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Weighted polynomial approximation on the sphere and related domains

Feng Dai

CHAPTER 1

Weighted Polynomial Inequalities with doubling or A_∞ weights

Polynomial inequalities have been playing crucial roles in approximation theory and other related fields. The main purposes in this chapter is to establish various important weighted polynomial inequalities on the sphere, such as Bernstein, Marcinkiewicz-Zygmund, Nikolskii, Remez, under the doubling condition or the slightly A_∞ condition on the weights. Definitions and examples of doubling weights and A_∞ weights are given in the first section, together with some useful properties of these weights. Section 2 is devoted to establishing useful highly localized estimates for a reproducing kernel and spherical polynomials. The fundamental tool for the investigation of weighted polynomial inequalities is a maximal function for spherical polynomials, which we introduce and study in the third section. It turns out that this maximal function can be controlled pointwisely by the Hardy-Littlewood maximal function, based on which several useful corollaries are deduced. Weighted Marcinkiewicz-Zygmund (MZ) inequalities are established in the fourth section, whereas weighted positive cubature formulas are studied in connection with MZ inequalities in the fifth section. The sixth section deals with weighted Bernstein and Nikolskii inequalities. Finally, in the seventh section, we establish weighted Remez type inequalities with A_∞ weights.

1. Preliminaries

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\|^2 := x_1^2 + \cdots + x_d^2 = 1\}$ denote the unit sphere of $\mathbb{R}^d$ equipped with the usual rotation invariant Lebesgue measure $d\sigma(x)$. Let $d(x, y) = \arccos x \cdot y$ denote the geodesic distance of $x, y \in \mathbb{S}^{d-1}$, and let $\mathsf{c}(x, r) := \{y \in \mathbb{S}^{d-1} : d(x, y) \le r\}$ denote the spherical cap centered at $x \in \mathbb{S}^{d-1}$ having radius $r > 0$. Given a set $E \subset \mathbb{S}^{d-1}$, we denote by χ_E and $\mathrm{meas}(E) \equiv |E|$ the characteristic function of E and the Lebesgue measure $\sigma(E)$ of E, respectively. We shall use the notation $A \sim B$ to mean that there exists an inessential constant $c > 0$, called the constant of equivalence, such that $c^{-1}A \le B \le cA$.

A spherical harmonic of degree k on $\mathbb{S}^{d-1}$ is the restriction to the sphere $\mathbb{S}^{d-1}$ of a homogeneous harmonic polynomial of degree k, while a spherical polynomial of degree at most N on $\mathbb{S}^{d-1}$ is the restriction to $\mathbb{S}^{d-1}$ of a polynomial in d variables of degree at most N. We denote by $\mathcal{H}_k^d$ the space of all real spherical harmonics of degree k on $\mathbb{S}^{d-1}$, and by Π_n^d the space of all real spherical polynomials of degree at most n on $\mathbb{S}^{d-1}$.

The spaces $\mathcal{H}_k^d$, $k = 0, 1, \cdots$ of spherical harmonics are mutually orthogonal with respect to the inner product

$$\langle f, g \rangle := \int_{\mathbb{S}^{d-1}} f(x) g(x)\, d\sigma(x). \tag{1.1}$$

In fact, each space $\mathcal{H}_n^d$ is the orthogonal complement of Π_{n-1}^d in the Hilbert space Π_n^d with respect to the inner product (1.1). Thus, the space Π_n^d has an orthogonal decomposition $\Pi_n^d = \bigoplus_{k=0}^n \mathcal{H}_k^d$. The dimension of $\mathcal{H}_k^d$ is given by

$$a_k^d := \dim \mathcal{H}_k^d = \frac{(2k+d-2)\Gamma(k+d-1)}{(k+d-2)\Gamma(k+1)\Gamma(d-1)} \sim k^{d-2}, \quad \text{as } k \to \infty,$$

which also implies

$$\dim \Pi_n^d = \sum_{k=0}^n a_k^d = \binom{n+d-1}{n} \sim n^{d-1}.$$

Since the space of spherical polynomials is dense in $L^2 \equiv L^2(\mathbb{S}^{d-1})$, we also have the orthogonal decomposition for $L^2(\mathbb{S}^{d-1})$: $L^2(\mathbb{S}^{d-1}) = \bigoplus_{k=0}^\infty \mathcal{H}_k^d$. By the addition formula for spherical harmonics, the orthogonal projection $\operatorname{proj}_k f$ of $f \in L^2$ onto $\mathcal{H}_k^d$ can be expressed as

$$\operatorname{proj}_k f(x) = c_k \int_{\mathbb{S}^{d-1}} f(y) P_k^{(\frac{d-3}{2}, \frac{d-3}{2})}(x \cdot y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{1.2}$$

where

$$c_k = \frac{\Gamma(\frac{d-1}{2})}{\Gamma(d-1)|\mathbb{S}^{d-1}|} \frac{(2k+d-2)\Gamma(k+d-2)}{\Gamma(k+\frac{d-1}{2})},$$

and $P_k^{(\alpha,\beta)}(t)$ denotes the usual Jacobi polynomial of degree k and indices α, β.

The Laplace -Beltrami operator Δ_0 on $\mathbb{S}^{d-1}$ is defined by

$$\Delta_0 f(x) := \Delta_{\mathbb{R}^d} F(x), \quad x \in \mathbb{S}^{d-1}, \quad f \in C^2(\mathbb{S}^{d-1}),$$

with $\Delta_{\mathbb{R}^d} = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ and $F(y) := f(y|y|^{-1})$. The operator Δ_0 is an elliptic, unbounded, selfadjoint operator on $L^2(\mathbb{S}^{d-1})$. More importantly, each $\mathcal{H}_k^d$ is the space of eigenfunctions of Δ_0 corresponding to the eigenvalue $\lambda_k = -k(k+d-2)$; that is,

$$\mathcal{H}_k^d = \Big\{ f \in C^2(\mathbb{S}^{d-1}) : \quad \Delta_0 f = \lambda_k f \Big\}, \quad k = 0, 1, \cdots. \tag{1.3}$$

The Laplace-Beltrami operator Δ_0 can be decomposed as

$$\Delta_0 = \sum_{1 \le i < j \le d} D_{i,j}^2, \tag{1.4}$$

where

$$D_{i,j} := x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}, \quad 1 \le i \ne j \le d. \tag{1.5}$$

A remarkable fact is that each differential operator $D_{i,j}$ commutes with the Laplace-Beltrami operator Δ_0. The differential operators $D_{i,j}$ will play an important role in our analysis. Thus, we collect some of the properties of these operators in the following lemma.

Lemma 1.1. *(i) Each $D_{i,j}$ preserves spherical harmonics, i.e., $D_{i,j}\mathcal{H}_k^d \subset \mathcal{H}_k^d$.*
(ii) Each $D_{i,j}$ commutes with the Laplacian-Beltrami operator Δ_0.
(iii) $\langle D_{i,j} f, g \rangle = -\langle f, D_{i,j} g \rangle$ holds for all $f, g \in C^1(\mathbb{S}^{d-1})$.
(iv) For $r \in \mathbb{N}$,

$$D_{1,2}^r f(x) = \left(-\frac{\partial}{\partial \phi} \right)^r f(s\cos\phi, s\sin\phi, x_3, \dots, x_d), \tag{1.6}$$

where $(x_1, x_2) = (s\cos\phi, s\sin\phi)$. Other $D_{i,j}^r$ can be expressed likewise.

(v) Every tangential derivative on the sphere can be expressed in terms of the differential operators $D_{i,j}$:

$$\frac{\partial}{\partial x_j} \left[f\left(\frac{x}{\|x\|} \right) \right]_{\|x\|=1} = - \sum_{\{i:\ i=1,\cdots,d,\ i \ne j\}} x_i D_{i,j} f.$$

(vi) For any $f, g \in C^1(\mathbb{S}^{d-1})$, we have

$$\langle \nabla_0 f, \nabla_0 g \rangle = \sum_{1 \le i < j \le d} \langle D_{i,j} f, D_{i,j} g \rangle,$$

where $\nabla_0 f$ denotes the tangential gradient given by

$$\nabla_0 f = \nabla \left[f \left(\frac{x}{\|x\|} \right) \right]_{\|x\|=1}.$$

A nonnegative integrable function on $\mathbb{S}^{d-1}$ is called a weight function.

Given a weight function w (i.e., a nonnegative integrable function) on $\mathbb{S}^{d-1}$, we denote by $L_{p,w} \equiv L_{p,w}(\mathbb{S}^{d-1})$ the Lebesgue space on $\mathbb{S}^{d-1}$ endowed with the quasi-norm

$$\|f\|_{p,w} = \left(\int_{\mathbb{S}^{d-1}} |f(x)|^p w(x)\, d\sigma(x) \right)^{\frac{1}{p}}, \quad 0 < p < \infty.$$

We write $w(E) = \int_E w(x)\, d\sigma(x)$ for $E \subset \mathbb{S}^{d-1}$ and a weight function w. Given a spherical cap $B := \mathsf{c}(x, r)$ and $c > 0$, we denote by cB the spherical cap $\mathsf{c}(x, cr)$ with the same center as B but with c times the radius.

A weight function w on $\mathbb{S}^{d-1}$ is said to satisfy the doubling condition if there exists a constant $L > 0$ such that

$$w(2B) \le L w(B) \quad \text{for all spherical caps } B \subset \mathbb{S}^{d-1}. \tag{1.7}$$

The least constant L for which (1.7) is satisfies is called the doubling constant of w, and is denoted by L_w. We will use the letter s_w to denote a number which satisfies $0 \le s_w \le \log L_w / \log 2$ and

$$\sup_B \frac{w(2^m B)}{w(B)} \le C_{L_w} 2^{m s_w}, \quad m = 1, 2, \cdots, \tag{1.8}$$

for some constant C_{L_w} depending only on L_w, with the supremum being taken over all spherical caps $B \subset \mathbb{S}^{d-1}$. Evidently, such a number s_w satisfies

$$\varlimsup_{m \to \infty} \frac{1}{m} \log_2 \left(\sup_B \frac{w(2^m B)}{w(B)} \right) \le s_w \le \frac{\log L_w}{\log 2}. \tag{1.9}$$

Note that according to (1.7), (1.8) is satisfies with $s_w = \log L_w / \log 2$ and $C_{L_w} = 1$. For $n = 1, 2, \cdots$, we set

$$w_n(x) = n^{d-1} \int_{\mathsf{c}(x, \frac{1}{n})} w(y)\, d\sigma(y), \text{ and } w_0(x) = w_1(x). \tag{1.10}$$

The following lemma sketches several useful properties of doubling weights.

Lemma 1.2. *Let w be a doubling weight on $\mathbb{S}^{d-1}$.*
(i) If $0 < r < t$ and $x \in \mathbb{S}^{d-1}$, then

$$w(\mathsf{c}(x, t)) \le C_{L_w} \left(\frac{t}{r} \right)^{s_w} w(\mathsf{c}(x, r)). \tag{1.11}$$

(ii) For $x, y \in \mathbb{S}^{d-1}$ and $n = 0, 1, \cdots$,

$$w_n(x) \le C_{L_w} (1 + n d(x, y))^{s_w} w_n(y). \tag{1.12}$$

Proof. (i) Assuming that $2^{m-1} \le t/r \le 2^m$, we have

$$w(\mathsf{c}(x, t)) \le w(\mathsf{c}(x, 2^m r)) \le L_w C_{L_w} 2^{(m-1) s_w} w(\mathsf{c}(x, r)) \le L_w C_{L_w} \left(\frac{t}{r} \right)^{s_w} w(\mathsf{c}(x, r)),$$

which proves (i).

(ii) For $x, y \in \mathbb{S}^{d-1}$, we have

$$w_n(x) = n^{-(d-1)} \int_{\mathsf{c}(x,n^{-1})} w(z)\, d\sigma(z) \le n^{-(d-1)} \int_{\mathsf{c}(y,n^{-1}+d(x,y))} w(z)\, dz$$
$$\le C_{L_w} n^{-(d-1)} \Big(\frac{n^{-1} + d(x,y)}{n^{-1}} \Big)^{s_w} w(\mathsf{c}(y, n^{-1})) \le C_{L_w} (1 + n d(x,y))^{s_w} w_n(y),$$

where the third step uses (1.11). This completes the proof of (ii). □

A weight function w on $\mathbb{S}^{d-1}$ is called an A_∞ weight if there exists a constant $\beta \ge 1$ such that

$$\frac{w(B)}{w(E)} \le \beta \Big(\frac{\operatorname{meas} B}{\operatorname{meas} E} \Big)^{\beta} \tag{1.13}$$

for every spherical cap $B \subset \mathbb{S}^{d-1}$ and every measurable subset E of B. The least constant β in (1.13) is called the A_∞ constant of w, and is denoted by $A_\infty(w)$.

Directly by their definitions, an A_∞ weight must be a doubling weight. Below we give two important examples of A_∞-weights on $\mathbb{S}^{d-1}$.

Example 1.3. *Let $\kappa = (\kappa_1, \cdots, \kappa_d) \in \mathbb{R}^d_+$ and $|\kappa| = \kappa_1 + \ldots + \kappa_d$. The weight function*

$$w_\kappa(x) = \prod_{j=1}^d |x_j|^{\kappa_j}, \quad x = (x_1, \cdots, x_d) \in \mathbb{S}^{d-1}. \tag{1.14}$$

satisfies the A_∞ condition. Furthermore, the least constant s_{w_κ} for which (1.8) *is satisfied is given by*

$$s_{w_\kappa} := d - 1 + |\kappa| - \min_{1\le j\le d} \kappa_j. \tag{1.15}$$

Moreover, if $x \in \mathbb{S}^{d-1}$ and $\theta \in (0, \pi)$, then

$$w_\kappa(\mathsf{c}(x,\theta)) \sim \theta^{d-1} \prod_{j=1}^d (|x_j| + \theta)^{\kappa_j}. \tag{1.16}$$

Proof. Let E be a measurable subset of a spherical cap $B = \mathsf{c}(x,\theta) \subset \mathbb{S}^{d-1}$, and let $\gamma := \operatorname{meas} E / \operatorname{meas} B$. We claim that

$$c_1 \gamma^\beta \theta^{d-1} \prod_{j=1}^d (|x_j| + \theta)^{\kappa_j} \le w_\kappa(E) \le c_2 \gamma \theta^{d-1} \prod_{j=1}^d (|x_j| + \theta)^{\kappa_j}, \tag{1.17}$$

with $\beta = 1 + |\kappa| - \min_{1\le j\le d} \kappa_j$.

For the moment, we take (1.17) for granted and proceed with our proof. Clearly, (1.16) follows directly by (1.17) applied to $E = B$. For the proof of (1.15), using (1.8) and (**??**), it suffices to verify the following two inequalities:

$$\frac{w_\kappa(2^m B)}{w_\kappa(B)} \le C 2^{m s'_\kappa}, \quad \forall B = \mathsf{c}(x,\theta), \quad x \in \mathbb{S}^{d-1}, \tag{1.18}$$

$$\varlimsup_{m\to\infty} \frac{1}{m} \log_2 \left(\sup_B \frac{w_\kappa(2^m B)}{w_\kappa(B)} \right) \ge s'_\kappa, \tag{1.19}$$

where $s'_\kappa = d - 1 + |\kappa| - \min_{1\le j\le d} \kappa_j$. To show (1.18), without loss of generality, we may assume that $|x_1| = \max_{1\le j\le d} |x_j|$. Then using (1.16), we obtain, for $0 < \theta \le 2^{-m}$,

$$\frac{w_\kappa(2^m B)}{w_\kappa(B)} \sim \frac{2^{m(d-1)} \prod_{j=2}^d (|x_j| + 2^m\theta)^{\kappa_j}}{\prod_{j=2}^d (|x_j| + \theta)^{\kappa_j}} \le 2^{m(d-1)} 2^{m(|\kappa| - \kappa_1)} \le 2^{m s'_\kappa},$$

while for $2^{-m} < \theta \le \pi$,

$$\frac{w_\kappa(2^m B)}{w_\kappa(B)} \sim \frac{1}{\theta^{d-1}\prod_{j=2}^d (|x_j|+\theta)^{\kappa_j}} \le \Big(\frac{1}{\theta}\Big)^{d-1+|\kappa|-\kappa_1} \le 2^{ms'_\kappa}.$$

This proves (1.18). To show (1.19), without loss of generality, we may assume that $\kappa_1 = \min_{1\le j\le d}\kappa_j$. Then taking $x = e_1 := (1,0,\cdots,0) \in \mathbb{S}^{d-1}$, and using (1.16), we deduce

$$\begin{aligned}\overline{\lim_{m\to\infty}}\frac{1}{m}\log_2\left(\sup_B \frac{w_\kappa(2^m B)}{w_\kappa(B)}\right) &\ge \overline{\lim_{m\to\infty}}\frac{1}{m}\log_2\left(\sup_{\theta\in(0,2^{-m})}\frac{w_\kappa(\mathsf{c}(e_1,2^m\theta))}{w_\kappa(\mathsf{c}(e_1,\theta))}\right)\\ &= \overline{\lim_{m\to\infty}}\frac{1}{m}\log_2\left(\sup_{\theta\in(0,2^{-m}]}\frac{\prod_{j=2}^d 2^{m\kappa_j}\theta^{\kappa_j}}{\prod_{j=2}^d\theta^{\kappa_j}}\right) = s'_\kappa.\end{aligned}$$

This proves (1.19). Finally, we show that w_κ satisfies the A_∞ condition. Indeed, using (1.17) and (1.17), we obtain

$$c_1\gamma^\beta = c_1\left(\frac{\operatorname{meas} E}{\operatorname{meas} B}\right)^\beta \le \frac{w_\kappa(E)}{w_\kappa(B)} \le c_2\frac{\operatorname{meas} E}{\operatorname{meas} B} = c_2\gamma. \tag{1.20}$$

The A_∞ property of the weights w_κ follows by the definition.

It remains to prove the claim (1.17). Without loss of generality, we may assume that $\theta \in (0, 99^{-1}d^{-\frac12})$, since (1.17) holds trivially if $\theta \ge 99^{-1}d^{-\frac12}$. Let $\varepsilon \in (0,\frac12)$ be a sufficiently small absolute constant to be determined later, and let $K_j := \{y \in \mathsf{c}(x,\theta):\ \ |y_j| \le \varepsilon\gamma\theta\}$ for $j = 1,2,\cdots,d$. We then assert that

$$\operatorname{meas} K_j \le C_d\varepsilon \operatorname{meas} E, \quad 1 \le j \le d. \tag{1.21}$$

To see this, it is enough to consider the case $j = 1$. Write $x = (x_1,\cdots,x_d) = (\cos t_0, \xi\sin t_0)$ for some $t_0 \in [0,\pi]$ and $\xi \in \mathbb{S}^{d-2}$. If $|x_1| = |\cos t_0| > 2\theta$ then $|y_1| \ge |x_1| - d(x,y) \ge \theta$ for all $y \in \mathsf{c}(x,\theta)$, which implies that $K_1 = \emptyset$. So, without loss of generality, we may assume that $|x_1| \le 2\theta$. Since $\theta \in (0, 99^{-1}d^{-\frac12})$,

$$|t-t_0| + \|\eta-\xi\| \sim |t-t_0| + (\sin t\,\sin t_0)^{1/2}\|\eta-\xi\| \sim d(x,y) \le \theta$$

whenever $y = (\cos t, \eta\sin t) \in \mathsf{c}(x,\theta)$ with $t\in[0,\pi]$ and $\eta \in \mathbb{S}^{d-2}$. Hence,

$$\begin{aligned}\operatorname{meas} K_1 &= \int_0^\pi \sin^{d-2} t\left(\int_{\mathbb{S}^{d-2}} \chi_{K_1}(\cos t, \eta\sin t)\,d\sigma(\eta)\right)dt\\ &\le \int_{\pi/2-\arcsin(\varepsilon\gamma\theta)}^{\pi/2+\arcsin(\varepsilon\gamma\theta)}\left(\int_{\{\eta\in\mathbb{S}^{d-2}:\ d(\eta,\xi)\le c\theta\}} d\sigma(\eta)\right)dt \le C_d\varepsilon\ \operatorname{meas} E,\end{aligned}$$

which proves the assertion (1.21).

Next, we choose $\varepsilon = \frac{1}{2dC_d}$ so that by (1.21), $\sum_{j=1}^d \operatorname{meas} K_j \le dC_d\varepsilon \operatorname{meas} E \le \frac12 \operatorname{meas} E$. In addition, setting $I_1 := \{j:\ 1\le j\le d,\ \ |x_j| \ge 4\theta\}$ and $I_2 := \{j:\ 1 \le j \le d,\ \ |x_j| < 4\theta\}$, and observing that $|x_j| \sim |y_j|$ whenever $j \in I_1$ and $y \in \mathsf{c}(x,\theta)$, we deduce

$$\begin{aligned}w_\alpha(E) &= \int_E \prod_{j=1}^d |y_j|^{\alpha_j}\,d\sigma(y) \sim \left(\prod_{j\in I_1}|x_j|^{\alpha_j}\right)\int_E\prod_{j\in I_2}|y_j|^{\alpha_j}\,d\sigma(y) \qquad (1.22)\\ &\ge \left(\prod_{j\in I_1}|x_j|^{\alpha_j}\right)\int_{E\setminus\cup_{j=1}^d K_j}\prod_{j\in I_2}(\varepsilon\gamma\theta)^{\alpha_j}\,d\sigma(y) \ge c\gamma^\beta\theta^{d-1}\prod_{j=1}^d(|x_j|+\theta)^{\alpha_j}.\end{aligned}$$

Moreover, using (1.22) directly, we have the following upper estimates:

$$w_\alpha(E) \le C\left(\prod_{j\in I_1}|x_j|^{\alpha_j}\right)\int_E\prod_{j\in I_2}\theta^{\alpha_j}\,d\sigma(y) \sim \gamma\theta^{d-1}\prod_{j=1}^d(|x_j|+\theta)^{\alpha_j}.$$

This completes the proof of the claim (1.17). $\square$

EXAMPLE 1.4. *Let* $\kappa = (\kappa_1, \ldots, \kappa_m) \in \mathbb{R}_+^m$, $v = (v_1, \ldots, v_m)$ *with* $v_j \in \mathbb{S}^{d-1}$, $1 \le j \le m$. *Then the weight function*

$$w_{\kappa,v}(x) = \prod_{j=1}^m |\langle x, v_j\rangle|^{\kappa_j},$$

is an A_∞ *weight on* $\mathbb{S}^{d-1}$. *Furthermore, if* $x = (x_1, \cdots, x_d) \in \mathbb{S}^{d-1}$ *and* $\theta \in (0, \pi)$, *then*

$$w_{\kappa,v}(\mathsf{c}(x,\theta)) \sim \theta^{d-1} \prod_{j=1}^m (|\langle x, v_j\rangle| + \theta)^{\kappa_j}. \tag{1.23}$$

PROOF. Following the proof in the above example, we can show that if E is a measurable subset of a spherical cap $B \subset \mathbb{S}^{d-1}$, then

$$c_1 \left(\frac{\operatorname{meas} E}{\operatorname{meas} B}\right)^{1+|\kappa|} \le \frac{w_\alpha(E)}{w_\alpha(B)} \le c_2 \frac{\operatorname{meas} E}{\operatorname{meas} B}$$

for some positive constants c_1, c_2 depending only on d, m and κ, from which it will follow that $w_{\kappa,v}$ is an A_∞ weight. The proof of this last equation relies on (1.23). To illustrate the idea, we give a detailed proof of (1.23) below.

Without loss of generality, we may assume that $v_i \ne v_j$ if $i \ne j$. Set

$$\mathcal{A} = \{i : \quad 1 \le i \le m, \quad |\langle x, v_i\rangle| < 4\theta\},$$
$$\mathcal{B} = \{i : \quad 1 \le i \le m, \ |\langle x, v_i\rangle| \ge 4\theta\}.$$

Note that, for $y \in \mathsf{c}(x, 2\theta)$,

$$w_{\kappa,\mathbf{v}}(y) \sim \left(\prod_{i\in\mathcal{A}} |\langle y, v_i\rangle|^{\kappa_i}\right)\left(\prod_{j\in\mathcal{B}} |\langle x, v_j\rangle|^{\kappa_j}\right) \tag{1.24}$$

$$\le \left(\prod_{i\in\mathcal{A}} (6\theta)^{\kappa_i}\right)\left(\prod_{j\in\mathcal{B}} |\langle x, v_j\rangle|^{\kappa_j}\right). \tag{1.25}$$

The upper estimate of (1.23) then follows.

To show the lower estimate, let

$$E_j = \Big\{y \in \mathsf{c}\Big(x, \frac{\theta}{4}\Big) : \quad \Big|d(y, v_j) - \frac{\pi}{2}\Big| \le \varepsilon_{d,m}\theta\Big\}, \quad 1 \le j \le m,$$

with $\varepsilon_{d,m}$ be a sufficiently small positive constant depending only on d and m. A straightforward calculation shows that

$$\sum_{j=1}^m \operatorname{meas}(E_j) \le C_d \varepsilon_{d,m} m \theta^{d-1} \le \frac{1}{2} \operatorname{meas}\Big(\mathsf{c}\Big(x, \frac{\theta}{4}\Big)\Big)$$

provided that $\varepsilon_{d,m}$ is small enough. Thus, there must exist a point $y_0 \in \mathsf{c}(x, 4^{-1}\theta)$ such that

$$|\langle y_0, v_j\rangle| \ge \sin(\varepsilon_{d,m}\theta), \quad \text{for all } 1 \le j \le m.$$

It follows that $\mathsf{c}(y_0, \frac{\varepsilon_{d,m}\theta}{2}) \subset \mathsf{c}(x, \theta)$ and that for any $y \in \mathsf{c}(y_0, \frac{\varepsilon_{d,m}\theta}{2})$ and $i \in \mathcal{A}$,

$$5\theta > |\langle y, v_i\rangle| \ge \sin\left(\frac{\varepsilon_{d,m}\theta}{2}\right) \ge \frac{\varepsilon_{d,m}\theta}{\pi}.$$

Using (1.24), we obtain

$$w_{\kappa,\mathbf{v}}(y) \sim \left(\prod_{i\in\mathcal{A}} \theta^{\kappa_i}\right) \prod_{j\in\mathcal{B}} |\langle x, v_j\rangle|^{\kappa_j}, \quad \forall y \in \mathsf{c}\Big(y_0, \frac{\varepsilon_{d,m}\theta}{2}\Big). \tag{1.26}$$

Integrating over $\mathsf{c}(y_0, \frac{\varepsilon_{d,m}\theta}{2}) \subset \mathsf{c}(x, \theta)$ then gives the desired lower estimate of (1.23). This completes the proof. $\square$

2. Highly localized reproducing kernels

Let η be a C^∞-function on $[0,\infty)$ such that $\eta(x)=1$ for $x\in[0,1]$ and $\eta(x)=0$ for $x\geq 2$. For $\lambda=\frac{d-2}{2}$ and $N=1,2,\cdots$, we define

$$L_N(t)=\frac{1}{\omega_d}\sum_{k=0}^{2N}\eta(\frac{k}{N})\frac{k+\lambda}{\lambda}C_k^\lambda(t). \tag{1.27}$$

It follows by (**??**) that

$$f(x)=\int_{\mathbb{S}^{d-1}}f(y)L_N(\langle x,y\rangle)\,d\sigma(y),\quad x\in\mathbb{S}^{d-1},\quad \forall f\in\Pi_N^d. \tag{1.28}$$

This means that $L_N(\langle x,y\rangle)$ is a reproducing kernel for the space Π_N^d. We shall keep the notations L_N and η throughout this chapter.

The kernels L_N will play crucial roles in our later discussions. Our main goal in this section is to show that these kernels and their derivatives are highly localized at $t=1$. More precisely, we shall prove the following theorem.

Theorem 2.1. *Given positive integer ℓ, we have, for $N\geq 1$ and $\theta\in[0,\pi]$,*

$$|L_N^{(i)}(\cos\theta)|\leq C_{\ell,i}\|\eta^{(3\ell-1)}\|_\infty N^{d-1+2i}(1+N\theta)^{-\ell},\quad i=0,1,\cdots,$$

where $L_N^{(0)}(t)=L_N(t)$, and $L_N^{(i)}(t)=\left(\frac{d}{dt}\right)^i L_N(t)$ for $i\geq 1$.

