \in \mathbb{Z}} \mathcal{R}(\ell^*) \right) \cup \mathbb{S}^1 \times \{0\}.$$

So, the statement holds for  $j = 0$ . Now assume that it holds for some  $j \geq 0$ , and prove it for  $j + 1$ . By Definition 3.4 and (d),

$$\begin{aligned} \text{Graph}\left(\gamma_{j+1} \Big|_{\mathbb{S}^1 \setminus Z_{j+1}^*}\right) &\subset \mathcal{R}(j^*) \cup \mathcal{R}((-j)^*) \cup \text{Graph}\left(\gamma_j \Big|_{\mathbb{S}^1 \setminus Z_j^*}\right) \\ &\subset \mathcal{R}(j^*) \cup \mathcal{R}((-j)^*) \cup \left( \bigcup_{\ell \in \mathbb{Z}} \mathcal{R}(\ell^*) \right) \cup \mathbb{S}^1 \times \{0\} \\ &\subset \left( \bigcup_{\ell \in \mathbb{Z}} \mathcal{R}(\ell^*) \right) \cup \mathbb{S}^1 \times \{0\}. \end{aligned}$$

To end the proof of (b) it remains to show the claim.

Let  $\ell \in D$  and  $m \in Z_{|\ell|}$ ,  $m \neq \ell$ . Then, either

$$\begin{cases} B_\ell^\rceil[\ell^*] \cap B_m^\rceil[m^*] = \emptyset & \text{or} \\ |\ell| > |m|, m < 0 & \text{and } B_\ell^\rceil[\ell^*] \subset B_m^\rceil[m^*] \setminus B_{\alpha_{|m|}}[m^*]. \end{cases} \tag{16}$$

To see this, observe that if  $B_\ell^\rceil[\ell^*] \cap B_m^\rceil[m^*] \neq \emptyset$  then, by (g),  $|\ell| > |m|$  and either  $\mathcal{R}^\rceil(\ell^*) \subset \mathcal{R}(m^*)$  or  $m < 0$  and  $B_\ell^\rceil[\ell^*] \subset B_m^\rceil[m^*] \setminus B_{\alpha_{|m|}}[m^*]$ , and the first possibility is ruled out because  $\ell \in D$ .

By using iteratively the dichotomy (16) we get that, for every  $\ell \in D$ , there exists a sequence  $m_0, m_1, \dots, m_k = \ell \in \mathbb{Z}$  with  $k \geq 0$  such that  $B_{m_0}^\rceil[(m_0)^*] \cap B_q^\rceil[q^*] = \emptyset$  for every  $q \in Z_{|m_0|}$ ,  $q \neq m_0$  and, in the case  $k > 0$ ,  $|m_0| < |m_1| < \dots < |m_k| = |\ell|$  and, for every  $p = 0, 1, \dots, k - 1$ ,

- $m_p < 0$ ,
- $B_{m_{p+1}}^\rceil[(m_{p+1})^*] \subset B_{m_p}^\rceil[(m_p)^*] \setminus B_{\alpha_{|m_p|}}[(m_p)^*]$  and
- $B_{m_{p+1}}^\rceil[(m_{p+1})^*] \cap B_q^\rceil[q^*] = \emptyset$  for every  $q \in Z_{|m_{p+1}|}$ ,  $q \neq m_p, m_{p+1}$  and  $|m_p| \leq |q|$ .

The condition  $B_{m_0}^\rceil[(m_0)^*] \cap B_q^\rceil[q^*] = \emptyset$  for every  $q \in Z_{|m_0|}$ ,  $q \neq m_0$  implies

$$\gamma_{|m_0|-1} \Big|_{B_{m_0}^\rceil[(m_0)^*]} = \gamma_{|m_0|-2} \Big|_{B_{m_0}^\rceil[(m_0)^*]} = \dots = \gamma_0 \Big|_{B_{m_0}^\rceil[(m_0)^*]} \equiv 0$$

by Definition 3.4(R.6) and Remark 3.5(R.6) (with  $\ell = m_0$ ). This ends the proof of the Claim when  $k = 0$ .

Assume now that  $k > 0$ . As before we have

$$\gamma_{|m_0|-1} \Big|_{B_{m_0}^\rceil[(m_0)^*] \setminus B_{\alpha_{|m_0|}}((m_0)^*)} = \gamma_{|m_0|} \Big|_{B_{m_0}^\rceil[(m_0)^*] \setminus B_{\alpha_{|m_0|}}((m_0)^*)}.$$

This, together with the inclusion,

$$B_{m_1}^\rceil[(m_1)^*] \subset B_{m_0}^\rceil[(m_0)^*] \setminus B_{\alpha_{|m_0|}}[(m_0)^*]$$

implies that

$$\gamma_{|m_0|} \Big|_{B_{m_1}^\rceil[(m_1)^*]} \equiv 0.$$

Then, by Definition 3.4(R.6.i) and Remark 3.5(R.6.i) with  $\ell = m_1$ ,

$$0 \equiv \gamma_{|m_0|} \Big|_{B_{m_1}^\rceil[(m_1)^*]} = \gamma_{|m_0|+1} \Big|_{B_{m_1}^\rceil[(m_1)^*]} = \dots = \gamma_{|m_1|-1} \Big|_{B_{m_1}^\rceil[(m_1)^*]}.$$

If  $k = 1$  we are done. Otherwise,  $k \geq 2$  and, as above,

$$\gamma_{|m_1|} \Big|_{B_{m_2}^\rceil[(m_2)^*]} \equiv 0.$$

By iterating the above arguments at most  $k$  times the Claim holds. This ends the proof of (b).

(e) By Definition 3.4(R.2) and Remark 3.5(R.2) it follows that

$$\theta \notin Z_{j+1}^* \cup B_{\alpha_j}(\ell^*) \cup B_{-\ell}^\rceil((-\ell)^*) \quad \text{for every } \theta \in B_\ell^\rceil[\ell^*] \setminus B_{\alpha_j}(\ell^*).$$

So, by (a),  $\gamma_{j-1}(\theta)$  is well defined and  $\gamma_{j-1}$  is continuous at  $\theta$ . Thus, by the definition of  $\gamma_j$  (Definition 3.4) and the continuity of  $\gamma_{j-1}$  at  $\theta$ ,  $\gamma_j(\theta) = \gamma_{j-1}(\theta)$ .

Now assume that  $\theta \in \text{Bd}(B_\ell^\rceil[\ell^*] \setminus B_{\alpha_j}(\ell^*)) = \text{Bd}(B_{\alpha_j}[\ell^*]) \cup \text{Bd}(B_\ell^\rceil[\ell^*])$ . By (g),  $\theta \notin B_n^\rceil[n^*] \cup B_{-n}^\rceil[(-n)^*]$  for every  $n > j$ . So, by the iterative use of the definition of  $\gamma_i$  for  $i = j + 1, j + 2, \dots, n$  (Definition 3.4) we get

$$\gamma_j(\theta) = \gamma_{j+1}(\theta) = \dots = \gamma_{n-1}(\theta) = \gamma_n(\theta).$$

Now we prove the part of (e) concerning  $R_\omega(B_\ell^\rceil[\ell^*] \setminus B_{\alpha_j}(\ell^*))$ . We first assume that  $\ell = j \geq 0$ . Then,

$$B_j^\rceil[j^*] = B_{\alpha_j}[j^*], \quad \theta \in \text{Bd}(B_{\alpha_j}[j^*]) \quad \text{and} \quad R_\omega(\theta) \in \text{Bd}(B_{\alpha_j}[(j+1)^*]).$$

Again by Definition 3.4(R.2),  $R_\omega(\theta) \notin Z_{j+1}^* \cup B_{\alpha_j}[j^*] \cup B_{-j}^\rceil[(-j)^*]$ . So, by (a) and the definition of  $\gamma_j$  (Definition 3.4),  $\gamma_{j-1}(R_\omega(\theta))$  is well defined and  $\gamma_j(R_\omega(\theta)) = \gamma_{j-1}(R_\omega(\theta))$ . By Definition 3.4(R.3) (with  $j = n$  and  $k = \ell = j$ ),  $R_\omega(\theta) \notin B_{\alpha_n}[n^*] \cup B_{\alpha_n}[(-n)^*]$  for every  $n > j$ . So,  $\gamma_n(R_\omega(\theta)) = \gamma_j(R_\omega(\theta))$  as above.

Assume now that  $\ell = -j < 0$ . In this case we have  $B_\ell^\rceil[\ell^*] = B_{\alpha_{|\ell+1}}[\ell^*]$  and, hence,  $R_\omega(\theta) \in B_{\alpha_{|\ell+1}}[(\ell+1)^*] \setminus B_{\alpha_j}((\ell+1)^*)$ . By Definition 3.4(R.1) we have

$$B_{\alpha_j}[(\ell+1)^*] \subset B_{\alpha_{|\ell+1}}[(\ell+1)^*] \subset B_{\ell+1}^\rceil[(\ell+1)^*].$$

Thus,  $R_\omega(\theta) \in B_{\ell+1}^\rceil[(\ell+1)^*] \setminus \{(\ell+1)^*\}$ . Again by Definition 3.4(R.2) and Remark 3.5(R.2) (with  $j$  replaced by  $-(\ell+1)$ ),

$$R_\omega(\theta) \notin Z_\ell^* \cup B_{\alpha_{-(\ell+1)}}[(-\ell)^*] \cup B_\ell^\rceil[\ell^*] \supset Z_j^* \cup B_{\alpha_j}[j^*] \cup B_{-j}^\rceil[(-j)^*].$$

So, by (a) and the definition of  $\gamma_j$  (Definition 3.4),  $\gamma_{j-1}(R_\omega(\theta))$  is well defined and  $\gamma_j(R_\omega(\theta)) = \gamma_{j-1}(R_\omega(\theta))$ .

To end the proof of (e), assume as above that  $\theta \in \text{Bd}(B_{\alpha_j}[\ell^*]) \cup \text{Bd}(B_\ell^\rceil[\ell^*])$  and, hence,  $R_\omega(\theta) \in \text{Bd}(B_{\alpha_j}[(\ell+1)^*]) \cup \text{Bd}(B_{\alpha_{|\ell+1}}[(\ell+1)^*])$ . We have to show that, in this case,  $R_\omega(\theta) \notin B_{\alpha_n}[n^*] \cup B_{\alpha_n}[(-n)^*]$  for every  $n > j$  (the fact that  $\gamma_n(R_\omega(\theta)) = \gamma_j(R_\omega(\theta))$  follows as above). When  $R_\omega(\theta) \in \text{Bd}(B_{\alpha_j}[(\ell+1)^*])$  this follows from Definition 3.4(R.3) as before. Assume now that  $R_\omega(\theta) \in \text{Bd}(B_{\alpha_{|\ell+1}}[(\ell+1)^*])$ . Then, by (g),  $R_\omega(\theta) \notin B_n^\rceil[n^*] \cup B_{-n}^\rceil[(-n)^*]$  for every  $n > j$ .

(f) If  $\ell \geq 0$  then the first two statements of (f) follow directly from the definitions. Moreover, by Remarks 3.2(2) and 3.5(R.1),

$$\text{diam}(\mathcal{R}^\rceil(\ell^*)) = \text{diam}(\mathcal{R}(\ell^*)) = \text{diam}(\mathcal{R}((-\ell)^*)) = 2 \cdot 2^{-n_\ell} \leq 2 \cdot 2^{-(\ell+1)} = 2^{-\ell}.$$

Assume that  $\ell < 0$ . From Definition 3.4(R.2) and Remark 3.5(R.2) we get  $(B_\ell^\rceil[\ell^*] \setminus B_{\alpha_{|\ell}}(\ell^*)) \cap Z_{|\ell}^* = \emptyset$  and, hence,  $\gamma_{|\ell}$  is continuous in an open neighbourhood of  $B_\ell^\rceil[\ell^*] \setminus B_{\alpha_{|\ell}}(\ell^*)$  by (a). On the other hand, by (d),  $(\theta, \gamma_{|\ell}(\theta)) \in \mathcal{R}(\ell^*)$  for every  $\theta \in \text{Bd}(B_{\alpha_{|\ell}}[\ell^*]) \subset B_\ell^\rceil[\ell^*] \setminus B_{\alpha_{|\ell}}(\ell^*)$ . Thus,

$$\mathcal{R}^\rceil(\ell^*) = \mathcal{R}(\ell^*) \cup \text{Graph}\left(\gamma_{|\ell} \Big|_{B_\ell^\rceil[\ell^*] \setminus B_{\alpha_{|\ell}}(\ell^*)}\right)$$

is closed, connected and projects onto the whole  $B_\ell^\rceil[\ell^*]$ .

On the other hand, by (e) and (a) (since  $\ell < 0$ ,  $|\ell+1| = |\ell| - 1$ ),

$$\begin{aligned} \gamma_{|\ell}(B_\ell^\rceil[\ell^*] \setminus B_{\alpha_{|\ell}}(\ell^*)) &= \gamma_{|\ell-1}(B_{\alpha_{|\ell+1}}[\ell^*] \setminus B_{\alpha_{|\ell}}(\ell^*)) \\ &\subset [\gamma_{|\ell-1}(\ell^*) - 2^{-n_{|\ell-1}}, \gamma_{|\ell-1}(\ell^*) + 2^{-n_{|\ell-1}}]. \end{aligned}$$

Thus, by Remark 3.2(1), (c) and Definition 3.4(R.1),

$$\begin{aligned} \mathcal{R}^\rceil(\ell^*) &= \mathcal{R}(\ell^*) \cup \text{Graph}\left(\gamma_{|\ell} \Big|_{B_\ell^\rceil[\ell^*] \setminus B_{\alpha_{|\ell}}(\ell^*)}\right) \\ &\subset B_{\alpha_{|\ell}}[\ell^*] \times [\gamma_{|\ell-1}(\ell^*) - 2^{-n_{|\ell}}, \gamma_{|\ell-1}(\ell^*) + 2^{-n_{|\ell}}] \\ &\cup (B_{\alpha_{|\ell+1}}[\ell^*] \setminus B_{\alpha_{|\ell}}(\ell^*)) \times [\gamma_{|\ell-1}(\ell^*) - 2^{-n_{|\ell-1}}, \gamma_{|\ell-1}(\ell^*) + 2^{-n_{|\ell-1}}] \\ &\subset B_{\alpha_{|\ell+1}}[\ell^*] \times [\gamma_{|\ell-1}(\ell^*) - 2^{-n_{|\ell-1}}, \gamma_{|\ell-1}(\ell^*) + 2^{-n_{|\ell-1}}]. \end{aligned}$$

Hence, by Definition 3.4(R.1) and Remark 3.5(R.1),

$$\text{diam}(\mathcal{R}^\rceil(\ell^*)) \leq 2 \cdot \max\{\alpha_{|\ell+1|}, 2^{-n_{|\ell-1}}\} = 2 \cdot 2^{-n_{|\ell-1}} \leq 2 \cdot 2^{-|\ell|}.$$

This ends the proof of Lemma 3.6.

A.2. Proof of Proposition 7.3 in the case  $m = 0$

This section is devoted to prove Proposition 7.3 for  $m = 0$ ; that is, to study the map  $T_0$ . It is the first technical counterpart of Section 7.

To prove Proposition 7.3 for  $T_0$  we will need some more notation and a technical lemma.

Given a skew product  $F(\theta, x) = (R_\omega(\theta), \zeta(\theta, x))$  from  $\Omega = \mathbb{S}^1 \times \mathbb{I}$  to itself we define the fibre map function of  $F$ ,  $\text{fib}(F): \mathbb{S}^1 \rightarrow \mathcal{C}(\mathbb{I}, \mathbb{I})$  by  $\text{fib}(F)(\theta) := \zeta(\theta, \cdot)$ . A simple exercise shows that  $F$  is continuous if and only if  $\zeta(\theta, \cdot)$  is continuous for every  $\theta \in \mathbb{S}^1$ , and  $\text{fib}(F)$  is continuous.

**Lemma A.2.1.** *Let  $\theta \in \text{Bd}(B_i^-[i^*])$  for some  $i \in \mathfrak{D}_0$ . Then,  $m_i(\theta) = M_i(\theta) = 0$ ,  $g_i(\theta, m_i(\theta)) = \gamma(R_\omega(\theta))$ , and  $f_{0,\theta}$  is the unique piecewise affine map with two affine pieces whose graph joins the point  $(-2, 2)$  with  $(0, \gamma(R_\omega(\theta)))$ , and this with the point  $(2, -2)$ .*

**Proof.** By Lemma 4.1(d) and Definition 7.1, we have  $m_i(\theta) = M_i(\theta)$ . Hence,  $f_{0,\theta}$  is the piecewise affine map with two affine pieces whose graph joins the point  $(-2, 2)$  with  $(m_i(\theta), g_{i,\theta}(m_i(\theta)))$ , and this with the point  $(2, -2)$ . So, we need to show that  $m_i(\theta) = 0$ , and  $g_{i,\theta}(m_i(\theta)) = \gamma(R_\omega(\theta))$ .

Lemma 3.6(g) and the fact that  $\text{depth}i = 0$ ,  $B_i^-[i^*] \cap B_\ell^-[i^*] = \emptyset$  for every  $\ell \in Z_{|i|}$ ,  $i \neq \ell$ . Consequently, by Definition 3.4(R.6),  $m_i(\theta) = M_i(\theta) = a_i^- = 0$ .

Now we show that  $g_{i,\theta}(m_i(\theta)) = \gamma(R_\omega(\theta))$ . From the definition of the map  $g_i$  (Definitions 4.2 and 4.4), Lemma 3.6(e) and Definitions 3.4 and 3.8(R.1), we get

$$g_{i,\theta}(m_i(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)) = \gamma(R_\omega(\theta)).$$

This ends the proof of the lemma.  $\square$

**Proof of Proposition 7.3 for  $m = 0$ .** By Lemma 3.6(b),

$$-1 \leq m_{\mathfrak{b}^-(\theta,0)}(\theta) \leq M_{\mathfrak{b}^-(\theta,0)}(\theta) \leq 1$$

for every  $\theta \in \mathbb{B}_0^-$ . So,  $T_0$  is well defined.

(b) If  $\theta \in \mathbb{S}^1 \setminus \mathbb{B}_0^-$ , then the statement follows directly from Definition 7.1. Now assume that  $\theta \in \mathbb{B}_0^-$  and let  $i = \mathfrak{b}^-(\theta, 0)$ . From the definition of the maps  $g_{i,\theta}$  (Definitions 4.2 and 4.4) and Definition 7.1, it follows that  $f_{0,\theta}|_{\mathbb{I}_{i,\theta}}$  is piecewise affine and non-increasing. On the other hand, again by Definition 7.1,  $f_{0,\theta}|_{[-2, m_i(\theta)]}$  and  $f_{0,\theta}|_{[M_i(\theta), 2]}$  are affine with negative slope and  $f_{0,\theta}(2) = -2$  and  $f_{0,\theta}(-2) = 2$ . The fact that

$$-1 \leq f_{0,\theta}(M_{\mathfrak{b}^-(\theta,0)}(\theta)) \leq f_{0,\theta}(m_{\mathfrak{b}^-(\theta,0)}(\theta)) \leq 1$$

for every  $\theta \in \mathbb{B}_0^-$  follows from Definition 7.1 and Lemmas 4.3(a) and 4.5(a). This ends the proof of (b).

(c) Recall that

$$\mathcal{R}^-(i^*) = \bigcup_{\theta \in B_i^-[i^*]} \{\theta\} \times \mathbb{I}_{i,\theta}.$$

Hence, from Definition 7.1 and the definition of  $G_i$  (Definitions 4.2 and 4.4) it follows that

$$T_m(\theta, x) = (R_\omega(\theta), f_m(\theta, x)) = (R_\omega(\theta), g_{i,\theta}(x)) = G_i(\theta, x),$$

for every  $(\theta, x) \in \mathcal{R}^-(i^*)$ . Thus,  $T_0(A_{|i|}^{i^*}) = A_{|i+1|}^{(i+1)^*}$  from Lemmas 3.10(b), 4.3(c) and 4.5(c). On the other hand, Lemma 5.3(b) implies that  $i^* \in \mathbb{B}_0^-$  but  $i^* \notin \mathbb{B}_k^-$  for every  $k \in \mathbb{N}$ . Then, we get  $f_{k,i^*} = f_{0,i^*}$  from Definition 7.2.

(a) Since  $T_0$  is a skew product with base  $R_\omega$  we only have to prove that  $f_0$  is continuous.

By [Definition 7.1](#), for every  $\theta \in \mathbb{S}^1$ , the map  $f_{0,\theta}$  is continuous. So we have to prove that the map  $\text{fib}(T_0)$  (that is, the map  $s \mapsto f_{0,s}$ ) is continuous.

In the rest of the proof we will denote

$$\mathbb{I}\mathbb{B}_0^\sim := \bigcup_{i \in \mathcal{D}_0} B_i^\sim(i^*) \subset \mathbb{B}_0^\sim.$$

Clearly, since for every  $i \in \mathbb{Z}$ , the maps  $m_i$  and  $M_i$  are continuous on  $B_i^\sim[i^*]$ , it follows that the map  $s \mapsto f_{0,s}$  is continuous on  $\mathbb{I}\mathbb{B}_0^\sim$ . Thus, we have to see that the fibre map function is continuous at every  $\theta \in \mathbb{S}^1 \setminus \mathbb{I}\mathbb{B}_0^\sim$ ; that is,  $\lim_{j \rightarrow \infty} f_{0,\theta_j} = f_{0,\theta}$  for every  $\{\theta_j\}_{j=1}^\infty \subset \mathbb{S}^1$  converging to  $\theta$ . Given  $\alpha > 0$ , we can consider four sets associated to such a sequence:

$$\begin{aligned} & \{j \in \mathbb{N} : \theta_j \in \mathbb{S}^1 \setminus \mathbb{I}\mathbb{B}_0^\sim\}, & \{j \in \mathbb{N} : \theta_j \in \mathbb{I}\mathbb{B}_0^\sim \setminus B_\alpha(\theta)\}, \\ & \{j \in \mathbb{N} : \theta_j \in (\theta, \theta + \alpha) \cap \mathbb{I}\mathbb{B}_0^\sim\} & \text{and } \{j \in \mathbb{N} : \theta_j \in (\theta - \alpha, \theta) \cap \mathbb{I}\mathbb{B}_0^\sim\}. \end{aligned}$$

Observe that the second set  $\{j \in \mathbb{N} : \theta_j \in \mathbb{I}\mathbb{B}_0^\sim \setminus B_\alpha(\theta)\}$  is always finite and that any of the other three sets gives rise to a subsequence of  $\{\theta_j\}_{j=1}^\infty$  converging to  $\theta$ , when it is infinite. Consequently, the continuity of the fibre map function  $s \mapsto f_{0,s}$  at  $\theta$  is equivalent to the fact that  $\lim_{j \rightarrow \infty} f_{0,\theta_j} = f_{0,\theta}$  for every  $\{\theta_j\}_{j=1}^\infty$  converging to  $\theta$  and such that, for some  $\alpha > 0$ ,  $\{\theta_j\}_{j=1}^\infty$  is contained either in  $\mathbb{S}^1 \setminus \mathbb{I}\mathbb{B}_0^\sim$ , or  $(\theta, \theta + \alpha) \cap \mathbb{I}\mathbb{B}_0^\sim$ , or  $(\theta - \alpha, \theta) \cap \mathbb{I}\mathbb{B}_0^\sim$ . We will only deal with the first two cases since the proof in the last case (for  $(\theta - \alpha, \theta)$ ) can be done symmetrically.

**Case 1:**  $\lim_{j \rightarrow \infty} \theta_j = \theta$  and  $\{\theta_j\}_{j=1}^\infty \subset \mathbb{S}^1 \setminus \mathbb{I}\mathbb{B}_0^\sim$ .

By [Definition 7.1](#) and [Lemma A.2.1](#),  $f_{0,\theta_j}$  (respectively  $f_{0,\theta}$ ) is the unique piecewise affine map with two affine pieces whose graph joins the point  $(-2, 2)$  with  $(0, \gamma(R_\omega(\theta_j)))$  (respectively  $(0, \gamma(R_\omega(\theta)))$ ), and this with the point  $(2, -2)$ . By [Lemma 5.3\(c\)](#) and [Definition 3.8](#) the function  $\gamma$  is continuous at  $R_\omega(\theta) \notin O^*(\omega)$ . Hence,  $\lim_{j \rightarrow \infty} \gamma(R_\omega(\theta_j)) = \gamma(R_\omega(\theta))$  and, thus,  $\lim_{j \rightarrow \infty} f_{0,\theta_j} = f_{0,\theta}$ .

**Case 2:**  $\lim_{j \rightarrow \infty} \theta_j = \theta$  and  $\{\theta_j\}_{j=1}^\infty \subset (\theta, \theta + \alpha) \cap \mathbb{I}\mathbb{B}_0^\sim$ .

If there exists  $i \in \mathcal{D}_0$  such that  $\theta$  is the left endpoint of  $B_i^\sim[i^*] \subset \mathbb{B}_0^\sim$  then the result follows from [Definition 7.1](#), the continuity of the maps  $m_i$  and  $M_i$  and the continuity of the maps  $g_i$  ([Lemmas 4.3\(a\)](#) and [4.5\(a\)](#)).

Assume now that  $\theta$  is not the left endpoint of  $B_i^\sim(i^*)$  for every  $i \in \mathcal{D}_0$ . For every  $j \in \mathbb{N}$  we set  $i_j := \mathfrak{b}^\sim(\theta_j, 0) \in \mathcal{D}_0$  (that is,  $\theta_j \in B_{i_j}^\sim((i_j)^*)$ ).

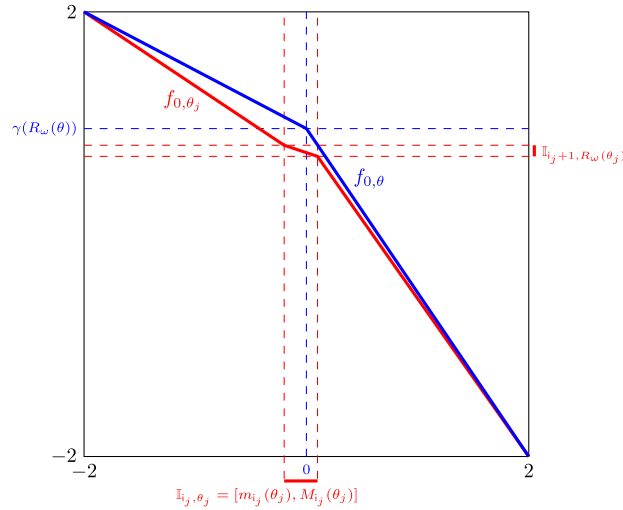
We claim that  $\lim_{j \rightarrow \infty} |i_j| = \infty$  and consequently, by [Definition 3.4\(R.1\)](#),

$$\lim_{j \rightarrow \infty} 2^{-n|i_j+1|} = \lim_{j \rightarrow \infty} 2^{-n|i_j|} = 0. \tag{17}$$

To prove this claim, assume by way of contradiction that there exists  $L$  such that for every  $k \in \mathbb{N}$  there exists  $j_k \geq k$  such that  $|i_{j_k}| \leq L$ . Then,

$$\{\theta_{j_k}\}_{k=1}^\infty \subset \bigcup_{k=1}^\infty B_{i_{j_k}}^\sim((i_{j_k})^*)$$

and, since  $\{i_{j_k} : k \in \mathbb{N}\}$  is finite, it follows that there exists  $i \in \{i_{j_k} : k \in \mathbb{N}\} \subset \mathcal{D}_0$  and a subsequence of  $\{\theta_{j_k}\}_{k=1}^\infty$ , that by abuse of notation will also be called  $\{\theta_{j_k}\}$ , such that  $\{\theta_{j_k}\}_{k=1}^\infty \subset B_i^\sim(i^*)$ . So,



**Fig. 6.** A symbolic representation of the maps  $f_{0,\theta}$  and  $f_{0,\theta_j}$  in Case 2 of the proof of Proposition 7.3 for  $m = 0$ . The map  $f_{0,\theta}$  and the points  $0$  and  $\gamma(R_\omega(\theta))$  are drawn in blue. The map  $f_{0,\theta_j}$  and the corresponding intervals  $\mathbb{I}_{i_j, \theta_j}$  and  $\mathbb{I}_{i_j+1, R_\omega(\theta_j)}$  are drawn in red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$\theta = \lim_{k \rightarrow \infty} \theta_{j_k} \in B_i^\infty [i^*];$$

a contradiction. So, the claim (and hence (17)) holds.

Next we claim that the conditions

$$\lim_{j \rightarrow \infty} M_{i_j}(\theta_j) = \lim_{j \rightarrow \infty} m_{i_j}(\theta_j) = 0, \quad \text{and} \tag{18}$$

there exists a sequence  $\{x_j\}_{j=1}^\infty$  with  $x_j \in \mathbb{I}_{i_j, \theta_j} = [m_{i_j}(\theta_j), M_{i_j}(\theta_j)]$  for every  $j$ ,  
 such that  $\lim_{j \rightarrow \infty} f_{0,\theta_j}(x_j) = \gamma(R_\omega(\theta))$  (19)

imply

$$\lim_{j \rightarrow \infty} f_{0,\theta_j} = f_{0,\theta}.$$

To prove the claim notice that, by Definition 7.1 and Lemma A.2.1,  $f_{0,\theta}$  is the unique piecewise affine map with two affine pieces whose graph joins the point  $(-2, 2)$  with  $(0, \gamma(R_\omega(\theta)))$ , and this with the point  $(2, -2)$ . On the other hand, for every  $j$ ,

- $f_{0,\theta_j}|_{[-2, m_{i_j}(\theta_j)]}$  is the affine map joining the point  $(-2, 2)$  with the point  $(m_{i_j}(\theta_j), g_{i_j}(\theta_j, m_{i_j}(\theta_j)))$ , and
- $f_{0,\theta_j}|_{[M_{i_j}(\theta_j), 2]}$  is the affine map joining the point  $(M_{i_j}(\theta_j), g_{i_j}(\theta_j, M_{i_j}(\theta_j)))$  with the point  $(2, -2)$

(see Fig. 6). Moreover, from the part of the proposition already proven we know that  $f_{0,\theta_j}$  is non-increasing and continuous. Therefore, the claim holds provided that

$$\lim_{j \rightarrow \infty} \text{diam}(f_{0,\theta_j}(\mathbb{I}_{i_j, \theta_j})) = 0$$

(see again Fig. 6).

When  $\theta_j \in B_{\alpha_{i_j}}[(i_j)^*] \setminus B_{\alpha_{i_j+1}}((i_j)^*)$  and  $i_j \geq 0$ , by Definitions 7.1 and 4.2,

$$\text{diam}(f_{0,\theta_j}(\mathbb{I}_{i_j, \theta_j})) = \text{diam}(g_{i_j, \theta_j}(\mathbb{I}_{i_j, \theta_j})) = \text{diam}(\{\gamma_{i_j+1}(R_\omega(\theta_j))\}) = 0.$$

Otherwise, by Definition 7.1, and Lemmas 4.3(b) and 4.5(b),

$$\begin{aligned} \{R_\omega(\theta_j)\} \times f_{0,\theta_j}(\mathbb{I}_{i_j,\theta_j}) &= \{R_\omega(\theta_j)\} \times g_{i_j,\theta_j}(\mathbb{I}_{i_j,\theta_j}) = G_{i_j}(\mathcal{R}((i_j)^*)^{\theta_j}) \\ &\subset \mathcal{R}((i_j + 1)^*)^{R_\omega(\theta_j)}. \end{aligned}$$

So, by Remark 3.2(2),

$$\text{diam}(f_{0,\theta_j}(\mathbb{I}_{i_j,\theta_j})) \leq \text{diam}(\mathcal{R}((i_j + 1)^*)) \leq 2 \cdot 2^{-n|i_j+1|}.$$

In any case,

$$0 \leq \text{diam}(f_{0,\theta_j}(\mathbb{I}_{i_j,\theta_j})) \leq 2 \cdot 2^{-n|i_j+1|} \quad \text{for every } j \in \mathbb{N}$$

and, by (17),  $\lim_{j \rightarrow \infty} \text{diam}(f_{0,\theta_j}(\mathbb{I}_{i_j,\theta_j})) = 0$ . This ends the proof of the claim.