Indeed, we shall prove below a more general result for kernels of the form

$$G_{N,\varphi}^{(\alpha,\beta)}(t):=\sum_{k=0}^{\infty}\varphi(\frac{k}{N})\frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\beta+1)}P_k^{(\alpha,\beta)}(t) \tag{1.29}$$

for some smooth cutoff functions $\varphi\colon[0,\infty)\to\mathbb{C}$, and $\alpha\geq\beta\geq-\frac12$. It turns out that such kernels and their derivatives are also highly localized at $t=1$ under certain conditions on φ. Since, by (A.6) in Appendix A,

$$L_N(t)=\frac{2\pi^{\frac d2}}{\Gamma(\frac d2)}\frac{\Gamma(\frac{d-1}{2})}{\Gamma(d-1)}G_{N,\eta}^{(\frac{d-3}{2},\frac{d-3}{2})}(t),$$

Theorem 2.1 follows directly from the following more general estimates on the kernels $G_{N,\varphi}^{(\alpha,\beta)}(t)$:

Proposition 2.2. *Let ℓ be a positive integer, and let $\varphi\in C^{3\ell-1}[0,\infty)$ be such that supp $\varphi\subset[0,2]$ and $\eta^{(j)}(0)=0$ for $j=1,2,\cdots,3\ell-2$. Then for the kernel function $G_N\equiv G_{N,\varphi}^{(\alpha,\beta)}$ defined by (1.29) with $\alpha\geq\beta\geq-1/2$,*

$$|G_N^{(i)}(\cos\theta)|\leq C_{\ell,i,\alpha}\|\eta^{(3\ell-1)}\|_\infty N^{2\alpha+2i+2}(1+N\theta)^{-\ell},\quad i=0,1,\cdots, \tag{1.30}$$

where $\theta\in[0,\pi]$, $N\in\mathbb{N}$, $G_N^{(0)}(t)=G_{N,\varphi}^{(\alpha,\beta)}(t)$ and $G_N^{(i)}(t)=\left(\frac{d}{dt}\right)^i\{G_{N,\varphi}^{(\alpha,\beta)}(t)\}$ for $i\geq 1$.

Proof. By (A.3) in Appendix A, it follows that

$$G_N^{(i)}(t)=2^{-i}\sum_{k=0}^{2N}\varphi(\frac{k+i}{N})\frac{(2k+\alpha+\beta+2i+1)\Gamma(k+\alpha+\beta+2i+1)}{\Gamma(k+i+\beta+1)}P_k^{(\alpha+i,\beta+i)}(t).$$

Thus, using summation by parts ℓ times, as well as (A.4) in Appendix A with $(\alpha+i+j,\beta+i)$ in place of (α,β) for $j=0,1,\cdots,\ell$, we conclude that

$$G_N^{(i)}(t)=2^{-\ell}\sum_{k=0}^{2N}a_{N,\ell}(k)\frac{\Gamma(k+\alpha+\beta+2i+\ell+1)}{\Gamma(k+\beta+i+1)}P_k^{(\alpha+i+\ell,\beta+i)}(t), \tag{1.31}$$

where $\{a_{N,j}\}_{j=0}^{\infty}$ is a sequence of functions on $[0,\infty)$ defined by

$$a_{N,0}(s) = (2s+\alpha+\beta+2i+1)\varphi\Big(\frac{s+i}{N}\Big),$$
$$a_{N,j+1}(s) = \frac{a_{N,j}(s)}{2s+\alpha+\beta+2i+j+1} - \frac{a_{N,j}(s+1)}{2s+\alpha+\beta+2i+j+3}, \quad j \geq 0.$$

Next, we claim that if $m+j \leq \ell$ and $j \geq 1$, then

$$|a_{N,j}^{(m)}(s)| \leq c_{\ell,i}(s+1)^{-m-2j+1}\Big(\frac{s+1}{N}\Big)^{2\ell-1}\|\varphi^{(2\ell+m+j-1)}\|_{L^\infty[0,\frac{s+j+i}{N}]}, \tag{1.32}$$

which, in particular, implies

$$|a_{N,\ell}(k)| \leq c_{\ell,i}\|\varphi^{(3\ell-1)}\|_\infty N^{-2\ell+1}. \tag{1.33}$$

For the moment, we take (1.32) for granted, and proceed with the proof of (1.30). Indeed, using (1.33) and (1.31),

$$|G_N^{(i)}(\cos\theta)| \leq c_{\ell,i}\|\varphi^{(3\ell-1)}\|_\infty N^{-2\ell+1}\sum_{k=0}^{2N}(k+1)^{\alpha+i+\ell}|P_k^{(\alpha+i+\ell,\beta+i)}(\cos\theta)|.$$

By (A.5) in Appendix A, this implies that for $\theta \in [0,\pi/2]$,

$$\begin{aligned}|G_N^{(i)}(\cos\theta)| &\leq c_{\ell,i}\|\varphi^{(3\ell-1)}\|_\infty N^{-2\ell+1}\Big[\sum_{0\leq k\leq \max\{\theta^{-1},2N\}}(k+1)^{2\alpha+2i+2\ell}\\ &\quad + \sum_{\max\{\theta^{-1},2N\}<k\leq 2N}(k+1)^{\alpha+i+\ell-\frac12}\theta^{-\alpha-i-\ell-\frac12}\Big]\\ &\leq c_{\ell,i}\|\varphi^{(3\ell-1)}\|_\infty N^{2\alpha+2i+2}(1+N\theta)^{-(\alpha+i+\ell+\frac12)},\end{aligned}$$

and that for $\theta \in [\frac{\pi}{2},\pi]$,

$$\begin{aligned}|G_N^{(i)}(\cos\theta)| &\leq c_{\ell,i}\|\varphi^{(3\ell-1)}\|_\infty N^{-2\ell+1}\sum_{k=0}^{2N}(k+1)^{\alpha+i+\ell}(k+1)^{\beta+i}\\ &\leq c_{\ell,i}\|\varphi^{(3\ell-1)}\|_\infty N^{2\alpha+2i+2}N^{-\ell},\end{aligned}$$

where the last step uses the assumption $\alpha \geq \beta$. Putting the above together, and recalling that $\alpha \geq -\frac12$, we deduce the desired estimate (1.30).

It remains to prove the claim (1.32). We first observe that by Taylor's theorem,

$$\|\varphi^{(m)}\|_{L^\infty[0,t]} \leq \frac{t^k}{k!}\|\varphi^{(m+k)}\|_{L^\infty[0,t]}, \quad t \geq 0 \tag{1.34}$$

whenever $m \geq 1$ and $m+k \leq 3\ell-1$. Next, we show (1.32) for the case of $j=1$. Since $a_{N,1}$ is supported in $[0,2N]$, without loss of generality, we may assume that $0 \leq s \leq 2N$. By definition, we have

$$a_{N,1}(s) = \varphi\Big(\frac{s+i}{N}\Big) - \varphi\Big(\frac{s+i+1}{N}\Big) = -\int_{\frac{i}{N}}^{\frac{i+1}{N}}\varphi'\Big(\frac{s}{N}+t\Big)\,dt.$$

This combined with (1.34) implies that

$$\begin{aligned}|a_{N,1}^{(m)}(s)| &\leq N^{-m}\Big|\int_{\frac{i}{N}}^{\frac{i+1}{N}}\varphi^{(m+1)}(\frac{s}{N}+t)\,dt\Big|\\ &\leq N^{-m-1}\frac{1}{(2\ell-1)!}\Big(\frac{s+i+1}{N}\Big)^{2\ell-1}\|\varphi^{(m+2\ell)}\|_{L^\infty[0,\frac{s+i+1}{N}]}\\ &\leq c_{\ell,i}(s+1)^{-m-1}\Big(\frac{s+1}{N}\Big)^{2\ell-1}\|\varphi^{(m+2\ell)}\|_{L^\infty[0,\frac{s+i+1}{N}]},\end{aligned}$$

where the last step uses the assumption $0 \le s \le 2N$. This proves (1.32) for the case of $j = 1$. Finally, assuming that (1.32) has been proven for some $j \ge 1$, and observing that

$$a_{N,j+1}^{(m)}(s) = -\int_0^1 \Big(\frac{d}{dt}\Big)^{m+1} \Big(\frac{a_{N,j}(s+t)}{2s+2t+\alpha+\beta+2i+j+1}\Big)\,dt,$$

we obtain that for $m+j+1 \le \ell$,

$$\begin{aligned}|a_{N,j+1}^{(m)}(s)| &\le \int_0^1 \max_{\substack{k_1+k_2=m+1\\ k_1,k_2\in\mathbb{Z}_+}} |a_{N,j}^{(k_1)}(s+t)|(s+1)^{-k_2-1}\,dt\\ &\le c_{\ell,i}(s+1)^{-m-2j-1}\Big(\frac{s+1}{N}\Big)^{2\ell-1}\|\varphi^{(2\ell+m+j)}\|_{L^\infty[0,\frac{s+j+i+1}{N}]}.\end{aligned}$$

This proves (1.32) for the case of $j+1$, and hence completes the induction. □

3. A maximal function for spherical polynomials

Let us first introduce some notations for the rest of this section. Let w be a doubling weight on $\mathbb{S}^{d-1}$, and let s_w denote a positive number satisfying (1.8) and $0 \le s_w \le \frac{\log L_w}{\log 2}$. The weighted Hardy-Littlewood maximal function M_w is defined by

$$M_w g(x) = \sup_{0<r\le\pi} \frac{1}{w(\mathsf{c}(x,r))}\int_{\mathsf{c}(x,r)} |g(y)|w(y)\,d\sigma(y).$$

Since w satisfies the doubling condition, it follows that

$$\|M_w g\|_{p,w} \le C_p\|g\|_{p,w}, \quad 1 < p \le \infty, \tag{1.35}$$

where $\|\cdot\|_{p,w}$ denotes the weighted quasi-norm defined by

$$\|f\|_{p,w} = \begin{cases}\Big(\int_{\mathbb{S}^{d-1}} |f(x)|^p w(x)\,d\sigma(x)\Big)^{\frac{1}{p}}, & 0<p<\infty,\\ \operatorname*{esssup}_{x\in\mathbb{S}^{d-1}} |f(x)|, & p=\infty.\end{cases}$$

Definition 3.1. *Given $\beta > 0$, $f \in C(\mathbb{S}^{d-1})$ and $n \in \mathbb{Z}_+$, we define*

$$f_{\beta,n}^*(x) = \max_{y\in\mathbb{S}^{d-1}} |f(y)|(1+nd(x,y))^{-\beta}, \quad x\in\mathbb{S}^{d-1}. \tag{1.36}$$

The maximal function $f_{\beta,n}^*$ will be the main tool for us to study the weighted spherical polynomial inequalities in the next few sections. Our main goal in this section is to show that for any spherical polynomial $f \in \Pi_n^d$, the maximal function $f_{\beta,n}^*$ can be controlled pointwisely by $(M_w(|f|^\gamma))^{1/\gamma}$ with $\gamma = s_w/\beta$.

Theorem 3.2. *For $f \in \Pi_n^d$, $\beta > 0$ and $\gamma = s_w/\beta$,*

$$f_{\beta,n}^*(x) \le C_{\beta,L_w}\Big(M_w(|f|^\gamma)(x)\Big)^{1/\gamma}, \quad x\in\mathbb{S}^{d-1}. \tag{1.37}$$

Proof. Let L_n be the kernel as defined in (1.27), and set

$$A_{n,\delta}(y,u) := \sup_{z\in\mathsf{c}(y,\frac{\delta}{n})} |L_n(\langle y,u\rangle) - L_n(\langle z,u\rangle)| \tag{1.38}$$

for $\delta > 0$ and $y, u \in \mathbb{S}^{d-1}$.

Using Theorem 2.1 with $i = 0, 1$, it is easily seen that for any $\ell > 0$, and $\theta = d(y,u)$,

$$A_{n,\delta}(y,u) \le C_{d,\ell}\begin{cases} n^{d-1}, & \text{if } \theta\in[0,\frac{4\delta}{n}],\\ \delta n^{d-1}(1+n\theta)^{-\ell}, & \text{if } \theta\in[\frac{4\delta}{n},\pi].\end{cases} \tag{1.39}$$

Thus, using (1.28) and (1.39), we obtain that for $x, y \in \mathbb{S}^{d-1}$,

$$\max_{z\in \mathsf{c}(y,\frac{\delta}{n})} \frac{|f(y)-f(z)|}{(1+nd(x,y))^\beta} \le f^*_{\beta,n}(x) \int_{\mathbb{S}^{d-1}} \left(\frac{1+nd(x,u)}{1+nd(x,y)}\right)^\beta A_{n,\delta}(y,u)\,d\sigma(u)$$
$$\le f^*_{\beta,n}(x)\int_{\mathbb{S}^{d-1}} (1+nd(y,u))^\beta A_{n,\delta}(y,u)\,d\sigma(u) \le C_\beta \delta f^*_{\beta,n}(x).$$

It follows that for $x, y \in \mathbb{S}^{d-1}$ and $\delta \in (0, \frac{1}{4})$,

$$|f(y)|^\gamma (1+nd(x,y))^{-s_w} \le 2^\gamma (1+nd(x,y))^{-s_w} \min_{z\in\mathsf{c}(y,\frac{\delta}{n})} |f(z)|^\gamma + \Big(2C_\beta\delta f^*_{\beta,n}(x)\Big)^\gamma$$
$$\le 2^\gamma(1+nd(x,y))^{-s_w}\left(\int_{\mathsf{c}(y,\frac{\delta}{n})} w(z)\,d\sigma(z)\right)^{-1}\int_{\mathsf{c}(y,\frac{\delta}{n})}|f(z)|^\gamma w(z)\,d\sigma(z)$$
$$+\Big(2C_\beta\delta f^*_{\beta,n}(x)\Big)^\gamma =: I + \Big(2C_\beta\delta f^*_{\beta,n}(x)\Big)^\gamma. \tag{1.40}$$

If $d(x,y)\le \frac{\delta}{n}$, then $\mathsf{c}(y,\frac{\delta}{n}) \subset \mathsf{c}(x,\frac{2\delta}{n}) \subset \mathsf{c}(y,\frac{3\delta}{n})$, and

$$I \le 2^\gamma L_w^2 \left(\int_{\mathsf{c}(x,\frac{2\delta}{n})} w(z)\,d\sigma(z)\right)^{-1}\int_{\mathsf{c}(x,\frac{2\delta}{n})} |f(z)|^\gamma w(z)\,d\sigma(z)$$
$$\le 2^\gamma L_w^2 M_w(|f|^\gamma)(x).$$

On the other hand, if $\frac{\delta}{n} \le \theta := d(x,y) \le \pi$, then $\mathsf{c}(y,\frac{\delta}{n}) \subset \mathsf{c}(x,2\theta) \subset \mathsf{c}(y,3\theta)$ and using (1.8),

$$\int_{\mathsf{c}(y,\frac{\delta}{n})} w(z)\,d\sigma(z) \ge C'_{L_w}\left(\frac{3\theta n}{\delta}\right)^{-s_w}\int_{\mathsf{c}(y,3\theta)} w(z)\,d\sigma(z)$$
$$\ge C'_{L_w}\left(\frac{3\theta n}{\delta}\right)^{-s_w}\int_{\mathsf{c}(x,2\theta)} w(z)\,d\sigma(z),$$

which in turn implies that

$$I \le C_{L_w} 2^\gamma \left(\frac{3\theta n}{\delta}\right)^{s_w} (1+n\theta)^{-s_w}\left(\int_{\mathsf{c}(x,2\theta)} w(z)\,d\sigma(z)\right)^{-1}\int_{\mathsf{c}(x,2\theta)}|f(z)|^\gamma w(z)\,d\sigma(z)$$
$$\le C_{L_w}2^\gamma\left(\frac{3}{\delta}\right)^{s_w} M_w(|f|^\gamma)(x).$$

Therefore, in either case, we have shown that

$$I \le 2^\gamma C_{L_w}\delta^{-s_w} M_w(|f|^\gamma)(x). \tag{1.41}$$

Substituting (1.41) into (1.40), letting $\delta = (4C_\beta)^{-1}$, and taking the supremum over all $y \in \mathbb{S}^{d-1}$, we deduce

$$\Big(f^*_{\beta,n}(x)\Big)^\gamma \le C_{L_w} 2^\gamma (4C_\beta)^{s_w} M_w(|f|^\gamma)(x) + 2^{-\gamma}(f^*_{\beta,n}(x))^\gamma.$$

The desired inequality (1.37) then follows with $C_{\beta,L_w} = 4^{\beta+1} C_{L_w}^{1/\gamma} C_\beta^\beta (2^\gamma-1)^{-1/\gamma}$. This completes the proof. $\square$

As a simple consequence of (1.35) and Theorem 3.2, we have the following useful corollary.

COROLLARY 3.3. *If* $0 < p \le \infty$, $f \in \Pi_n^d$ *and* $\beta > \frac{s_w}{p}$, *then*

$$\|f\|_{p,w} \le \|f^*_{\beta,n}\|_{p,w} \le C\|f\|_{p,w},$$

where $C > 0$ *depends only on* d, L_w *and* β *when* β *is big or close to* $\frac{s_w}{p}$.

DEFINITION 3.4. *For $f \in C(\mathbb{S}^{d-1})$ and $r > 0$, we define*

$$\operatorname{osc}(f)(x,r) = \sup_{y,z\in \mathsf{c}(x,r)} |f(z) - f(y)|, \quad x \in \mathbb{S}^{d-1}. \tag{1.42}$$

We conclude this section with the following useful lemma.

LEMMA 3.5. *If $f \in \Pi_n^d$ and $\delta \in (0,1]$, then for any $\beta > 0$,*

$$\operatorname{osc}(f)(x, n^{-1}\delta) \le C_\beta \delta f^*_{\beta,n}(x), \quad x \in \mathbb{S}^{d-1},$$

where the constant C_β depends only on d and β when β is big.

PROOF. Using (1.28) and (1.38), we have

$$\sup_{y,z\in \mathsf{c}(x,\frac{\delta}{n})} |f(y) - f(z)| \le 2\int_{\mathbb{S}^{d-1}} |f(u)| A_{n,\delta}(x,u)\, d\sigma(u).$$

However, by (1.39),

$$\begin{aligned}\operatorname{osc}(f)(x, n^{-1}\delta) &\le 2f^*_{\beta,n}(x) \int_{\mathbb{S}^{d-1}} (1 + nd(u,x))^\beta A_{n,\delta}(x,u)\, d\sigma(u)\\ &\le C_\beta \delta f^*_{\beta,n}(\omega).\end{aligned}$$

□

4. Marcinkiewicz-Zygmund (MZ) inequalities

We start with the following definition.

DEFINITION 4.1. *A subset Λ of $\mathbb{S}^{d-1}$ is called ε-separated for some $\varepsilon > 0$ if $d(\omega,\omega') \ge \varepsilon$ for any two distinct points $\omega,\omega' \in \Lambda$. A ε-separated subset Λ of $\mathbb{S}^{d-1}$ is called maximal if $\mathbb{S}^{d-1} = \bigcup_{\omega\in\Lambda} \mathsf{c}(\omega,\varepsilon)$.*

The following lemma collects some useful properties of separated subsets.

LEMMA 4.2. *(i) If $\Lambda \subset \mathbb{S}^{d-1}$ is ε-separated, then $\#\Lambda \le c_d\varepsilon^{-d+1}$, where $\#\Lambda$ denotes the cardinality of the set Λ. If, in addition, Λ is maximal, then $c_1\varepsilon^{-d+1} \le \#\Lambda \le c_2\varepsilon^{-d+1}$ for some positive constants c_1 and c_2 depending only on d.*

(ii) If $\Lambda \subset \mathbb{S}^{d-1}$ is ε-separated, and $\beta \ge 1$ then $\sum_{\omega\in\Lambda} \chi_{\mathsf{c}(\omega,\beta\varepsilon)}(x) \le C_d\beta^{d-1}$ for every $x \in \mathbb{S}^{d-1}$. If, in addition, Λ is maximal, then

$$1 \le \sum_{\omega\in\Lambda} \chi_{\mathsf{c}(\omega,\beta\varepsilon)}(x) \le C_1\beta^{d-1} \quad \textit{for all } x \in \mathbb{S}^{d-1}, \tag{1.43}$$

where C_1 depends only on d.

PROOF. If $\Lambda \subset \mathbb{S}^{d-1}$ is ε-separated, then the spherical caps $\mathsf{c}(\omega, \frac{\varepsilon}{2})$, $\omega \in \Lambda$ are pairwise disjoint, hence

$$\sum_{\omega\in\Lambda} \operatorname{meas} \mathsf{c}(\omega, \frac{\varepsilon}{2}) = \operatorname{meas} \bigcup_{\omega\in\Lambda} \mathsf{c}\Big(\omega, \frac{\varepsilon}{2}\Big) \le \operatorname{meas}(\mathbb{S}^{d-1}),$$

which implies that $\#\Lambda \le c_d\varepsilon^{-d+1}$. If, in addition, Λ is maximal, then $\operatorname{meas}(\mathbb{S}^{d-1}) \le \sum_{\omega\in\Lambda} \operatorname{meas} \mathsf{c}(\omega,\varepsilon)$, which implies the lower estimate $c_1\varepsilon^{-d+1} \le \#\Lambda$. This proves (i). Assertion (ii) can be proven using a similar volume comparison argument. In fact, let $A_x := \{\omega \in \Lambda : \ x \in \mathsf{c}(\omega,\beta\varepsilon)\}$ for each $x \in \mathbb{S}^{d-1}$. Then $\bigcup_{\omega\in A_x} \mathsf{c}(\omega, \frac{1}{2}\varepsilon) \subset \mathsf{c}(x, (\beta + \frac{1}{2})\varepsilon)$, and hence

$$\sum_{\omega\in A_x} \operatorname{meas} \mathsf{c}(\omega, \frac{1}{2}\varepsilon) \le \operatorname{meas} \mathsf{c}\Big(x, \Big(\beta + \frac{1}{2}\Big)\varepsilon\Big).$$

It follows that

$$\sum_{\omega\in\Lambda} \chi_{\mathsf{c}(\omega,\beta\varepsilon)}(x) = \#A_x \le C_1\beta^{d-1},$$

which proves (ii). $\square$

THEOREM 4.3. *Let Λ be a $\frac{\delta}{n}$ -separated subset of $\mathbb{S}^{d-1}$ with $\delta \in (0,1)$. Then for all $f \in \Pi_n^d$ and $0 < p < \infty$,*

$$\Big(\sum_{\omega\in\Lambda} |\mathrm{osc}(f)(\omega, n^{-1}\delta)|^p w\Big(\mathsf{c}\Big(\omega, \frac{\delta}{n}\Big)\Big)\Big)^{\frac{1}{p}} \le K_p \delta \|f\|_{p,w}, \tag{1.44}$$

where $K_p \equiv K_{p,L_w}$ depends only on L_w and p when p is close to 0.

PROOF. By Lemma 3.5, we have, for $f \in \Pi_n^d$ and $\omega \in \Lambda$,

$$\mathrm{osc}(f)(\omega, n^{-1}\delta) \le C\delta f^*_{\frac{2s_w}{p},n}(\omega),$$

where $C > 0$ depends only on L_w and p when p is small. Since

$$f^*_{2s_w/p,n}(y) \sim f^*_{2s_w/p,n}(\omega), \quad \text{for } y \in \mathsf{c}(\omega, \frac{\delta}{n}),$$

it follows that

$$\begin{aligned}\sum_{\omega\in\Lambda} |\mathrm{osc}(f)(\omega, n^{-1}\delta)|^p \int_{\mathsf{c}(\omega,\frac{\delta}{n})} w(y)\,d\sigma(y) &\le (C\delta)^p \sum_{\omega\in\Lambda} \int_{\mathsf{c}(\omega,\frac{\delta}{n})} (f^*_{2s_w/p,n}(y))^p w(y)\,d\sigma(y)\\ &\le (C\delta)^p \int_{\mathbb{S}^{d-1}} (f^*_{2s_w/p,n}(y))^p w(y)\,d\sigma(y)\\ &\le (K_p\delta)^p \int_{\mathbb{S}^{d-1}} |f(y)|^p w(y)\,d\sigma(y),\end{aligned}$$

where the last two steps use Lemma 4.2 (ii), and Corollary 3.3, respectively. This completes the proof. $\square$

THEOREM 4.4. *For $f \in \Pi_n^d$ and $0 < p < \infty$,*

$$C^{-1}\|f\|_{p,w_n} \le \|f\|_{p,w} \le C\|f\|_{p,w_n},$$

where $C > 0$ depends only on d, L_w and p when p is small.

PROOF. Since each w_n is again a doubling weight with $L_{w_n} \sim L_w$, according to Corollary 3.3, it suffices to prove that

$$\|f^*_{2s/p,n}\|_{p,w} \sim \|f^*_{2s/p,n}\|_{p,w_n}, \tag{1.45}$$

where $s = \max\{s_w, s_{w_n}\}$. To this end, let $\Lambda \subset \mathbb{S}^{d-1}$ be a maximal $\frac{1}{n}$ -separated subset, and observe that for each $x \in \mathsf{c}(\omega, \frac{1}{n})$,

$$f^*_{2s/p,n}(x) \sim f^*_{2s/p,n}(\omega) \quad \text{and} \quad w_n(x) \sim w_n(\omega).$$

It follows by Lemma 4.2 that

$$\begin{aligned}\|f^*_{2s/p,n}\|^p_{p,w} &\sim \sum_{\omega\in\Lambda} \int_{\mathsf{c}(\omega,\frac{1}{n})} (f^*_{2s/p,n}(x))^p w(x)\,d\sigma(x) \sim n^{-(d-1)} \sum_{\omega\in\Lambda} \Big(f^*_{2s/p,n}(\omega)\Big)^p w_n(\omega)\\ &\sim \sum_{\omega\in\Lambda} \int_{\mathsf{c}(\omega,\frac{1}{n})} \Big(f^*_{2s/p,n}(x)\Big)^p w_n(x)\,d\sigma(x) \sim \|f^*_{2s/p,n}\|_{p,w_n},\end{aligned}$$

which proves (1.45). $\square$

The MZ inequality for spherical polynomials can now be stated as follows.

THEOREM 4.5. *(MZ inequality). (i) Let Λ be a $\frac{\delta}{n}$-separated subset of $\mathbb{S}^{d-1}$ with $\delta \in (0,1]$. Then for all $0 < p < \infty$ and $f \in \Pi_m^d$ with $m \ge n$,*

$$\sum_{\omega\in\Lambda} \Big(\max_{x\in\mathsf{c}(\omega,n^{-1}\delta)} |f(x)|^p\Big) w(\mathsf{c}(\omega, n^{-1}\delta)) \le C_{L_w,p} \Big(\frac{m}{n}\Big)^{s_w} \|f\|^p_{p,w}, \tag{1.46}$$

where $C_{L_w,p}$ depends only on L_w and p when p is close to 0.

(ii) Let $0 < r \le 1$, and let Λ be a maximal $\frac{\delta}{n}$-separated subset of $\mathbb{S}^{d-1}$ with $\delta \in (0, (4K_r)^{-1})$. If $f \in \Pi_n^d$, then $\|f\|_\infty \sim \max_{\omega\in\Lambda} |f(\omega)|$, and for all $r \le p < \infty$,

$$\|f\|_{p,w} \sim \Big(\sum_{\omega\in\Lambda} w\Big(\mathsf{c}\Big(\omega, \frac{\delta}{n}\Big)\Big) \min_{x\in\mathsf{c}\left(\omega,\frac{\delta}{n}\right)} |f(x)|^p\Big)^{1/p} \tag{1.47}$$

$$\sim \Big(\sum_{\omega\in\Lambda} w\Big(\mathsf{c}\Big(\omega, \frac{\delta}{n}\Big)\Big) \max_{x\in\mathsf{c}\left(\omega,\frac{\delta}{n}\right)} |f(x)|^p\Big)^{1/p}, \tag{1.48}$$

where the constants of equivalence depend only on L_w and r when r is close to 0.

The proof of (3.26) in the above theorem relies on the following general lemma.