By the last claim, to end the proof of the proposition in the case  $m = 0$  it is enough to show that (18)–(19) hold. We start by proving (18). By Lemma A.2.1,

$$m_{i_j}(\text{Bd}(B_{i_j}^\sim[(i_j)^*])) = M_{i_j}(\text{Bd}(B_{i_j}^\sim[(i_j)^*])) = 0,$$

and from the definition of the maps  $m_{i_j}$  and  $M_{i_j}$ , Definition 3.1 (or Lemma 4.1) and Remark 3.2(2), for every  $s \in B_{i_j}^\sim((i_j)^*)$  we get

$$\begin{aligned} -1 \leq m_{i_j}(s) < 0 < M_{i_j}(s) \leq 1, \quad \text{and} \\ M_{i_j}(s) - m_{i_j}(s) = \text{diam}(\mathbb{I}_{i_j,s}) \leq 2 \cdot 2^{-n|i_j|}. \end{aligned} \tag{20}$$

So, (18) holds by (17). Now we prove (19).

By (1), (2) and (8), it follows that

$$\begin{aligned} m_{i_j}(\theta_j) < \gamma_{|i_j|}(\theta_j) < M_{i_j}(\theta_j) & \quad \text{if } \theta_j \neq (i_j)^*, \text{ and} \\ m_{i_j}(\theta_j) < \gamma_{|i_j|-1}(\theta_j) = 0 < M_{i_j}(\theta_j) & \quad \text{if } \theta_j = (i_j)^*. \end{aligned}$$

Also, from Definition 7.1, the definitions of  $G_i$  and  $g_{i,\theta}$  (Definitions 4.2 and 4.4), and Lemmas 4.3(c) and 4.5(c) we get

$$\begin{aligned} f_{0,\theta_j}(\gamma_{|i_j|}(\theta_j)) &= g_{i_j,\theta_j}(\gamma_{|i_j|}(\theta_j)) = \gamma_{|i_j+1|}(R_\omega(\theta_j)) & \quad \text{if } \theta_j \neq (i_j)^*, \\ f_{0,\theta_j}(\gamma_{i_j-1}(\theta_j)) &= g_{i_j,\theta_j}(\gamma_{i_j-1}(\theta_j)) = \gamma_{i_j}(R_\omega(\theta_j)) & \quad \text{if } \theta_j = (i_j)^* \text{ and } i_j \geq 0, \text{ and} \\ f_{0,\theta_j}(\gamma_{|i_j|-1}(\theta_j)) &= g_{i_j,\theta_j}(\gamma_{|i_j+1|}(\theta_j)) = \gamma_{|i_j+2|}(R_\omega(\theta_j)) & \quad \text{if } \theta_j = (i_j)^* \text{ and } i_j < 0. \end{aligned}$$

Thus, to prove (19), we have to show that

$$\begin{cases} \lim_{j \rightarrow \infty} \gamma_{|i_j+1|}(R_\omega(\theta_j)) = \gamma(R_\omega(\theta)) & \text{if } \theta_j \neq (i_j)^*, \\ \lim_{j \rightarrow \infty} \gamma_{i_j}(R_\omega(\theta_j)) = \gamma(R_\omega(\theta)) & \text{if } \theta_j = (i_j)^* \text{ and } i_j \geq 0, \text{ and} \\ \lim_{j \rightarrow \infty} \gamma_{|i_j+2|}(R_\omega(\theta_j)) = \gamma(R_\omega(\theta)) & \text{if } \theta_j = (i_j)^* \text{ and } i_j < 0 \end{cases} \tag{21}$$

(that is, we take  $x_j := \gamma_{|i_j|}(\theta_j)$  if  $\theta_j \neq (i_j)^*$ ,  $x_j := \gamma_{i_j-1}(\theta_j)$  if  $\theta_j = (i_j)^*$  and  $i_j \geq 0$ , and  $x_j := \gamma_{|i_j|-1}(\theta_j)$  if  $\theta_j = (i_j)^*$  and  $i_j < 0$ ).

Let  $\varepsilon > 0$ . By Lemma 5.3(c) and Definition 3.4(R.1) we have that  $\theta \notin O^*(\omega)$  and, hence,  $R_\omega(\theta) \notin O^*(\omega)$ . By the continuity of  $\gamma$  on  $\mathbb{S}^1 \setminus O^*(\omega)$  and the fact that  $\lim_{i \rightarrow \infty} \gamma_i = \gamma$ , there exist  $\delta > 0$  and  $L \in \mathbb{N}$  such that

$$\begin{aligned} |\gamma(R_\omega(\theta)) - \gamma(\hat{\theta})| &< \varepsilon/2 \quad \text{for every } \hat{\theta} \in B_\delta(R_\omega(\theta)) \setminus O^*(\omega), \text{ and} \\ d_\infty(\gamma, \gamma_i) &< \varepsilon/2 \quad \text{for every } i \geq L. \end{aligned}$$

Then, since  $\lim_{j \rightarrow \infty} \theta_j = \theta$  and  $\lim_{j \rightarrow \infty} |i_j| = \infty$ , there exists  $N \in \mathbb{N}$  such that  $|\theta - \theta_j| < \delta/2$ , and  $|i_j| \geq L+2$  for every  $j \geq N$ .

First we will show that

$$\left| \gamma(R_\omega(\theta)) - \gamma_{|i_j+1|}(R_\omega(\theta_j)) \right| \leq \varepsilon$$

for every  $j \geq N$  such that  $\theta_j \neq (i_j)^*$ . To see it observe that, by [Definition 3.4\(R.2\)](#) and [Remark 3.5\(R.2\)](#),  $\theta_j, R_\omega(\theta_j) \notin Z_{|i_j+1|}^*$  whenever  $\theta_j \neq (i_j)^*$ . Thus,  $\gamma_{|i_j+1|}$  is continuous at  $R_\omega(\theta_j)$  by [Lemma 3.6\(a\)](#).

Also, there exists a sequence  $\{\hat{\theta}_{j\ell}\}_{\ell=1}^\infty \subset (B_{\delta/2}(\theta_j) \cap B_{i_j}^{\rceil}((i_j)^*)) \setminus O^*(\omega)$  converging to  $\theta_j$ , because  $\mathbb{S}^1 \setminus O^*(\omega)$  is dense in  $\mathbb{S}^1$ . Clearly, for every  $j \geq N$ , we have  $\{R_\omega(\hat{\theta}_{j\ell})\}_{\ell=1}^\infty \subset B_\delta(R_\omega(\theta)) \setminus O^*(\omega)$  and  $\lim_{\ell \rightarrow \infty} R_\omega(\hat{\theta}_{j\ell}) = R_\omega(\theta_j)$ . Moreover, since  $\{R_\omega(\hat{\theta}_{j\ell})\}_{\ell=1}^\infty \subset \mathbb{S}^1 \setminus O^*(\omega) \subset \mathbb{S}^1 \setminus Z_{|i_j+1|}^*$ ,  $\gamma_{|i_j+1|}$  is defined for every  $R_\omega(\hat{\theta}_{j\ell})$ . Then, for every  $j \geq N$  and  $\ell \in \mathbb{N}$ , we have

$$\begin{aligned} \left| \gamma(R_\omega(\theta)) - \gamma_{|i_j+1|}(R_\omega(\hat{\theta}_{j\ell})) \right| &\leq \left| \gamma(R_\omega(\theta)) - \gamma(R_\omega(\hat{\theta}_{j\ell})) \right| + \left| \gamma(R_\omega(\hat{\theta}_{j\ell})) - \gamma_{|i_j+1|}(R_\omega(\hat{\theta}_{j\ell})) \right| \\ &< \frac{\varepsilon}{2} + d_\infty(\gamma, \gamma_{|i_j+1|}) < \varepsilon. \end{aligned}$$

Consequently,

$$\left| \gamma(R_\omega(\theta)) - \gamma_{|i_j+1|}(R_\omega(\theta_j)) \right| = \lim_{\ell \rightarrow \infty} \left| \gamma(R_\omega(\theta)) - \gamma_{|i_j+1|}(R_\omega(\hat{\theta}_{j\ell})) \right| \leq \varepsilon.$$

This ends the proof of the first equality of [\(21\)](#). The second and third equalities of [\(21\)](#) follow as above by replacing  $\gamma_{|i_j+1|}$  by  $\gamma_{i_j}$  (respectively  $\gamma_{|i_j+2|}$ ), and noting that

$$R_\omega(\theta_j) = R_\omega((i_j)^*) = \begin{cases} ((i_j + 1))^* \notin Z_{i_j}^* & \text{if } i_j \geq 0, \text{ and} \\ ((-(|i_j| - 1))^*) \notin Z_{|i_j|-2}^* & \text{if } i_j < 0. \end{cases}$$

This ends the proof of the continuity of  $T_0$ , and the proposition for the case  $m = 0$ .  $\square$

### A.3. Proof of [Proposition 7.3](#) for $m > 0$

This section is the second technical counterpart of [Section 7](#) and is devoted to prove [Proposition 7.3](#) for every map  $T_m$  with  $m > 0$ . To do this we will need some more technical results. Also we will use the notion of fibre map function introduced in the previous section.

The next two lemmas establish some basic properties of the maps  $T_m|_{\mathbb{V}_m^{\rceil}}$  and clarify some aspects of [Definition 7.2](#).

**Lemma A.3.1.** *For every  $m \in \mathbb{N}$  and for every  $\theta \in \mathbb{B}_m^{\rceil}$ ,*

$$f_{m,\theta}|_{\mathbb{I}_{i,\theta}} = g_{i,\theta}|_{\mathbb{I}_{i,\theta}},$$

where  $i = \mathbf{b}^{\rceil}(\theta, m)$ . Moreover, assume that  $\theta \in \mathbb{WB}_m^{\rceil} \setminus \mathbb{WIB}_m$ . Then,

$$f_{m,\theta}(x) = \begin{cases} g_{i,\theta}(x) & \text{if } x \in \mathbb{I}_{i,\theta}, \\ \frac{2 - g_{i,\theta}(m_i(\theta))}{2 - f_{m-1,\theta}(m_i(\theta))} (f_{m-1,\theta}(x) - 2) + 2 & \text{if } x \in [-2, m_i(\theta)], \\ \frac{2 + g_{i,\theta}(M_i(\theta))}{2 + f_{m-1,\theta}(M_i(\theta))} (f_{m-1,\theta}(x) + 2) - 2 & \text{if } x \in [M_i(\theta), 2]. \end{cases}$$



**Proof.** We start by proving the first statement. When  $\theta \in \mathbb{B}_m$  there is nothing to prove. So, assume that  $\theta \in \mathbb{B}_m^{\leftarrow} \setminus \mathbb{B}_m$ . By Definition 6.2,  $\theta \in \mathbb{W}\mathbb{B}_m^{\leftarrow}$ ,  $i < 0$  and  $\theta \in B_i^{\leftarrow} [i^*] \setminus B_{\alpha_{|i|}} (i^*)$ . By Lemma 6.3(b),

$$\mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} \subset \mathbb{I}\mathbb{W}_{m,\theta}.$$

Consequently, by Definition 7.2 and the definition of the maps  $g_{i,\theta}$  for  $i < 0$  (Definition 4.4—notice that  $\mathbb{I}_{i,\theta} \subset \mathcal{R}^{\leftarrow}(i^*)$  by definition),

$$f_{m,\theta}(\gamma_{|i|}(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)) = g_{i,\theta}(\gamma_{|i|}(\theta)).$$

So, the first statement holds. Now we prove the second one. By Lemma 6.3(b),

$$\mathbb{I}_{i,\theta} = \{m_i(\theta)\} = \{M_i(\theta)\} = \{\gamma_{|i|}(\theta)\} = \{\lambda_m(\theta)\} = \{\tau_m(\theta)\} = \mathbb{I}\mathbb{W}_{m,\theta}.$$

Thus, by the part already proven, the formulas

$$\begin{cases} g_{i,\theta}(x) & \text{if } x \in \mathbb{I}_{i,\theta}, \\ \frac{2 - g_{i,\theta}(m_i(\theta))}{2 - f_{m-1,\theta}(m_i(\theta))}(f_{m-1,\theta}(x) - 2) + 2 & \text{if } x \in [-2, m_i(\theta)], \\ \frac{2 + g_{i,\theta}(M_i(\theta))}{2 + f_{m-1,\theta}(M_i(\theta))}(f_{m-1,\theta}(x) + 2) - 2 & \text{if } x \in [M_i(\theta), 2], \end{cases}$$

and

$$\begin{cases} \gamma_{|i+1|}(R_\omega(\theta)) & \text{if } x \in \mathbb{I}\mathbb{W}_{m,\theta}, \\ \frac{2 - \gamma_{|i+1|}(R_\omega(\theta))}{2 - f_{m-1,\theta}(\lambda_m(\theta))}(f_{m-1,\theta}(x) - 2) + 2 & \text{if } x \in [-2, \lambda_m(\theta)], \\ \frac{2 + \gamma_{|i+1|}(R_\omega(\theta))}{2 + f_{m-1,\theta}(\tau_m(\theta))}(f_{m-1,\theta}(x) + 2) - 2 & \text{if } x \in [\tau_m(\theta), 2], \end{cases}$$

coincide.  $\square$

**Lemma A.3.2.** *The following statements hold for every  $m \in \mathbb{N}$  and  $i \in \mathfrak{D}_m$ :*

- (a) *The map  $T_m|_{V_{i^*}^{\leftarrow}}$  is well defined and continuous.*
- (b) *For every  $\theta \in B_i^{\leftarrow} [i^*]$ ,*
  - (b.i)  $f_{m,\theta}(2) = -2$  and  $f_{m,\theta}(-2) = 2$ ,
  - (b.ii)  $f_{m,\theta}$  is piecewise affine and non-increasing, and
  - (b.iii)  $-1 \leq f_{m,\theta}(M_i(\theta)) \leq f_{m,\theta}(m_i(\theta)) \leq 1$ .
- (c)  $T_m|_{\mathcal{R}^{\leftarrow}(i^*)} = G_i$  and  $T_m(A_{|i|}^{i^*}) = A_{|i+1|}^{(i+1)^*}$ .

**Proof.** Clearly,  $T_m|_{V_{i^*}^{\leftarrow}}$  is well defined and continuous if and only if so is  $f_m|_{V_{i^*}^{\leftarrow}}$ .

We will prove by induction on  $m \in \mathbb{Z}^+$  that, (a), (b) and

- (b.iv)  $f_{m,\theta}|_{[-2,-1]}$  and  $f_{m,\theta}|_{[1,2]}$  are affine,  $f_{m,\theta}(-1) < 2$  and  $f_{m,\theta}(1) > -2$

hold for every  $\theta \in B_i^{\leftarrow} [i^*]$ .

First we will show that (a), (b) and (b.iv) hold for  $m = 0$  and  $i \in \mathfrak{D}_0$  (we are including the map  $f_0$  studied earlier to correctly start the induction process). By Proposition 7.3(a), (b) for  $m = 0$  we have that  $T_0|_{V_{i^*}^{\leftarrow}}$  is well defined and continuous and (b) holds. By Definition 7.1, we also know that  $f_{m,\theta}|_{[-2,m_i(\theta)]}$  and  $f_{m,\theta}|_{[M_i(\theta),2]}$  are affine. Then, (b.iv) follows from  $-1 \leq m_i(\theta) \leq M_i(\theta) \leq 1$  (see Lemma 4.1(a)) and (b.iii).

Assume now that (a), (b) and (b.iv) hold for some  $m - 1 \in \mathbb{Z}_+$  and prove it for  $m$  and  $i \in \mathfrak{D}_m$ . By Lemma 5.2(a),  $\theta \in B_i^\sim [i^*] \not\subset B_k^\sim [k^*]$  for some  $k \in \mathfrak{D}_{m-1}$ . Consequently,  $V_{i^*}^\sim \subset V_{k^*}^\sim$  and  $f_{m-1}|_{V_{i^*}^\sim}$  is well defined and continuous.

By Lemma 4.1(a) and Definition 6.2,

$$\begin{aligned} -1 \leq m_i(\theta) \leq M_i(\theta) \leq 1 & \quad \text{for } \theta \in B_i^\sim [i^*], \text{ and} \\ -1 \leq \lambda_m(\theta) \leq \tau_m(\theta) \leq 1 & \quad \text{for } \theta \in B_i^\sim [i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset \mathbb{W}\mathbb{B}_m^\sim \ (i < 0). \end{aligned} \tag{22}$$

Consequently, by (b.ii) and (b.iv) for  $m - 1$ ,

$$-2 < f_{m-1,\theta}(1) \leq f_{m-1,\theta}(M_i(\theta)) \leq f_{m-1,\theta}(m_i(\theta)) \leq f_{m,\theta}(-1) < 2$$

for every  $\theta \in B_i^\sim [i^*]$ , and

$$-2 < f_{m-1,\theta}(1) \leq f_{m-1,\theta}(\tau_m(\theta)) \leq f_{m-1,\theta}(\lambda_m(\theta)) \leq f_{m,\theta}(-1) < 2$$

for  $\theta \in B_i^\sim [i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset \mathbb{W}\mathbb{B}_m^\sim$  when  $i < 0$ .

On the other hand, as it was observed in Definition 7.2,  $f_{m,\theta}$  is defined in two different ways when  $\theta \in \mathbb{W}\mathbb{B}_m^\sim \cap \mathbb{B}_m$ . In such a case, by Lemmas 6.3(a), (b) and A.3.1,  $\theta \notin \mathbb{W}\mathbb{I}\mathbb{B}_m$  and both definitions for  $f_{m,\theta}$  coincide. Hence,  $f_m|_{V_{i^*}^\sim}$  is well defined.

Now we prove that  $f_m|_{V_{i^*}^\sim}$  is continuous by using the continuity of  $f_{m-1}|_{V_{i^*}^\sim}$ . Since  $B_{\alpha_{|i|}}[i^*] \subset \mathbb{B}_m$ , by Definition 7.2, the continuity of the maps  $m_i$  and  $M_i$  (see Lemma 4.1(b)), and the continuity of the maps  $g_i$  (Lemmas 4.3(a) and 4.5(a)),  $f_m|_{\uparrow\uparrow B_{\alpha_{|i|}}[i^*]}$  are continuous. Now we assume that  $i < 0$  and we study the continuity of  $f_m|_{\uparrow\uparrow U}$  on a connected component  $U$  of  $B_i^\sim [i^*] \setminus B_{\alpha_{|i|}}(i^*)$ . Observe that, by Definition 6.2 and Lemma 5.2(b),  $U$  is a connected component of  $\mathbb{W}\mathbb{B}_m^\sim$ . Then, again by Definition 7.2, the continuity of the maps  $\lambda_m|_U$  and  $\tau_m|_U$  (Lemma 6.4), and the continuity of the map  $\gamma_{|i|}|_U$  (Lemma 3.6(a) and Definition 3.4(R.2) and Remark 3.5(R.2)),  $f_m|_{\uparrow\uparrow U}$  is continuous. Therefore,  $f_m|_{V_{i^*}^\sim}$  is continuous because it is well defined on  $\uparrow\uparrow ((B_i^\sim [i^*] \setminus B_{\alpha_{|i|}}(i^*)) \cap B_{\alpha_{|i|}}[i^*])$ .

Let  $\theta \in B_{\alpha_{|i|}}[i^*] \subset \mathbb{B}_m$ . By Definition 7.2, and the definition of the maps  $g_{i,\theta}$  (Definitions 4.2 and 4.4),  $f_{m,\theta}|_{\mathbb{I}_{i,\theta}}$  is piecewise affine and non-increasing. So, by Lemma A.3.1 for  $m - 1$  and Definition 7.2,  $f_{m,\theta}(2) = -2$ ,  $f_{m,\theta}(-2) = 2$ , and  $f_{m,\theta}|_{[-2,m_i(\theta)]}$  and  $f_{m,\theta}|_{[M_i(\theta),2]}$  are affine transformations of the map  $f_{m-1,\theta}$  with positive slope. Hence, (b.i,ii) hold for  $f_{m,\theta}$  in this case. Moreover, (b.iv) is verified by (22) and (b.iv) for  $m - 1$ .

Consider  $\theta \in B_i^\sim [i^*] \setminus B_{\alpha_{|i|}}(i^*) \subset \mathbb{W}\mathbb{B}_m^\sim$ . Again by Definition 7.2,  $f_{m,\theta}|_{\mathbb{I}\mathbb{W}_{m,\theta}^\sim}$  is constant. Then, (b.i,ii) and (b.iv) hold for  $f_{m,\theta}$  as above by replacing  $m_i(\theta)$  and  $M_i(\theta)$  by  $\lambda_m(\theta)$  and  $\tau_m(\theta)$ , respectively.

By (b.ii) and (22) we have  $f_{m,\theta}(M_i(\theta)) \leq f_{m,\theta}(m_i(\theta))$ . Hence, (b.iii) follows from Lemma A.3.1, Definition 7.2, Lemmas 4.3(b) and 3.10(c), Definition 3.4(R.2) and Remark 3.5(R.2), Lemma 4.5(b) and Lemma 3.6(b).

(c) In a similar way to the proof of Proposition 7.3 for the case  $m = 0$ ,

$$\mathcal{R}^\sim(i^*) = \bigcup_{\theta \in B_i^\sim [i^*]} \{\theta\} \times \mathbb{I}_{i,\theta} \subset V_{i^*}^\sim \subset V_m^\sim$$

and, by Definition 7.2, Lemma A.3.1 and the definition of  $G_i$  (Definitions 4.2 and 4.4) it follows that

$$T_m(\theta, x) = (R_\omega(\theta), f_m(\theta, x)) = (R_\omega(\theta), g_{i,\theta}(x)) = G_i(\theta, x),$$

for every  $(\theta, x) \in \mathcal{R}^\sim(i^*)$ . Thus,  $T_m(A_{|i|}^{i^*}) = A_{|i+1|}^{(i+1)^*}$  from Lemmas 3.10(b), 4.3(c) and 4.5(c).  $\square$

The next technical lemma compares the images of  $f_{m,\theta}$  and  $f_{m-1,\theta}$  on a point. It is an extension of Lemma 5.4.

**Lemma A.3.3.** *Assume that  $B_i^\sim [i^*] \subset B_k^\sim [k^*]$  for some  $i \in \mathfrak{D}_m$ ,  $k \in \mathfrak{D}_{m-1}$  and  $m \in \mathbb{N}$ . Then, for every  $\theta \in B_i^\sim [i^*] \setminus B_{\alpha|i} (i^*)$ ,  $m_i(\theta) = M_i(\theta) = \gamma_i(\theta)$  and*

$$\begin{aligned} f_{m,\theta}(m_i(\theta)) &= g_{i,\theta}(m_i(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)), \quad \text{and} \\ f_{m-1,\theta}(m_i(\theta)) &= g_{k,\theta}(m_i(\theta)) = \gamma_{|k+1|}(R_\omega(\theta)). \end{aligned}$$

**Proof.** The fact that  $m_i(\theta) = M_i(\theta) = \gamma_i(\theta)$  follows directly from the definitions. The first equation follows from Lemma A.3.1, and the definition of the map  $g_{i,\theta}$  (Definitions 4.2 and 4.4).

By Lemma 5.4,  $\mathbb{I}_{i,\theta} = \{m_i(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta}$ . Moreover, as in the proof of Lemma 5.4,  $\theta \neq k^*$ . Consequently, by Definition 7.1, Lemma A.3.1, Lemmas 4.3(c) and 4.5(c) and (1) (alternatively, for the last equality check directly the proofs of Lemmas 4.3(c) and 4.5(c)),

$$f_{m-1,\theta}(m_i(\theta)) = g_{k,\theta}(m_i(\theta)) = g_{k,\theta}(\gamma_{|k|}(\theta)) = \gamma_{|k+1|}(R_\omega(\theta)). \quad \square$$

The following lemma is the analogue of Lemma A.2.1 for  $m \geq 1$ . To state it we will use the set

$$\uparrow\uparrow \mathbb{E}\mathbb{B}_m^\sim = \mathbb{E}\mathbb{B}_m^\sim \times \mathbb{I} \subset \mathbb{V}_m^\sim.$$

**Lemma A.3.4.**  *$T_m|_{\uparrow\uparrow \mathbb{E}\mathbb{B}_m^\sim} = T_{m-1}|_{\uparrow\uparrow \mathbb{E}\mathbb{B}_m^\sim}$  for every  $m \in \mathbb{N}$ . Equivalently,  $f_{m,\theta} = f_{m-1,\theta}$  for every  $m \in \mathbb{N}$  and  $\theta \in \mathbb{E}\mathbb{B}_m^\sim$ .*

**Proof.** Fix  $m \in \mathbb{N}$  and  $\theta \in \mathbb{E}\mathbb{B}_m^\sim \subset \mathbb{B}_m^\sim$ . By Lemma 5.2(a), (b), there exist  $i \in \mathfrak{D}_m$  and  $k \in \mathfrak{D}_{m-1}$  such that  $\theta \in \text{Bd}(B_i^\sim [i^*]) \subset B_i^\sim [i^*] \subsetneq B_k^\sim [k^*]$ . So, we are in the assumptions of Lemmas 5.4 and A.3.3 and, hence,

$$\begin{aligned} \mathbb{I}_{i,\theta} &= \{m_i(\theta)\} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta}, \\ f_{m,\theta}(m_i(\theta)) &= g_{i,\theta}(m_i(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)), \quad \text{and} \\ f_{m-1,\theta}(m_i(\theta)) &= g_{k,\theta}(m_i(\theta)) = \gamma_{|k+1|}(R_\omega(\theta)). \end{aligned}$$

Thus, if  $i \geq 0$ ,  $\theta \in \mathbb{B}_m$  and, by Definition 7.2 and Lemma A.3.2(a), to prove that  $f_{m,\theta} = f_{m-1,\theta}$  we only have to show that

$$g_{i,\theta}(m_i(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)) = \gamma_{|k+1|}(R_\omega(\theta)) = f_{m-1,\theta}(m_i(\theta)).$$

When  $i < 0$ ,  $\theta \in \mathbb{W}\mathbb{B}_m^\sim \cap \mathbb{E}\mathbb{B}_m^\sim$  and, by Lemma 6.3(a),  $\theta \notin \mathbb{W}\mathbb{I}\mathbb{B}_m$ . Then, by Lemma A.3.1, we get again that

$$g_{i,\theta}(m_i(\theta)) = \gamma_{|i+1|}(R_\omega(\theta)) = \gamma_{|k+1|}(R_\omega(\theta)) = f_{m-1,\theta}(m_i(\theta))$$

implies  $f_{m,\theta} = f_{m-1,\theta}$ .

If  $|k+1| = |i+1|$  there is nothing to prove. So, by Lemma 5.4, we can assume that  $|k+1| < |i+1|$  and we have

$$\gamma_{|k+1|}(R_\omega(\theta)) = \gamma_{|k+1|+1}(R_\omega(\theta)) = \dots = \gamma_{|i+1|-1}(R_\omega(\theta)).$$

Hence, we have to show that  $\gamma_{|i+1|}(R_\omega(\theta)) = \gamma_{|i+1|-1}(R_\omega(\theta))$ . If  $i \geq 0$  we get

$$\gamma_{|i+1|}(R_\omega(\theta)) = \gamma_{i+1}(R_\omega(\theta)) = \gamma_i(R_\omega(\theta)) = \gamma_{|i+1|-1}(R_\omega(\theta))$$

by Lemma 3.6(e). Otherwise we have  $i < 0$ ,  $\theta \in \text{Bd}(B_i^\rceil [i^*]) = \text{Bd}(B_{\alpha_{|i+1|}} [i^*])$  and, consequently,  $R_\omega(\theta) \in \text{Bd}(B_{\alpha_{|i+1|}} [(i+1)^*])$ . Again by Lemma 3.6(e) for  $j = |i+1|$ ,

$$\gamma_{|i+1|}(R_\omega(\theta)) = \gamma_{|i+1|-1}(R_\omega(\theta)).$$

This ends the proof of the lemma.  $\square$

Now we aim at computing two different kind of upper bounds for  $\|f_{m,\theta} - f_{m-1,\theta}\|$  (Lemma A.3.6 and Proposition 7.4). This will be a key tool in the proof of Propositions 7.3 for  $m > 0$  and 7.4. The next two lemmas and remark will be useful to automate and simplify the proofs of these two results.

**Lemma A.3.5.**

$$\|f_{m,\theta} - f_{m-1,\theta}\| = \begin{cases} \left\| f_{m,\theta}|_{\mathbb{B}_{\mathbf{b}^\rceil(\theta,m),\theta}} - f_{m-1,\theta}|_{\mathbb{B}_{\mathbf{b}^\rceil(\theta,m),\theta}} \right\| & \text{when } \theta \in \mathbb{B}_m^\rceil \setminus \text{WIB}_m, \text{ and} \\ \left\| f_{m,\theta}|_{\mathbb{IW}_{m,\theta}} - f_{m-1,\theta}|_{\mathbb{IW}_{m,\theta}} \right\| & \text{when } \theta \in \text{WIB}_m, \end{cases}$$

for every  $m \geq 2$  and  $\theta \in \mathbb{B}_m^\rceil$ .

**Proof.** Set  $i = \mathbf{b}^\rceil(\theta, m) \in \mathfrak{D}_m$ , so that  $\theta \in B_i^\rceil [i^*]$ .

When  $\theta \in \mathbb{B}_m^\rceil \setminus \text{WIB}_m = \mathbb{B}_m \cup \text{WB}_m^\rceil \setminus \text{WIB}_m$ , by Definition 7.2 and Lemma A.3.1, it is enough to show that

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(m_i(\theta)) - f_{m-1,\theta}(m_i(\theta))|$$

for every  $x \in [-2, m_i(\theta)]$ , and

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(M_i(\theta)) - f_{m-1,\theta}(M_i(\theta))|$$

for every  $x \in [M_i(\theta), 2]$ . We will prove the first statement. The second one follows similarly.

Definition 7.2 and Lemma A.3.1 give

$$\begin{aligned} f_{m,\theta}(x) - f_{m-1,\theta}(x) &= \frac{2 - g_{i,\theta}(m_i(\theta))}{2 - f_{m-1,\theta}(m_i(\theta))} (f_{m-1,\theta}(x) - 2) + 2 - f_{m-1,\theta}(x) \\ &= \frac{2 - f_{m,\theta}(m_i(\theta))}{2 - f_{m-1,\theta}(m_i(\theta))} (f_{m-1,\theta}(x) - 2) - (f_{m-1,\theta}(x) - 2) \\ &= (f_{m-1,\theta}(x) - 2) \left( \frac{2 - f_{m,\theta}(m_i(\theta))}{2 - f_{m-1,\theta}(m_i(\theta))} - 1 \right) \\ &= (2 - f_{m-1,\theta}(x)) \frac{f_{m,\theta}(m_i(\theta)) - f_{m-1,\theta}(m_i(\theta))}{2 - f_{m-1,\theta}(m_i(\theta))}. \end{aligned}$$

By Lemma A.3.2(b),  $2 \geq f_{m-1,\theta}(x) \geq f_{m-1,\theta}(m_i(\theta))$  and  $1 \geq f_{m-1,\theta}(m_i(\theta))$ . Hence,

$$\begin{aligned} |f_{m,\theta}(x) - f_{m-1,\theta}(x)| &= (2 - f_{m-1,\theta}(x)) \frac{|f_{m,\theta}(m_i(\theta)) - f_{m-1,\theta}(m_i(\theta))|}{2 - f_{m-1,\theta}(m_i(\theta))} \\ &\leq |f_{m,\theta}(m_i(\theta)) - f_{m-1,\theta}(m_i(\theta))|. \end{aligned}$$

Now assume that  $\theta \in \text{WIB}_m \subset \text{WB}_m^\rceil$ . By Definition 7.2 it is enough to show that

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(\lambda_m(\theta)) - f_{m-1,\theta}(\lambda_m(\theta))|$$

for every  $x \in [-2, \lambda_m(\theta)]$ , and

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(\tau_m(\theta)) - f_{m-1,\theta}(\tau_m(\theta))|$$

for every  $x \in [\tau_m(\theta), 2]$ . As before, we will prove the first statement. The second one follows similarly. We have

$$f_{m,\theta}(x) - f_{m-1,\theta}(x) = (2 - f_{m-1,\theta}(x)) \frac{f_{m,\theta}(\lambda_m(\theta)) - f_{m-1,\theta}(\lambda_m(\theta))}{2 - f_{m-1,\theta}(\lambda_m(\theta))}.$$

By Lemma A.3.2(b),  $2 \geq f_{m-1,\theta}(x) \geq f_{m-1,\theta}(\lambda_m(\theta))$  and hence,

$$|f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq |f_{m,\theta}(m_i(\theta)) - f_{m-1,\theta}(m_i(\theta))|$$

provided that  $2 - f_{m-1,\theta}(\lambda_m(\theta)) \neq 0$ . Assume by way of contradiction that we have  $f_{m-1,\theta}(\lambda_m(\theta)) = 2$ . Then, by Definition 6.2 and Lemma A.3.2(b),  $-1 \leq \lambda_m(\theta)$  and

$$2 \geq f_{m-1,\theta}(-1) \geq f_{m-1,\theta}(\lambda_m(\theta)) = 2;$$

which contradicts statement (b.iv) from the proof of Lemma A.3.2.  $\square$

Next we compute an upper bound for  $\|f_{m,\theta} - f_{m-1,\theta}\|$  for every  $\theta \in B_i^{\leftarrow}[i^*]$  and  $i \in \mathfrak{D}_m$  such that  $\text{diam}(B_i^{\leftarrow}[i^*])$  is small enough.