LEMMA 4.6. *If μ is a finite nonnegative measure on $\mathbb{S}^{d-1}$ satisfying*

$$\mu\Big(\mathsf{c}\Big(x, \frac{1}{n}\Big)\Big) \le K w\Big(\mathsf{c}\Big(x, \frac{1}{n}\Big)\Big), \quad x \in \mathbb{S}^{d-1} \tag{1.49}$$

for some positive integer n, then for all $0 < p < \infty$ and $f \in \Pi_m^d$ with $m \ge n$, we have

$$\int_{\mathbb{S}^{d-1}} |f(x)|^p \, d\mu(x) \le CK\Big(\frac{m}{n}\Big)^{s_w} \|f\|_{p,w}^p,$$

where C depends only on L_w and p when p is close to 0.

PROOF. Let Λ be a maximal $\frac{1}{m}$-separated subset of $\mathbb{S}^{d-1}$, and set $\beta = \frac{s_w}{p} + 1$. Then for $f \in \Pi_m^d$, we have

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} |f(x)|^p \, d\mu(x) &\le C \sum_{\omega\in\Lambda} (f_{\beta,m}^*(\omega))^p \int_{\mathsf{c}(\omega,\frac{1}{m})} d\mu(x) \\
&\le CK \sum_{\omega\in\Lambda} (f_{\beta,m}^*(\omega))^p w(\mathsf{c}(\omega, n^{-1})) \\
&\le cK\Big(\frac{m}{n}\Big)^{s_w} \sum_{\omega\in\Lambda} (f_{\beta,m}^*(\omega))^p w\big(\mathsf{c}(\omega, m^{-1})\big) \\
&\le CK\Big(\frac{m}{n}\Big)^{s_w} \int_{\mathbb{S}^{d-1}} (f_{\beta,m}^*(y))^p w(y) \, d\sigma(y) \\
&\le cK\Big(\frac{m}{n}\Big)^{s_w} \|f\|_{p,w}^p,
\end{aligned}$$

where we used (1.43) in the first and the fourth steps, (1.49) in the second step, (1.11) in the third step, and Corollary 3.3 in the last step. This completes the proof. $\square$

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. We start with the proof of (3.26). For convenience, we set n_1 to be the integer such that $\frac{n}{2\delta} < n_1 \le \frac{n}{\delta}$. Let $\xi_\omega \in \mathsf{c}(\omega, n^{-1}\delta)$ be such that $f(\xi_\omega) = \max_{x\in\mathsf{c}(\omega,n^{-1}\delta)} |f(x)|^p$ for each $\omega \in \Lambda$. Let μ be a nonnegative measure supported in the set $\{\xi_\omega : \omega \in \Lambda\}$ and such that $\mu(\xi_\omega) = w(\mathsf{c}(\omega, n^{-1}\delta))$ for each $\omega \in \Lambda$. Then for any $x \in \mathbb{S}^{d-1}$,

$$\begin{aligned}
\mu(\mathsf{c}(x, {n_1}^{-1})) &\le \sum_{\substack{\omega\in\Lambda \\ \xi_\omega\in\mathsf{c}(x,n_1^{-1})}} w\Big(\mathsf{c}\Big(\omega, n_1^{-1}\Big)\Big) \le L_w^2 \sum_{\omega\in\Lambda\cap\mathsf{c}(x,\frac{2}{n_1})} w\Big(\mathsf{c}\Big(\omega, \frac{1}{4n_1}\Big)\Big) \\
&\le L_w^2 \int_{\mathsf{c}(x,3n_1^{-1})} w(y) \, d\sigma(y) \\
&\le L_w^4 w(\mathsf{c}(x, n_1^{-1})).
\end{aligned}$$

It follows by Lemma 4.6 that for $f \in \Pi_m^d$ with $m \ge n$,

$$\begin{aligned}\sum_{\omega\in\Lambda}\Big(\max_{x\in \mathsf{c}(\omega,n^{-1}\delta)}|f(x)|^p\Big)w(\mathsf{c}(\omega,n^{-1}\delta)) &= \int_{\mathbb{S}^{d-1}}|f(x)|^p\,d\mu(x)\\ &\le C_{L_w}\Big(1+\frac{m}{n_1}\Big)^{s_w}\|f\|_{p,w}^p\\ &\le L_wC_{L_w}\Big(\frac{m}{n}\Big)^{s_w}\|f\|_{p,w}^p,\end{aligned}$$

which proves (3.26).

Next, we show that if $r \le p < \infty$, $\delta \in (0, \delta_r)$ and Λ is maximal $\frac{\delta}{n}$ -separated, then for $f \in \Pi_n^d$,

$$\|f\|_{p,w} \le C_2\bigg(\sum_{\omega\in\Lambda}\Big(\min_{x\in \mathsf{c}(\omega,n^{-1}\delta)}|f(x)|^p\Big)w(\mathsf{c}(\omega,n^{-1}\delta))\mathcal{B}igr)^{1/p}, \tag{1.50}$$

where C_2 depends only on r and L_w. Once (1.50) is proved, using (3.26), we deduce (1.47) and (1.47). Since the constant c in (1.50) is independent of p, we deduce the equivalence for the case of $p = \infty$ from (1.50) as well.

For the proof of (1.50), we observe that

$$\begin{aligned}\|f\|_{p,w}^p &\le \sum_{\omega\in\Lambda}\int_{\mathsf{c}(\omega,n^{-1}\delta)}|f(x)|^pw(x)\,d\sigma(x)\\ &\le 2^p\sum_{\omega\in\Lambda}|\mathrm{osc}(f)(\omega,n^{-1}\delta)|^pw(\mathsf{c}(\omega,n^{-1}\delta))\\ &\quad+2^p\sum_{\omega\in\Lambda}\Big(\min_{y\in \mathsf{c}(\omega,n^{-1}\delta)}|f(y)|^p\Big)w(\mathsf{c}(\omega,n^{-1}\delta)).\end{aligned}$$

Using Theorem 4.3, we then obtain that for $r < p < \infty$,

$$\|f\|_{p,w}^p \le (2K_r\delta)^p\|f\|_{p,w}^p + 2^p\sum_{\omega\in\Lambda}\min_{y\in \mathsf{c}(\omega,n^{-1}\delta)}|f(y)|^pw(\mathsf{c}(\omega,n^{-1}\delta)).$$

Since $\delta \in (0, (4K_r)^{-1})$, the desired inequality (1.50) then follows. This completes the proof of Theorem 4.1. □

5. Positive Cubature formulas

Our main goal in this section is to show the existence of weighted positive cubature formulas on the sphere, using Theorem 4.3 and Theorem 4.1. Let C_1 and C_2 denote the constants in (1.43) and (1.50) with $p = 1$ respectively, and let $K_1 \equiv K_{L_w}$ denote the constant in (1.44) with $p = 1$. Set $\delta_0 := \frac{1}{2C_1C_2K_1}$. clearly, δ_0 depends only on L_w. With these notations, we can state our second result as follows.

Theorem 5.1. *Given a maximal $\frac{\delta}{n}$-separated subset $\Lambda \subset \mathbb{S}^{d-1}$ with $\delta \in (0, \delta_0)$, there are positive numbers λ_ω, $\omega \in \Lambda$ such that $\lambda_\omega \sim w\big(\mathsf{c}(\omega, \frac{\delta}{n})\big)$ for all $\omega \in \Lambda$*

$$\int_{\mathbb{S}^{d-1}} f(x)w(x)\,d\sigma(x) = \sum_{\omega\in\Lambda}\lambda_\omega f(\omega), \quad f \in \Pi_n^d. \tag{1.51}$$

The proof of Theorem 5.1 relies on a series of lemmas.

Lemma 5.2. *(Gordan) Let V be a finite dimensional real Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Then for any elements $a^0, a^1, \cdots, a^m$ of V, exactly one of the* *following systems has a solution:*

$$\sum_{i=0}^{m} \lambda_i a^i = 0, \quad \sum_{i=0}^{m} \lambda_i = 1, \quad 0 \le \lambda_0, \lambda_1, \cdots, \lambda_m \in \mathbb{R}, \tag{1.52}$$

$$\langle a^i, x\rangle > 0 \quad \textit{for } i = 0, 1, \cdots, m, \quad x \in V. \tag{1.53}$$

Geometrically, Gordan's lemma says that the origin does not lie in the convex hull of the set $\{a^0, a^1, \cdots, a^m\}$ if and only if there is an open halfspace $\{y \in V : \langle y, x\rangle > 0\}$ which contains $\{a^0, a^1, \cdots, a^m\}$.

PROOF. Clearly, if (1.52) is solvable, then (1.53) has no solution. Conversely, we assume that (1.53) has no solution, and without loss of generality $V = \mathbb{R}^N$. Consider the function

$$f(x) := \log\biggl(\sum_{i=0}^{m} \exp(\langle a^i, x\rangle)\biggr), \quad x \in \mathbb{R}^N.$$

f is bounded below since $f(x) \ge \log\Bigl(\max_{0\le i\le m} \exp(\langle a^i, x\rangle)\Bigr) \ge 0$. Next, let $\varphi \in C^\infty(\mathbb{R}^N)$ be such that $\varphi(x) = 0$ for $\|x\| \le \frac{1}{2}$, and $\varphi(x) = 1$ for $\|x\| \ge 1$, and define

$$F_k(x) := f(x) + k^{-1}\varphi(kx)\|x\|$$

for $k = 1, 2, \cdots$. Since f is bounded below by zero, we have $\lim_{\|x\|\to\infty} F_k(x) = \infty$, which in turn implies that F_k attains a global minimum at some $x^k \in \mathbb{R}^N$, and therefore,

$$0 = \nabla F_k(x^k) = \nabla f(x^k) + \|x^k\|\nabla\varphi(kx^k) + k^{-1}\varphi(kx^k)x^k/\|x^k\|.$$

Since $\nabla\varphi$ is supported in $\{x : 1/2 \le \|x\| \le 1\}$, it follows that

$$|\nabla f(x^k)| = \left|\sum_{j=0}^{m} \lambda_j^k a^j\right| \le ck^{-1} \tag{1.54}$$

where $\lambda_j^k = \exp(\langle a^j, x^k\rangle)/\sum_{i=0}^m \exp(\langle a^i, x^k\rangle) > 0$, and $\sum_{j=0}^m \lambda_j^k = 1$. Since the sequence $\{(\lambda_0^k, \lambda_1^k, \cdots, \lambda_m^k) : \ k = 1, 2, \cdots\}$ is bounded in $\mathbb{R}^{m+1}$, according to the Weierstrass theorem, it has a convergent subsequence, and by (1.54), the limit of this convergent subsequence solves the system (1.52). □

LEMMA 5.3. *(Farkas). Let V be a finite-dimensional real Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Then for any points $a^1, a^2, \cdots, a^m$ and ζ in V, exactly one of the following systems has a solution:*

$$\sum_{j=1}^{m} \mu_j a^j = \zeta, \quad 0 \le \mu_1, \mu_2, \cdots, \mu_m \in \mathbb{R}, \tag{1.55}$$

$$\langle a^j, x\rangle \ge 0 \quad \textit{for } j = 1, 2, \cdots, m, \quad \langle\zeta, x\rangle < 0, \quad x \in V. \tag{1.56}$$

Geometrically, Farkas' lemma says that any point ζ not lying in the finitely generated cone $\mathcal{C} = \{\sum_{j=1}^m \mu_j a^j : \ 0 \le \mu_1, \mu_2, \cdots, \mu_m \in \mathbb{R}\}$ can be separated from $\mathcal{C}$ by a hyperplane.

PROOF. It is immediate that if (1.55) has a solution then (1.56) has no solution. Conversely, we assume that (1.56) has no solution, and deduce that (1.55) has a solution by using induction on m. Clearly, when $m = 0$, the assertion is trivial and there's nothing to prove. Suppose then that the result holds in any finite-dimensional real Hilbert space for any set of $m - 1$ elements and any element ζ. Define $a^0 = -\zeta$. The unsolvability of (1.56) then implies the unsolvability of (1.53). Hence, applying the Gordan lemma shows that there are nonnegative scalars

$\lambda_0, \cdots, \lambda_m$ in $\mathbb{R}$, not all zero, satisfying $\lambda_0 \zeta = \sum_{j=1}^m \lambda_j a^j$. If $\lambda_0 > 0$, the proof is complete, so suppose $\lambda_0 = 0$ and without loss of generality, $\lambda_m > 0$. Then

$$a^m = -\lambda_m^{-1} \sum_{j=1}^{m-1} \lambda_j a^j. \tag{1.57}$$

Define $Y = \{y \in V : \langle y, a^m \rangle = 0\}$, and let $P_Y : V \to Y$ denote the orthogonal projection onto Y. By assumption, the system

$$\langle a^j, y \rangle \geq 0 \quad \text{for } j = 1, 2, \cdots, m-1, \quad \langle \zeta, y \rangle < 0, \quad y \in Y$$

or equivalently,

$$\langle P_Y a^j, y \rangle \geq 0 \quad \text{for } j = 1, 2, \cdots, m-1, \quad \langle P_Y \zeta, y \rangle < 0, \quad y \in Y$$

has no solution. By the induction hypothesis applied to the subspace Y, there are nonnegative real numbers $\mu_1, \cdots, \mu_{m-1}$ satisfying $\sum_{j=1}^{m-1} \mu_j P_Y a^j = P_Y \zeta$. This means that $\zeta - \sum_{j=1}^{m-1} \mu_j a^j$ is orthogonal to the space Y, the orthogonal complement of span $\{a^m\}$ in V. Thus, there is a number $\mu_m \in \mathbb{R}$ such that

$$\mu_m a^m = \zeta - \sum_{j=1}^{m-1} \mu_k a^j. \tag{1.58}$$

If $\mu_m \geq 0$, we immediately obtain a solution of (1.55), whereas if $\mu_m < 0$, we can substitute (1.57) into (1.58) to obtain

$$(-\mu_m)\lambda_m^{-1} \sum_{j=1}^{m-1} \lambda_j a^j = \zeta - \sum_{j=1}^{m-1} \mu_j a^j.$$

which again gives a solution of (1.55). □

LEMMA 5.4. *If Λ is a maximal $\frac{\delta}{n}$-separated subset of $\mathbb{S}^{d-1}$ with $\delta \in (0, 1]$, and $f \in \Pi_n^d$ satisfies that $\min_{\omega \in \Lambda} f(\omega) \geq 0$, then*

$$\int_{\mathbb{S}^{d-1}} f(x) w(x)\, d\sigma(x) \geq (C_1^{-1} - C_2 K_1 \delta) \sum_{\omega \in \Lambda} f(\omega) w(\mathsf{c}(\omega, n^{-1}\delta)).$$

PROOF. Setting $N(x) := \sum_{\omega \in \Lambda} \chi_{\mathsf{c}(\omega, n^{-1}\delta)}(x)$, and using Lemma 4.2 (ii), we obtain

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} f(x) w(x)\, d\sigma(x) &= \sum_{\omega \in \Lambda} \int_{\mathsf{c}(\omega, n^{-1}\delta)} f(x) \frac{w(x)}{N(x)}\, d\sigma(x) \\
&\geq C_1^{-1} \sum_{\omega \in \Lambda} f(\omega) \int_{\mathsf{c}(\omega, n^{-1}\delta)} w(x)\, d\sigma(x) \\
&\quad - \sum_{\omega \in \Lambda} \int_{\mathsf{c}(\omega, n^{-1}\delta)} |f(x) - f(\omega)| w(x)\, d\sigma(x) \\
&\geq C_1^{-1} \sum_{\omega \in \Lambda} f(\omega) w(\mathsf{c}(\omega, n^{-1}\delta)) - K_1 \delta \|f\|_{1,w} \\
&\geq (C_1^{-1} - C_2 K_1 \delta) \sum_{\omega \in \Lambda} f(\omega) w(\mathsf{c}(\omega, n^{-1}\delta)),
\end{aligned}$$

where the last two steps use Theorem 4.3, and (1.50) with $p = 1$ respectively. □

LEMMA 5.5. *If μ is a nonnegative measure on $\mathbb{S}^{d-1}$ such that the equation*

$$\int_{\mathbb{S}^{d-1}} f(x)\, w(x)\, d\sigma(x) = \int_{\mathbb{S}^{d-1}} f(x)\ d\mu(x) \tag{1.59}$$

holds for all $f \in \Pi_n^d$ and some positive integer n, then for all $x \in \mathbb{S}^{d-1}$,

$$\mu\Big(\mathsf{c}\Big(x, \frac{2}{n}\Big)\Big) \le C w\Big(\mathsf{c}\Big(x, \frac{2}{n}\Big)\Big), \tag{1.60}$$

with the constant C depending only on L_w.

Proof. For $m = [\frac{d-1}{2} + \frac{s_w}{2}] + 1$, and $n_1 = \left[\frac{n}{2m}\right]$, we define

$$T_n(\cos\theta) = \gamma_n \left(\frac{\sin(n_1 + \frac{1}{2})\theta}{\sin\frac{\theta}{2}}\right)^{2m}, \tag{1.61}$$

where γ_n is chosen so that $\int_0^\pi T_n(\cos\theta)\sin^{d-2}\theta\, d\theta = 1$. A straightforward computation shows that $\gamma_n \sim n^{d-1-2m}$, which in turn implies that

$$0 \le T_n(\cos\theta) \le C_{L_w} n^{d-1}(1 + n\theta)^{-2m}, \quad \theta \in [0, 2\pi], \tag{1.62}$$

and

$$T_n(\cos\theta) \ge c n^{d-1}, \quad \theta \in \Big[0, \frac{2}{n}\Big]. \tag{1.63}$$

Since $T_n(\cos\theta)$ is a polynomial in $\cos\theta$ of degree at most n, it follows that $T_n(\langle x, \cdot\rangle)$ is a spherical polynomial of degree at most n for each fixed $x \in \mathbb{S}^{d-1}$. Thus, using (1.59), we have, for each $x \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}} T_n(\langle x, y\rangle)\, d\mu(y) = \int_{\mathbb{S}^{d-1}} T_n(\langle x, y\rangle) w(y)\, d\sigma(y).$$

However, using (1.63) and the positivity of T_n, we have

$$\int_{\mathbb{S}^{d-1}} T_n(\langle x, y\rangle)\, d\mu(y) \ge c n^{d-1} \mu\Big(\mathsf{c}\Big(x, \frac{2}{n}\Big)\Big),$$

whereas using Corollary 4.4, (1.62), and (1.12), we deduce

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} T_n(\langle x, y\rangle) w(y)\, d\sigma(y) &\le C \int_{\mathbb{S}^{d-1}} T_n(\langle x, y\rangle) w_n(y)\, d\sigma(y)\\ &\le C n^{d-1} w_n(x) \int_{\mathbb{S}^{d-1}} (1 + n d(x, y))^{s_w - 2m}\, d\sigma(y)\\ &\le C w_n(x).\end{aligned}$$

Putting these together, and using the doubling property of the weight w, we deduce the desired estimate (1.62). □

We are now in a position to prove Theorem 5.1.

Proof of Theorem 5.1. Without loss of generality, we may assume that $\frac{1}{4}\delta_0 \le \delta \le \delta_0$, since otherwise we may replace n by the largest integer $\le \frac{n\delta_0}{2\delta}$. We shall use Lemma 5.2 with V being the real Hilbert space Π_n^d endowed with the inner product

$$\langle f, g\rangle := \int_{\mathbb{S}^{d-1}} f(x) g(x)\, d\sigma(x), \quad f, g \in \Pi_n^d.$$

Define $G_n(x, y) := \frac{1}{\omega_d}\sum_{k=0}^n \frac{k+\lambda}{\lambda} C_k^\lambda(x \cdot y)$ for $x, y \in \mathbb{S}^{d-1}$, where $\lambda = \frac{d-2}{2}$, and $x \cdot y$ denotes the dot product of $x, y \in \mathbb{R}^d$. G_n is a reproducing kernel for the Hilbert space V in the sense that

$$\langle f, G_n(x, \cdot)\rangle = f(x), \quad x \in \mathbb{S}^{d-1}, \quad f \in V. \tag{1.64}$$

Next, we let $\omega_1, \cdots, \omega_N$ be the list of elements in the set Λ, and define the functions ζ and a^j in the space V as follows:

$$\begin{aligned}\zeta(x) &:= \int_{\mathbb{S}^{d-1}} G_n(x,y)w(y)\,d\sigma(y) - (2C_1)^{-1}\sum_{j=1}^N G_n(x,\omega_j)w(\mathsf{c}(\omega_j, n^{-1}\delta)),\\ a^j(x) &:= G_n(x,\omega_j), \quad j=1,2,\cdots,N.\end{aligned}$$

On one hand, using (1.64), we have, for all $f\in V$,

$$\langle f, a_j\rangle = f(\omega_j), \quad j=1,2,\cdots,N, \tag{1.65}$$

$$\langle f,\zeta\rangle = \int_{\mathbb{S}^{d-1}} f(y)w(y)\,d\sigma(y) - (2C_1)^{-1}\sum_{j=1}^N f(\omega_j)w(\mathsf{c}(\omega_j,n^{-1}\delta)). \tag{1.66}$$

On the other hand, however, using Lemma 5.4, we conclude that if $0<\delta\le \frac{1}{C_1(C_2K_1+1)}$, and $f\in V$ satisfies $\min_{1\le j\le N} f(\omega_j)\ge 0$, then

$$\langle f,\zeta\rangle \ge ((2C_1)^{-1} - C_2K_1\delta)\sum_{j=1}^N f(\omega_j)w(\mathsf{c}(\omega_j,n^{-1}\delta))\ge 0.$$

Thus, the system (1.56) is not solvable, and by Lemma 5.2 applied to the Hilbert space $V=\Pi_n^d$ and the functions $a^1,\cdots,a^N$ and ζ, there are $\mu_1,\cdots,\mu_m\ge 0$ such that $\zeta=\sum_{j=1}^N \mu_j a^j$. Using (1.65) and (1.66), this further implies that for $f\in\Pi_n^d$,

$$\int_{\mathbb{S}^{d-1}} f(y)w(y)\,d\sigma(y) - (2C_1)^{-1}\sum_{j=1}^N f(\omega_j)w(\mathsf{c}(\omega_j,n^{-1}\delta)) = \sum_{j=1}^N \mu_j f(\omega_j);$$

or equivalently,

$$\int_{\mathbb{S}^{d-1}} f(y)w(y)\,d\sigma(y) = \sum_{j=1}^N \lambda_j f(\omega_j) \tag{1.67}$$

with

$$\lambda_j := \mu_j + (2C_1)^{-1}w(\mathsf{c}(\omega_j,n^{-1}\delta)), \quad 1\le j\le N. \tag{1.68}$$

To complete the proof, we just note that the lower estimate $\lambda_j\ge C_{L_w}w(\mathsf{c}(\omega_j,n^{-1}\delta))$ is an immediate consequence of (1.68), whereas the desired upper estimate $\lambda_j\le C_{L_w}w(\mathsf{c}(\omega_j,n^{-1}\delta))$ follows by (1.67) and Lemma 5.5 applied to a finite measure μ supported on the finite set Λ and satisfying $\mu\{\omega_j\}=w(\mathsf{c}(\omega_j,n^{-1}\delta))$ for $1\le j\le N$.

Our last result in this section connects positive cubature formulas with MZ inequalities.

THEOREM 5.6. *If μ is a nonnegative finite measure on $\mathbb{S}^{d-1}$ for which the formula,*

$$\int_{\mathbb{S}^{d-1}} f(x)\,w(x)\,d\sigma(x) = \int_{\mathbb{S}^{d-1}} f(x)\;d\mu(x), \tag{1.69}$$

holds for all $f\in\Pi_{3n}^d$ and some positive integer n, then for all $0<p<\infty$ and $f\in\Pi_n^d$,

$$\|f\|_{p,w}\sim\Big(\int_{\mathbb{S}^{d-1}} |f(x)|^p\,d\mu(x)\Big)^{\frac1p}, \tag{1.70}$$

with the constants of equivalence depend only on the doubling constant of w, and the constant p when p is close to 0.

For the proof of Theorem 5.6, we need the following useful lemma.

LEMMA 5.7. *If $f : \mathbb{S}^{d-1} \to [0, \infty)$ is a nonnegative function satisfying*

$$f(x) \le C_f (1 + nd(x, y))^\alpha f(y), \qquad x, y \in \mathbb{S}^{d-1} \tag{1.71}$$

for some fixed positive integer n, and some nonnegative number α, then for each $0 < p < \infty$, there exists a nonnegative spherical polynomial $g \in \Pi_n^d$ such that

$$C^{-1} f(x)^{\frac{1}{p}} \le g(x) \le C f(x)^{\frac{1}{p}}, \quad \textit{for all } x \in \mathbb{S}^{d-1}, \tag{1.72}$$

where the constant C depends only on C_f, α and p when p is close to zero. If, in addition, $f(x) := F(\langle x, e\rangle)$ is a zonal function on $\mathbb{S}^{d-1}$ for some fixed $e \in \mathbb{S}^{d-1}$, then we may choose the function g in (1.72) to be a zonal polynomial of the form $G(\langle x, e\rangle)$ as well.

PROOF. As in the proof of Lemma 5.5, we define a function $T_n(\cos\theta)$ as in (1.61) with $m = [\alpha/p] + d + 1$, and $n_1 = \left[\frac{n}{2m}\right]$. Then $T_n(\cos\theta)$ a polynomial in $\cos\theta$ of degree at most n satisfying (1.62) and (1.63) with the constants depending only on α and p when p is close to zero. Next, we define

$$g(x) = \int_{\mathbb{S}^{d-1}} f(y)^{\frac{1}{p}} T_n(x \cdot y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{1.73}$$

and show the function g has the desired properties. Clearly, g is a nonnegative spherical polynomial of degree at most n, and if f is a zonal function, so is g. On the other hand, using (1.62) and (1.73), we obtain

$$g(x) \le C_f^{1/p} f(x)^{1/p} \int_{\mathbb{S}^{d-1}} (1 + nd(x, y))^{\alpha/p} T_n(x \cdot y)\, d\sigma(y) \le C f(x)^{1/p},$$

and using (1.63), we deduce

$$g(x) \ge \int_{d(x,y) \le \frac{1}{2n}} f(y)^{\frac{1}{p}} T_n(x \cdot y)\, d\sigma(y) \ge C f(x)^{1/p} \int_0^{\frac{1}{2n}} n^{d-1} \theta^{d-2}\, d\theta \ge C f(x)^{1/p}.$$

This shows that g satisfies (1.72), and completes the proof. $\square$

Now we are in a position to prove Theorem 5.6.

Proof of Theorem 5.6. We first note that the inequality

$$\int_{\mathbb{S}^{d-1}} |f(x)|^p\, d\mu(x) \le C^p \int_{\mathbb{S}^{d-1}} |f(x)|^p w(x)\, d\sigma(x), \quad f \in \Pi_{2n}^d \tag{1.74}$$

follows directly from Lemma 5.5 and Lemma 4.6. Thus, it remains to prove the inverse inequality

$$\int_{\mathbb{S}^{d-1}} |f(x)|^p w(x)\, d\sigma(x) \le C^p \int_{\mathbb{S}^{d-1}} |f(x)|^p\, d\mu(x). \tag{1.75}$$

Using (1.28) and Hölder's inequality, we obtain that for $x \in \mathbb{S}^{d-1}$ and any $f \in \Pi_n^d$,

$$|f(x)| \le C \bigg(\int_{\mathbb{S}^{d-1}} |f(y)|^2 |L_n(\langle x, y\rangle)|\, d\sigma(y) \bigg)^{\frac{1}{2}},$$

which, using (1.12), implies

$$\begin{aligned} &|f(x)|^p w_n(x) \\ &\le C \bigg(\int_{\mathbb{S}^{d-1}} |f(y)|^2 |L_n(\langle x, y\rangle)| (1 + nd(x, y))^{\frac{2}{p} s_w} (w_n(y))^{\frac{2}{p}}\, d\sigma(y) \bigg)^{\frac{p}{2}}. \end{aligned} \tag{1.76}$$

However, using Lemma 5.7 and (1.12), there exist a nonnegative spherical polynomial $Q_1 \in \Pi^d_{[n/2]}$ and a nonnegative zonal spherical polynomial $Q_2(\langle x, y\rangle) \in \Pi^d_{[n/2]}$ such that

$$Q_1(y) \sim \Big(w_n(y)\Big)^{\frac{2}{p}-1}, \quad Q_2(\langle x, y\rangle) \sim n^{d-1}(1+nd(y,x))^{-\ell}, \tag{1.77}$$

where $\ell > (d-1)\max\{\frac{2}{p}, 1\}$ is a fixed integer. Thus, using (1.76) and Theorem 2.1, we deduce

$$\begin{aligned} |f(x)|^p w_n(x) &\le C\Big(\int_{\mathbb{S}^{d-1}} |f(y)|^2 Q_2(x\cdot y) Q_1(y) w_n(y)\, d\sigma(y)\Big)^{\frac{p}{2}} \\ &\le C\Big(\int_{\mathbb{S}^{d-1}} |f(y)|^2 Q_2(x\cdot y) Q_1(y) w(y)\, d\sigma(y)\Big)^{\frac{p}{2}} \\ &= C\Big(\int_{\mathbb{S}^{d-1}} |f(y)|^2 Q_2(x\cdot y) Q_1(y)\, d\mu(y)\Big)^{\frac{p}{2}}, \end{aligned} \tag{1.78}$$

where the last two steps uses Corollary 4.4 and (1.69) respectively.