**Lemma A.3.6.** *Assume that  $T_{m-1}$  is continuous for some  $m \geq 2$  and let  $\varepsilon$  be positive. Then, there exist  $\varrho_m(\varepsilon) \in \mathbb{N}$  such that*

$$\|f_{m,\theta} - f_{m-1,\theta}\| \leq \varepsilon$$

for every  $\theta \in B_i^{\leftarrow}[i^*]$  and  $i \in \mathfrak{D}_m$  (that is,  $B_i^{\leftarrow}[i^*] \subset \mathbb{B}_m^{\leftarrow}$ ) such that  $|i| \geq \varrho_m(\varepsilon)$ .

**Proof.** Since  $T_{m-1}$  is uniformly continuous, there exists  $\delta_{m-1} = \delta_{m-1}(\varepsilon) > 0$  such that  $d_\Omega(T_{m-1}(\theta, x), T_{m-1}(\nu, y)) < \varepsilon$  provided that  $d_\Omega((\theta, x), (\nu, y)) < \delta_{m-1}$ . We choose  $\varrho_m = \varrho_m(\varepsilon) \in \mathbb{N}$  such that

$$3 \cdot 2^{-\varrho_m} < \min\{\delta_{m-1}(\varepsilon/2), \varepsilon/2\}.$$

Assume that  $i \in \mathfrak{D}_m$  verifies  $|i| \geq \varrho_m(\varepsilon)$  and let  $(\theta, x) \in V_{i^*}^{\leftarrow} = B_i^{\leftarrow}[i^*] \times \mathbb{I}$ . When  $\theta \in B_i^{\leftarrow}[i^*] \setminus \text{WIB}_m$  we can use Lemma A.3.5 with  $\mathbb{I}_{i,\theta}$  to compute  $\|f_{m,\theta} - f_{m-1,\theta}\|$ . We have to show that  $|f_{m,\theta}(x) - f_{m-1,\theta}(x)| < \varepsilon$  for every  $x \in \mathbb{I}_{i,\theta}$ .

Let  $\nu \in \text{Bd}(B_i^{\leftarrow}[i^*]) \subset \mathbb{EB}_m^{\leftarrow}$ . We have  $(\theta, x), (\nu, m_i(\nu)) \in \mathcal{R}^{\leftarrow}(i^*)$  and, by Lemmas A.3.2(c) and 3.6(f),

$$\begin{aligned} d_\Omega(T_m(\theta, x), T_m(\nu, m_i(\nu))) &= d_\Omega(G_i(\theta, x), G_i(\nu, m_i(\nu))) \\ &\leq \text{diam}(G_i(\mathcal{R}^{\leftarrow}(i^*))), \quad \text{and} \\ d_\Omega((\theta, x), (\nu, m_i(\nu))) &\leq \text{diam}(\mathcal{R}^{\leftarrow}(i^*)) \leq 2 \cdot 2^{-|i|} < 3 \cdot 2^{-\varrho_m} < \delta_{m-1}(\varepsilon/2). \end{aligned}$$

Thus,

$$d_\Omega(T_{m-1}(\theta, x), T_{m-1}(\nu, m_i(\nu))) < \varepsilon/2.$$

Consequently, by Lemma A.3.4,

$$\begin{aligned} |f_{m,\theta}(x) - f_{m-1,\theta}(x)| &= d_\Omega(T_m(\theta, x), T_{m-1}(\theta, x)) \\ &\leq d_\Omega(T_m(\theta, x), T_{m-1}(\nu, m_i(\nu))) + d_\Omega(T_{m-1}(\nu, m_i(\nu)), T_{m-1}(\theta, x)) \\ &< d_\Omega(T_m(\theta, x), T_m(\nu, m_i(\nu))) + \varepsilon/2 \\ &< \text{diam}(G_i(\mathcal{R}^{\leftarrow}(i^*))) + \varepsilon/2. \end{aligned}$$

Now we look at the size of  $G_i(\mathcal{R}^\sim(i^*))$ . When  $i < 0$ , from [Lemmas 4.5\(b\)](#) and [3.6\(f\)](#), we obtain

$$\text{diam}(G_i(\mathcal{R}^\sim(i^*))) \leq \text{diam}(\mathcal{R}((i+1)^*)) \leq 2^{-(|i|-1)} < 2 \cdot 2^{-|i|}. \tag{23}$$

When  $i \geq 0$ , from [Lemma 4.3\(b\)](#) we get

$$G_i(\mathcal{R}^\sim(i^*)) = G_i(\mathcal{R}(i^*)) \subset \mathcal{R}((i+1)^*) \cup A_{i+1}^{\uparrow(B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*))}.$$

Moreover, as in the proof of [Lemma 3.6\(f\)](#) for  $\ell < 0$ , the set

$$\mathcal{R}((i+1)^*) \cup A_{i+1}^{\uparrow(B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*))}$$

is connected. So, by [Lemma 3.6\(f\)](#),

$$\begin{aligned} \text{diam}(G_i(\mathcal{R}^\sim(i^*))) &\leq \text{diam}\left(\mathcal{R}((i+1)^*) \cup A_{i+1}^{\uparrow(B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*))}\right) \\ &\leq \text{diam}(\mathcal{R}((i+1)^*)) + \text{diam}\left(A_{i+1}^{\uparrow(B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*))}\right) \\ &\leq 2^{-(i+1)} + \text{diam}\left(A_{i+1}^{\uparrow(B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*))}\right). \end{aligned}$$

As noticed earlier,  $B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*)$  is disjoint from

$$B_{\alpha_{i+1}}((i+1)^*) \cup B_{-(i+1)}^\sim[(-(i+1))^*] \cup Z_{i+1}^*$$

by [Definition 3.4\(R.2\)](#) and [Remark 3.5\(R.2\)](#). So, by [Lemma 3.10\(c\)](#), [Definition 3.4](#) and [Lemma 3.6\(a\)](#),

$$\begin{aligned} A_{i+1}^\nu &= \{(\nu, \gamma_{i+1}(\nu))\} = \{(\nu, \gamma_i(\nu))\} \\ &\subset \{\nu\} \times [\gamma_i((i+1)^*) - 2^{-n_i}, \gamma_i((i+1)^*) + 2^{-n_i}]. \end{aligned}$$

for every  $\nu \in B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*)$ . On the other hand,  $\gamma_i((i+1)^*) \in \mathbb{I}_{i+1, (i+1)^*}$  by [Lemma 3.6\(c\)](#). Hence, by [Remark 3.2\(2\)](#), [Definition 3.4\(R.1\)](#) and [Remark 3.5\(R.1\)](#),

$$\begin{aligned} \text{diam}\left(A_{i+1}^{\uparrow(B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*))}\right) &\leq \max\{\text{diam}(B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*)), 2 \cdot (2^{-n_i} + 2^{-n_{i+1}})\} \\ &\leq 2 \cdot \max\{\alpha_i, 2^{-n_i} + 2^{-n_{i+1}}\} = 2 \cdot (2^{-n_i} + 2^{-n_{i+1}}) \\ &< 4 \cdot 2^{-n_i} \leq 2 \cdot 2^{-i}. \end{aligned}$$

Summarizing, when  $i \geq 0$ ,

$$\text{diam}(G_i(\mathcal{R}^\sim(i^*))) \leq 2^{-(i+1)} + 2 \cdot 2^{-i} < 3 \cdot 2^{-i}$$

and, from [\(23\)](#),

$$\text{diam}(G_i(\mathcal{R}^\sim(i^*))) < 3 \cdot 2^{-|i|} \leq 3 \cdot 2^{-\ell m} < \varepsilon/2$$

for every  $i \in \mathbb{Z}^+$ . Thus, for every  $x \in \mathbb{I}_{i, \theta}$ ,

$$|f_{m, \theta}(x) - f_{m-1, \theta}(x)| < \text{diam}(G_i(\mathcal{R}^\sim(i^*))) + \varepsilon/2 < \varepsilon.$$

Now assume that  $\theta \in B_i^- [i^*] \cap \mathbb{W}\mathbb{I}\mathbb{B}_m$ . We can use [Lemma A.3.5](#) with  $\mathbb{I}\mathbb{W}_{m, \theta}$  to compute  $\|f_{m, \theta} - f_{m-1, \theta}\|$ . We have to show that  $|f_{m, \theta}(x) - f_{m-1, \theta}(x)| < \varepsilon$  for every  $x \in \mathbb{I}\mathbb{W}_{m, \theta}$ . Since  $\theta \in \mathbb{W}\mathbb{I}\mathbb{B}_m$ , by [Definition 6.2](#) and [Lemma 6.3\(b\)](#),  $i < 0$ ,  $\theta \in \mathbb{W}\mathbb{B}_m^-$  and

$$\mathbb{I}_{i, \theta} = \{\gamma_{|i|}(\theta)\} \subset \mathbb{I}\mathbb{W}_{m, \theta} = \mathbb{I}_{\ell, \theta} \ni x$$

with  $\ell = \mathbf{b}^-(\theta, \text{led}(\theta, m)) \in \mathfrak{W}\mathfrak{F}\mathfrak{D}_m$ . In this case we will consider the points  $(\theta, x) \in \mathcal{R}(\ell^*)$  and  $(\nu, m_i(\nu)), (\theta, \gamma_{|i|}(\theta)) \in \mathcal{R}^-(i^*)$  with  $\nu \in \text{Bd}(B_i^-(i^*)) \subset \mathbb{E}\mathbb{B}_m^-$ . By Lemma 6.1(b), Remark 3.2(2) and Lemma 3.6(f),  $|i| < |\ell|$  and

$$\begin{aligned} d_\Omega((\theta, x), (\nu, m_i(\nu))) &\leq d_\Omega((\theta, x), (\theta, \gamma_{|i|}(\theta))) + d_\Omega((\theta, \gamma_{|i|}(\theta)), (\nu, m_i(\nu))) \\ &\leq |x - \gamma_{|i|}(\theta)| + \text{diam}(\mathcal{R}^-(i^*)) \\ &\leq \text{diam}(\mathcal{R}(\ell^*)) + \text{diam}(\mathcal{R}^-(i^*)) \\ &\leq 2^{-|\ell|} + 2 \cdot 2^{-|i|} < 3 \cdot 2^{-|i|} \leq 3 \cdot 2^{-\ell_m} < \delta_{m-1}(\varepsilon/2). \end{aligned}$$

Thus,

$$d_\Omega(T_{m-1}(\theta, x), T_{m-1}(\nu, m_i(\nu))) < \varepsilon/2.$$

On the other hand, by Lemma A.3.2(c), Definition 7.2 and (23),

$$\begin{aligned} d_\Omega(T_m(\theta, x), T_m(\nu, m_i(\nu))) &\leq d_\Omega(T_m(\theta, x), T_m(\theta, \gamma_{|i|}(\theta))) + d_\Omega(T_m(\theta, \gamma_{|i|}(\theta)), T_m(\nu, m_i(\nu))) \\ &\leq |f_{m,\theta}(x) - f_{m,\theta}(\gamma_{|i|}(\theta))| + d_\Omega(G_i(\theta, \gamma_{|i|}(\theta)), G_i(\nu, m_i(\nu))) \\ &= d_\Omega(G_i(\theta, x), G_i(\nu, m_i(\nu))) \leq \text{diam}(G_i(\mathcal{R}^-(i^*))) < 2 \cdot 2^{-|i|} \\ &\leq 3 \cdot 2^{-\ell_m} < \varepsilon/2. \end{aligned}$$

So, in a similar way as before, Lemma A.3.4 gives

$$\begin{aligned} |f_{m,\theta}(x) - f_{m-1,\theta}(x)| &= d_\Omega(T_m(\theta, x), T_{m-1}(\theta, x)) \\ &\leq d_\Omega(T_m(\theta, x), T_{m-1}(\nu, m_i(\nu))) + d_\Omega(T_{m-1}(\nu, m_i(\nu)), T_{m-1}(\theta, x)) \\ &< \varepsilon. \quad \square \end{aligned}$$

**Proof of Proposition 7.3 for  $m > 0$ .** (a) We start by proving by induction on  $m$  that  $T_m$  is continuous for every  $m \in \mathbb{Z}^+$ .

By Proposition 7.3(a) for  $m = 0$ ,  $T_0$  is continuous. So, we may assume that  $T_{m-1}$  is continuous for some  $m \in \mathbb{N}$  and prove that  $T_m$  is continuous.

Let  $\varepsilon > 0$  be fixed but arbitrary, and let  $(\theta, x), (\nu, y) \in \Omega$ . We have to show that there exists  $\delta(\varepsilon) > 0$  such that

$$d_\Omega(T_m(\theta, x), T_m(\nu, y)) < \varepsilon \quad \text{when } d_\Omega((\theta, x), (\nu, y)) < \delta.$$

We start by defining  $\delta(\varepsilon)$ . To this end we need to introduce some more notation and establish some facts about the maps  $T_m$  and  $T_{m-1}$ .

Since  $T_{m-1}$  is uniformly continuous, we know that

$$\begin{aligned} \text{there exists } \delta_{m-1} = \delta_{m-1}(\varepsilon) > 0 \text{ such that } d_\Omega(T_{m-1}(\theta, x), T_{m-1}(\nu, y)) &< \varepsilon \\ \text{provided that } d_\Omega((\theta, x), (\nu, y)) &< \delta_{m-1}. \end{aligned} \tag{24}$$

On the other hand, Lemma A.3.2(a) tells us that  $T_m|_{V_{i^*}^-}$  is uniformly continuous for every  $i \in \mathfrak{D}_m$ . So, for every  $i \in \mathfrak{D}_m$ ,

$$\begin{aligned} \text{there exists } \delta_{m,i} = \delta_{m,i}(\varepsilon) > 0 \text{ such that } d_\Omega(T_m(\theta, x), T_m(\nu, y)) &< \varepsilon \\ \text{for every } (\theta, x), (\nu, y) \in V_{i^*}^- \subset V_m^- \text{ verifying } d_\Omega((\theta, x), (\nu, y)) &< \delta_{m,i}(\varepsilon). \end{aligned} \tag{25}$$

Then, by using the numbers  $\delta_{m-1}(\varepsilon/7)$  given by (24),  $\delta_{m,i}(\varepsilon/7)$  given by (25) and  $\varrho_m(\varepsilon/7)$  given by Lemma A.3.6, we set

$$\delta = \delta(\varepsilon) := \min \{ \delta_{m-1}(\varepsilon/7), \min \{ \delta_{m,i}(\varepsilon/7) : i \in \mathfrak{D}_m \cap Z_{\varrho_m(\varepsilon/7)} \} \}.$$

Clearly,  $\delta > 0$  because the set  $\mathfrak{D}_m \cap Z_{\varrho_m(\varepsilon/7)}$  is finite.

Now we will show that if  $d_\Omega((\theta, x), (\nu, y)) < \delta$ , then  $d_\Omega(T_m(\theta, x), T_m(\nu, y)) < \varepsilon$ .

Assume first that  $(\theta, x), (\nu, y) \in V_{\ell^*}$  for some  $\ell \in \mathfrak{D}_m \cap Z_{\varrho_m(\varepsilon/7)}$ . We have

$$d_\Omega((\theta, x), (\nu, y)) < \delta \leq \min \{ \delta_{m,i}(\varepsilon/7) : i \in \mathfrak{D}_m \cap Z_{\varrho_m(\varepsilon/7)} \} \leq \delta_{m,\ell}(\varepsilon/7).$$

Hence, by (25),

$$d_\Omega(T_m(\theta, x), T_m(\nu, y)) < \varepsilon/7 < \varepsilon.$$

Next we assume that  $(\theta, x), (\nu, y) \in V_{\ell^*}$  for some  $\ell \in \mathfrak{D}_m$  such that  $|\ell| > \varrho_m(\varepsilon/7)$  (in particular,  $\theta, \nu \in B_{\ell^*}^{\leftarrow}[\ell^*]$ ). In this situation we have

$$d_\Omega((\theta, x), (\nu, y)) < \delta \leq \delta_{m-1}(\varepsilon/7)$$

and, by (24) and Lemma A.3.6,

$$\begin{aligned} d_\Omega(T_m(\theta, x), T_m(\nu, y)) &\leq d_\Omega(T_m(\theta, x), T_{m-1}(\theta, x)) + d_\Omega(T_{m-1}(\theta, x), T_{m-1}(\nu, y)) + d_\Omega(T_{m-1}(\nu, y), T_m(\nu, y)) \\ &= |f_{m,\theta}(x) - f_{m-1,\theta}(x)| + d_\Omega(T_{m-1}(\theta, x), T_{m-1}(\nu, y)) + |f_{m,\nu}(y) - f_{m-1,\nu}(y)| \\ &\leq \|f_{m,\theta} - f_{m-1,\theta}\| + d_\Omega(T_{m-1}(\theta, x), T_{m-1}(\nu, y)) + \|f_{m,\nu} - f_{m-1,\nu}\| \\ &< \frac{3}{7}\varepsilon < \varepsilon. \end{aligned}$$

In summary, we have proved that

$$d_\Omega(T_m(\theta, x), T_m(\nu, y)) < \frac{3}{7}\varepsilon$$

when  $d_\Omega((\theta, x), (\nu, y)) < \delta$  and  $(\theta, x), (\nu, y) \in V_{\ell^*}$  for some  $\ell \in \mathfrak{D}_m$ .