Now we deduce (1.75) for the case of $0 < p \le 2$ from (1.78). Taking a maximal $\frac{1}{n}$-separated subset Λ of $\mathbb{S}^{d-1}$, and using (1.78), we obtain

$$\begin{aligned} &|f(x)|^p w_n(x) \\ &\quad \le C\sum_{\omega\in\Lambda}\Big|\int_{\mathsf{c}(\omega,\frac{1}{n})} |f(y)|^2 Q_2(\langle x, y\rangle) Q_1(y)\, d\mu(y)\Big|^{\frac{p}{2}} \\ &\quad \le C\sum_{\omega\in\Lambda}\Big(f^*_{2/p,n}(\omega)\Big)^{(2-p)\frac{p}{2}} (Q_2(x\cdot\omega))^{\frac{p}{2}} (w_n(\omega))^{1-\frac{p}{2}} \Big(\int_{\mathsf{c}(\omega,\frac{1}{n})} |f(y)|^p\, d\mu(y)\Big)^{\frac{p}{2}}. \end{aligned}$$

Integrating with respect to $x \in \mathbb{S}^{d-1}$, and then using Hölder's inequality, we obtain

$$\begin{aligned} &\|f\|^p_{p,w} \le C\|f\|^p_{p,w_n} \\ &\le Cn^{(d-1)(\frac{p}{2}-1)} \sum_{\omega\in\Lambda}\Big(f^*_{2/p,n}(\omega)\Big)^{(2-p)\frac{p}{2}} (w_n(\omega))^{1-\frac{p}{2}} \Big(\int_{\mathsf{c}(\omega,\frac{1}{n})} |f(y)|^p\, d\mu(y)\Big)^{\frac{p}{2}} \\ &\le C\Big(\int_{\mathbb{S}^{d-1}} |f(y)|^p\, d\mu(y)\Big)^{\frac{p}{2}} \Big(\sum_{\omega\in\Lambda}\int_{\mathsf{c}(\omega,\frac{1}{n})} |f^*_{2/p,n}(y)|^p w_n(y)\, d\sigma(y)\Big)^{1-\frac{p}{2}} \\ &\le C\Big(\int_{\mathbb{S}^{d-1}} |f(y)|^p\, d\mu(y)\Big)^{\frac{p}{2}} \|f^*_{2/p,n}\|^{p(1-\frac{p}{2})}_{p,w_n} \le C\Big(\int_{\mathbb{S}^{d-1}} |f(y)|^p\, d\mu(y)\Big)^{\frac{p}{2}} \|f\|^{p(1-\frac{p}{2})}_{p,w}, \end{aligned}$$

where the last step uses Corollaries 3.3 and 4.4. The desired inequality (1.75) in the case of $0 < p \le 2$ then follows.

Finally, we show (1.75) for $2 < p < \infty$. In this case, using (1.78) and Hölder's inequality, we obtain

$$\begin{aligned} &|f(x)|^p w_n(x) \\ &\le C\Big(\int_{\mathbb{S}^{d-1}} |f(y)|^p Q_2(\langle x, y\rangle)\, d\mu(y)\Big)\Big(\int_{\mathbb{S}^{d-1}} Q_2(\langle x, y\rangle)|Q_1(y)|^{\frac{p}{p-2}}\, d\mu(y)\Big)^{\frac{p}{2}-1}. \end{aligned} \tag{1.79}$$

On the other hand, however, using Lemma 5.7 and (1.77), there exists a nonnegative spherical polynomial $Q_3 \in \Pi^d_n$ such that

$$Q_3(y) \sim Q_1(y)^{\frac{p}{p-2}} \sim w_n(y)^{-1}, \quad \text{for all } y \in \mathbb{S}^{d-1}.$$

Thus, using (1.74) and (1.77), we deduce

$$\begin{aligned}\Big(\int_{\mathbb{S}^{d-1}} Q_2(\langle x,y\rangle)|Q_1(y)|^{\frac{p}{p-2}}\,d\mu(y)\Big)^{\frac{p}{2}-1}&\\ \le C\Big(\int_{\mathbb{S}^{d-1}} Q_2(\langle x,y\rangle)Q_3(y)w_n(y)\,d\sigma(y)\Big)^{\frac{p}{2}-1}&\\ \le C\Big(\int_{\mathbb{S}^{d-1}} Q_2(\langle x,y\rangle)\,d\sigma(y)\Big)^{\frac{p}{2}-1}\le C.& \qquad (1.80)\end{aligned}$$

Now combining (1.79) with (1.80), integrating with respect to x over $\mathbb{S}^{d-1}$, and using (1.77), we conclude that

$$\|f\|_{p,w}^p\le C\|f\|_{p,w_n}^p\le C\int_{\mathbb{S}^{d-1}}|f(y)|^p\,d\mu(y),$$

which proves (1.75) for $2<p<\infty$.

6. Nikolskii and Bernstein inequalities

Throughout this section, w denotes a doubling weight on $\mathbb{S}^{d-1}$ normalized by $w(\mathbb{S}^{d-1})=1$. We start with the following weighted Nikolskii inequality:

Theorem 6.1. *(Nikolskii's inequality) If $0<p<q\le\infty$, and $f\in\Pi_n^d$, then*

$$\|f\|_{q,w}\le Cn^{(\frac{1}{p}-\frac{1}{q})s_w}\|f\|_{p,w},$$

where C depends only on d, p, q, and L_w.

Proof. Let us first consider the case $0<p<q=\infty$. Let Λ be a maximal $\frac{1}{12K_pn}$-separated subset of $\mathbb{S}^{d-1}$ with K_p being the same constant as in Theorem 4.1. Using Theorem 4.1, we obtain that for $f\in\Pi_n^d$,

$$\begin{aligned}\|f\|_\infty&\le C\max_{\omega\in\Lambda}|f(\omega)|\le C\Big(\min_{\omega\in\Lambda}\lambda_\omega\Big)^{-\frac{1}{p}}\Big(\sum_{\omega\in\Lambda}\lambda_\omega|f(\omega)|^p\Big)^{\frac{1}{p}}\\ &\le C\|f\|_{p,w}\max_{\omega\in\Lambda}\Big(w(B(\omega,n^{-1}))\Big)^{-\frac{1}{p}}. \qquad (1.81)\end{aligned}$$

Let m be a positive integer such that $2^{m-1}\le n\pi\le 2^m$. Then using (1.12), we have, for any $\omega\in\Lambda$,

$$\begin{aligned}1=w(\mathbb{S}^{d-1})&=w(B(\omega,\pi))\le C_{L_w}2^{ms_w}w(B(\omega,n^{-1}))\\ &\le C_{L_w}(2\pi n)^{s_w}w(B(\omega,n^{-1})),\end{aligned}$$

which implies that $(w(B(\omega,n^{-1})))^{-\frac{1}{p}}\le cn^{s_w/p}$. Inserting this last estimate into (1.81) then gives the desired Nikolskii's inequality for the case of $0<p<q=\infty$:

$$\|f\|_\infty\le Cn^{s_w/p}\|f\|_{p,w}. \qquad (1.82)$$

Finally, for $0<p<q<\infty$, using (1.82), one has

$$\|f\|_{q,w}^q\le\|f\|_\infty^{q-p}\|f\|_{p,w}^p\le Cn^{s_w(q-p)/p}\|f\|_{p,w}^q,$$

which in turn implies the desired inequality $\|f\|_{q,w}\le Cn^{(\frac{1}{p}-\frac{1}{q})s_w}\|f\|_{p,w}$. □

The next goal in this section is to show a weighted Bernstein inequality. Given $f\in C^\ell(\mathbb{S}^{d-1})$, we define its ℓth tangential derivative in the direction of $\xi\in T_x:=\{y\in\mathbb{S}^{d-1}:\langle x,y\rangle=0\}$ at a point $x\in\mathbb{S}^{d-1}$ by

$$\Big(\frac{\partial}{\partial\xi}\Big)^\ell f(x)=\Big(\frac{\partial}{\partial\theta}\Big)^\ell\Big(f(x\cos\theta+\xi\sin\theta)\Big)\Big|_{\theta=0}.$$

Then the weighted Bernstein inequality can be sated as follows.

THEOREM 6.2. *(Bernstein's inequality.) If $\ell \in \mathbb{N}$, $0 < p < \infty$, and $f \in \Pi_n^d$, then*

$$\left(\int_{\mathbb{S}^{d-1}} \sup_{\xi \in T_x} \left| \left(\frac{\partial}{\partial \xi} \right)^{\ell} f(x) \right|^p w(x)\, d\sigma(x) \right)^{\frac{1}{p}} \le C n^{\ell} \|f\|_{p,w}$$

where $C > 0$ depends only on d, ℓ, the doubling constant of w and p when p is small.

Recall that $L_n(x, y)$ denotes the kernel defined in (1.27) with $\eta \in C^{\infty}[0, \infty$ satisfying $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ for $|x| \ge 2$. To show Theorem 6.2, we need the following estimate on the tangential derivative of L_n.

LEMMA 6.3. *If $x \in \mathbb{S}^{d-1}$, $y \in T_x$ and*

$$\varphi(\theta) \equiv \varphi_{x,y,\xi}(\theta) = K_n(x \cdot y \cos\theta + \xi \cdot y \sin\theta),$$

then for any $v, m \in \mathbb{N}$,

$$|\varphi^{(v)}(0)| \le C n^{d-1+v} \min\{1, (nd(x, y))^{-m}\}, \tag{1.83}$$

with C depending only on v and m.

PROOF. Using induction on v, we can write $\varphi^{(v)}(\theta)$ in the form

$$\varphi^{(v)}(\theta) = \sum_{i=1}^{v} \sum_{(j_0,j_1,j_2,j_3)\in\Lambda} C_{j_0,j_1,j_2,j_3} L_n^{(i)}\Big(t(\theta)\Big)\Big(t(\theta)\Big)^{j_0}\Big(t'(\theta)\Big)^{j_1}\Big(t''(\theta)\Big)^{j_2}\Big(t'''(\theta)\Big)^{j_3}, \tag{1.84}$$

where $\Lambda := \{(j_0, j_1, j_2, j_3) \in \mathbb{Z}_+^4 : \ j_0 + j_1 + j_2 + j_3 = i, \ j_1 + j_3 \ge 2i - v\}$, $t(\theta) = x \cdot y \cos\theta + y \cdot \xi \sin\theta$, and C_{j_0,j_1,j_2,j_3} are some absolute constants. Note that $|y \cdot \xi| \le \sqrt{1 - (x \cdot y)^2}$ for any $y \in \mathbb{S}^{d-1}$ and $\xi \in T_x$. Thus, using Theorem 2.1 with $\ell = m + v$, we deduce that for $1 \le i \le v$ and each $(j_0, j_1, j_2, j_3) \in \Lambda$,

$$\left| L_n^{(i)}\Big(t(0)\Big)\Big(t(0)\Big)^{j_0}\Big(t'(0)\Big)^{j_1}\Big(t''(0)\Big)^{j_2}\Big(t'''(0)\Big)^{j_3} \right| \le C n^{d-1+v} \min\{1, (nd(x, y))^{-m}\}.$$

The desired inequality (1.83) then follows by (1.84). This completes the proof. $\square$

Proof of Theorem 6.2. By the definition and (1.28), we have, for $f \in \Pi_n^d$ and $x, \xi \in \mathbb{S}^{d-1}$ with $x \cdot \xi = 0$,

$$\left(\frac{\partial}{\partial \xi} \right)^{\ell} f(x) = \int_{\mathbb{S}^{d-1}} f(y) \varphi_{x,y,\xi}^{(\ell)}(0)\, d\sigma(y)$$

with $\varphi_{x,y,\xi}$ as defined in Lemma 5.3. It then follows by Lemma 6.3 with $m > \frac{2}{p} s_w + d - 1$ that

$$\begin{aligned}
\left| \left(\frac{\partial}{\partial \xi} \right)^{\ell} f(x) \right| &\le C n^{d-1+\ell} \int_{\mathbb{S}^{d-1}} |f(y)| (1 + nd(x, y))^{-m}\, d\sigma(y) \\
&\le C n^{d-1+\ell} f_{2s_w/p,n}^{*}(x) \int_{\mathbb{S}^{d-1}} (1 + nd(x, y))^{-m+\frac{2}{p}s_w}\, d\sigma(y) \\
&\le C n^{\ell} f_{2s_w/p,n}^{*}(x).
\end{aligned}$$

This combined with Corollary 3.3 gives the desired Bernstein's inequality and therefore completes the proof.

Note that

$$D_{i,j} := x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} = \frac{\partial}{\partial \xi_{i,j}} f(x)$$

for $f \in C^1(\mathbb{S}^{d-1})$ and $1 \le i \neq j \le d$, where $\xi_{i,j} = x + (x_j - x_i)e_i + (x_i - x_j)e_j \in T_x$, and

$$e_1 = (1, 0, \cdots, 0), e_2 = (0, 1, 0, \dots, 0) \cdots, e_d = (0, \cdots, 0, 1).$$

Thus, as an immediate consequence of Theorem 6.2, we deduce the following.

COROLLARY 6.4. *If $f \in \Pi_n^d$, $\ell \in \mathbb{N}$, and $0 < p < \infty$, then*

$$\max_{1 \le i < j \le d} \|D_{i,j}^\ell f\|_{p,w} \le C n^\ell \|f\|_{p,w},$$

where C depends on L_w, but is independent of f, n and p when p is bounded away from zero.

Recalling that $\Delta_0 = \sum_{1 \le i < j \le d} D_{i,j}^2$ for the Laplace-Beltrami operator Δ_0 on $\mathbb{S}^{d-1}$, we then deduce from (6.4) the following weighted inequality:

COROLLARY 6.5. *If $\ell \in \mathbb{N}$, $0 < p < \infty$, and $f \in \Pi_n^d$, then*

$$\|\triangle_0^\ell f\|_{p,w} \le C n^{2\ell} \|f\|_{p,w},$$

where $\triangle_0^{i+1} f = \triangle_0(\triangle_0^i f)$ for $i \ge 1$, $C > 0$ depends only on ℓ, L_w and p when p is close to zero.

7. Remez-type inequalities with A_∞ weights

For convenience, throughout this section, we normalize the Lebesgue measure $d\sigma(x)$ on $\mathbb{S}^{d-1}$ by $\int_{\mathbb{S}^{d-1}} d\sigma(x) = 1$. Our main goal in this section is to show the following Remez-type inequality:

THEOREM 7.1. *Let w be an A_∞ weight on $\mathbb{S}^{d-1}$, and let $0 < p < \infty$. If $f \in \Pi_n^d$, $E \subset \mathbb{S}^{d-1}$, and $\operatorname{meas}(E) = t^{d-1} \le \frac{1}{2}$, then*

$$\int_{\mathbb{S}^{d-1}} |f(x)|^p w(x)\, d\sigma(x) \le C^{nt+1} \int_{\mathbb{S}^{d-1} \setminus E} |f(x)|^p w(x)\, d\sigma(x),$$

where $C > 0$ depends only on d, p and the A_∞ constant of w.

PROOF. Firstly, we show that for any $f \in \Pi_n^d$,

$$\|f\|_{C(\mathbb{S}^{d-1})} \le C^{nt} \sup_{x \in \mathbb{S}^{d-1} \setminus E} |f(x)|, \tag{1.85}$$

where $E \subset \mathbb{S}^{d-1}$ and $\operatorname{meas}(E) = t^{d-1} \le \frac{4}{5}$. Let $x_0 \in \mathbb{S}^{d-1}$ be such that $|f(x_0)| = \|f\|_{C(\mathbb{S}^{d-1})}$. We denote by $C(x_0, y)$ the great circle on $\mathbb{S}^{d-1}$ passing through x_0 and $y \in \mathbb{S}^{d-1} \setminus \{x_0\}$, and by $d\gamma_{x_0,y}$ the one-dimensional Lebesgue measure on $C(x_0, y)$ normalized by $\gamma_{x_0,y}(C(x_0, y)) = 2\pi$. We claim that if $E \subset \mathbb{S}^{d-1}$, and $\operatorname{meas}(E) = t^{d-1} \le \frac{4}{5}$, then there must exist a point $y_0 \in \mathbb{S}^{d-1} \setminus \{x_0\}$ such that

$$\gamma_{x_0,y_0}\big(E \bigcap C(x_0, y_0)\big) \le \min\big\{C_d t, \quad 2\pi - \varepsilon_d\big\}, \tag{1.86}$$

where $C_d > 0$ and $\varepsilon_d \in (0, \pi)$ denote two constants depending only on d. Indeed, once (1.86) is proved, then

$$\begin{aligned} \|f\|_{C(\mathbb{S}^{d-1})} = |f(x_0)| &= \max_{y \in C(x_0, y_0)} |f(y)| \\ &\le C^{nt} \max_{y \in C(x_0, y_0) \setminus E} |f(y)| \le C^{nt} \sup_{x \in \mathbb{S}^{d-1} \setminus E} |f(x)|, \end{aligned}$$

where the third step uses (1.86), the Remez-type inequality for trigonometric polynomials on the circle ([**40**]), and the fact that the restriction of $f \in \Pi_n^d$ to the great circle $C(x_0, y_0)$ is a trigonometric polynomial of degree at most n. Thus, (1.85) will follow from (1.86).

For the proof of (1.86), we set $E^c = \mathbb{S}^{d-1} \backslash E$ and

$$S(x_0) = \{y \in \mathbb{S}^{d-1} : \quad \langle y, x_0 \rangle = 0\}.$$

We denote by $d\sigma_{x_0}(y)$ the Lebesgue measure on $S(x_0)$ normalized by $\int_{S(x_0)} d\sigma_{x_0}(y) = 1$. We then assert that (1.86) is a consequence of the following two inequalities:

$$\int_{S(x_0)} \Big(\gamma_{x_0,y}\big(E \cap C(x_0, y)\big)\Big)^{d-1} d\sigma_{x_0}(y) \le C_d' t^{d-1}, \tag{1.87}$$

and

$$\int_{S(x_0)} \gamma_{x_0,y}\big(E^c \cap C(x_0, y)\big)\, d\sigma_{x_0}(y) \ge \varepsilon_d' > 0. \tag{1.88}$$

Indeed, using (1.87),

$$\sigma_{x_0}\Big\{y \in S(x_0) : \ \gamma_{x_0,y}\Big(E \cap C(x_0, y)\Big) > C_d t\Big\} \le \frac{C_d'}{C_d^{d-1}},$$

whereas using (1.88),

$$\begin{aligned}
&\sigma_{x_0}\Big\{y \in S(x_0) : \ \gamma_{x_0,y}(E^c \cap C(x_0, y)) \le \varepsilon_d\Big\} \\
&= \sigma_{x_0}\Big\{y \in S(x_0) : \quad \gamma_{x_0,y}\Big(E \cap C(x_0, y)\Big) \ge 1 - \varepsilon_d\Big\} \\
&\le \frac{1-\varepsilon_d'}{1-\varepsilon_d}.
\end{aligned}$$

Thus, letting $\varepsilon_d = \frac{1}{2}\varepsilon_d'$, and choosing C_d sufficiently large so that

$$\frac{C_d'}{C_d^{d-1}} + \frac{1-\varepsilon_d'}{1-\varepsilon_d} < 1,$$

we conclude that there must exist a $y_0 \in S(x_0)$ such that

$$\gamma_{x_0,y_0}\big(E \cap C(x_0, y_0)\big) \le C_d t,$$

and

$$\gamma_{x_0,y_0}\big(E^c \cap C(x_0, y_0)\big) = 1 - \gamma_{x_0,y_0}\big(E \cap C(x_0, y_0)\big) \ge \varepsilon_d > 0.$$

The desired inequality (1.86) then follows.

Thus, we have reduced the proof of (1.86) to showing (1.87) and (1.88).

To show (1.87), we note that

$$\gamma_{x_0,y}(E \cap C(x_0, y)) = \int_{-\pi}^{\pi} \chi_{_E}(x_0 \cos\theta + y \sin\theta)\, d\theta.$$

Hence, setting

$$E(x_0, y) = \Big\{\theta \in [-\pi, \pi] : \quad |\sin\theta| \ge \sin\big[8^{-1}\gamma_{x_0,y}\big(C(x_0, y) \cap E\big)\big]\Big\},$$

we deduce

$$\begin{aligned}
t^{d-1} = \operatorname{meas}(E) &= C_d'' \int_{S(x_0)} \int_{-\pi}^{\pi} \chi_{_E}(x_0 \cos\theta + y \sin\theta) |\sin^{d-2}\theta|\, d\theta\, d\sigma_{x_0}(y) \\
&\ge C_d'' \int_{S(x_0)} \int_{E(x_0,y)} \chi_{_E}(x_0 \cos\theta + y \sin\theta) |\sin^{d-2}\theta|\, d\theta\, d\sigma_{x_0}(y) \\
&\ge \frac{1}{2} C_d'' \int_{S(x_0)} \Big(\sin\Big(\frac{\gamma_{x_0,y}\big(E \cap C(x_0, y)\big)}{8}\Big)\Big)^{d-2} \gamma_{x_0,y}\big(E \cap C(x_0, y)\big)\, d\sigma_{x_0}(y),
\end{aligned}$$

which implies (1.87). To show (1.88), we recall that $\sigma(\mathbb{S}^{d-1}) = 1$, and obtain

$$\frac{1}{5} \le \text{meas}(E^c) = C_d'' \int_{S(x_0)} \int_{-\pi}^{\pi} \chi_{E^c}(x_0 \cos\theta + y \sin\theta) |\sin^{d-2}\theta| \, d\theta \, d\sigma_{x_0}(y)$$
$$\le C_d'' \int_{S(x_0)} \gamma_{x_0,y}\big(E^c \bigcap C(x_0, y)\big) \, d\sigma_{x_0}(y).$$

This proves (1.88).

Next, we show that for $0 < p < \infty$ and $f \in \Pi_n^d$,

$$\int_{\mathbb{S}^{d-1}} |f(x)|^p \, d\sigma(x) \le C^{nt+1} \int_{\mathbb{S}^{d-1} \setminus E} |f(x)|^p \, d\sigma(x), \tag{1.89}$$

where $E \subset \mathbb{S}^{d-1}$ and $\text{meas}(E) = t^{d-1} \le \frac{3}{4}$. Set $\alpha = \left(\frac{16}{15}\right)^{\frac{1}{d-1}}$, and

$$F = \Big\{x \in \mathbb{S}^{d-1} : \quad |f(x)| \ge \|f\|_\infty C^{-\alpha nt}\Big\},$$

with $C > 0$ being the same as in (1.85). Then by the already proven inequality (1.85) it follows that $\text{meas}(F) \ge (\alpha t)^{d-1}$ and hence $\text{meas}(F \cap E^c) \ge \frac{1}{15} \text{meas}(E)$. Therefore, we have

$$\int_E |f(x)|^p \, d\sigma(x) \le |E| \|f\|_\infty^p \le 15 C^{\alpha ntp} \int_{E^c \cap F} |f(x)|^p \, d\sigma(x)$$
$$\le 15 C^{\alpha pnt} \int_{\mathbb{S}^{d-1} \setminus E} |f(x)|^p \, d\sigma(x)$$

and (1.89) then follows.

Finally, we show that for any A_∞ weight w, $0 < p < \infty$ and all $f \in \Pi_n^d$,

$$\int_{\mathbb{S}^{d-1}} |f(x)|^p w(x) \, d\sigma(x) \le C^{nt+1} \int_{\mathbb{S}^{d-1} \setminus E} |f(x)|^p w(x) \, d\sigma(x), \tag{1.90}$$

where $\text{meas}(E) = t^{d-1} \le \frac{1}{2}$. Let $\{\omega_i\}_{i=1}^{M(n,\delta)}$ be a maximal $\frac{\delta}{n}$-separated subset of $\mathbb{S}^{d-1}$ with $\delta > 0$ to be specified later, and let

$$B_1^* = B\Big(\omega_1, \frac{\delta}{n}\Big) - \bigcup_{j=2}^{M(n,\delta)} B\Big(\omega_j, \frac{\delta}{4n}\Big)$$

and

$$B_i^* = B\Big(\omega_i, \frac{\delta}{n}\Big) - \left[\Big(\bigcup_{k=1}^{i-1} B_k^*\Big) \bigcup \Big(\bigcup_{j=i+1}^{M(n,\delta)} B\Big(\omega_j, \frac{\delta}{4n}\Big)\Big)\right], \quad 2 \le i \le M(n, \delta).$$

Then the following properties can be easily verified:

$$B_i^* \bigcap B_j^* = \emptyset \quad \text{if } i \ne j;$$
$$B\Big(\omega_i, \frac{\delta}{4n}\Big) \subset B_i^* \subset B\Big(\omega_i, \frac{\delta}{n}\Big), \quad \text{for} \quad 1 \le i \le M(n, \delta);$$
$$\bigcup_{i=1}^{M(n,\delta)} B_i^* = \mathbb{S}^{d-1}.$$

Now setting

$$\Lambda^* = \Big\{i : \quad 1 \le i \le M(n, \delta), \quad \Big|B_i^* \bigcap E\Big| > \frac{2}{3}|B_i^*|\Big\},$$

we have

$$\sum_{i \in \Lambda^*} |B_i^*| \le \frac{3}{2}|E| = \frac{3}{2} t^{d-1},$$

and hence, using Theorem 4.4, Lemma 5.7 and the already proven inequality (1.89), we obtain

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} |f(x)|^p w(x)\,d\sigma(x) &\le C^{nt+1}\int_{\mathbb{S}^{d-1}\setminus \bigcup_{i\in\Lambda^*} B_i^*} |f(x)|^p w_n(x)\,d\sigma(x)\\ &\le C^{nt+1}\sum_{i\notin\Lambda^*}\int_{B(\omega_i,\frac{\delta}{n})}|f(x)|^p w_n(x)\,d\sigma(x)\\ &\le C^{nt+1}\sum_{i\notin\Lambda^*}|f(\xi_i)|^p\int_{B(\omega_i,\frac{\delta}{n})} w_n(x)\,d\sigma(x)\\ &\quad + C^{nt+1}\sum_{i\notin\Lambda^*}|\mathrm{osc}(f)(\omega_i)|^p\int_{B(\omega_i,\frac{\delta}{n})} w_n(x)\,d\sigma(x),\end{aligned}$$

where

$$|f(\xi_i)| = \min_{x\in B(\omega_i,\frac{\delta}{n})}|f(x)|$$

and

$$\mathrm{osc}(f)(\omega_i) = \max_{x,y\in B(\omega_i,\frac{\delta}{n})}|f(x)-f(y)|.$$

Since w_n is a doubling weight with the doubling constant depending only on that of w, it follows by Theorems 4.3 and 4.4 that

$$\begin{aligned}\sum_{i\notin\Lambda^*}|\mathrm{osc}(f)(\omega_i)|^p\int_{B(\omega_i,\frac{\delta}{n})} w_n(x)\,d\sigma(x) &\le C\delta^p\int_{\mathbb{S}^{d-1}}|f(x)|^p w_n(x)\,d\sigma(x)\\ &\le C\delta^p\int_{\mathbb{S}^{d-1}}|f(x)|^p w(x)\,d\sigma(x).\end{aligned}$$

On the other hand, however, by the A_∞-property of w it follows that for $i\notin\Lambda^*$,

$$\begin{aligned}|f(\xi_i)|^p\int_{B(\omega_i,\frac{\delta}{n})} w_n(x)\,d\sigma(x) &\le C\delta^{d-1}|f(\xi_i)|^p\int_{B(\omega_i,\frac{1}{n})} w(x)\,d\sigma(x)\\ &\le C\delta^{d-1-s}|f(\xi_i)|^p\int_{B(\omega_i,\frac{\delta}{n})} w(x)\,d\sigma(x)\\ &\le C\delta^{d-1-s}|f(\xi_i)|^p\int_{B(\omega_i,\frac{\delta}{n})\setminus E} w(x)\,d\sigma(x)\\ &\le C\delta^{d-1-s}\int_{B(\omega_i,\frac{\delta}{n})\setminus E}|f(x)|^p w(x)\,d\sigma(x),\end{aligned}$$

where in the first inequality we have used the fact that $w_n(x)\sim w_n(\omega_i)$ for $x\in B(\omega_i,\frac{\delta}{n})$, in the second inequality we have used the doubling property of w, and in the third inequality we have used the definition of Λ^* and the A_∞ property of w.

Therefore, noticing that $\sum_{i=1}^{M(n,\delta)}\chi_{B(\omega_i,\frac{\delta}{n})}(x)\le C_d$, we deduce

$$\begin{aligned}\int_{\mathbb{S}^{d-1}}|f(x)|^p w(x)\,d\sigma(x) &\le C^{nt+1}\delta^{d-1-s}\int_{\mathbb{S}^{d-1}\setminus E}|f(x)|^p w(x)\,d\sigma(x)\\ &\quad+\delta^p C^{nt+1}\int_{\mathbb{S}^{d-1}}|f(x)|^p w(x)\,d\sigma(x).\end{aligned}$$

Now letting $\delta = (\frac{1}{2})^{\frac{1}{p}}C^{-(nt+1)\frac{1}{p}}$, we obtain the desired inequality (1.90) and therefore complete the proof. $\square$

8. Notes and further results

1. Theorem 2.1, the fast decaying of the kernel, was established in [**10, 55, 64, 70**]. The proof of Proposition 2.2 follows along [**50**] and [**10**]. Under additional assumptions on the cut-off function, the rate of decay can be improved to sub-exponential estimate [**49**],

$$\left|L_n^{(j)}(\cos\theta)\right| \le c_1 n^{2\alpha+2j+2} \exp\left\{-\frac{c_2 n\theta}{[\ln(e+n\theta)]^{1+\epsilon}}\right\}, \quad 0 \le \theta \le \pi, \tag{1.91}$$

where $c_2 = c'\epsilon$ with $c' > 0$ an absolute constant and $c_1 = c'' 8^j$ with $c'' > 0$ depending only on α, β, and ϵ.

2. In one-dimensional case, various important, weighted polynomial inequalities, such as Bernstein, Marcinkiewicz-Zygmund, Nikolskii, Schur, Remez, etc, have been proved under the doubling condition or the slightly stronger A_∞ condition on the weights in the pioneering work of Mastroianni and Totik [**59**]. Most of these weighted inequalities of [**59**] hold for $0 < p < 1$ as well, as observed by Erdélyi [**41**]. Weighted Markov-Berntein-type inequalities for trigonometric polynomials with respect to doubling weights on a finite interval were established in [**42**].
3. A good reference for polynomial inequalities is [**8**]. For polynomial approximation with doubling weights, we refer to [**58**], [**60**] and [**13**]. For orthogonal polynomials with doubling weights, we refer to [**61**].
4. For positive cubature formulas and MZ inequalities on unweighted sphere $\mathbb{S}^{d-1}$, we refer to [**56, 64, 10**]. For local MZ inequalities and cubature formulas on the sphere, we refer to [**57, 21**].
5. Most of the weighted results in the third, the fourth, the sixth and the seventh sections were proved in [**12**]. In the unweighted case, the maximal function in the second section was introduced and studied in [**14**].

6 The proof of the Remez inequality (Theorem 7.1) follows the ideas of [**59**], and [**40**]. It was shown by Mastroianni and Totik [**59**] that weighted Remez type inequality, in general, does not hold for general doubling weights.

7. Weighted polynomial inequalities on the unit ball $\mathbb{B}^d := \{x \in \mathbb{R}^d : \ \|x\| \le 1\}$ can be deduced from the corresponding inequalities on the sphere (see [**12**]). To be precise, let ρ denote the following metric defined on $\mathbb{B}^d$:

$$\rho(x,y) = \sqrt{\|x-y\|^2 + (\sqrt{1-\|x\|^2} - \sqrt{1-\|y\|^2})^2}, \quad \text{for } x, y \in \mathbb{B}^d.$$

Set

$$B_\rho(x,r) = \{y \in \mathbb{B}^d : \ \ \rho(x,y) \le r\}, \quad r > 0, \quad x \in \mathbb{B}^d.$$

A weight function w on $\mathbb{B}^d$ is said to be a doubling weight if

$$\int_{B_\rho(x,2r)} w(y)\,dy \le L \int_{B_\rho(x,r)} w(y)\,dy, \quad x \in \mathbb{B}^d, \quad r > 0.$$

An A_∞-weight on $\mathbb{B}^d$ can be defined likewise. All the classical weights of the form

$$w_\alpha(x) = |x_1|^{\alpha_1} \cdots |x_d|^{\alpha_d} (1-|x|^2)^{\alpha_{d+1}-\frac{1}{2}}, \quad x \in \mathbb{B}^d \tag{1.92}$$

with $\alpha = (\alpha_1, \cdots, \alpha_d, \alpha_{d+1})$, $\alpha_i \ge 0$, $1 \le i \le d+1$, have the A_∞-property on $\mathbb{B}^d$. Connections between doubling weights (resp. A_∞ weights) on the unit ball and doubling weights (resp. A_∞ weights) on the sphere can be found in [**22**]. We denote by $\Pi_n(\mathbb{B}^d)$ the space of all real algebraic polynomials on $\mathbb{B}^d$ of degree $\le n$. The following weighted inequalities were proved in [**12**]:

THEOREM 8.1. *(MZ inequality) Given a doubling weight w on $\mathbb{B}^d$, there exists a positive constant $\gamma \equiv \gamma_{d,L_w}$ with the following property: If Λ is a maximal $\frac{\delta}{n}$-separated subset of $\mathbb{B}^d$ (with respect to the metric ρ) with $0 < \delta < \gamma$, then there exists a sequence of positive numbers $\{\lambda_\omega : \omega \in \Lambda\}$ such that $\lambda_\omega \sim \int_{B_\rho(\omega, \frac{\delta}{n})} w(y)\,dy$, and for any $f \in \Pi_{2n}(\mathbb{B}^d)$,*

$$\int_{\mathbb{B}^d} f(x)w(x)dx = \sum_{\omega\in\Lambda} \lambda_\omega f(\omega),$$

and moreover, for $f \in \Pi_n^d$ and $0 < p < \infty$,

$$\int_{\mathbb{B}^d} |f(x)|^p w(x)\,dx \sim \sum_{\omega\in\Lambda} |f(\omega)|^p \Big(\int_{B_\rho(\omega, \frac{\delta}{n})} w(y)\,dy \Big),$$

where the constants of equivalence depend only on d, the doubling constant of w and p.

THEOREM 8.2. *(Bernstein inequality) If w is a doubling weight on $\mathbb{B}^d$ and $0 < p < \infty$, then for all $f \in \Pi_n(\mathbb{B}^d)$,*

$$\left(\int_{\mathbb{B}^d} \Big(\varphi(|x|)\Big)^{|\alpha| p} |D^\alpha f(x)|^p w(x)\,dx \right)^{\frac{1}{p}} \le C n^{|\alpha|} \left(\int_{\mathbb{B}^d} |f(x)|^p w(x)\,dx \right)^{\frac{1}{p}},$$

where $\varphi(t) = \sqrt{1-t^2}$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in \mathbb{Z}_+^d$, $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$, and $|\alpha| = \sum_{j=1}^{d} \alpha_j$.

THEOREM 8.3. *(Remez inequality) If w is an A_∞ weight on $\mathbb{B}^d$, $0 < p < \infty$, $E \subset \mathbb{B}^d$, and $|E| = \left(\frac{A}{n}\right)^d \le \frac{1}{2}|\mathbb{B}^d|$ for some $A > 0$, then for any $f \in \Pi_n(\mathbb{B}^d)$,*

$$\int_{\mathbb{B}^d} |f(x)|^p w(x)\,dx \le C^{\sqrt{nA}+1} \int_{\mathbb{B}^d \setminus E} |f(x)|^p w(x)\,dx.$$

CHAPTER 2

Marcinkiewicz multiplier theorem for h-spherical harmonic expansions on $\mathbb{S}^{d-1}$

1. Introduction

Given a nonzero vector $\alpha \in \mathbb{R}^d$, we denote by σ_α the reflection with respect to the hyperplane perpendicular to α; that is, $\sigma_\alpha x = x - 2(\langle x, \alpha\rangle/\|\alpha\|^2)\alpha$ for all $x \in \mathbb{R}^d$. A reduced root system in $\mathbb{R}^d$ is a finite subset R of $\mathbb{R}^d \setminus \{0\}$ with the properties $\sigma_\alpha R = R$ and $R \cap \{t\alpha : t \in \mathbb{R}\} = \{\pm\alpha\}$ for all $\alpha \in R$.

Let R be a reduced root system in $\mathbb{R}^d$ normalized so that $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in R$. Let G denote the finite subgroup of the orthogonal group $O(d)$ generated by the reflections σ_α, $\alpha \in R$. Let $\kappa\colon R \to \mathbb{R}_+$ be a nonnegative multiplicity function on R with the property $\kappa(g\alpha) = \kappa(\alpha)$ for all $\alpha \in R$ and $g \in G$. Associated with the reflection group G and the function κ is the weight function h_κ defined by

$$h_\kappa(x) := \prod_{\alpha \in R_+} |\langle x, \alpha\rangle|^{\kappa(\alpha)}, \quad x \in \mathbb{R}^d, \tag{2.1}$$

where R_+ is an arbitrary but fixed positive subsystem of R. The function h_κ is a homogeneous function of degree $\gamma_\kappa := \sum_{\alpha\in R_+} \kappa(\alpha)$, and is invariant under the reflection group G. From now on, we shall set $\lambda_\kappa = \frac{d-2}{2} + \gamma_\kappa$.

The Dunkl operators associated with G and κ are defined by

$$\mathcal{D}_{\kappa,i} f(x) = \partial_i f(x) + \sum_{\alpha\in R_+} \kappa(\alpha) \frac{f(x) - f(\sigma(\alpha)x)}{\langle x, \alpha\rangle} \langle \alpha, e_i\rangle, \quad 1 \le i \le d, \tag{2.2}$$

where $e_1 = (1, 0, \cdots, 0), \cdots, e_d = (0, \cdots, 0, 1)$ are the standard unit vectors of $\mathbb{R}^d$. Those operators mutually commute, and map $\mathcal{P}_n^d$ to $\mathcal{P}_{n-1}^d$, where $\mathcal{P}_n^d$ is the space of homogeneous polynomials of degree n in d variables. We denote by $\Pi^d := \Pi(\mathbb{R}^d)$ the $\mathbb{C}$-algebra of polynomial functions on $\mathbb{R}^d$. An important result in Dunkl theory states that there exists a linear operator $V_\kappa\colon \Pi^d \to \Pi^d$ determined uniquely by

$$V_\kappa(\mathcal{P}_n^d) \subset \mathcal{P}_n^d, \quad V_\kappa(1) = 1, \quad \text{and} \quad \mathcal{D}_{\kappa,i} V_\kappa = V_\kappa \partial_i, \quad 1 \le i \le d. \tag{2.3}$$

Such an operator is called the Dunkl intertwining operator. We have the following important result of Rösler [**72**] on the Dunkl intertwining operator:

Lemma 1.1. [**72**, Th. 1.2 and Cor. 5.3] *For every $x \in \mathbb{R}^d$, there exists a unique probability measure μ_x^κ on the Borel σ-algebra of $\mathbb{R}^d$ such that*

$$V_\kappa P(x) = \int_{\mathbb{R}^d} P(\xi)\, d\mu_x^\kappa(\xi), \quad P \in \Pi^d. \tag{2.4}$$

Furthermore, the representing measures μ_x^κ are compactly supported in the convex hull $C(x) := co\{gx : g \in G\}$ of the orbit of x under G, and satisfy

$$\mu_{rx}^\kappa(E) = \mu_x^\kappa(r^{-1}E), \quad \text{and } \mu_{gx}^\kappa(E) = \mu_x^\kappa(g^{-1}E) \tag{2.5}$$

for all $r > 0$, $g \in G$ and each Borel subset E of $\mathbb{R}^d$.

In particular, the above lemma asserts that the intertwining operator V_κ is positive. By means of (2.4), V_κ can be extended to the space $C(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d$. We denote this extension by V_κ again.

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ denote the unit sphere of $\mathbb{R}^d$ equipped with the usual Lebesgue measure $d\sigma(x)$. For the weight function h_κ given in (4.1), we consider the weighted Lebesgue space $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ of functions on $\mathbb{S}^{d-1}$ endowed with the finite norm

$$\|f\|_{L^p(h_\kappa^2;\mathbb{S}^{d-1})} \equiv \|f\|_{\kappa,p} := \Big(\int_{\mathbb{S}^{d-1}} |f(y)|^p h_\kappa^2(y) d\sigma(y)\Big)^{1/p}, \qquad 1 \le p < \infty,$$

and for $p = \infty$ we assume that L^∞ is replaced by $C(\mathbb{S}^{d-1})$, the space of continuous functions on $\mathbb{S}^{d-1}$ with the usual uniform norm $\|f\|_\infty$.

A homogeneous polynomial is called an h-harmonic if it is orthogonal to all polynomials of lower degree with respect to the inner product of $L^2(h_\kappa^2; \mathbb{S}^{d-1})$. We denote by $\mathcal{H}_n^d(h_\kappa^2)$ denote the space of all real h-harmonics of degree n. Thus, $\mathcal{H}_0^d(h_\kappa^2)$ is the space of constant functions on $\mathbb{S}^{d-1}$, and for $n \in \mathbb{N}$, $\mathcal{H}_n^d(h_\kappa^2)$ is the orthogonal complement of Π_{n-1}^d in the space Π_n^d with respect to the inner product

$$\langle f, g\rangle_{L^2(h_\kappa^2;\mathbb{S}^{d-1})} := \int_{\mathbb{S}^{d-1}} f(x)g(x)h_\kappa^2(x)\, d\sigma(x).$$

Let $\operatorname{proj}_n^\kappa \colon L^2(h_\kappa^2; \mathbb{S}^{d-1}) \to \mathcal{H}_n^d(h_\kappa^2)$ denote the orthogonal projection operator. The projection $\operatorname{proj}_n^\kappa$ has an integral representation

$$\operatorname{proj}_n^\kappa f(x) := \int_{\mathbb{S}^{d-1}} f(y)P_n^\kappa(x,y)h_\kappa^2(y)\, d\sigma(y),\ x \in \mathbb{S}^{d-1}. \tag{2.6}$$

where $P_n^\kappa(x, y)$ is the reproducing kernel of $\mathcal{H}_n^d(h_\kappa^2)$; that is, if $\{Y_{n,j}\}_{j=1}^{m_n}$ is an orthonormal basis in $\mathcal{H}_n^d(h_\kappa^2)$, then

$$P_n^\kappa(x,y) = \sum_{j=1}^{m_n} Y_{n,j}(x)Y_{n,j}(y), \quad x, y \in \mathbb{S}^{d-1}.$$

A remarkable fact in the theory of h-spherical harmonics is that $P_n^\kappa(x, y)$ has a compact representation in terms of the Dunkl interwining operator V_κ:

$$P_n^\kappa(x,y) = \frac{n+\lambda_k}{\lambda_\kappa} V_\kappa\left[C_n^{\lambda_k}(\langle x, \cdot\rangle)\right](y), \qquad x, y \in \mathbb{S}^{d-1} \tag{2.7}$$

with $\lambda_\kappa := \gamma_\kappa + \frac{d-2}{2}$. Here C_n^λ denotes the standard Gegenbauer polynomial of degree n and index λ. By means of (4.6) and (4.7), the projection $\operatorname{proj}_n^\kappa f$ can be extended to all $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$.

Our main goal in this chapter is to show the following Marcinkiewicz type multiplier theorem for h-spherical harmonic expansions:

THEOREM 1.2. *Let $\{\mu_j\}_{j=0}^\infty$ be a sequence of real numbers that satisfies*

(i) $\sup_j |\mu_j| \le c < \infty$,

(ii) $\sup_{j\ge1} 2^{j(r-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^r \mu_l| \le c < \infty$, *with r being the smallest integer $\ge \lambda_\kappa + 1$,*

where $\triangle\mu_l = \mu_l - \mu_{l+1}$ and $\triangle^{j+1}\mu_l = \triangle^j\mu_l - \triangle^j\mu_{l+1}$. Then $\{\mu_j\}$ defines an $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ multiplier for all $1 < p < \infty$; that is,

$$\Big\|\sum_{j=0}^\infty \mu_j \operatorname{proj}_j^\kappa f\Big\|_{\kappa,p} \le A_p c\|f\|_{\kappa,p}, \qquad 1 < p < \infty,$$

where A_p is independent of $\{\mu_j\}$ and f.

We organize this chapter as follows. In section 2, we collect two general results on semigroups of operators on spaces of homogeneous type: the Hopf-Dunford-Schwartz ergodic theorem, and the Stein theorem on Littlewood-Paley functions. These general theorems will play fundamental roles in Section 4, where weighted Littlewood-Paley theory on the sphere are developed. In Section 3, we establish several useful results on h-spherical harmonic analysis. Various properties of Cesàro means, Poisson integrals and the maximal functions of Yuan Xu are given in Section 3. After that, in Section 4, we introduce three useful Littlewood-Paley functions on the sphere, and their equivalence in weighted L^p-norm is established. Section 5 is devoted to the proof of Theorem 2.2. The Littlewood-Paley theory on the sphere developed in Section 4, and the results on h-spherical harmonic analysis proved in Section 3 will play crucial roles in this section.

2. Analysis on homogeneous spaces

Both the Euclidean space $\mathbb{R}^d$ and the sphere $\mathbb{S}^{d-1}$ are homogeneous spaces. We start with the definition of homogeneous spaces in general.

DEFINITION 2.1. *A homogeneous space is a measure space (X, μ, ρ) with a positive measure μ and a metric ρ such that all open balls $B(x, r) := \{y \in X : \rho(x, y) < r\}$ are measurable, and μ is a regular measure satisfying the doubling property*

$$\mu(B(x, 2r)) \le C\mu(B(x, r)), \quad \forall x \in X, \quad \forall r > 0, \tag{2.8}$$

where C is independent of x and r. The least constant C in (2.8) is called the doubling constant, and is referred to as the geometric constant of the space.

Most of the measures in analysis satisfy the doubling condition. In the rest of this section, we assume that (X, μ, ρ) is a fixed homogeneous space and state several results without proof. Our main reference for this section is [**76**], where the full proof can be found.

In this section, we shall state with two general results on semi-groups of operators without proofs. The definition of such operators is given in [**78**, p. 2]:

DEFINITION 2.2. *Let (X, μ) be a measure space with a positive measure μ. A family of operators $\{T_t\}_{t\ge 0}$ is said to form a symmetric diffusion semi-group if*

$$T^{t_1}T^{t_2} = T^{t_1+t_2}, \qquad T^0 = id,$$

and it satisfies the following assumptions:

(*i*) *T^t are contractions on $L^p(X, \mu)$, i.e., $\|T^t f\|_p \le \|f\|_p$, $1 \le p \le \infty$;*
(*ii*) *T^t are symmetric, i.e., each T^t is self-adjoint on $L^2(M, d\mu)$;*
(*iii*) *T^t are positive preserving, i.e., $T^t f \ge 0$ if $f \ge 0$;*
(*iv*) *$T^t f_0 = f_0$ if $f_0(x) = 1$.*

THEOREM 2.3. *Suppose that $\|T^t f\|_p \le \|f\|_p$ for all $f \in L^p(X, \mu)$ and for each $1 \le p \le \infty$. Then the function*

$$Mf(x) = \sup_{s\ge 0} \left(\frac{1}{s} \int_0^s T^t f(x) dt \right)$$

satisfies the inequalities

(*a*) *$\|Mf\|_p \le c_p \|f\|_p$ for each p with $1 < p \le \infty$;*
(*b*) *$\mu(\{x \in X : Mf(x) > \alpha\}) \le (c/\alpha)\|f\|_1$ for each $\alpha > 0$ and $f \in L^1(X, \mu)$, where c is independent of f and α.*

This statement in [**78**, p. 48] and it is a special case of the Hopf-Dunford-Schwartz ergodic theorem.

Given $f \in L^p(X,\mu)$, its Littlewood-Paley function in terms of $\{T^t\}$ is defined by

$$\widetilde{g}(f) := \Big(\int_0^\infty t \Big| \frac{\partial}{\partial t} T^t f \Big|^2 \, dt \Big)^{\frac{1}{2}}. \tag{2.9}$$

The following theorem is due to Stein [**78**, Theorem 10, p. 111]:

THEOREM 2.4. *For $f \in L^p(d\mu)$, $1 < p < \infty$,*

$$c_p^{-1} \|f\|_{L^p(d\mu)} \le \|\widetilde{g}(f)\|_{L^p(d\mu)} \le c_p \|f\|_{L^p(d\mu)},$$

where the first inequality holds under the additional assumption $\int_X f \, d\mu = 0$ and the constant C_p is independent of f.

3. h-spherical harmonic analysis

We start with the following important formula of Yuan Xu [**95**]:

PROPOSITION 3.1. *For a continuous function $g : \mathbb{B}^d \mapsto \mathbb{R}$,*

$$\int_{\mathbb{S}^{d-1}} V_\kappa g(x) h_\kappa^2(x) d\sigma(x) = b_\kappa \int_{\mathbb{B}^d} g(x)(1 - \|x\|^2)^{|\kappa|-1} dx, \tag{2.10}$$

where b_κ is a constant such that the above equation holds when $g = 1$.

PROOF. We give a different proof here. Firstly, we show that (2.10) holds whenever $g \colon \mathbb{B}^d \to \mathbb{R}$ is of the form $g(x) = p(\langle x, y \rangle)$ for some polynomial $p \colon [-1,1] \to \mathbb{R}$ and $y \in \mathbb{S}^{d-1}$. Indeed, if p is an algebraic polynomial of degree n on $[-1,1]$, then it can be expanded in terms of Gegenbauer polynomials:

$$p(t) = \sum_{j=0}^n a_{j,p} \frac{\lambda_\kappa + j}{\lambda_\kappa} C_j^{\lambda_\kappa}(t), \quad t \in [-1,1],$$

hence

$$V_\kappa[p(\langle \cdot, y \rangle)](x) = \sum_{j=0}^n a_{j,p} \frac{\lambda_\kappa + j}{\lambda_\kappa} V_\kappa[C_j^{\lambda_\kappa}(\langle \cdot, y \rangle)](x), \quad x, y \in \mathbb{S}^{d-1}.$$

Recall that $V_\kappa[C_j^{\lambda_\kappa}(\langle \cdot, y \rangle)] \in \mathcal{H}_j^d(h_\kappa^2)$ for each j and each fixed $y \in \mathbb{S}^{d-1}$, and that the spaces $\mathcal{H}_j^d(h_\kappa^2)$, $j \ge 1$ are orthogonal to $\mathcal{H}_0^d(h_\kappa^2)$ (consisting of constant functions) in $L^2(h_\kappa^2; \mathbb{S}^{d-1})$. Thus,

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} V_\kappa[p(\langle \cdot, y \rangle)](x) h_\kappa^2(x) d\sigma(x) &= a_{0,p} \int_{\mathbb{S}^{d-1}} V_\kappa[C_0^{\lambda_\kappa}(\langle \cdot, y \rangle)](x) h_\kappa^2(x) d\sigma(x) = c_\kappa' a_0 \\ &= c_\kappa' \int_{-1}^1 p(t)(1-t^2)^{\lambda_\kappa - \frac{1}{2}} \, dt, \quad \forall y \in \mathbb{S}^{d-1}. \end{aligned}$$

On the other hand, however, for $y \in \mathbb{S}^{d-1}$,

$$\begin{aligned} \int_{\mathbb{B}^d} p(\langle x, y \rangle)(1 - \|x\|^2)^{|\kappa|-1} dx &= \int_{\mathbb{B}^d} p(x_1)(1 - \|x\|^2)^{|\kappa|-1} \, dx \\ &= c_d \int_{-1}^1 p(t)(1-t^2)^{|\kappa|-1+\frac{d-1}{2}} \, dt \\ &= c_d \int_{-1}^1 p(t)(1-t^2)^{\lambda_\kappa - \frac{1}{2}} \, dt. \end{aligned}$$

Thus, we obtain the desired result in this case.

Secondly, we claim that if g is a polynomial of degree m on $\mathbb{B}^d$, then it can be written in the form

$$g(x) = \sum_{j=1}^{r_m} p_j(\langle x, \xi_j \rangle) \tag{2.11}$$

for some suitably chosen polynomials $p_j : [-1,1] \to \mathbb{R}$ and points $\xi_j \in \mathbb{S}^{d-1}$, where $r_m := \binom{m+d-1}{d-1}$. By linearity, this will imply that (2.10) holds for all polynomials g on $\mathbb{B}^d$, which, by the bounded convergence theorem, will further imply that (2.10) holds for all continuous functions $g \in C(\mathbb{B}^d)$.

To show (2.11), we let $\mathcal{P}_m^d$ denote the space of all real homogeneous polynomials of degree m on $\mathbb{R}^d$. endowed with the following inner product: for $f(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha$ and $g(x) = \sum_{|\alpha|=m} b_\alpha x^\alpha \in \mathcal{P}_m^d$,

$$\langle f, g\rangle_{\mathcal{P}_m^d} := f(\partial)g = \sum_{|\alpha|=m} a_\alpha b_\alpha \alpha!,$$

where $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_d!$. Then $r_m = \dim \mathcal{P}_m^d$, and there exist $\xi_{m,1}, \cdots, \xi_{m,r_m} \in \mathbb{S}^{d-1}$ such that for $f \in \mathcal{P}_m^d$,

$$f(\xi_{m,j}) = 0 \ \text{ for all } 1 \le j \le r_m \ \text{ if and only if } \quad f = 0. \tag{2.12}$$

For convenience, we set $f_{m,j}(x) := (x \cdot \xi_{m,j})^m$ for $1 \le j \le r_m$, where $x \cdot y$ denotes the Euclidean dot product of $x, y \in \mathbb{R}^d$. Then

$$f_{m,j}(x) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha \xi_{m,j}^\alpha \in \mathcal{P}_m^d,$$

and hence, for each $f(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha \in \mathcal{P}_m^d$,

$$\langle f_{m,j}, f\rangle_{\mathcal{P}_m^d} = m! \sum_{|\alpha|=m} a_\alpha \xi_{m,j}^\alpha = m! f(\xi_{m,j}).$$

This together with (2.12) implies that for $f \in \mathcal{P}_m^d$,

$$\langle f_{m,j}, f\rangle_{\mathcal{P}_m^d} = 0 \ \text{ for all } 1 \le j \le r_m \ \text{ if and only if } \quad f = 0.$$

Thus,

$$\mathcal{P}_m^d = \operatorname{span}\big\{(x \cdot \xi_{m,j})^m : \quad 1 \le j \le r_m\big\}.$$

To complete the proof of (2.11), it suffices to show that

$$\mathcal{P}_n^d = \operatorname{span}\{(x \cdot \xi_{m,j})^n : \quad 1 \le j \le r_m\}, \quad \text{for all } 0 \le n \le m. \tag{2.13}$$

Indeed, setting $f_{m,n,j}(x) = (x \cdot \xi_{m,j})^n$, we have, for any $f \in \mathcal{P}_n^d$,

$$\langle f, f_{m,n,j}\rangle_{\mathcal{P}_n^d} = n! f(\xi_{m,j}) \ \text{ for all } 1 \le j \le r_m.$$

Note that if $n \le m$ then $fg \in \mathcal{P}_m^d$ for any $g \in \mathcal{P}_{m-n}^d$. Thus, using (2.12), we conclude that if $n \le m$, and $\langle f, f_{m,n,j}\rangle_{\mathcal{P}_n^d} = 0$ for all $1 \le j \le r_m$, then $fg = 0$ for all $g \in \mathcal{P}_{m-n}^d$, and hence $f = 0$. This proves the desired equation (2.13). □

Using Proposition 3.1 and positivity of V_κ, we have, for any $g \in C(\mathbb{B}^d)$,

$$\|V_\kappa g\|_{L^1(h_\kappa^2;\mathbb{S}^{d-1})} \le b_\kappa \int_{\mathbb{B}^d} |g(x)|(1-\|x\|^2)^{|\kappa|-1} dx \equiv b_\kappa \|g\|_{L^1((1-\|x\|^2)^{|\kappa|-1};\mathbb{B}^d)}.$$

This last equation allows us to extend V_κ to a positive, bounded operator from $L^1((1-\|x\|^2)^{|\kappa|-1}; \mathbb{B}^d)$ to $L^1(h_\kappa^2; \mathbb{S}^{d-1})$ so that (2.10) holds for all $g \in L^1((1-\|x\|^2)^{|\kappa|-1}; \mathbb{B}^d)$. In particular, we have the following useful corollary:

Corollary 3.2. *If $p : [-1,1] \to \mathbb{R}$ satisfies $\int_{-1}^1 |g(t)|(1-t^2)^{\lambda_\kappa - \frac12}\, dt < \infty$, then*

$$\int_{\mathbb{S}^{d-1}} V_\kappa \Big[p(\langle \cdot, y\rangle)\Big](x) h_\kappa^2(x) d\sigma(x) = b_\kappa' \int_{-1}^1 g(t)(1-t^2)^{\lambda_\kappa - \frac12} dt.$$

For simplicity, we set $w_{\lambda_\kappa}(t) = (1-t^2)^{\lambda_k - \frac{1}{2}}$ for $t \in [-1,1]$, and denote by $L^1(w_{\lambda_\kappa};[-1,1])$ denote the weighted Lebesgue space of functions $f\colon [-1,1] \to \mathbb{R}$ with $\int_{-1}^1 |f(t)| w_{\lambda_\kappa}(t)\, dt < \infty$. Corollary 3.2 allows us to introduce the following definition of convolutions:

Definition 3.3. [**107**] *For $f \in L^1(h_\kappa^2;\mathbb{S}^{d-1})$ and $g \in L^1(w_{\lambda_\kappa};[-1,1])$, define*

$$f \star_\kappa g(x) := a_\kappa \int_{\mathbb{S}^{d-1}} f(y) V_\kappa[g(\langle x,\cdot\rangle)](y) h_\kappa^2(y) d\sigma(y), \tag{2.14}$$

where a_κ is chosen so that $a_\kappa \int_{\mathbb{S}^{d-1}} h_\kappa^2(y) d\sigma(y) = 1$.