Next we assume that  $(\theta, x), (\nu, y) \in V_m$  but  $(\theta, x), (\nu, y) \notin V_{\ell^*}$  for every  $\ell \in \mathfrak{D}_m$ . By Lemma 5.2(a), (b), there exist  $i = b^-(\theta, m), k = b^-(\nu, m) \in \mathfrak{D}_m, i \neq k$ , such that  $\theta \in B_i^{\leftarrow}[i^*], (\theta, x) \in V_{i^*}, \nu \in B_k^{\leftarrow}[k^*]$  and  $(\nu, y) \in V_{k^*}$ . Then, there exist

$$\tilde{\theta} \in A \cap \text{Bd}(B_i^{\leftarrow}[i^*]) \subset \mathbb{E}B_m^{\leftarrow} \quad \text{and} \quad \tilde{\nu} \in A \cap \text{Bd}(B_k^{\leftarrow}[k^*]) \subset \mathbb{E}B_m^{\leftarrow},$$

where  $A$  denotes the closed arc of  $\mathbb{S}^1$  such that

$$\text{diam}(A) = d_{\mathbb{S}^1}(\theta, \nu) \quad \text{and} \quad \text{Bd}(A) = \{\theta, \nu\}.$$

Clearly we have,  $(\theta, x), (\tilde{\theta}, x) \in V_{i^*}, (\nu, y), (\tilde{\nu}, y) \in V_{k^*}$  and, by the previous case,

$$\begin{aligned} d_\Omega((\theta, x), (\tilde{\theta}, x)) &= d_{\mathbb{S}^1}(\theta, \tilde{\theta}) \leq d_{\mathbb{S}^1}(\theta, \nu) \leq d_\Omega((\theta, x), (\nu, y)) < \delta, \quad d_\Omega(T_m(\theta, x), T_m(\tilde{\theta}, x)) < \frac{3}{7}\varepsilon \\ d_\Omega((\nu, y), (\tilde{\nu}, y)) &= d_{\mathbb{S}^1}(\nu, \tilde{\nu}) \leq d_{\mathbb{S}^1}(\theta, \nu) \leq d_\Omega((\theta, x), (\nu, y)) < \delta, \quad \text{and} \quad d_\Omega(T_m(\nu, y), T_m(\tilde{\nu}, y)) < \frac{3}{7}\varepsilon. \end{aligned}$$

On the other hand,  $(\tilde{\theta}, x), (\tilde{\nu}, y) \in \mathbb{E}B_m^{\leftarrow} \subset V_m \subset V_{m-1}$  and, by Lemma A.3.4 and (24),

$$\begin{aligned} d_\Omega((\tilde{\theta}, x), (\tilde{\nu}, y)) &= \max \{ d_{\mathbb{S}^1}(\tilde{\theta}, \tilde{\nu}), |x - y| \} \leq \max \{ d_{\mathbb{S}^1}(\theta, \nu), |x - y| \} \\ &= d_\Omega((\theta, x), (\nu, y)) < \delta \leq \delta_{m,i}(\varepsilon/7), \quad \text{and} \\ d_\Omega(T_m(\theta, x), T_m(\nu, y)) &\leq d_\Omega(T_m(\theta, x), T_m(\tilde{\theta}, x)) + d_\Omega(T_m(\tilde{\theta}, x), T_m(\tilde{\nu}, y)) + d_\Omega(T_m(\tilde{\nu}, y), T_m(\nu, y)) \\ &< \frac{3}{7}\varepsilon + d_\Omega(T_{m-1}(\tilde{\theta}, x), T_{m-1}(\tilde{\nu}, y)) + \frac{3}{7}\varepsilon = \varepsilon. \end{aligned}$$



If  $(\theta, x), (\nu, y) \notin \mathbb{V}_m^\sim$  then, by Definition 7.2 and (24),

$$d_\Omega(T_m(\theta, x), T_m(\nu, y)) = d_\Omega(T_{m-1}(\theta, x), T_{m-1}(\nu, y)) < \varepsilon/7 < \varepsilon$$

because  $d_\Omega((\theta, x), (\nu, y)) < \delta \leq \delta_{m-1}(\varepsilon/7)$ .

Lastly, assume that  $(\nu, y) \notin \mathbb{V}_m^\sim$  but  $(\theta, x) \in \mathbb{V}_{i^*}^\sim \subset \mathbb{V}_m^\sim$ , for some  $i \in \mathfrak{D}_m$  (that is,  $\theta \in B_i^\sim[i^*]$ ). In this situation, as before, there exists  $\tilde{\theta} \in \text{Bd}(B_i^\sim[i^*]) \subset \mathbb{E}\mathbb{B}_m^\sim$  such that, by Lemma A.3.4 and Definition 7.2 ( $(\tilde{\theta}, x) \in \mathbb{E}\mathbb{B}_m^\sim \subset \mathbb{V}_m^\sim \subset \mathbb{V}_{m-1}^\sim$ ), and (24),

$$\begin{aligned} d_\Omega((\theta, x), (\tilde{\theta}, x)) &< \delta, \\ d_\Omega((\tilde{\theta}, x), (\nu, y)) &< \delta \leq \delta_{m-1}(\varepsilon/7), \\ d_\Omega(T_m(\theta, x), T_m(\tilde{\theta}, x)) &< \frac{3}{7}\varepsilon, \quad \text{and} \\ d_\Omega(T_m(\theta, x), T_m(\nu, y)) &\leq d_\Omega(T_m(\theta, x), T_m(\tilde{\theta}, x)) + d_\Omega(T_m(\tilde{\theta}, x), T_m(\nu, y)) \\ &< \frac{3}{7}\varepsilon + d_\Omega(T_{m-1}(\tilde{\theta}, x), T_{m-1}(\nu, y)) < \varepsilon. \end{aligned}$$

This ends the proof of the continuity of  $T_m$  and, hence, of (a).

(b) When  $\theta \in \mathbb{B}_m^\sim$  the statement follows from Lemma A.3.2(b). When  $\theta \in \mathbb{S}^1 \setminus \mathbb{B}_m^\sim$ , it follows from the part already proven and the continuity of  $T_m$ .

(c) The first two statements follow from Lemma A.3.2(c) and statement (a). On the other hand, as in the proof of Proposition 7.3(c) for  $m = 0$ , Lemma 5.3(b) implies that  $i^* \in \mathbb{B}_m^\sim$  but  $i^* \notin \mathbb{B}_k^\sim$  for every  $k > m$ . Then, we get  $f_{k,i^*} = f_{m,i^*}$  from Definition 7.2.  $\square$

#### A.4. Proof of Proposition 7.4

This section is devoted to prove Proposition 7.4. It is the third technical counterpart of Section 7. In contrast to Lemma A.3.6 the bound given by Proposition 7.4. is valid for every  $\theta \in \mathbb{B}_m^\sim$ .

Before starting the proof of this proposition we will state and prove a number of very simple lemmas that will help in automating the proof of Proposition 7.4.

**Lemma A.4.1.** *Assume that  $B_i^\sim[i^*] \subset B_k^\sim[k^*]$  for some  $i \in \mathfrak{D}_m$ ,  $k \in \mathfrak{D}_{m-1}$  and  $m \geq 2$ , and assume that either*

$$i < 0 \text{ and } \theta \in B_i^\sim[i^*] \setminus \{i^*\} \quad \text{or} \quad i \geq 0 \text{ and } \theta \in B_{\alpha_i}[i^*] \setminus B_{\alpha_{i+1}}(i^*).$$

Then,

$$|\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))| \leq 2^{-|k|}.$$

**Proof.** The lemma holds trivially when  $|k + 1| = |i + 1|$ . Thus, we may assume that  $|k + 1| \neq |i + 1|$ . Then by Lemma 5.4,  $|k| < |i|$ ,  $|k + 1| < |i + 1|$  and

$$\gamma_{|k+1|}(R_\omega(\theta)) = \gamma_{|k+1|+1}(R_\omega(\theta)) = \dots = \gamma_{|i+1|-1}(R_\omega(\theta)).$$

By assumption we have

$$\theta \in \begin{cases} B_{\alpha_i}[i^*] \setminus B_{\alpha_{i+1}}(i^*) & \text{when } i \geq 0, \text{ and} \\ B_i^\sim[i^*] \setminus \{i^*\} = B_{\alpha_{|i+1|}}[i^*] \setminus \{i^*\} & \text{when } i < 0, \end{cases}$$

and, hence,

$$R_\omega(\theta) \in \begin{cases} B_{\alpha_i} [(i + 1)^*] \setminus B_{\alpha_{i+1}} ((i + 1)^*) & \text{when } i \geq 0, \text{ and} \\ B_{\alpha_{|i+1|}} [(i + 1)^*] \setminus \{(i + 1)^*\} & \text{when } i < 0. \end{cases}$$

Thus, in the case  $i \geq 0$  we have

$$R_\omega(\theta) \notin B_{\alpha_{i+1}} ((i + 1)^*) \cup B_{-(i+1)}^\sim [(-(i + 1))^*] \cup Z_{i+1}^*$$

by Definition 3.4(R.2) and Remark 3.5(R.2). So, by Definition 3.4,

$$\gamma_{i+1}(R_\omega(\theta)) = \gamma_i(R_\omega(\theta)) = \gamma_{|k+1|}(R_\omega(\theta)).$$

This ends the proof of the lemma in this case.

Assume now that  $i < 0$ . By Lemma 3.6(c), (d), (f) and Definition 3.4(R.2) and Remark 3.5(R.2),

$$\begin{aligned} |\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))| &= |\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|i+1|-1}(R_\omega(\theta))| \\ &\leq \text{diam}(\mathcal{R}((i + 1)^*)) \leq 2^{-|i+1|} \leq 2^{-|k|} \end{aligned}$$

(observe that  $|i + 1| > |k + 1| \geq |k| - 1$ ).  $\square$

**Lemma A.4.2.** *Let  $s, t \in \mathbb{Z}$ ,  $s \neq t$  be such that  $\theta \in B_s^\sim(s^*) \setminus B_{\alpha_{|s|}}(s^*)$ , and either  $t < 0$  and  $\theta \in B_{\alpha_{|t|}}(t^*)$  or  $t \geq 0$  and  $\theta \in B_{\alpha_{t+1}}(t^*)$ . Then, the following statements hold:*

- (a)  $R_\omega(\theta) \in B_{\alpha_{|s+1|}}((s + 1)^*) \cap B_{\alpha_{|t+1|}}((t + 1)^*)$ .
- (b) Let  $u, v \in \mathbb{Z}$  be such that  $\{u, v\} = \{s, t\}$  and  $|u + 1| \leq |v + 1|$ .  
Then,  $\mathbb{I}_{v+1, R_\omega(\theta)} \subset \mathbb{I}_{u+1, R_\omega(\theta)}$ .
- (c)

$$|x - y| \leq 2 \cdot 2^{-|u|}$$

for every  $x \in \mathbb{I}_{t+1, R_\omega(\theta)}$  and  $y \in \mathbb{I}_{s+1, R_\omega(\theta)}$ .

**Proof.** By assumption we have

$$\theta \in \begin{cases} B_{\alpha_{t+1}}(t^*) & \text{when } t \geq 0, \text{ and} \\ B_{\alpha_{|t|}}(t^*) \subset B_t^\sim(t^*) = B_{\alpha_{|t+1|}}(t^*) & \text{when } t < 0. \end{cases}$$

Hence,  $R_\omega(\theta) \in B_{\alpha_{|t+1|}}((t + 1)^*)$ . Moreover, as in the proof of Lemma A.4.1,  $s < 0$  and  $R_\omega(\theta) \in B_{\alpha_{|s+1|}}((s + 1)^*)$ . This proves (a).

Now we prove (b). From (a) we have

$$\begin{aligned} R_\omega(\theta) &\in B_{\alpha_{|u+1|}}((u + 1)^*) \cap B_{\alpha_{|v+1|}}((v + 1)^*) \\ &\subset B_{\alpha_{|u+1|}}((u + 1)^*) \cap B_{v+1}^\sim[(v + 1)^*]. \end{aligned}$$

Moreover,  $s \neq t$  implies  $u + 1 \neq v + 1$  and we have  $|u + 1| \leq |v + 1|$  by assumption. Consequently, by Lemma 3.6(g), (d) and Definition 3.4(R.2) and Remark 3.5(R.2),  $|u + 1| < |v + 1|$  and

$$\mathcal{R}((v + 1)^*) \subset \text{Int}(\mathcal{R}((u + 1)^*) \setminus \uparrow\uparrow\{(u + 1)^*\})$$

which implies (b).

Thus,  $x, y \in \mathbb{I}_{u+1, R_\omega(\theta)}$  and, by Lemma 3.6(f),

$$|x - y| \leq \text{diam}(\mathcal{R}((u + 1)^*)) \leq 2^{-|u+1|} \leq 2^{-(|u|-1)} = 2 \cdot 2^{-|u|}. \quad \square$$

Now we are ready to start the proof of Proposition 7.4.

**Proof of Proposition 7.4.** We start by showing that  $\{T_m\}_{k=0}^\infty$  is a Cauchy sequence, assuming that the bound (15) holds for every  $m \geq 2$  and  $\theta \in \mathbb{S}^1$ .

We start by estimating  $d_\infty(T_m, T_{m+1})$  for every  $m \in \mathbb{N}$ . From (15) and the definition of  $\mu_m$

$$d_\infty(T_m, T_{m+1}) = \sup_{\theta \in \mathbb{S}^1} \|f_{m,\theta} - f_{m+1,\theta}\| \leq 2 \cdot \sup_{\theta \in \mathbb{S}^1} 2^{-|b^\rceil(\theta,m)|} \leq 2 \cdot 2^{-\mu_m}.$$

By Lemma 5.3(a)  $\{\mu_m\}_{m=0}^\infty$  is strictly increasing (and  $\lim_{m \rightarrow \infty} \mu_m = \infty$ ). Therefore, for every  $\varepsilon > 0$ , there exists  $N \geq 2$ , such that  $4 \cdot 2^{-\mu_m} < \varepsilon$  for every  $m \geq N$ . Hence,

$$\begin{aligned} d_\infty(T_m, T_{m+i}) &\leq \sum_{\ell=m}^{m+i-1} d_\infty(T_\ell, T_{\ell+1}) \leq 2 \cdot \sum_{\ell=m}^{m+i-1} 2^{-\mu_\ell} \\ &\leq 2 \cdot 2^{-\mu_m} \sum_{\ell=0}^\infty 2^{-\ell} = 4 \cdot 2^{-\mu_m} \leq 4 \cdot 2^{-\mu_N} < \varepsilon \end{aligned}$$

for every  $m \geq N$  and  $i \in \mathbb{N}$ . So,  $\{T_m\}_{k=0}^\infty$  is a Cauchy sequence.

Now we prove (15). That is,

$$\|f_{m,\theta} - f_{m-1,\theta}\| \leq 2 \cdot 2^{-|b^\rceil(\theta,m-1)|}$$

for every  $m \geq 2$  and  $\theta \in \mathbb{S}^1$ .

From Definition 7.2 and Lemma A.3.4 we know that  $f_{m,\theta} = f_{m-1,\theta}$  for every  $\theta \in (\mathbb{S}^1 \setminus \mathbb{B}_m^\rceil) \cup \mathbb{E}\mathbb{B}_m^\rceil$ . Then, (15) holds in this case.