This convolution satisfies the usual Young's inequality, which is a simple consequence of Corollary 3.2 and the Riesz-Thorin interpolation theorem.

Proposition 3.4. [**107**] *If $p,q,r \geq 1$ and $p^{-1} = r^{-1} + q^{-1} - 1$, then for $f \in L^q(h_\kappa^2;\mathbb{S}^{d-1})$ and $g \in L^r(w_{\lambda_\kappa};[-1,1])$,*

$$\|f \star_\kappa g\|_{k,p} \leq \|f\|_{k,q} \|g\|_{w_{\lambda_\kappa},r}.$$

Definition 3.5. [**38**] *For $f \in L^1(h_\kappa^2, \mathbb{S}^{d-1})$, the Poisson integral of f is defined by*

$$P_r^\kappa f(\xi) := \frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} f(y) P_r^\kappa(\xi, y) h_\kappa^2(y) d\sigma(y), \quad \xi \in \mathbb{S}^{d-1}, \tag{2.15}$$

where the kernel $P_r^\kappa(x,\cdot)$ is given by, for $0<r<1$,

$$P_r^\kappa(x,y) := V_\kappa\left[\frac{1-r^2}{(1-2r\langle\cdot,y\rangle + r^2)^{\lambda_\kappa+1}}\right](x). \tag{2.16}$$

Lemma 3.6. [**38**] *For $0<r<1$, the Poisson kernel satisfies the following properties:*

(1) For $x,y \in \mathbb{S}^{d-1}$, $P_r^\kappa(x,y) = \sum_{n=0}^\infty r^n \frac{n+\lambda_\kappa}{\lambda_\kappa} V_\kappa\left[C_n^{\lambda_\kappa}(\langle x,\cdot\rangle)\right](y)$;
(2) $P_r^\kappa f = \sum_{n=0}^\infty r^n \operatorname{proj}_n^\kappa f$;
(3) $P_r^\kappa(x,y) \geq 0$ and $\frac{1}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} P_r^\kappa(x,y) h_\kappa^2(y) d\sigma(y) = 1$.

We define

$$\mathtt{b}(x,\theta) := \{y : \langle x,y\rangle \geq \cos\theta\}, \quad x \in \mathbb{S}^{d-1}, \quad 0 \leq \theta \leq \pi.$$

Let χ_E denote the characteristic function of the set E. The following maximal function was introduced by Yuan Xu [**105**]:

Definition 3.7. [**105**] *For $f \in L^1(h_\kappa^2;\mathbb{S}^{d-1})$, define the maximal function*

$$\begin{aligned} \mathcal{M}_\kappa f(x) &= \sup_{0<\theta\leq\pi} \frac{\int_{\mathbb{S}^{d-1}} |f(y)| V_\kappa[\chi_{\mathtt{b}(x,\theta)}](y) h_\kappa^2(y) d\sigma(y)}{\int_{\mathbb{S}^{d-1}} V_\kappa[\chi_{\mathtt{b}(x,\theta)}](y) h_\kappa^2(y) d\sigma(y)} \\ &= \sup_{0<\theta\leq\pi} \frac{(|f| *_\kappa \chi_{[\cos\theta,1]})(x)}{c_{\lambda_\kappa} \int_0^\theta (\sin\phi)^{2\lambda_\kappa} d\phi}. \end{aligned} \tag{2.17}$$

This maximal function can be used to study the h-harmonic expansions, since we can often prove $|(f \star_\kappa g)(x)| \leq c\mathcal{M}_\kappa f(x)$. Using Corollary 3.2 leads to

$$\int_{\mathbb{S}^{d-1}} V_\kappa[\chi_{[\cos\theta,1]}(\langle x,\cdot\rangle)](y) h_\kappa^2(y) d\sigma(y) = \int_0^\theta (\sin\phi)^{2\lambda_\kappa}\, d\phi \sim \theta^{2\lambda_\kappa+1}. \tag{2.18}$$

To state the weak type inequality, we define, for any measurable subset E of $\mathbb{S}^{d-1}$, the measure with respect to h_κ^2 as

$$\operatorname{meas}_\kappa E := \int_E h_\kappa^2(y) d\sigma(y).$$

The following estimates of $\mathcal{M}_\kappa f$ was proved in [**24**]:

THEOREM 3.8. *If $f \in L^1(h_\kappa^2, \mathbb{S}^{d-1})$, then $\mathcal{M}_\kappa f$ satisfies*

$$\mathrm{meas}_\kappa\{x : \mathcal{M}_\kappa f(x) \ge \alpha\} \le c\frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0. \tag{2.19}$$

Furthermore, if $f \in L^p(h_\kappa^2, \mathbb{S}^{d-1})$ for $1 < p \le \infty$, then $\|\mathcal{M}_k f\|_{\kappa,p} \le c\|f\|_{\kappa,p}$.

In order to prove Theorem 3.8, we use Theorem 2.3. Recall that P_r^κ denotes the Poisson integral. By Lemma 3.6, it is easy to verify that $T^t := P_r^\kappa f$ with $r = e^{-t}$ satisfies all requirements in the definition. We will need another semi-group, which is the discrete analog of the heat operator,

$$H_t^\kappa f := f *_\kappa q_t^\kappa, \qquad q_t^\kappa(s) := \sum_{n=0}^\infty e^{-n(n+2\lambda_\kappa)t}\frac{n+\lambda_\kappa}{\lambda_\kappa} C_n^{\lambda_\kappa}(s). \tag{2.20}$$

LEMMA 3.9. *The family of operators $\{H_\kappa^t\}$ is a symmetric diffusion semi-group.*

PROOF. The kernel q_t^κ is known to be nonnegative [**48**], from which it follows immediately that H_κ^t are positive and that $\|q_t^\kappa\|_{\lambda,1} = 1$ by the orthogonality of the Gegenbauer polynomials. Hence, by Young's inequality, $\|H_t^\kappa f\|_{\kappa,p} \le \|f\|_{k,p}$. Other requirements in Definition 2.2 can be directly verified. □

LEMMA 3.10. *The Poisson and the heat semi-groups are connected by*

$$P_{e^{-t}}^\kappa f(x) = \int_0^\infty \phi_t(s) H_s^\kappa f(x) ds, \tag{2.21}$$

where

$$\phi_t(s) := \frac{t}{2\sqrt{\pi}} s^{-3/2} e^{-(\frac{t}{2\sqrt{s}} - \lambda_\kappa\sqrt{s})^2}.$$

Furthermore, assume that $f(x) \ge 0$ for all x, then for all $t > 0$, then

$$P_*^\kappa f(x) := \sup_{0<r<1} P_r^\kappa f(x) \le c \sup_{s>0}\frac{1}{s}\int_0^s H_u^\kappa f(x) du. \tag{2.22}$$

Consequently, $P_*^\kappa f$ is bounded on $L^p(h_\kappa^2; S^d)$ for $1 < p \le \infty$ and of weak type $(1,1)$.

PROOF. The fact that $\{H_t^\kappa\}$ is a semi-group of operators allows us to apply the Hopf-Dunford-Schwartz ergodic theorem ([**78**, p.48]), which shows that the maximal operator $\sup_{s>0}\left(\frac{1}{s}\int_0^s H_u^\kappa f(x)du\right)$ is bounded on $L^p(h_\kappa^2, \mathbb{S}^{d-1})$ for $1 < p \le \infty$ and of week type $(1,1)$. Therefore, it is sufficient to prove (2.21) and (2.22).

First we prove (2.21). Applying the well known identity ([**78**, p.46])

$$e^{-v} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-v^2/4u} du, \qquad v > 0,$$

with $v = (n+\lambda_\kappa)t$ and making a change of variable $s = t^2/4u$, we obtain that

$$\begin{aligned} e^{-nt} &= e^{\lambda_\kappa t}\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{n(n+2\lambda_\kappa)t^2}{4u}} e^{-\frac{\lambda_\kappa^2 t^2}{4u}} du \\ &= \frac{t}{2\sqrt{\pi}}\int_0^\infty e^{-n(n+2\lambda_\kappa)s} s^{-3/2} e^{-(\frac{t}{2\sqrt{s}} - \lambda_\kappa\sqrt{s})^2} ds \\ &= \int_0^\infty e^{-n(n+2\lambda_\kappa)s}\phi_t(s)\, ds. \end{aligned}$$

Multiplying by $\mathrm{proj}_n^\kappa f$ and summing up over n proves the integral relation (2.21).

For the proof of (2.22), we use (2.21) and integration by parts to obtain

$$\begin{aligned} P^\kappa_{e^{-t}} f(x) &= -\int_0^\infty \left(\int_0^s H^\kappa_u f(x) du\right) \phi_t'(s) ds \\ &\le \sup_{s>0} \left(\frac{1}{s}\int_0^s H^\kappa_u f(x) du\right) \int_0^\infty s|\phi_t'(s)| ds, \end{aligned}$$

where the derivative of $\phi_t'(s)$ is taken with respect to s. Furthermore, since $\mathcal{P}^\kappa_r f = f *_k p^\kappa_r$ and $|p^\kappa_r(t)| \le c$ for $0 < r \le e^{-1}$, it follows that

$$\sup_{0<r\le e^{-1}} P^\kappa_r f(x) \le c\|f\|_{1,\kappa} = c \lim_{s\to\infty} \frac{1}{s}\int_0^s H^\kappa_u(|f|)(x)\, du.$$

Therefore, to finish the proof of (2.22), it suffices to show that $\sup_{0<t\le 1}\int_0^\infty s|\phi_t'(s)|ds$ is bounded by a constant. A quick computation shows that $\phi_t'(s) > 0$ if $s < \alpha_t$ and $\phi_t'(s) < 0$ if $s > \alpha_t$, where

$$\alpha_t := \frac{t^2}{3+\sqrt{9+4\lambda_\kappa^2 t^2}} \sim t^2, \qquad 0 \le t \le 1.$$

Since the integral of $\phi_t(s)$ over $[0,\infty)$ is 1 and $\phi_t(s) \ge 0$, integration by parts gives

$$\begin{aligned} \int_0^\infty s|\phi_t'(s)|ds &= 2\alpha_t\phi_t(\alpha_t) - \int_0^{\alpha_t} \phi_t(s)ds + \int_{\alpha_t}^\infty \phi_t(s)ds \\ &\le 2\alpha_t\phi_t(\alpha_t) + 1 = \frac{t}{\sqrt{\pi\alpha_t}} e^{-\frac{(t-2\lambda_\kappa\alpha_t)^2}{4\alpha_t}} + 1 \le c \end{aligned}$$

as desired. □

We are now in a position to prove Theorem 3.8.

Proof of Theorem 3.8. From the definition of p^κ_r in (2.16), if $1-r \sim \theta$, then

$$\begin{aligned} p^\kappa_r(\cos\theta) &= \frac{1-r^2}{\left((1-r)^2 + 4r\sin^2\frac{\theta}{2}\right)^{\lambda_\kappa+1}} \\ &\ge c\frac{1-r^2}{\left((1-r)^2 + r\theta^2\right)^{\lambda_\kappa+1}} \ge c\,(1-r)^{-(2\lambda_\kappa+1)}. \end{aligned}$$

For $j \ge 0$ define $r_j := 1 - 2^{-j}\theta$ and set $B_j := \left\{y \in \mathbb{B}^d : 2^{-j-1}\theta \le d(x,y) \le 2^{-j}\theta\right\}$. The lower bound of p^κ_r proved above shows that

$$\chi_{B_j}(y) \le c\,(2^{-j}\theta)^{2\lambda_k+1} p^\kappa_{r_j}(\langle x,y\rangle),$$

which implies immediately that

$$\chi_{\mathtt{b}(x,\theta)}(y) \le \sum_{j=0}^\infty \chi_{B_j}(y) \le c\,\theta^{2\lambda_k+1}\sum_{j=0}^\infty 2^{-j(2\lambda_\kappa+1)} p^\kappa_{r_j}(\langle x,y\rangle).$$

Since V_κ is a positive linear operator, applying V_κ to the above inequality gives

$$\begin{aligned} &\int_{S^{d-1}} |f(y)| V_\kappa\left[\chi_{\mathtt{b}(x,\theta)}\right](y) h^2_\kappa(y) d\sigma(y) \\ &\quad\le c\,\theta^{2\lambda_\kappa+1}\sum_{j=0}^\infty 2^{-j(2\lambda_\kappa+1)} \int_{S^{d-1}} |f(y)| V_\kappa\left[p_{r_j}(\langle x,y\rangle)\right](y) h^2_\kappa(y) d\sigma(y) \\ &\quad= c\,\theta^{2\lambda_\kappa+1}\sum_{j=0}^\infty 2^{-j(2\lambda_\kappa+1)} P^\kappa_{r_j}(|f|;x) \\ &\quad\le c\,\theta^{2\lambda_\kappa+1}\sup_{0<r<1} P^\kappa_r(|f|;x). \end{aligned}$$

Dividing by $\theta^{2\lambda_\kappa+1}$ and using the fact that

$$\frac{1}{\omega_d^\kappa}\int_{\mathbb{S}^{d-1}} V_\kappa[\chi_{\mathtt{b}(x,\theta)}](y)h_\kappa^2(y)d\sigma(y) = c_{\lambda_\kappa}\int_0^\theta (\sin\phi)^{2\lambda_\kappa}d\phi \sim \theta^{2\lambda_\kappa+1}.$$

we have proved that $\mathcal{M}_\kappa f(x) \le cP_*^\kappa|f|(x)$. The desired result now follows from Lemma 3.10. □

Definition 3.11. *For $\delta > -1$, the Cesàro (C,δ) means of the h-spherical harmonic expansions are defined by*

$$S_n^\delta(h_\kappa^2;f,x) := (A_n^\delta)^{-1}\sum_{k=0}^n A_{n-k}^\delta \operatorname{proj}_k^\kappa f(x), \qquad A_j^\delta = \frac{\Gamma(j+\delta+1)}{\Gamma(j+1)\Gamma(\delta+1)}.$$

These means can be written as

$$S_n^\delta(h_\kappa^2;f) = (f\star_\kappa K_n^\delta(h_\kappa^2))(x), \tag{2.23}$$

where

$$K_n^\delta\left(h_\kappa^2;t\right) := \frac{1}{A_n^\delta}\sum_{k=0}^n A_{n-k}^\delta\frac{k+\lambda_\kappa}{\lambda_\kappa}C_k^{\lambda_\kappa}(t) = k_n^\delta\left(w_{\lambda_\kappa};1,t\right), \tag{2.24}$$

in which $k_n^\delta(w_{\lambda_\kappa};\cdot,\cdot)$ is the kernel of the (C,δ) means of the Fourier orthogonal series in the Gegenbauer polynomials.

The following theorem is a consequence of Corollary 3.2:

Theorem 3.12. [**96**] *The Cesàro means of the h-spherical harmonics series satisfy*

1. *If $\delta \ge 2\lambda_k+1$ then $S_n^\delta(h_\kappa^2)$ is a nonnegative operator;*
2. *If $\delta > \lambda_\kappa$ then $S_n^\delta(h_\kappa^2;f)$ converges to f in $L^p(h_\kappa^2;\mathbb{S}^{d-1})$ for $1\le p\le\infty$.*

The following result was proved in [**104**]:

Corollary 3.13. *If $\delta > \lambda_\kappa$ and $f\in L^1(h_\kappa^2,\mathbb{S}^{d-1})$ then for every $x\in\mathbb{S}^{d-1}$,*

$$\sup_{n\ge 0}|S_n^\delta(h_\kappa^2;f(x)| \le c\left[\mathcal{M}_\kappa f(x) + \mathcal{M}_\kappa f(-x)\right]. \tag{2.25}$$

If, in addition, $\delta \ge 2\lambda_k+1$, then the term $Mf(-x)$ in (2.25) can be dropped.

We conclude this section with the following result:

Theorem 3.14. *For $\delta > \lambda_\kappa$, $1<p<\infty$ and any sequence $\{n_j\}$ of positive integers,*

$$\left\|\left(\sum_{j=0}^\infty\left|S_{n_j}^\delta(h_\kappa^2;f_j)\right|^2\right)^{1/2}\right\|_{\kappa,p} \le c\left\|\left(\sum_{j=0}^\infty\left|f_{n_j}\right|^2\right)^{1/2}\right\|_{\kappa,p}. \tag{2.26}$$

Proof. The proof of (2.26) follows the approach of [**78**, p.104-5] that uses a generalization of the Riesz convexity theorem for sequences of functions. Let $L^p(\ell^q)$ denote the space of all sequences $\{f_k\}$ of functions for which the norm

$$\|(f_k)\|_{L^p(\ell^q)} := \left(\int_{\mathbb{S}^{d-1}}\left(\sum_{j=0}^\infty|f_j(x)|^q\right)^{p/q}h_\kappa^2(x)d\sigma(x)\right)^{1/p}$$

are finite. If T is bounded as operator on $L^{p_0}(\ell^{q_0})$ and on $L^{p_1}(\ell^{q_1})$, then the Riesz convexity theorem states that T is also bounded on $L^{p_t}(\ell^{q_t})$, where

$$\frac{1}{p_t} = \frac{1-t}{p_0}+\frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0}+\frac{t}{q_1}, \quad 0\le t\le 1.$$

We apply this theorem on the operator T that maps the sequence $\{f_j\}$ to the sequence $\{S^\delta_{n_j}(h^2_\kappa; f_j)\}$ with $\delta > \lambda_\kappa$. By Corollary 3.13, T is bounded on $L^p(\ell^p)$. It is also bounded on $L^p(\ell^\infty)$ since

$$\Big\|\sup_{j\ge 0}|S^\delta_{n_j}(h^2_\kappa; f_j)|\Big\|_{\kappa,p} \le c\Big\|\mathcal{M}_\kappa\Big(\sup_{j\ge0}|f_j|\Big)\Big\|_{\kappa,p} \le c\Big\|\sup_{j\ge 0}|f_j|\Big\|_{\kappa,p}.$$

Hence, the Riesz convexity theorem shows that T is bounded on $L^p(\ell^q)$ if $1 < p \le q \le \infty$. In particular, T is bounded on $L^p(\ell^2)$ if $1 < p \le 2$. The case $2 < p < \infty$ follows by the standard duality argument, since the dual space of $L^p(\ell^2)$ is $L^{p'}(\ell^2)$, where $1/p + 1/p' = 1$, under the paring

$$\langle (f_j), (g_j)\rangle := \int_{\mathbb{S}^{d-1}} \sum_j f_j(x) g_j(x) h^2_\kappa(x) d\sigma(x)$$

and T is self-adjoint under this paring as $S^\delta_n(h^2_\kappa)$ is self-adjoint in $L^2(h^2_\kappa; \mathbb{S}^{d-1})$. $\square$

4. The Littlewood-Paley theory on the sphere

For simplicity, from now now, we write $S^\delta_n f$ for $S^\delta_n(h^2_\kappa; f)$. For functions f on the sphere $\mathbb{S}^{d-1}$, we define

$$g(f) := \left(\int_0^1 (1-r)\Big|\frac{\partial}{\partial r} P^\kappa_r f\Big|^2\, dr\right)^{\frac12}, \tag{2.27}$$

where $P^\kappa_r f$ denotes the Poisson integral of f. The general result of Stein, Theorem 2.4, then leads to the following useful corollary:

Corollary 4.1. *If $1 < p < \infty$ and $f \in L^p(h^2_\kappa; \mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}} f(x) h^2_\kappa(x)\, d\sigma(x) = 0$, then for $g(f)$ given in (2.27), we have*

$$C_p^{-1}\|f\|_{\kappa,p} \le \|g(f)\|_{\kappa,p} \le C_p \|f\|_{\kappa,p}, \tag{2.28}$$

where the constant C_p is independent of f.

Proof. Applying Theorem 2.4 to the semigroup $\{P^\kappa_{e^{-t}} : t \ge 0\}$, we deduce that $\|\widetilde{g}(f)\|_{\kappa,p} \sim \|f\|_{\kappa,p}$, where

$$\widetilde{g}(f) := \left(\int_0^\infty \Big|\frac{\partial}{\partial t}(P^\kappa_{e^{-t}} f)\Big|^2 t\, dt\right)^{\frac12} = \left(\int_0^1 \Big|\frac{\partial}{\partial r} P^\kappa_r f\Big|^2 r|\log r|\, dr\right)^{\frac12}.$$

The desired equation (2.28) then follows by using the fact that

$$\max_{0<r<\frac12}\Big\|\frac{\partial}{\partial r}P^\kappa_r f\Big\|_{\kappa,p} \le \sum_{k=1}^\infty k2^{-k+1}\|\operatorname{proj}^\kappa_k f\|_{\kappa,p} \le c\|f\|_{\kappa,p}\sum_{k=1}^\infty 2^{-k}k^{\lambda_\kappa+2} \le c\|f\|_{\kappa,p}.$$

$\square$

We also need a refined version of the Littlewood-Paley function $g(f)$ defined as follows:

$$g_\delta(f) = \left(\sum_{n=1}^\infty |S^{\delta+1}_n f - S^\delta_n f|^2 n^{-1}\right)^{\frac12}, \tag{2.29}$$

where $\delta \ge 0$ and $f \in L(h^2_\kappa; \mathbb{S}^{d-1})$. In this section, we shall prove the following result concerning the equivalence of $\|g_\delta(f)\|_{\kappa,p}$ and $\|f\|_{\kappa,p}$:

Theorem 4.2. *If $1 < p < \infty$, and $f \in L^p(h^2_\kappa; \mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}} f(x) h^2_\kappa(x)\, d\sigma(x) = 0$, then $\|f\|_{\kappa,p} \le C_p \|g_\delta(f)\|_{\kappa,p}$ holds for all $\delta \ge 0$. Conversely, if $\delta \ge 0$, $1 < p < \infty$, and the inequality*

$$\left\|\left(\sum_{k=1}^\infty |S^\delta_{N_k} f_k|^2\right)^{\frac12}\right\|_{\kappa,p} \le C_p \left\|\left(\sum_{k=1}^\infty |f_k|^2\right)^{\frac12}\right\|_{\kappa,p} \tag{2.30}$$

holds for all $\{N_k\} \subset \mathbb{Z}_+$ *and* $\{f_k\} \subset L(h_\kappa^2; \mathbb{S}^{d-1})$*, then* $\|g_\delta f\|_{\kappa,p} \le C_p \|f\|_{\kappa,p}$ *holds for all* $f \in L^p(h_\kappa^2; \mathbb{S}^{d-1})$.

The second part of the above theorem 4.2 follows directly from a more general result, Theorem 4.3 below.

THEOREM 4.3. *Assume that* $\delta \ge 0$, $1 < p < \infty$, *and that the inequality* (2.30) *holds for all* $\{N_k\} \subset \mathbb{Z}_+$ *and* $\{f_k\} \subset L(h_\kappa^2; \mathbb{S}^{d-1})$. *Let* $\{v_k\}_{k=1}^\infty$ *be a sequence of positive numbers satisfying* $\sup_N N^{-1} \sum_{k=1}^N v_k = M < \infty$. *Then for* $f \in L^p(h_\kappa^2; \mathbb{S}^{d-1})$, *and the function* $g_\delta^*(f) := \Big(\sum_{n=1}^\infty |S_n^{\delta+1} f - S_n^\delta f|^2 n^{-1} v_n\Big)^{\frac12}$, *the inequality* $\|g_\delta^*(f)\|_{\kappa,p} \le M C_p \|f\|_{\kappa,p}$ *holds, with the constant* C_p *being independent of* f *and* v_k.

The rest of this section is devoted to the proofs of Theorems 4.2 and 4.3.

4.1. Properties of the Cesàro coefficients. Let us first recall some basic properties of the Cesàro coefficients A_j^δ, $j = 0, 1, \cdots$, which are defined for all $\delta \in \mathbb{R}$ by $(1-s)^{-1-\delta} = \sum_{j=0}^\infty A_j^\delta s^j$. The following facts can be easily verified from the definition:

$$\text{(i)} \qquad A_j^\delta - A_{j-1}^\delta = A_j^{\delta-1}, \qquad \sum_{j=0}^n A_j^\delta = A_n^{\delta+1}, \quad \sum_{j=0}^n A_{n-j}^\alpha A_j^\beta = A_n^{\alpha+\beta+1} \tag{2.31}$$

$$\text{(ii)} \qquad |A_j^\delta| \sim (j+1)^\delta, \quad \text{whenever } j+\delta+1 > 0. \tag{2.32}$$

(iii) If $\{a_k\}_{k=0}^\infty$ and $\{b_k\}_{k=0}^\infty$ are two sequences of complex numbers, and n is a positive integer, then

$$\triangle^n (a_k b_k) = \sum_{j=0}^n \binom{n}{j} (\triangle^j a_k)(\triangle^{n-j} b_{k+j}). \tag{2.33}$$

LEMMA 4.4. *If* $\{a_j\}_{j=0}^\infty$ *is a bounded sequence of complex numbers satisfying* $\sum_{j=1}^\infty |\triangle^{\ell+1} a_j| j^\ell < \infty$ *for some nonnegative integer* ℓ, *then* $\lim_{n\to\infty} a_n =: L$ *exists, and the series* $\sum_{j=0}^\infty (a_j - L)$ *converges and satisfies* $\sum_{j=0}^\infty (a_j - L) = \sum_{j=0}^\infty (\triangle^{\ell+1} a_j) A_j^\ell (s_j^\ell - L)$, *where* $s_n^\ell = (A_n^\ell)^{-1} \sum_{j=0}^n A_{n-j}^\ell a_j$.

PROOF. We first claim that for each positive integer i,

$$\sum_{j=0}^\infty |\triangle^i a_j| A_j^{i-1} \le \sum_{j=0}^\infty |\triangle^{i+1} a_j| A_j^i. \tag{2.34}$$

Without loss of generality, we may assume that the infinite sum on the right hand side of this last equation is finite. Then for each $k, m \in \mathbb{N}$,

$$|\triangle^i a_k - \triangle^i a_{k+m}| = \Big|\sum_{j=k}^{k+m-1} \triangle^{i+1} a_j\Big| \le \sum_{j=k}^{k+m-1} |\triangle^{i+1} a_j| A_j^i \to 0, \quad \text{as } k \to \infty.$$

This means that $\{\triangle^i a_k\}$ is a Cauchy sequence in $\mathbb{C}$, and therefore, is convergent. Since

$$\left|\sum_{j=k}^{2k-1} \triangle^i a_j\right| = \left|\triangle^{i-1} a_k - \triangle^{i-1} a_{2k}\right| \le C_i \sup_j |a_j| < \infty,$$

we must have $\lim_{n\to\infty} \triangle^i a_n = 0$, which further implies $|\triangle^i a_n| = |\sum_{j=n}^\infty \triangle^{i+1} a_j|$. The claim (2.34) then follows.

Now applying (2.34) ℓ times yields $\sum_{j=0}^\infty |\triangle a_j| \le \sum_{k=0}^\infty |\triangle^{\ell+1} a_k| A_k^\ell < \infty$. This, in particular, implies that $\lim_{n\to\infty} a_n = L$ exists, and $\lim_{n\to\infty} \triangle^i a_n = 0$ for all positive integers i. Thus, to complete the proof, we just need to apply summation by parts $\ell+1$ times to the partial sums $s_n := \sum_{j=0}^n (a_j - L)$ and then letting $n \to \infty$. □

4.2. Three crucial Lemmas. In this subsection, we shall establish three crucial lemmas for the proofs of Theorems 4.2 and 4.3.

LEMMA 4.5. *If $\delta \ge 0$, $1 - \frac{1}{N} \le r < 1$ and $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$ then*

$$S_N^\delta f = r^{-N} P_r^\kappa (S_N^\delta f) + \sum_{j=0}^{N-1} a_{j,N}^\delta P_r^\kappa (S_j^\delta f)$$

for some real numbers $a_{j,N}^\delta$ satisfying

$$\max_{0\le j\le N-1} |a_{j,N}^\delta| \le C(1-r), \tag{2.35}$$

where $C > 0$ is independent of f and N.

PROOF. Recalling that $(1-s)^{-\delta-1} = \sum_{j=0}^\infty A_j^\delta s^j$ for $0 \le s < 1$, we obtain

$$(1-s)^{-\delta-1} P_s^\kappa f = \sum_{n=0}^\infty (A_n^\delta S_n^\delta f) s^n. \tag{2.36}$$

On the other hand, since $P_s^\kappa f = P_{s/r}^\kappa P_r^\kappa f$ for $0 < s < r < 1$, we have

$$\begin{aligned}(1-s)^{-\delta-1} P_s^\kappa f &= (1-s)^{-\delta-1} (1 - \frac{s}{r})^{1+\delta} \Big(1 - \frac{s}{r}\Big)^{-1-\delta} P_{s/r}^\kappa (P_r^\kappa f) \\ &= \Big(\sum_{j=0}^\infty A_j^\delta s^j\Big)\Big(\sum_{j=0}^\infty A_j^{-\delta-2} (s/r)^j\Big)\Big(\sum_{j=0}^\infty A_j^\delta P_r^\kappa (S_j^\delta f)(s/r)^j\Big) \\ &= \sum_{n=0}^\infty s^n \Bigg[\sum_{j=0}^n A_j^\delta P_r^\kappa (S_j^\delta f) r^{-j} \sum_{k+\ell=n-j} A_k^\delta A_\ell^{-\delta-2} r^{-\ell}\Bigg], \end{aligned} \tag{2.37}$$

where the second step uses (2.36) with s/r in place of s. Thus, comparing the coefficients of s^n in (2.36) and (2.37) yields that

$$S_N^\delta f = \sum_{j=0}^N a_{j,N}^\delta P_r^\kappa (S_j^\delta f),$$

where

$$a_{j,N}^\delta = (A_N^\delta)^{-1} A_j^\delta r^{-j} \Bigg(\sum_{k+\ell=N-j} A_k^\delta A_\ell^{-\delta-2} r^{-\ell}\Bigg).$$

Clearly, $a_{N,N}^\delta = r^{-N}$. Thus, it remains to verify (2.35) for $1 - N^{-1} \le r \le 1$. Observing that $r^{-\ell} \le r^{-N} \le c$ for all $0 \le \ell \le N$, we obtain that for $0 \le j \le N-1$,

$$|a_{j,N}^\delta| \le c \max_{1\le M\le N} \Bigg|\sum_{\ell=0}^M A_{M-\ell}^\delta A_\ell^{-\delta-2} r^{M-\ell}\Bigg| = c \max_{1\le M\le N} \Bigg|\sum_{\ell=0}^M A_{M-\ell}^\delta A_\ell^{-\delta-2} (1 - r^{M-\ell})\Bigg|,$$

where the last step uses the identity $\sum_{\ell=0}^M A_{M-\ell}^\delta A_\ell^{-\delta-2} = A_M^{-1} = 0$. Thus, for the proof of (2.35), it suffices to show that for $1 \le M \le N$

$$\Bigg|\sum_{\ell=0}^M A_{M-\ell}^\delta A_\ell^{-\delta-2} \big(1 - r^{M-\ell}\big)\Bigg| \le C(1-r). \tag{2.38}$$

To this end, let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta(x) = 1$ for $|x| \le \frac{1}{4}$, and $\eta(x) = 0$ for $|x| \ge \frac{1}{2}$. We then split the sum on the left hand side of (2.38) into two parts: $\Sigma_1 + \Sigma_2$, where

$$\Sigma_1 = \sum_{j=0}^{M} \eta\Big(\frac{j}{M}\Big) A_{M-j}^\delta A_j^{-\delta-2}(1 - r^{M-j}),$$
$$\Sigma_2 = \sum_{j=0}^{M} \Big(1 - \eta\Big(\frac{j}{M}\Big)\Big) A_{M-j}^\delta A_j^{-\delta-2}(1 - r^{M-j}).$$

Using (2.32), and direct computations show that

$$|\Sigma_2| \le c(1-r) \sum_{M/4 \le j \le M} (M-j+1)^{\delta+1} j^{-\delta-2} \le c(1-r).$$

To estimate Σ_1, let k be a positive integer $\ge \delta$, and for convenience, we set $A_j^\delta = 0$ for $j < 0$. Then using (2.31), (2.32), and summation by parts k times, we obtain

$$\begin{aligned} |\Sigma_1| &\le \sum_{0 \le j \le M/2} \Big| \triangle^k \Big(A_{M-j}^\delta \eta\Big(\frac{j}{M}\Big)(1 - r^{M-j})\Big)\Big| A_j^{-\delta-2+k} \\ &\le c \sum_{0 \le j \le M/2} \Big(M^{\delta-k}(1 - r^{M-j}) + \max_{\substack{i_1+i_2=k \\ 1 \le i_2 \le k}} M^{\delta-i_1}(1-r)^{i_2}\Big)(j+1)^{-\delta-2+k} \\ &\le c(1-r)M^{\delta-k+1} \sum_{j=1}^{M} j^{-\delta-2+k} \le c(1-r), \end{aligned}$$

where the third step uses the facts that $1 - r^{M-j} \le (M-j)(1-r)$ and $1 - r \le N^{-1} \le M^{-1}$. Putting the above together, we deduce (2.35), and hence complete the proof. $\square$

LEMMA 4.6. *If $\delta \ge 0$, $0 < r < 1$ and $f \in L(h_\kappa^2; \mathbb{S}^{d-1})$ then*

$$P_r^\kappa(S_N^\delta f) = \sum_{j=0}^{N} b_{j,N}^\delta S_j^\delta f,$$

for some real numbers $b_{j,N}^\delta$ independent of f and satisfying

$$\sum_{j=0}^{N} |b_{j,N}^\delta| \le C_\delta. \tag{2.39}$$

PROOF. Using (2.36), we have, for $0 < s, r < 1$,

$$(1-s)^{-\delta-1} P_s^\kappa P_r^\kappa f = \sum_{n=0}^{\infty} A_n^\delta (P_r^\kappa S_n^\delta f) s^n.$$

On the other hand, however,

$$\begin{aligned} (1-s)^{-1-\delta} P_s^\kappa P_r^\kappa f &= (1-s)^{-\delta-1} P_{sr}^\kappa f = (1-s)^{-\delta-1}(1-sr)^{1+\delta}(1-sr)^{-\delta-1} P_{sr}^\kappa f \\ &= \Big(\sum_{j=0}^{\infty} A_j^\delta s^j\Big)\Big(\sum_{j=0}^{\infty} A_j^{-\delta-2} s^j r^j\Big)\Big(\sum_{j=0}^{\infty} A_j^\delta (S_j^\delta f)(sr)^j\Big) \\ &= \sum_{n=0}^{\infty} s^n \Big[\sum_{j=0}^{n} A_j^\delta (S_j^\delta f) r^j \sum_{k+\ell=n-j} A_k^\delta A_\ell^{-\delta-2} r^\ell\Big]. \end{aligned}$$

Thus, comparing the coefficients of s^N yields $P_r^\kappa(S_N^\delta f) = \sum_{j=0}^N b_{j,N}^\delta S_j^\delta f$, where

$$b_{j,N}^\delta := (A_N^\delta)^{-1} A_j^\delta r^j \Big[\sum_{\ell=0}^{N-j} A_{N-j-\ell}^\delta A_\ell^{-\delta-2} r^\ell \Big].$$

It remains to show that the $b_{j,N}^\delta$ satisfy (2.39).

Clearly, $b_{N,N}^\delta = r^N$. For $0 \le j \le N-1$, we claim that

$$\Big| \sum_{\ell=0}^{N-j} A_{N-j-\ell}^\delta A_\ell^{-\delta-2} r^\ell \Big| \le C(1-r) + C(N-j)^\delta (1-r)^{1+\delta}. \tag{2.40}$$

To see this, we set $M = N-j$, and let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta(x) = 1$ for $|x| \le \frac{1}{4}$, and $\eta(x) = 0$ for $|x| \ge \frac{1}{2}$. Since $\sum_{\ell=0}^M A_{M-\ell}^\delta A_\ell^{-\delta-2} = A_M^{-1} = 0$, it follows that

$$\Big| \sum_{\ell=0}^{M} A_{M-\ell}^\delta A_\ell^{-\delta-2} r^\ell \Big| = \Big| \sum_{\ell=0}^{M} A_{M-\ell}^\delta A_\ell^{-\delta-2} (1-r^\ell) \Big| \le J_1 + J_2,$$

where

$$J_1 = \Big| \sum_{\ell=0}^{M} A_{M-\ell}^\delta A_\ell^{-\delta-2} \eta(\frac{\ell}{M})(1-r^\ell) \Big|,$$
$$J_2 = \Big| \sum_{\ell=0}^{M} A_{M-\ell}^\delta A_\ell^{-\delta-2} (1-\eta(\frac{\ell}{M}))(1-r^\ell) \Big|.$$

Clearly,

$$J_2 \le C \sum_{M/4 \le \ell \le M} (M-\ell+1)^\delta (\ell+1)^{-\delta-2} \ell (1-r) \le c(1-r).$$

To estimate J_1, we set $A_j^\delta = 0$ for $j<0$, and let k be the smallest integer such that $-\delta-2+k > -1$. Then summation by parts k times yields

$$\begin{aligned} J_1 &\le \sum_{\ell=0}^{M} \Big| \triangle^k \Big(A_{M-\ell}^\delta \eta\Big(\frac{\ell}{M}\Big)(1-r^\ell)\Big)\Big| A_\ell^{-\delta-2+k} \\ &\le c \sum_{\ell=0}^{M/2} \Big[M^{\delta-k}(1-r^\ell) + \max_{\substack{i+j=k \\ 1\le j\le k}} M^{\delta-i} r^\ell (1-r)^j \Big] A_\ell^{-\delta-2+k} \\ &\le c(1-r) + cM^\delta (1-r)^k \sum_{\ell=0}^{\infty} A_\ell^{-\delta-2+k} r^\ell \le c(1-r) + cM^\delta(1-r)^{1+\delta}. \end{aligned}$$

Putting these together, we prove the claim (2.40).

Now using (2.40), we deduce that for $0 \le j \le N-1$,

$$|b_{j,N}^\delta| \le (A_N^\delta)^{-1} A_j^\delta r^j \Big| \sum_{\ell=0}^{N-j} A_{N-j-\ell}^\delta A_\ell^{-\delta-2} r^\ell \Big| \le Cr^j(1-r) + cA_j^\delta r^j (1-r)^{1+\delta}.$$

It then follows that

$$\sum_{j=0}^N |b_{j,N}^\delta| \le 1 + c(1-r) \sum_{j=0}^\infty r^j + c(1-r)^{1+\delta} \sum_{j=0}^\infty A_j^\delta r^j \le c,$$

which proves (2.39), and hence completes the proof of the lemma. $\square$

We shall use the notation $|I|$ to denote the length of a given interval $I \subset \mathbb{R}$, and $\mathbb{Z}_+$ to denote the set of all nonnegative positive integers.

LEMMA 4.7. *Assume that $1 < p < \infty$, $\delta \ge 0$ and the inequality*

$$\Big\|\Big(\sum_{k=1}^{\infty}|S_{N_k}^{\delta}f_k|^2\Big)^{\frac12}\Big\|_{\kappa,p} \le C_p\Big\|\Big(\sum_{k=1}^{\infty}|f_k|^2\Big)^{\frac12}\Big\|_{\kappa,p} \tag{2.41}$$

holds for any $\{N_k\}\subset\mathbb{Z}_+$ and $\{f_k\}\subset L(h_\kappa^2;\mathbb{S}^{d-1})$. If $r_j\in(0,1)$ and I_j is a subinterval of $[r_j,1)$ for $j=1,2,\cdots$, then the inequality,

$$\Big\|\Big(\sum_{k=1}^{\infty}|S_{N_k}^{\delta}P_{r_k}^{\kappa}f_k|^2\Big)^{\frac12}\Big\|_{\kappa,p} \le C_p\Big\|\Big(\sum_{k=1}^{\infty}\frac{1}{|I_k|}\int_{I_k}|P_r^{\kappa}f_k|^2\,dr\Big)^{\frac12}\Big\|_{\kappa,p}, \tag{2.42}$$

holds for all $\{N_k\}_{k=1}^{\infty}\subset\mathbb{Z}_+$, and $\{f_k\}_{k=1}^{\infty}\subset L(h_\kappa^2;\mathbb{S}^{d-1})$, with the constant C_p being independent of $\{r_k\}$, $\{I_k\}$, $\{N_k\}$ and $\{f_k\}$.

PROOF. We first claim that for each $\{N_k\}\subset\mathbb{Z}_+$ and $\{f_k\}\subset L(h_\kappa^2;\mathbb{S}^{d-1})$,

$$\Big\|\Big(\sum_{k=1}^{\infty}|S_{N_k}^{\delta}P_{r_k}^{\kappa}f_k|^2\Big)^{\frac12}\Big\|_{\kappa,p} \le C_p\Big\|\Big(\sum_{k=1}^{\infty}|f_k|^2\Big)^{\frac12}\Big\|_{\kappa,p}. \tag{2.43}$$

To see this, we use Lemma 4.6 and obtain

$$|S_{N_k}^{\delta}P_{r_k}^{\kappa}f_k|^2 \le C\sum_{\ell=0}^{N_k}|b_{\ell,N_k}^{\delta}||S_\ell^{\delta}f_k|^2,\quad k=1,2,\cdots.$$

Summing over k, and invoking (2.53), we deduce

$$\begin{aligned}\Big\|\Big(\sum_{k=1}^{\infty}|S_{N_k}^{\delta}P_{r_k}^{\kappa}f_k|^2\Big)^{\frac12}\Big\|_{\kappa,p} &\le C_p\Big\|\Big(\sum_{k=1}^{\infty}\sum_{\ell=0}^{N_k}|b_{\ell,N_k}^{\delta}||f_k|^2\Big)^{\frac12}\Big\|_{\kappa,p}\\ &\le C\Big\|\Big(\sum_{k=1}^{\infty}|f_k|^2\Big)^{\frac12}\Big\|_{\kappa,p},\end{aligned}$$

which proves the claim (2.43).

Next, we show that the desired inequality (2.42) follows from (2.43). To see this, for each $k\ge 0$ and $n\ge 1$, we let $\{r_{k,i}\}_{i=0}^{2^n}\subset I_k$ be such that $r_{k,i}-r_{k,i-1}=2^{-n}|I_k|$ for all $1\le i\le 2^n$. Then for each $n\in\mathbb{N}$, $R_n := 2^{-n}\sum_{i=1}^{2^n}|P_{r_{k,i}}^{\kappa}f_k|^2$ is a Riemann sum of the integral $\frac{1}{|I_k|}\int_{I_k}|P_r^{\kappa}f_k|^2\,dr$, and moreover, the sequence $\{R_n\}_{n=1}^{\infty}$ increases to the integral $\frac{1}{|I_k|}\int_{I_k}|P_r^{\kappa}f_k|^2\,dr$. Thus, by the dominated convergence theorem, it follows that

$$\Big\|\Big(\sum_{k=1}^{\infty}\frac{1}{|I_k|}\int_{I_k}|P_r^{\kappa}f_k|^2\,dr\Big)^{\frac12}\Big\|_{\kappa,p} = \lim_{n\to\infty}\Big\|\Big(2^{-n}\sum_{k=1}^{\infty}\sum_{i=1}^{2^n}|P_{r_{k,i}}^{\kappa}f_k|^2\Big)^{\frac12}\Big\|_{\kappa,p}.$$

On the other hand, however, since for each fixed $n\in\mathbb{N}$, $r_k<r_{k,i}$ for all $1\le i\le n$ and $k\in\mathbb{N}$, using (2.43), we have

$$\begin{aligned}\Big\|\Big(\sum_{k=1}^{\infty}|S_{N_k}^{\delta}P_{r_k}^{\kappa}f_k|^2\Big)^{\frac12}\Big\|_{\kappa,p} &= \Big\|\Big(2^{-n}\sum_{i=1}^{2^n}\sum_{k=1}^{\infty}|S_{N_k}^{\delta}P_{r_k/r_{k,i}}^{\kappa}(P_{r_{k,i}}^{\kappa}f_k)|^2\Big)^{\frac12}\Big\|_{\kappa,p}\\ &\le C_p\Big\|\Big(2^{-n}\sum_{i=1}^{2^n}\sum_{k=1}^{\infty}|S_{N_k}^{\delta}(P_{r_{k,i}}^{\kappa}f_k)|^2\Big)^{\frac12}\Big\|_{\kappa,p}.\end{aligned}$$

Thus, letting $n\to\infty$, we prove the desired inequality (2.42), and completes the proof of the lemma. □

4.3. Proofs of Theorems 4.2 and 4.3. We start with the proof of Theorem 4.2.

Proof of Theorem 4.2. The second part of Theorem 4.2 follows directly from Theorem 4.3, whose proof will be given later. So, here, we shall only prove the inequality $\|f\|_{\kappa,p} \le C_p \|g_\delta(f)\|_{\kappa,p}$. By Corollary 4.1, it suffices to show that for the Littlewood-Paley function $g(f)$ given in (2.27), and for any $\delta \ge 0$, one has $g(f) \le C g_\delta(f)$. To see this, we note that

$$\begin{aligned}\Big|\frac{\partial}{\partial r} P_r^\kappa f\Big| &= (1-r)^{\delta+1}(1-r)^{-\delta-1}\Big|\sum_{k=0}^\infty k r^{k-1} \operatorname{proj}_k^\kappa f\Big| \\ &= (1-r)^{\delta+1}\Big|\sum_{n=1}^\infty \Big(\sum_{k=0}^n k A_{n-k}^\delta \operatorname{proj}_k^\kappa f\ r^{n-1}\Big)\Big|.\end{aligned}$$

Since

$$S_n^{\delta+1} f - S_n^\delta f = -(n+\delta+1)^{-1}(A_n^\delta)^{-1}\sum_{k=0}^n k A_{n-k}^\delta \operatorname{proj}_k^\kappa f,$$

it follows that

$$\Big|\frac{\partial}{\partial r} P_r^\kappa f\Big| \le c(1-r)^{\delta+1}\sum_{n=1}^\infty n A_n^\delta |S_n^{\delta+1} f - S_n^\delta f| r^{n-1},$$

which, using the Cauchy-Schwartz inequality, implies

$$\begin{aligned}\Big|\frac{\partial}{\partial r} P_r^\kappa f\Big|^2 &\le c(1-r)^{2\delta+2}\Big(\sum_{n=1}^\infty n A_n^\delta |S_n^{\delta+1} f - S_n^\delta f|^2 r^{n-1}\Big)\Big(\sum_{n=1}^\infty n A_n^\delta r^{n-1}\Big) \\ &= c(1+\delta)(1-r)^\delta \sum_{n=1}^\infty n A_n^\delta |S_n^{\delta+1} f - S_n^\delta f|^2 r^{n-1}.\end{aligned}$$

Thus,

$$\begin{aligned}|g(f)|^2 &= \int_0^1 \Big|\frac{\partial}{\partial r} P_r^\kappa f\Big|^2 (1-r)\, dr \le c\sum_{n=1}^\infty n A_n^\delta |S_n^{\delta+1} f - S_n^\delta f|^2 \int_0^1 (1-r)^{1+\delta} r^{n-1}\, dr \\ &\le c\sum_{n=1}^\infty n^{-1}\big|S_n^{\delta+1} f - S_n^\delta f\big|^2 = |g_\delta(f)|^2,\end{aligned}$$

where the third step uses the fact that $\int_0^1 (1-r)^{\delta+1} r^{n-1}\, dr = \frac{\Gamma(\delta+2)\Gamma(n)}{\Gamma(n+\delta+2)} \sim n^{-\delta-2}$. This proves the desired inequality $g(f) \le c g_\delta(f)$.

It remains to prove Theorem 4.3.

Proof of Theorem 4.3. Without loss of generality, we may assume that $n \le \sum_{k=1}^n v_k \le 2n$, since otherwise we may consider the sequences $\widetilde{v}_j = 1$ and $\widetilde{v}_j = M^{-1} v_j + 1$. For convenience, we define, for $n = 1, 2, \cdots$,

$$E_n f = -(n+1+\delta)^{-1}\sum_{k=0}^n k \operatorname{proj}_k^\kappa f.$$

It is easily seen that for $0 \le j \le n$,

$$S_j^\delta(E_n f) = \frac{j+\delta+1}{n+\delta+1}\Big(S_j^{\delta+1} f - S_j^\delta f\Big). \tag{2.44}$$

Using Lemma 4.5, we obtain that for any $r \in [1-n^{-1}, 1)$,

$$\begin{aligned} S_n^{\delta+1} f - S_n^\delta f &= S_n^\delta(E_n f) = r^{-n} P_r^\kappa(S_n^\delta(E_n f)) + \sum_{j=1}^{n-1} a_{j,n}^\delta P_r^\kappa(S_j^\delta(E_n f)) \\ &= r^{-n}\Big(S_n^{\delta+1}(P_r^\kappa f) - S_n^\delta(P_r^\kappa f)\Big) \\ &\quad + \sum_{j=1}^{n-1} \frac{j+\delta+1}{n+\delta+1} a_{j,n}^\delta \Big[S_j^{\delta+1}(P_r^\kappa f) - S_j^\delta(P_r^\kappa f)\Big], \qquad (2.45) \end{aligned}$$

where $|a_{j,n}^\delta| \le c(1-r) \le cn^{-1}$, and the last step uses (2.44). Now let $\mu_1 = 1$, and $\mu_n = 1 + \sum_{i=1}^{n-1} v_i$ for $n > 1$. Clearly, $r_n := 1 - \frac{1}{\mu_n} \in [1-n^{-1}, 1-(2n-1)^{-1}]$. Thus, applying (2.45) with $r = r_n$, and setting $f_n = P_{r_n}^\kappa f$, we deduce

$$|S_n^{\delta+1} f - S_n^\delta f| \le c|S_n^{\delta+1} f_n - S_n^\delta f_n| + cn^{-2} \sum_{j=1}^{n-1} j|S_j^{\delta+1}(f_n) - S_j^\delta(f_n)|,$$

which, using the Cauchy-Schwartz inequality, implies

$$|S_n^{\delta+1} f - S_n^\delta f|^2 \le c|S_n^{\delta+1} f_n - S_n^\delta f_n|^2 + +cn^{-3} \sum_{j=1}^{n-1} j^2|S_j^{\delta+1} f_n - S_j^\delta f_n|^2. \qquad (2.46)$$

Therefore, for the proof of Theorem 4.3, by (2.46) and Corollary 4.1, it suffices to show the following two inequalities:

$$\left\|\left(\sum_{n=1}^\infty n^{-1}|S_n^{\delta+1} f_n - S_n^\delta f_n|^2 v_n\right)^{\frac12}\right\|_{\kappa,p} \le C_p\|g(f)\|_{\kappa,p} \qquad (2.47)$$

and

$$\left\|\left(\sum_{n=1}^\infty \frac{v_n}{n^4} \sum_{j=1}^{n-1} j^2|S_j^{\delta+1} f_n - S_j^\delta f_n|^2\right)^{\frac12}\right\|_{\kappa,p} \le C_p\|g(f)\|_{\kappa,p}. \qquad (2.48)$$

To this end, let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ for $|x| \ge 2$. For $N = 1, 2, \cdots$, we define the operator V_N by $V_N(f) = \sum_{j=0}^\infty \eta(\frac{j}{N}) \operatorname{proj}_j^\kappa f$, and the operator D_N by $D_N f = -\sum_{j=0}^{2N} j\eta(\frac{j}{N}) \operatorname{proj}_j^\kappa f$. Note that summation by parts $d+1$ times shows that

$$\sup_N |V_N f| \le \sup_N \sum_{j=0}^{2N} \left|\triangle^{d+1}\eta\left(\frac{j}{N}\right)\right| A_j^d |\sigma_j^d(f)| \le c \sup_j |\sigma_j^d(f)| \le cM(f). \qquad (2.49)$$

Also, observe that for $1 \le j \le n \le N$,

$$S_j^{\delta+1} f_n - S_j^\delta f_n = (j+\delta+1)^{-1} P_{r_n}^\kappa(S_j^\delta(D_N f)). \qquad (2.50)$$

Thus, using Lemma 4.7 and (2.50) with $j = n$, we obtain

$$\begin{aligned} &\left\|\left(\sum_{n=1}^N n^{-1}|S_n^{\delta+1} f_n - S_n^\delta f_n|^2 v_n\right)^{\frac12}\right\|_{\kappa,p} \le c\left\|\left(\sum_{n=1}^N \frac{v_n}{n^3}|P_{r_n}^\kappa(S_n^\delta(D_N f))|^2\right)^{\frac12}\right\|_{\kappa,p} \\ &\le c\left\|\left(\sum_{n=1}^N \frac{v_n}{n^3} \frac{1}{r_{n+1}-r_n} \int_{r_n}^{r_{n+1}} |P_r^\kappa(D_N f)|^2\,dr\right)^{\frac12}\right\|_{\kappa,p}. \end{aligned}$$

However, using (2.49),

$$|P_r^\kappa(D_N f)| = r\left|V_N\left(\frac{\partial}{\partial r} P_r^\kappa f\right)\right| \le cM\left(\frac{\partial}{\partial r} P_r^\kappa f\right).$$

Thus, applying the Fefferman-Stein inequality to the Riemann sums of the integrals $\int_{r_n}^{r_{n+1}}$, we obtain

$$\begin{aligned}
&\left\|\left(\sum_{n=1}^{N}\frac{v_n}{n^3}\frac{1}{r_{n+1}-r_n}\int_{r_n}^{r_{n+1}}|P_r^\kappa(D_N f)|^2\,dr\right)^{\frac12}\right\|_{\kappa,p}\\
&\quad\le c\left\|\left(\sum_{n=1}^{N}\frac{v_n}{n^3}\frac{1}{r_{n+1}-r_n}\int_{r_n}^{r_{n+1}}\left|M\left(\frac{\partial}{\partial r}P_r^\kappa f\right)\right|^2 dr\right)^{\frac12}\right\|_{\kappa,p}\\
&\quad\le C_p\left\|\left(\sum_{n=1}^{N}\frac{v_n}{n^3}\frac{1}{r_{n+1}-r_n}\int_{r_n}^{r_{n+1}}\left|\frac{\partial}{\partial r}P_r^\kappa f\right|^2 dr\right)^{\frac12}\right\|_{\kappa,p}.
\end{aligned}$$

Since $r_{n+1}-r_n=\frac{v_n}{\mu_n\mu_{n+1}}\sim\frac{v_n}{n^2}$ and $1-r\sim\frac1n$ for all $r\in[r_n,r_{n+1}]$, it follows that

$$\begin{aligned}
&\left\|\left(\sum_{n=1}^{N}\frac{v_n}{n^3}\frac{1}{r_{n+1}-r_n}\int_{r_n}^{r_{n+1}}\left|\frac{\partial}{\partial r}P_r^\kappa f\right|^2 dr\right)^{\frac12}\right\|_{\kappa,p}\\
&\quad\le C_p\left\|\left(\sum_{n=1}^{\infty}\frac{1}{n}\int_{r_n}^{r_{n+1}}\left|\frac{\partial}{\partial r}P_r^\kappa f\right|^2 dr\right)^{\frac12}\right\|_{\kappa,p}\\
&\quad\le C_p\|g(f)\|_{\kappa,p}.
\end{aligned}$$

Putting the above together, and letting $N\to\infty$, we deduce the inequality (2.47).