In the rest of the involved proof we assume that  $\theta \in \mathbb{B}_m^\rceil \setminus \mathbb{E}\mathbb{B}_m^\rceil$ . Thus, by Lemmas 5.2(a), (b), 3.6(g) and 5.4,

$$\begin{aligned} \theta &\in B_i^\rceil(i^*) \subset B_k^\rceil(k^*) \setminus (\text{Bd}(B_{\alpha_{|k|}}[k^*]) \cup \{k^*\}) \text{ where} \\ i &= b^\rceil(\theta, m) \in \mathfrak{D}_m, \quad k = b^\rceil(\theta, m-1) \in \mathfrak{D}_{m-1}, \\ |k| &< |i|, \text{ and } |k+1| \leq |i+1|. \end{aligned}$$

Moreover,  $V_{i^*}^\rceil \subset V_{k^*}^\rceil \subset V_{m-1}^\rceil$ . Consequently, by Lemma A.3.2(a), (b), the maps  $f_{m,\theta}$  and  $f_{m-1,\theta}$  are well defined, continuous, piecewise affine and non-increasing, and  $f_{m,\theta}(2) = f_{m-1,\theta}(2) = -2$  and  $f_{m,\theta}(-2) = f_{m-1,\theta}(-2) = 2$  (see Figs. A.7–A.9 for some examples in generic cases).

We split the proof into three cases according to whether  $\theta$  belongs to

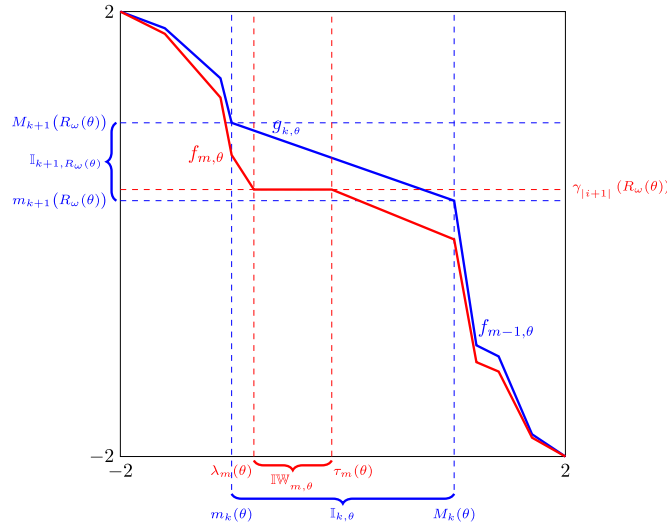
$$B_i^\rceil(i^*) \setminus B_{\alpha_{|i|}}(i^*), \quad B_{\alpha_{|i|}}(i^*) \subset B_k^\rceil(k^*) \setminus B_{\alpha_{|k|}}[k^*] \quad \text{or} \quad B_{\alpha_{|i|}}(i^*) \subset B_{\alpha_{|k|}}(k^*).$$

**Case 1.**  $\theta \in B_i^\rceil(i^*) \setminus B_{\alpha_{|i|}}(i^*)$ .

We have  $i < 0$  because  $B_i^\rceil(i^*) = B_{\alpha_i}(i^*)$  for  $i \geq 0$ . Moreover, by Definition 6.2,  $\theta \in \mathbb{W}\mathbb{B}_m^\rceil$ .

To deal with this case we consider three subcases.

**Subcase 1.1.**  $\theta \in (B_i^\rceil(i^*) \setminus B_{\alpha_{|i|}}(i^*)) \setminus \mathbb{W}\mathbb{I}\mathbb{B}_m$ .



**Fig. A.7.** A symbolic representation of the maps  $f_{m,\theta}$  and  $f_{m-1,\theta}$  in Subcase 1.3 of Proposition 7.4 ( $\theta \in (B_i^\sim(i^*) \setminus B_{\alpha_{|i|}}(i^*)) \cap \mathbb{WIB}_m$  and  $B_i^\sim(i^*) \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ ). The map  $f_{m-1,\theta}$  and the corresponding intervals  $\mathbb{I}_{k,\theta}$  and  $\mathbb{I}_{k+1,R_\omega(\theta)}$  are drawn in blue. The map  $f_{m,\theta}$ , the interval  $\mathbb{IW}_{m,\theta}$  and the point  $\gamma_{|i+1|}(R_\omega(\theta))$  are drawn in red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

By Lemmas 5.4, A.3.3, A.3.5 and A.4.1,

$$\begin{aligned} \mathbb{I}_{i,\theta} &= \{m_i(\theta)\} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{I}_{k,\theta}, \\ f_{m,\theta}(m_i(\theta)) &= \gamma_{|i+1|}(R_\omega(\theta)), \\ f_{m-1,\theta}(m_i(\theta)) &= \gamma_{|k+1|}(R_\omega(\theta)), \quad \text{and} \\ \|f_{m,\theta} - f_{m-1,\theta}\| &= \left\| f_{m,\theta}|_{\mathbb{I}_{i,\theta}} - f_{m-1,\theta}|_{\mathbb{I}_{i,\theta}} \right\| = |f_{m,\theta}(m_i(\theta)) - f_{m-1,\theta}(m_i(\theta))| \\ &= |\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))| \leq 2^{-|b^\top(\theta, m-1)|}. \end{aligned}$$

**Subcase 1.2.**  $\theta \in (B_i^\sim(i^*) \setminus B_{\alpha_{|i|}}(i^*)) \cap \mathbb{WIB}_m$  and  $B_i^\sim(i^*) \subset B_k^\sim(k^*) \setminus B_{\alpha_{|k|}}[k^*]$ .

In this subcase, by Definition 6.2 we have

$$\theta \in B_k^\sim(k^*) \setminus B_{\alpha_{|k|}}[k^*] \subset \mathbb{WIB}_{m-1}^\sim$$

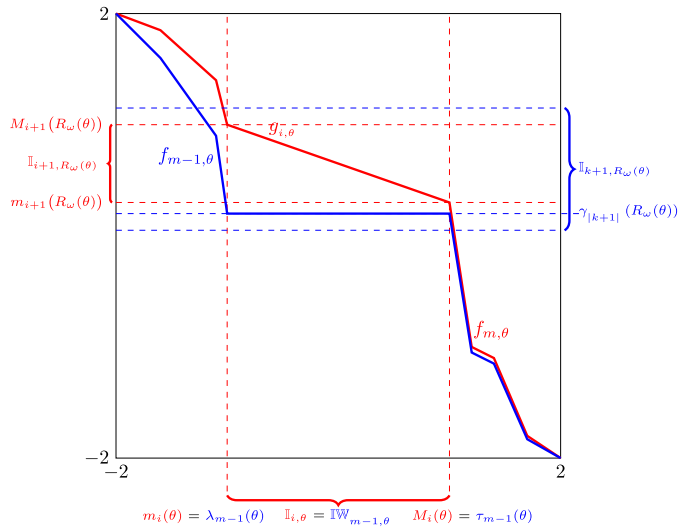
(recall that  $i < 0$ ). Then, by Lemmas 5.4 and 6.3(b), (c), Definition 7.2 and Lemmas A.3.5 and A.4.1,

$$\begin{aligned} \mathbb{I}_{i,\theta} &= \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \mathbb{IW}_{m,\theta} = \mathbb{IW}_{m-1,\theta}, \\ f_{m,\theta}(x) &= \gamma_{|i+1|}(R_\omega(\theta)) \quad \text{for every } x \in \mathbb{IW}_{m,\theta}, \\ f_{m-1,\theta}(x) &= \gamma_{|k+1|}(R_\omega(\theta)) \quad \text{for every } x \in \mathbb{IW}_{m-1,\theta}, \text{ and} \\ \|f_{m,\theta} - f_{m-1,\theta}\| &= \left\| f_{m,\theta}|_{\mathbb{IW}_{m,\theta}} - f_{m-1,\theta}|_{\mathbb{IW}_{m,\theta}} \right\| \\ &= |\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))| \leq 2^{-|b^\top(\theta, m-1)|}. \end{aligned}$$

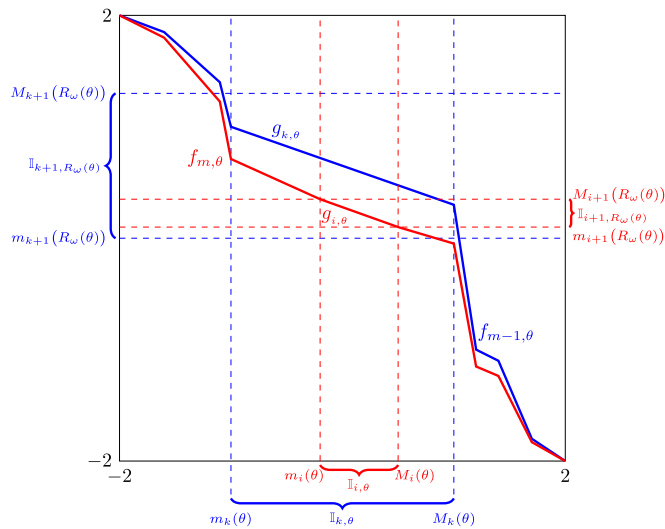
Observe that since  $B_i^\sim(i^*)$  is connected and

$$B_i^\sim(i^*) \subset B_k^\sim(k^*) \setminus (\text{Bd}(B_{\alpha_{|k|}}[k^*]) \cup \{k^*\}),$$

$B_i^\sim(i^*) \not\subset B_k^\sim(k^*) \setminus B_{\alpha_{|k|}}[k^*]$  implies  $B_i^\sim(i^*) \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$ .



**Fig. A.8.** A symbolic representation of the maps  $f_{m,\theta}$  and  $f_{m-1,\theta}$  in Case 2 ( $\theta \in B_{\alpha_{|i|}}(i^*) \subset \widetilde{B}_k(k^*) \setminus B_{\alpha_{|k|}}[k^*]$ ) of Proposition 7.4. The map  $f_{m-1,\theta}$  and the corresponding intervals  $\mathbb{IW}_{m-1,\theta}$  and  $\mathbb{I}_{k+1,R_\omega(\theta)}$  are drawn in blue. The map  $f_{m,\theta}$  and the corresponding intervals  $\mathbb{I}_{i,\theta} = \mathbb{IW}_{m-1,\theta}$  and  $\mathbb{I}_{i+1,R_\omega(\theta)}$  are drawn in red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)



**Fig. A.9.** A symbolic representation of the maps  $f_{m,\theta}$  and  $f_{m-1,\theta}$  in Subcase 3.1 from the proof of Proposition 7.4 ( $\theta \in B_{\alpha_{|i|}}(i^*)$  and  $\mathbb{I}_{i,\theta} \subset \mathbb{I}_{k,\theta}$  and either  $k < 0$  or  $k \geq 0$  and  $i^* \in B_{\alpha_{k+1}}[k^*]$ ). The map  $f_{m-1,\theta}$  and the corresponding intervals  $\mathbb{I}_{k,\theta}$  and  $\mathbb{I}_{k+1,R_\omega(\theta)}$  are drawn in blue. The map  $f_{m,\theta}$  and the corresponding intervals  $\mathbb{I}_{i,\theta}$  and  $\mathbb{I}_{i+1,R_\omega(\theta)}$  are drawn in red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

**Subcase 1.3.**  $\theta \in (B_i^-(i^*) \setminus B_{\alpha_{|i|}}(i^*)) \cap \text{WIB}_m$  and  $B_i^-(i^*) \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$  (see Fig. A.7 for a symbolic representation of this case).

By Lemmas 5.4 and 6.3(b) and Definition 7.2,

$$\mathbb{I}_{i,\theta} = \{\gamma_{|i|}(\theta)\} = \{\gamma_{|k|}(\theta)\} \subset \text{IW}_{m,\theta}, \quad \text{and}$$

$$f_{m,\theta}(x) = \gamma_{|i+1|}(R_\omega(\theta)) \quad \text{for every } x \in \text{IW}_{m,\theta}.$$

On the other hand, by Definition 6.2 and Lemma 6.1(a), (b),  $\theta \in \mathbb{WIB}_m \subset \mathbb{WDB}_m$ , and

$$\theta \in B_{\alpha_{|\ell|}}[\ell^*] \subset B_i^{\sim}(i^*) \setminus B_{\alpha_{|i|}}[i^*] \subset B_{\alpha_{|k|}}(k^*) \setminus \{k^*\}$$

with  $\ell = \mathbf{b}^{\sim}(\theta, \text{led}(\theta, m)) \in \mathfrak{W}\mathfrak{F}\mathfrak{D}_m$  and  $|\ell| > |i| > |k|$ . Then, by Lemma 3.6(g) and Definition 6.2,  $\mathcal{R}(\ell^*) \subset \text{Int}(\mathcal{R}(k^*) \uparrow\uparrow k^*)$  and

$$\mathbb{IW}_{m,\theta} = \mathbb{I}_{\ell,\theta} \subset \mathbb{I}_{k,\theta}.$$

Moreover, since  $\theta \in B_{\alpha_{|k|}}(k^*) \subset \mathbb{B}_m$ , Definition 7.2, Lemmas 4.3(b) and 4.5(b), and the definition of the maps  $g_{i,\theta}$  for  $i \geq 0$  (Definition 4.2) give

$$\begin{aligned} f_{m-1,\theta}(\mathbb{IW}_{m,\theta}) &\subset f_{m-1,\theta}(\mathbb{I}_{k,\theta}) \\ &\subset \begin{cases} \mathbb{I}_{k+1,R_\omega(\theta)} & \text{if } k < 0 \text{ or } k \geq 0 \text{ and } \theta \in B_{\alpha_{k+1}}(k^*), \\ \{\gamma_{k+1}(R_\omega(\theta))\} & \text{if } k \geq 0 \text{ and } \theta \in B_{\alpha_k}[k^*] \setminus B_{\alpha_{k+1}}(k^*). \end{cases} \end{aligned}$$

Now, as before, we will use Lemma A.3.5 to bound  $\|f_{m,\theta} - f_{m-1,\theta}\|$ . We start with the simplest case:  $k \geq 0$  and  $\theta \in B_{\alpha_k}[k^*] \setminus B_{\alpha_{k+1}}(k^*)$ . By Lemma A.4.1,

$$\begin{aligned} \|f_{m,\theta} - f_{m-1,\theta}\| &= \left\| f_{m,\theta}|_{\mathbb{IW}_{m,\theta}} - f_{m-1,\theta}|_{\mathbb{IW}_{m,\theta}} \right\| \\ &= |\gamma_{|i+1|}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))| \leq 2^{-|\mathbf{b}^{\sim}(\theta, m-1)|}. \end{aligned}$$

Now we assume that  $k < 0$  or  $k \geq 0$  and  $\theta \in B_{\alpha_{k+1}}(k^*)$ . In this case Lemma A.4.2 applies. By Lemmas A.4.2, 3.6(d) and Definition 3.4(R.2) and Remark 3.5(R.2), and Lemma A.3.5 we have

$$\begin{aligned} \gamma_{|i+1|}(R_\omega(\theta)) &\in \mathbb{I}_{i+1,R_\omega(\theta)} \subset \mathbb{I}_{k+1,R_\omega(\theta)}, \\ f_{m-1,\theta}(x) &\in \mathbb{I}_{k+1,R_\omega(\theta)} \quad \text{for every } x \in \mathbb{IW}_{m,\theta}. \end{aligned}$$

and

$$\begin{aligned} \|f_{m,\theta} - f_{m-1,\theta}\| &= \sup_{x \in \mathbb{IW}_{m,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)| \\ &= \sup_{x \in \mathbb{IW}_{m,\theta}} |\gamma_{|i+1|}(R_\omega(\theta)) - f_{m-1,\theta}(x)| \\ &\leq 2 \cdot 2^{-|k|} = 2 \cdot 2^{-|\mathbf{b}^{\sim}(\theta, m-1)|}. \end{aligned}$$

This ends the proof of the proposition in this case.

**Case 2.**  $\theta \in B_{\alpha_{|i|}}(i^*) \subset B_k^{\sim}(k^*) \setminus B_{\alpha_{|k|}}[k^*]$  (see Fig. A.8 for a symbolic representation of this case).

In this case we will use Lemma A.3.5 with  $\mathbb{I}_{i,\theta}$ . Thus, we need to compare the maps  $f_{m,\theta}|_{\mathbb{I}_{i,\theta}}$  and  $f_{m-1,\theta}|_{\mathbb{I}_{i,\theta}}$ .

Directly from the definitions we get  $k < 0$ ,  $B_{\alpha_{|i|}}[i^*] \subset \mathbb{B}_m$  and  $B_{\alpha_{|k|}}[k^*] \subset \mathbb{B}_{m-1}$ . Consequently, by Lemma 5.2(b) and Definition 6.2,

$$\theta \in \mathbb{B}_m \quad \text{and} \quad \theta \in \mathbb{B}_{m-1}^{\sim} \setminus \mathbb{B}_{m-1} \subset \mathbb{WDB}_{m-1} \subset \mathbb{WB}_{m-1}^{\sim}.$$

Moreover,  $\text{led}(\theta, m-1) = m$ ,  $i = \mathbf{b}^{\sim}(\theta, m) = \mathbf{b}^{\sim}(\theta, \text{led}(\theta, m-1)) \in \mathfrak{W}\mathfrak{F}\mathfrak{D}_{m-1}$  and, by Definition 6.2,  $\theta \in \mathbb{WIB}_{m-1}$  and

$$\mathbb{IW}_{m-1,\theta} = \mathbb{I}_{i,\theta}.$$

Furthermore, since  $k < 0$ , we obtain  $R_\omega(\theta) \in B_{\alpha_{|k+1|}}((k+1)^*)$  as in the proof of Lemma A.4.1. Thus, Definition 7.2, Lemma 3.6(d) and Definition 3.4(R.2) and Remark 3.5(R.2), give

$$f_{m-1,\theta}(x) = \gamma_{|k+1|}(R_\omega(\theta)) \in \mathbb{I}_{k+1,R_\omega(\theta)}$$

for every  $x \in \mathbb{I}_{i,\theta} = \mathbb{IW}_{m-1,\theta}$ .

Now we will use Lemma A.3.5 to bound the norm  $\|f_{m,\theta} - f_{m-1,\theta}\|$ . By Definition 6.2 and Lemma A.3.5,  $\theta \in \mathbb{B}_m \subset \mathbb{B}_m^\tau \setminus \mathbb{W}\mathbb{I}\mathbb{B}_m$ , and

$$\begin{aligned} \|f_{m,\theta} - f_{m-1,\theta}\| &= \sup_{x \in \mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)| \\ &= \sup_{x \in \mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - \gamma_{|k+1|}(R_\omega(\theta))|. \end{aligned}$$

Next we will compute  $f_{m,\theta}(\mathbb{I}_{i,\theta})$ . We start with the simplest case:  $i \geq 0$  and  $\theta \in B_{\alpha_i}(i^*) \setminus B_{\alpha_{i+1}}(i^*)$ . By Definition 7.2, the definition of the maps  $g_{i,\theta}$  for  $i \geq 0$  (Definition 4.2) and Lemma A.4.1,

$$\begin{aligned} \|f_{m,\theta} - f_{m-1,\theta}\| &= \sup_{x \in \mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - \gamma_{|k+1|}(R_\omega(\theta))| \\ &= |\gamma_{i+1}(R_\omega(\theta)) - \gamma_{|k+1|}(R_\omega(\theta))| \leq 2^{-|\mathbf{b}^\tau(\theta, m-1)|}. \end{aligned}$$

Assume that  $i < 0$  or  $i \geq 0$  and  $\theta \in B_{\alpha_{i+1}}(i^*)$ . Then, again by Definition 7.2 and Lemmas 4.3(b), 4.5(b) and A.4.2,

$$f_{m,\theta}(x) \in \mathbb{I}_{i+1,R_\omega(\theta)} \subset \mathbb{I}_{k+1,R_\omega(\theta)} \quad \text{for every } x \in \mathbb{I}_{i,\theta},$$

and

$$\begin{aligned} \|f_{m,\theta} - f_{m-1,\theta}\| &= \sup_{x \in \mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - \gamma_{|k+1|}(R_\omega(\theta))| \\ &\leq 2 \cdot 2^{-|k|} = 2 \cdot 2^{-|\mathbf{b}^\tau(\theta, m-1)|}. \end{aligned}$$

This ends the proof of the proposition in Case 2.

**Case 3.**  $\theta \in B_{\alpha_{|i|}}(i^*) \subset B_{\alpha_{|k|}}(k^*)$ .

In this case we have  $B_{\alpha_{|i|}}(i^*) \subset \mathbb{B}_m$  and  $B_{\alpha_{|k|}}(k^*) \subset \mathbb{B}_{m-1}$  so that,  $\theta \in \mathbb{B}_m \cap \mathbb{B}_{m-1}$ . Moreover, by Lemma 3.6(g),  $\mathcal{R}(i^*) \subset \text{Int}(\mathcal{R}(k^*) \setminus \uparrow\uparrow k^*)$  and, hence,

$$\mathbb{I}_{i,\theta} \subset \mathbb{I}_{k,\theta}.$$

Since  $\theta \in \mathbb{B}_m$ , by Definition 6.2 and Lemma A.3.5,  $\theta \in \mathbb{B}_m^\tau \setminus \mathbb{W}\mathbb{I}\mathbb{B}_m$ , and

$$\|f_{m,\theta} - f_{m-1,\theta}\| = \left\| f_{m,\theta}|_{\mathbb{I}_{i,\theta}} - f_{m-1,\theta}|_{\mathbb{I}_{i,\theta}} \right\| = \sup_{x \in \mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)|.$$

Thus, we need to compare the maps  $f_{m,\theta}|_{\mathbb{I}_{i,\theta}}$  and  $f_{m-1,\theta}|_{\mathbb{I}_{i,\theta}}$ . To do this we consider two subcases.

**Subcase 3.1.** Either  $k < 0$  or  $k \geq 0$  and  $\theta \in B_{\alpha_{k+1}}(k^*)$

(see Fig. A.9 for a symbolic representation of this case).

In this situation we aim at proving that

$$f_{m-1,\theta}(\mathbb{I}_{i,\theta}), f_{m,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

We start with  $f_{m-1,\theta}(\mathbb{I}_{i,\theta})$ . By Definition 7.2 and Lemmas 4.3(b) and 4.5(b) we obtain

$$f_{m-1,\theta}(\mathbb{I}_{i,\theta}) \subset f_{m-1,\theta}(\mathbb{I}_{k,\theta}) = g_{k,\theta}(\mathbb{I}_{k,\theta}) \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

Next we show that  $f_{m,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{k+1,R_\omega(\theta)}$ .

Since  $k < 0$  or  $k \geq 0$  and  $\theta \in B_{\alpha_{k+1}}(k^*)$ , by Definition 3.4(R.1) we obtain

$$R_\omega(\theta) \in \begin{cases} R_\omega(B_{\alpha_{|k|}}(k^*)) = B_{\alpha_{|k|}}((k+1)^*) \subset B_{\alpha_{|k+1|}}((k+1)^*) & \text{if } k < 0, \\ R_\omega(B_{\alpha_{k+1}}(k^*)) = B_{\alpha_{k+1}}((k+1)^*) & \text{if } k \geq 0 \text{ and } \theta \in B_{\alpha_{k+1}}(k^*). \end{cases} \tag{26}$$

Assume that  $i < 0$  or  $i \geq 0$  and  $\theta \in B_{\alpha_{i+1}}(i^*)$ . By (26) with  $k$  replaced by  $i$ ,

$$R_\omega(\theta) \in B_{\alpha_{|i+1|}}((i+1)^*) \cap B_{\alpha_{k+1}}((k+1)^*) \subset B_{i+1}^\sim[(i+1)^*] \cap B_{k+1}^\sim[(k+1)^*].$$

Therefore, since  $|k+1| \leq |i+1|$  and  $k+1 \neq i+1$ , from Lemma 3.6(g) we obtain  $|k+1| < |i+1|$ ,

$$B_{\alpha_{|i+1|}}[(i+1)^*] \subset B_{\alpha_{|k+1|}}((k+1)^*) \setminus \{(k+1)^*\}, \quad \text{and} \\ \mathcal{R}((i+1)^*) \subset \text{Int}(\mathcal{R}((k+1)^*) \setminus \uparrow\uparrow(k+1)^*).$$

Thus, by Definition 7.2 and Lemmas 4.3(b) and 4.5(b),

$$f_{m,\theta}(\mathbb{I}_{i,\theta}) = g_{i,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{i+1,R_\omega(\theta)} \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

Now we will consider the case  $i \geq 0$  and  $\theta \in B_{\alpha_i}(i^*) \setminus B_{\alpha_{i+1}}(i^*)$ . The fact that  $|k| < |i| = i$  implies  $|k+1| \leq |k| + 1 \leq i$ . We claim that

$$B_{\alpha_i}((i+1)^*) \subset B_{\alpha_{|k+1|}}((k+1)^*) \setminus \{(k+1)^*\}.$$

To prove the claim note that, by (26),

$$R_\omega(\theta) \in R_\omega(B_{\alpha_i}(i^*)) \cap B_{\alpha_{|k+1|}}((k+1)^*) \subset B_{\alpha_i}((i+1)^*) \cap B_{k+1}^\sim[(k+1)^*].$$

Moreover, the interval  $B_{\alpha_i}((i+1)^*)$  is disjoint from  $B_i^\sim[i^*]$  and  $B_{-i}^\sim[(-i)^*]$  by Definition 3.4(R.2). Thus,  $i \neq k+1, -(k+1)$  and, hence,  $|k+1| < i$  (that is,  $k+1 \in Z_{i-1}$ ). So, there exists  $q \in Z_{i-1}$  such that  $B_{\alpha_i}[(i+1)^*] \cap B_q^\sim[q^*] \neq \emptyset$  and  $|q| \geq |k+1|$  is maximal verifying these conditions. By Definition 3.4(R.4),

$$B_{\alpha_i}((i+1)^*) \subset B_q^\sim(q^*) \setminus (\text{Bd}(B_{\alpha_{|q|}}[q^*]) \cup \{q^*\}).$$

So, the claim holds when  $q = k+1$ . Assume that  $q \neq k+1$ . Then,

$$R_\omega(\theta) \in B_{\alpha_i}((i+1)^*) \cap B_{\alpha_{|k+1|}}((k+1)^*) \subset B_q^\sim(q^*) \cap B_{\alpha_{|k+1|}}((k+1)^*).$$

Hence, by Lemma 3.6(g),  $|q| > |k+1|$  and

$$B_{\alpha_i}((i+1)^*) \subset B_q^\sim[q^*] \subset B_{\alpha_{|k+1|}}((k+1)^*) \setminus \{(k+1)^*\}.$$

This ends the proof of the claim.

On the other hand, by Definition 3.4(R.2) and Remark 3.5(R.2),

$$(B_{\alpha_i}[(i+1)^*] \setminus B_{\alpha_{i+1}}((i+1)^*)) \cap Z_{i+1} = \emptyset.$$

Thus, by the claim,

$$R_\omega(\theta) \in R_\omega(B_{\alpha_i}(i^*) \setminus B_{\alpha_{i+1}}(i^*)) = B_{\alpha_i}((i+1)^*) \setminus B_{\alpha_{i+1}}((i+1)^*) \\ \subset B_{\alpha_{|k+1|}}((k+1)^*) \setminus Z_{i+1}.$$



By Definition 7.2, the definition of the maps  $g_{i,\theta}$  for  $i \geq 0$  (Definition 4.2) and Lemma 3.6(d) (with  $\ell = k + 1$  and  $n = i + 1$ ),

$$f_{m,\theta}(\mathbb{I}_{i,\theta}) = g_{i,\theta}(\mathbb{I}_{i,\theta}) = \{\gamma_{i+1}(R_\omega(\theta))\} \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

Summarizing, we have proved that

$$f_{m-1,\theta}(\mathbb{I}_{i,\theta}), f_{m,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{k+1,R_\omega(\theta)}.$$

So, by Lemma 3.6(f) (and the fact that  $|k + 1| \geq |k| - 1$ ),

$$\begin{aligned} \|f_{m,\theta} - f_{m-1,\theta}\| &= \sup_{x \in \mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)| \leq \text{diam}(\mathbb{I}_{k+1,R_\omega(\theta)}) \\ &\leq \text{diam}(\mathcal{R}((k + 1)^*)) \leq 2^{-|k+1|} \leq 2 \cdot 2^{-|k|} = 2 \cdot 2^{-|\mathbf{b}^\curvearrowright(\theta, m-1)|}. \end{aligned}$$

This ends the proof of the proposition in this subcase.

**Subcase 3.2.**  $k \geq 0$  and  $\theta \in B_{\alpha_k}(k^*) \setminus B_{\alpha_{k+1}}(k^*)$ .

We start by computing  $f_{m-1,\theta}(\mathbb{I}_{i,\theta})$ . By Definition 7.2 and the definition of the maps  $g_{k,\theta}$  for  $k \geq 0$  (Definition 4.2),

$$f_{m-1,\theta}(\mathbb{I}_{i,\theta}) \subset f_{m-1,\theta}(\mathbb{I}_{k,\theta}) = g_{k,\theta}(\mathbb{I}_{k,\theta}) = \{\gamma_{k+1}(R_\omega(\theta))\}.$$

Analogously, if  $i \geq 0$  and  $\theta \in B_{\alpha_i}(i^*) \setminus B_{\alpha_{i+1}}(i^*)$ ,

$$f_{m,\theta}(\mathbb{I}_{i,\theta}) = g_{i,\theta}(\mathbb{I}_{i,\theta}) = \{\gamma_{i+1}(R_\omega(\theta))\}.$$

Then, by Lemma A.4.1,

$$\begin{aligned} \|f_{m,\theta} - f_{m-1,\theta}\| &= \left\| f_{m,\theta}|_{\mathbb{I}_{i,\theta}} - f_{m-1,\theta}|_{\mathbb{I}_{i,\theta}} \right\| \\ &= |\gamma_{i+1}(R_\omega(\theta)) - \gamma_{k+1}(R_\omega(\theta))| \leq 2^{-|\mathbf{b}^\curvearrowright(\theta, m-1)|}. \end{aligned}$$

Assume now that  $i < 0$  or  $i \geq 0$  and  $\theta \in B_{\alpha_{|i+1|}}(i^*)$ . By (26), Definition 7.2 and Lemmas 4.3(b) and 4.5(b)

$$\begin{aligned} R_\omega(\theta) &\in B_{\alpha_{|i+1|}}((i + 1)^*), \quad \text{and} \\ f_{m,\theta}(\mathbb{I}_{i,\theta}) &= g_{i,\theta}(\mathbb{I}_{i,\theta}) \subset \mathbb{I}_{i+1,R_\omega(\theta)}. \end{aligned}$$

Moreover, if  $k + 1 < |i + 1|$ , by Lemmas 5.4(a) and 3.6(c), we have

$$f_{m-1,\theta}(\mathbb{I}_{i,\theta}) = \{\gamma_{k+1}(R_\omega(\theta))\} = \{\gamma_{|i+1|-1}(R_\omega(\theta))\} \subset \mathbb{I}_{i+1,R_\omega(\theta)}.$$

Therefore, by Lemma 3.6(f),

$$\begin{aligned} \|f_{m,\theta} - f_{m-1,\theta}\| &= \sup_{x \in \mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - f_{m-1,\theta}(x)| \\ &= \sup_{x \in \mathbb{I}_{i,\theta}} |f_{m,\theta}(x) - \gamma_{|i+1|-1}(R_\omega(\theta))| \\ &\leq \text{diam}(\mathbb{I}_{i+1,R_\omega(\theta)}) \leq \text{diam}(\mathcal{R}((i + 1)^*)) \leq 2^{-|i+1|} \\ &< 2^{-(k+1)} < 2^{-|\mathbf{b}^\curvearrowright(\theta, m-1)|}. \end{aligned}$$

So, to end the proof of the proposition we have to show that, in this subcase,  $k + 1 < |i + 1|$ . To prove this, notice that when  $i \geq 0$ ,  $k + 1 = |k| + 1 < |i| + 1 = |i + 1|$ . So, assume by way of contradiction that

$i < 0$  and  $k + 1 = |i + 1|$  (recall that  $k + 1 \leq |i + 1|$ ). Then,  $k + 1 = -(i + 1)$  and, hence,

$$\begin{aligned} R_\omega(\theta) &\in R_\omega(B_{\alpha_k}(k^*)) = B_{\alpha_k}((k + 1)^*), \quad \text{and} \\ R_\omega(\theta) &\in B_{\alpha_{|i+1|}}((i + 1)^*) = B_{\alpha_{k+1}}((-k + 1)^*) \subset B_{-(k+1)}^{\infty}((-k + 1)^*), \end{aligned}$$

which is a contradiction by [Definition 3.4\(R.2\)](#).  $\square$

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