The proof of the second inequality (2.48) is similar. In fact, using Lemma 4.7 and (2.50), we have

$$\begin{aligned}
&\left\|\left(\sum_{n=1}^{N}\frac{v_n}{n^4}\sum_{j=1}^{n-1}j^2|S_\ell^{\delta+1}f_n-S_\ell^\delta f_n|^2\right)^{\frac12}\right\|_{\kappa,p}\\
&\quad\le c\left\|\left(\sum_{n=1}^{N}\frac{v_n}{n^4}\sum_{j=1}^{n-1}|P_{r_n}^\kappa(S_j^\delta D_N f)|^2\right)^{\frac12}\right\|_{\kappa,p}\\
&\quad\le C_p\left\|\left(\sum_{n=1}^{\infty}\frac{v_n}{n^4}\sum_{j=1}^{n-1}\frac{1}{r_{n+1}-r_n}\int_{r_n}^{r_{n+1}}|\frac{\partial}{\partial r}P_r^\kappa f|^2\,dr\right)^{\frac12}\right\|_{\kappa,p}\\
&\quad\le C_p\left\|\left(\sum_{n=1}^{\infty}\frac{v_n}{n^3}\frac{1}{r_{n+1}-r_n}\int_{r_n}^{r_{n+1}}|\frac{\partial}{\partial r}P_r^\kappa f|^2\,dr\right)^{\frac12}\right\|_{\kappa,p}\le C_p\|g(f)\|_{\kappa,p}.
\end{aligned}$$

Letting $N\to\infty$, we deduce the desired inequality (2.48).

5. Proof of Theorem 2.2

Lemma 5.1. *If $\{\mu_j\}$ is a sequence of complex numbers satisfying (A_k) for some positive integer k, then $\{\mu_j\}$ satisfies (A_i) for all $1\le i\le k$, with a possible change of the absolute constant M.*

Proof. It suffices to show that (A_k) implies (A_{k-1}) for any $k\ge 2$. Indeed, if $k\ge2$ then (A_k) implies that $\sum_{\ell=1}^\infty|\triangle^k\mu_\ell|<\infty$, and hence, from the proof of Lemma 4.4, it follows that $\lim_{\ell\to\infty}\triangle^{k-1}\mu_\ell=0$. Thus,

$$\sum_{\ell=2^j+1}^{2^{j+1}}|\triangle^{k-1}\mu_\ell|\le\sum_{\ell=2^j+1}^{2^{j+1}}\sum_{i=\ell}^{\infty}|\triangle^k\mu_i|\le 2^j\sum_{i=2^j+1}^{\infty}|\triangle^k\mu_i|\le 2M2^{-j(k-2)},$$

which proves (A_{k-1}). □

LEMMA 5.2. *Let $\{a_j\}_{j=0}^{\infty}$ be a sequence of complex numbers, and let δ be a nonnegative integer. If $\{\mu_j\}_{j=0}^{\infty}$ is a sequence of complex numbers satisfying (A_0) and $(A_{\delta+1})$, then the Cesàro means $\sigma_N^\delta := (A_N^\delta)^{-1}\sum_{j=0}^{N} A_{N-j}^\delta \mu_j a_j$ of the sequence $\{\mu_j a_j\}$ can be written in the form $\sigma_N^\delta = \mu_N S_N^\delta + \sum_{\ell=0}^{N-1} C_{\ell,N}^\delta S_\ell^\delta$, where $S_\ell^\delta = (A_\ell^\delta)^{-1}\sum_{j=0}^{\ell} A_{\ell-j}^\delta a_j$, and the constants $C_{\ell,N}^\delta$ are independent of $\{a_j\}$ and satisfy*

$$|C_{\ell,N}^\delta| \le c\sum_{k=1}^{\delta+1}(\ell+1)^{k-1}|\triangle^k \mu_\ell|, \quad \ell = 0,1,\cdots,N-1. \tag{2.51}$$

PROOF. Using Lemma 4.4, we have, for $0<r<1$,

$$\sum_{n=0}^{\infty} A_n^\delta \sigma_n^\delta r^n = (1-r)^{-\delta-1}\sum_{n=0}^{\infty}\mu_n a_n r^n = (1-r)^{-\delta-1}\sum_{n=0}^{\infty}\triangle^{\delta+1}(\mu_n r^n)A_n^\delta S_n^\delta,$$

whereas, by (2.33), one has

$$\begin{aligned}\triangle^{\delta+1}(\mu_n r^n) &= \sum_{k=0}^{\delta+1}\binom{\delta+1}{k}(\triangle^k\mu_n)(\triangle^{\delta+1-k}r^{n+k})\\ &= \sum_{k=0}^{\delta+1}(-1)^k A_k^{-\delta-2}(\triangle^k\mu_n)r^{n+k}(1-r)^{\delta+1-k}.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} A_n^\delta\sigma_n^\delta r^n &= \sum_{k=0}^{\delta+1}(-1)^k A_k^{-\delta-2}(1-r)^{-k}\left(\sum_{n=0}^{\infty}(\triangle^k\mu_n)A_n^\delta S_n^\delta r^{n+k}\right)\\ &= \sum_{N=0}^{\infty}\left(\sum_{k=0}^{\delta+1}(-1)^k A_k^{-\delta-2}\sum_{\substack{i+j=N-k\\ i,j\ge 0}} A_j^{k-1}(\triangle^k\mu_i)A_i^\delta S_i^\delta\right)r^N.\end{aligned}$$

Comparing the coefficients of s^N yields

$$\begin{aligned}\sigma_N^\delta &= (A_N^\delta)^{-1}\sum_{k=0}^{\delta+1}(-1)^k A_k^{-\delta-2}\sum_{i=0}^{N-k} A_{N-k-i}^{k-1}(\triangle^k\mu_i)A_i^\delta S_i^\delta\\ &= (A_N^\delta)^{-1}\sum_{\ell=0}^{N} A_\ell^\delta S_\ell^\delta \sum_{k=0}^{\min\{N-\ell,\delta+1\}}(-1)^k A_k^{-\delta-2}A_{N-k-\ell}^{k-1}\triangle^k\mu_\ell\\ &= \mu_N S_N^\delta + \sum_{\ell=0}^{N-1} C_{\ell,N}^\delta S_\ell^\delta,\end{aligned}$$

where

$$C_{\ell,N}^\delta = (A_N^\delta)^{-1}A_\ell^\delta \sum_{k=1}^{\min\{N-\ell,\delta+1\}}(-1)^k A_k^{-\delta-2}A_{N-\ell-k}^{k-1}\triangle^k\mu_\ell$$

for $0\le\ell\le N-1$, and we have used the fact that $A_{N-\ell}^{-1}=0$ for $\ell<N$.

Finally, for $0\le\ell\le N-1$, we have

$$\begin{aligned}|C_{\ell,N}^\delta| &\le C\ell^\delta N^{-\delta}\sum_{k=1}^{\delta+1}(N-\ell)^{k-1}|\triangle^k\mu_\ell| \le C\sum_{k=1}^{\delta+1}(N-\ell)^{k-1}\left(\frac{\ell}{N}\right)^{k-1}|\triangle^k\mu_\ell|\\ &\le C\sum_{k=1}^{\delta+1}\ell^{k-1}|\triangle^k\mu_\ell|,\end{aligned}$$

which proves the desired inequality (2.51). $\square$

Now we are in a position to prove Theorem 2.2.

Proof of Theorem 2.2. Without loss of generality, we may assume $\mu_0 = 0$ and $M = 1$. Let δ be the smallest integer $> \lambda_\kappa$. Then (2.30) follows by Theorem 3.14. Now let $F = \sum_{j=1}^\infty \mu_j \operatorname{proj}_j^\kappa f$. By Theorems 4.2 and 4.3, it suffices to show that the inequality

$$g_\delta(F) \le C\bigg(\sum_{n=1}^\infty |S_n^{\delta+1}f - S_n^\delta f|^2 v_n n^{-1}\bigg)^{\frac{1}{2}} \tag{2.52}$$

holds for some sequence $\{v_n\}$ of positive numbers satisfying $\sup_N N^{-1}\sum_{j=1}^N v_j < \infty$.

To show (2.52), we use Lemma 5.2 to obtain

$$\begin{aligned} S_n^{\delta+1}F - S_n^\delta F &= \frac{-1}{n+\delta+1}(A_n^\delta)^{-1}\sum_{j=0}^n A_{n-j}^\delta \mu_j j \operatorname{proj}_j^\kappa f \\ &= \frac{1}{n+\delta+1}\bigg(\mu_n\sigma_n^\delta + \sum_{\ell=0}^{n-1} C_{\ell,n}^\delta \sigma_\ell^\delta\bigg), \end{aligned}$$

where

$$\sigma_\ell^\delta = -(A_\ell^\delta)^{-1}\sum_{j=0}^\ell A_{\ell-j}^\delta j \operatorname{proj}_j^\kappa f = (\ell+\delta+1)\big(S_\ell^{\delta+1}f - S_\ell^\delta f\big).$$

It then follows by (2.51) that

$$|S_n^{\delta+1}F - S_n^\delta F| \le |\mu_n||S_n^{\delta+1}f - S_n^\delta f| + Cn^{-1}\sum_{j=1}^{\delta+1}\sum_{\ell=1}^{n-1}\ell^j|\triangle^j\mu_\ell||S_\ell^{\delta+1}f - S_\ell^\delta f|.$$

Using Lemma 5.1 and $(A_{\delta+1})$, we have

$$\sum_{j=1}^{\delta+1}\sum_{\ell=1}^{n-1}\ell^j|\triangle^j\mu_\ell| \le cn. \tag{2.53}$$

Thus, using the Cauchy-Schwartz inequality, we deduce

$$\begin{aligned} |g_\delta(F)|^2 &\le c|g_\delta(f)|^2 + c\sum_{j=1}^{\delta+1}\sum_{n=1}^\infty n^{-2}\bigg(\sum_{\ell=1}^{n-1}\ell^j|\triangle^j\mu_\ell||S_\ell^{\delta+1}f - S_\ell^\delta f|^2\bigg) \\ &\le c|g_\delta(f)|^2 + c\sum_{\ell=1}^\infty |S_\ell^{\delta+1}f - S_\ell^\delta f|^2\sum_{j=1}^{\delta+1}\ell^{j-1}|\triangle^j\mu_\ell| \\ &\le c\sum_{n=1}^\infty |S_n^{\delta+1}f - S_n^\delta f|^2 v_n n^{-1}, \end{aligned}$$

where $v_n = 1 + \sum_{j=1}^{\delta+1}|\triangle^j\mu_n|n^j$. Finally, it follows directly from (2.53) that

$$n^{-1}\sum_{\ell=1}^n v_\ell = 1 + \sum_{j=1}^{\delta+1}n^{-1}\sum_{\ell=1}^n \ell^j|\triangle^j\mu_\ell| \le c.$$

This completes the proof of Theorem 2.2.

6. Notes and Further Results

The theory of h-harmonics is pioneered by C. Dunkl. The Dunkl operators were introduced in [**35**] and the intertwining operators and the integral kernel appeared in [**36**]. For more results on h-spherical harmonic analysis for general reflection groups, we refer to [**38**].

The first study of h-harmonic expansions appeared in [**95, 96**]. The multiplier theorem (Theorem 2.2) and its analogue on the unit ball and the simplex were proved in [**24**]. The Littlewood-Paley theory in Section 4 and the proof of Theorem 2.2 follow essentially the argument in Bonami and Clerc [**5**], The proof has been substantially simplified by using the operators defined via a cut-off function. A classical reference on general Littlewood-Paley theory is [**78**].

CHAPTER 3

Sharp Jackson and sharp inverse inequalities

1. Introduction

For trigonometric polynomials on $[-\pi, \pi] \equiv \mathbb{T}$, M. Timan [**83**] proved that for $1 < p < \infty$,

$$n^{-r}\left\{\sum_{k=1}^{n} k^{sr-1} E_k(f)_p^s\right\}^{1/s} \le C(r,p)\omega^r(f, n^{-1})_p, \quad s = \max(p, 2), \tag{3.1}$$

where $r \in \mathbb{N}$,

$$E_k(f)_p = \min\left\{\|f - T_n\|_{L_p(\mathbb{T})} : T_n \in \operatorname*{span}_{k<n}\{\sin kt, \cos kt\}\right\}$$

and

$$\omega^r(f,t)_p = \sup_{|h|\le t}\left\|\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} f(\cdot + jh)\right\|_{L_p(\mathbb{T})}.$$

Inequality (3.1) is clearly stronger than the classical Jackson inequality, $E_n(f)_p \le C\omega^r(f, 1/n)_p$, for $1 < p < \infty$. We call the generalization of Jackson-type inequality of the type given by (3.1) a sharp Jackson inequality.

An estimate of $\omega^r(f,t)_p$ in the direction opposite to (3.1) was also proved by M. Timan [**82**]: For $1 < p < \infty$ and $r \in \mathbb{N}$,

$$\omega^r(f, 1/n)_p \le C_1(r,p) n^{-r}\left\{\sum_{k=1}^{n} k^{rq-1} E_k(f)_p^q\right\}^{1/q}, \ q = \min(p, 2). \tag{3.2}$$

This estimate is essentially equivalent to the following: For $1 < p < \infty$ and $r \in \mathbb{N}$,

$$\omega^r(f,t)_p \le C_2(r,p)t^r\left\{\int_t^{1/2} u^{-qr-1}\omega^{r+1}(f,u)_p^q du\right\}^{1/q}, q = \min(p, 2). \tag{3.3}$$

Inequalities (3.2) and (3.3) are sometimes called a sharp inverse and a sharp Marchaud inequality respectively.

Similar to the sharp Marchaud inequality (3.3), one has in the other direction the formula: For $1 < p < \infty$,

$$t^r\left\{\int_t^{1/2} \frac{\omega^{r+1}(f,u)_p^s}{u^{sr+1}}\, du\right\}^{1/s} \le C\omega^r(f,t)_p, \quad s = \max(p, 2), \tag{3.4}$$

which is equivalent to (3.1).

Of particular interest is the case when $p = 2$, where, using (3.1) and (3.2), we have

$$\omega^r(f, 1/n)_2 \sim n^{-r}\left\{\sum_{k=1}^{n} k^{2r-1} E_k(f)_p^2\right\}^{1/2}.$$

The main goal in this chapter is to prove the multivariate analogues of the sharp Jackson (3.1) and the sharp inverse (3.2). While our method is applicable to more general settings, we shall focus only on ordinary spherical harmonic expansions for the sake of simplicity. The main tool in our argument is the multiplier theorem proved in the last chapter.

We organize the chapter as follows. Section 2 contains some preliminary results for spherical polynomial approximation for the rest of the chapter. In Section 3, we prove a refined version of the Littlewood-Paley inequality, which will be our main tool for the proof of the sharp Jackson inequality. The sharp Marchaud inequality and the sharp Jackson inequality are proved in Section 4 and Section 5 respectively. Finally, in Section 6, examples are given to show the optimality of the power q in the sharp Mauchaud inequality.

2. Preliminaries

In this section, we collect some of the known results on polynomial approximation on the sphere. Most of these results can be found in [**92**] and [**30**].

We start with the following definition of translation operators, which will be used to introduce a modulus of smoothness on the sphere:

DEFINITION 2.1. *The translation operator S_θ with step $\theta \in [0, \pi]$ is defined by*

$$S_\theta(f)(x) := \frac{1}{|\mathbb{S}^{d-2}| \sin^{d-2}\theta} \int_{\{y\in\mathbb{S}^{d-1}:\ \langle x,y\rangle=\cos\theta\}} f(y)\, d\ell_{x,\theta}(y), \quad x \in \mathbb{S}^{d-1},$$

where $f \in L^1(\mathbb{S}^{d-1})$ and $d\ell_{x,\theta}(y)$ denotes the Lebesgue measure element on the set $\{y \in \mathbb{S}^{d-1} :\ y \cdot x = \cos\theta\}$.

A significant fact on the operator S_θ lies in the fact that

$$\int_{\mathbb{S}^{d-1}} f(y)K(\langle x, y\rangle)\, d\sigma(y) = |\mathbb{S}^{d-2}| \int_0^\pi S_\theta(f)(x)K(\cos\theta) \sin^{d-2}\theta\, d\theta, \quad x \in \mathbb{S}^{d-1}.$$

We collect some useful results on S_θ in the following lemma.

LEMMA 2.2. *(i) For each θ, S_θ is a multiplier operator on $\mathbb{S}^{d-1}$ in the sense that*

$$\mathrm{proj}_k(S_\theta(f)) = \frac{C_k^\lambda(\cos\theta)}{C_k^\lambda(1)} \mathrm{proj}_k(f), \quad k = 0, 1, \cdots, \tag{3.5}$$

where $\lambda = \frac{d-2}{2}$.

(ii) Each S_θ is a positive operator satisfying

$$\|S_\theta f\|_p \le \|f\|_p, \quad 1 \le p \le \infty. \tag{3.6}$$

(iii) If the dimension d is even, then S_θ can be expressed as

$$S_\theta f(x) = \int_{SO(d)} f(Q^{-1}M_\theta Qx)\, dQ, \quad x \in \mathbb{S}^{d-1},\ \theta \in \mathbb{R}, \tag{3.7}$$

where $SO(d)$ denotes the group of $d \times d$ orthogonal matrices with determinant 1, dQ is the Haar measure on $SO(d)$ normalized by $\int_{SO(d)} dQ = 1$, and M_θ is a $d \times d$ matrix given by

$$M_\theta := \left\{\begin{matrix} \cos\theta & \sin\theta & & & \\ -\sin\theta & \cos\theta & & & \\ & & \ddots & & \\ & & & \cos\theta & \sin\theta \\ & & & -\sin\theta & \cos\theta \end{matrix}\right\}.$$

Next, we define the moduli of smoothness on the sphere.

Definition 2.3. *If $f \in L^p(\mathbb{S}^{d-1})$ and $1 \le p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ and $p = \infty$, then the rth order modulus of smoothness of f in L^p-metric is defined by*

$$\omega^r(f,t)_p = \sup_{0<\theta\le t} \|\triangle_\theta^r f\|_p,$$

where

$$\triangle_t^r := (I - S_t)^{\frac{r}{2}} \equiv \sum_{k=0}^{\infty} \frac{\frac{r}{2}(\frac{r}{2}-1)\cdots(\frac{r}{2}-k+1)}{k!}(S_t)^k. \tag{3.8}$$

The modulus of smoothness defined above is equivalent to a K-functional defined in terms of the fractional Laplacian-Beltrami operator. Recall that the Laplace -Beltrami operator Δ_0 on $\mathbb{S}^{d-1}$ is defined by

$$\Delta_0 f(x) := \Delta_{\mathbb{R}^d} F(x), \quad x \in \mathbb{S}^{d-1}, \quad f \in C^2(\mathbb{S}^{d-1}),$$

with $\Delta_{\mathbb{R}^d} = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ and $F(y) := f(y|y|^{-1})$, and that the space $\mathcal{H}_k^d$ of spherical harmonics of degree k on $\mathbb{S}^{d-1}$ is the space of eigenfunctions of Δ_0 corresponding to the eigenvalue $\lambda_k = -k(k+d-2)$:

$$\mathcal{H}_k^d = \Big\{ f \in C^2(\mathbb{S}^{d-1}) : \quad \Delta_0 f = \lambda_k f \Big\}, \quad k = 0, 1, \cdots .$$

From now on, we set $X^p = L^p(\mathbb{S}^{d-1})$ for $1 \le p < \infty$ and $X^p = C(\mathbb{S}^{d-1})$ for $p = \infty$.

Definition 2.4. *Give $\gamma \in \mathbb{R}$, we define the fractional Laplace -Beltrami operator $(-\Delta_0)^\gamma$ in a distributional sense by*

$$\mathrm{proj}_k(-\Delta_0)^\gamma f = \begin{cases} 0, & \text{if } k = 0, \\ (-k(k+d-2))^\gamma \,\mathrm{proj}_k(f), & \text{if } k = 1, 2, \cdots. \end{cases}$$

For $r > 0$ and $1 \le p \le \infty$, the Sobolev space W_p^r on $\mathbb{S}^{d-1}$ is defined to be

$$W_p^r := \Big\{ f \in X^p(\mathbb{S}^{d-1}) : \quad (-\Delta_0)^{r/2} f \in X^p(\mathbb{S}^{d-1}), \quad \int_{\mathbb{S}^{d-1}} f(x)\, d\sigma(x) = 0 \Big\}.$$

Definition 2.5. *For $r > 0$, $1 \le p \le \infty$ and $f \in X^p$, we define*

$$K_r(f,t)_p := \inf\Big\{ \|f-g\|_p + t^r \|(-\Delta_0)^{\frac{r}{2}} g\|_p : \ g \in W_p^r \Big\}. \tag{3.9}$$

We sometimes write $h^{(\alpha)} = (-\Delta_0)^{\frac{\alpha}{2}} h$ for $h \in W_1^\alpha$ for the sake of simplicity.

Theorem 2.6. *For $r > 0$, $1 \le p \le \infty$ and $f \in X^p$,*

$$\omega^r(f,t)_p \sim K_r(f,t^r)_p, \quad 1 \le p \le \infty, \quad t > 0, \tag{3.10}$$

where the constants of equivalence depend only on p, r and d.

Definition 2.7. *Let $f \in L^p(\mathbb{S}^{d-1})$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. For $n \ge 0$, the error of the best approximation to f by polynomials of degree at most n is defined by*

$$E_n(f)_p := \inf_{g \in \Pi_n(\mathbb{S}^{d-1})} \|f-g\|_p, \quad 1 \le p \le \infty. \tag{3.11}$$

The best approximation element exists, since $\Pi_n^d(\mathbb{S}^{d-1})$ is a finite dimensional space, by a general theorem in the Banach space ([**29**, p. 59]). Finding such a polynomial, however, is not easy. For most applications, fortunately, it is sufficient to find a polynomial that is near best approximation.

Definition 2.8. *Let η be a C^∞-function on $[0,\infty)$ such that $\eta(t) = 1$ for $0 \le t \le 1$ and $\eta(t) = 0$ for $t \ge \frac{3}{2}$. Define*

$$L_n(f) = \sum_{k=0}^{2n} \eta(\frac{k}{n}) \operatorname{proj}_k(f), \quad n > 0. \tag{3.12}$$

Theorem 2.9. *Let $f \in L^p$ if $1 \le p < \infty$ and $f \in C(\mathbb{S}^{d-1})$ if $p = \infty$. Then*
(1) *$L_n f \in \Pi^d_{2n-1}$ and $L_n f = f$ for $f \in \Pi^d_n$.*
(2) *For $n \in \mathbb{N}$, $\|L_n f\|_p \le c\|f\|_p$.*
(3) *For $n \in \mathbb{N}$,*

$$\|f - L_n f\|_p \le (1+c) E_n(f)_p. \tag{3.13}$$

We have the following very useful result:

Theorem 2.10. *For $1 \le p \le \infty$, $r \in \mathbb{N}$, and $n \in \mathbb{N}$,*

$$K_r(f, n^{-1})_p \sim \|f - L_n f\|_p + n^{-r} \|(-\Delta_0)^{\frac{r}{2}} L_n f\|_p, \tag{3.14}$$

where the constants of equivalence are independent of f and n.

Equivalence (3.14) is sometimes called the realization result for the K-functional. In general, such a result follows from the Cesàro summability of the orthogonal expansions.

Using the above theorem and the Bernstein inequality for spherical polynomials, we deduce the classical Jackson inequality and its Stechkin type inverse:

Theorem 2.11. *If $f \in L^p(\mathbb{S}^{d-1})$ and $1 \le p < \infty$ or $f \in C(\mathbb{S}^{d-1})$ and $p = \infty$, then for $r > 0$,*

$$E_n(f)_p \le C_{p,r} K_r(f, n^{-1})_p, \quad n = 1, 2, \cdots, \tag{3.15}$$

and

$$K_r(f, n^{-1})_p \le c_{p,r} n^{-r} \sum_{0 \le k \le n} (k+1)^{r-1} E_k(f)_p. \tag{3.16}$$

For general orthogonal expansions, Theorem 2.11 above follows from the Cesàro summability.

3. The Littlewood-Paley inequality

We start with the following definition.

Definition 3.1. *Let $\eta \in C^\infty[0,\infty)$ be as in Definition 2.8, and let $\theta(x) = \eta(2x) - \eta(4x)$. Define $\theta_0(f) = \operatorname{proj}_0(f)$, and*

$$\theta_j(f) := L_{2^{j-1}} f - L_{2^{j-2}} f = \sum_{n=0}^{\infty} \theta\left(\frac{n}{2^j}\right) \operatorname{proj}_n(f), \quad j = 1, 2, \cdots.$$

Since $\lim_{n\to\infty} L_n f = f$ in X^p for $1 \le p \le \infty$, it follows that

$$f = \sum_{j=0}^{\infty} \theta_j f, \quad \text{in } X^p. \tag{3.17}$$

Theorem 3.2. *For $1 < p < \infty$, $\gamma \ge 0$ and $f \in W^\gamma_p(\mathbb{S}^{d-1})$,*

$$\left\| \left\{ \sum_{j=1}^{\infty} 2^{2j\gamma} \left(\theta_j f\right)^2 \right\}^{1/2} \right\|_p \sim \|(-\Delta_0)^{\gamma/2} f\|_p, \tag{3.18}$$

where the constants of equivalence depend only on p, d and γ.

PROOF. We first prove the inequality

$$\Big\|\Big\{\sum_{j=1}^{\infty}2^{2j\gamma}\big(\theta_j f\big)^2\Big\}^{1/2}\Big\|_p \le C(p,\gamma)\|(-\Delta_0)^{\gamma/2}f\|_p,\quad \gamma\in\mathbb{R}. \tag{3.19}$$

Let $\{\xi_j\}_{j=0}^{\infty}$ be a sequence of independent, ±1-valued random variables with mean zero. Then

$$\Big(\mathbb{E}\Big|\sum_{j=0}^{\infty}a_j\xi_j\Big|^p\Big)^{1/p}\sim\Big(\sum_{j=0}^{\infty}|a_j|^2\Big)^{\frac12},\quad \forall a_j\in\mathbb{R},\ 0<p<\infty. \tag{3.20}$$

Consider the (random) linear operator

$$Tf=\sum_{j=0}^{\infty}2^{j\gamma}\xi_j\Big[\theta_j((-\Delta_0)^{-\gamma/2}f)\Big]. \tag{3.21}$$

Clearly, Tf can be rewritten in the form

$$Tf=\sum_{k=1}^{\infty}\mu_k\,\mathrm{proj}_k\,f,$$

where $\mu_k=A(k)$, and

$$A(u):=(-u(u+d-2))^{-\gamma/2}\sum_{j=1}^{\infty}2^{j\gamma}\theta\Big(\frac{u}{2^j}\Big)\xi_j.$$

A straightforward computation shows that

$$\Big\|\Big(\frac{d}{du}\Big)^r A(u)\Big\|\le C_r u^{-r},\quad u\ge1,\quad r=0,1,\cdots,$$

which implies that

$$\triangle^r\mu_k\le C_r'k^{-r},\quad r=0,1,\cdots,$$

where the constants C_r and C_r' are independent of the random variables ξ_j. Thus, by the Marcinkiewitcz multiplier theorem proved in last chapter, it follows that

$$\|Tf\|_p\le C_p\|f\|_p, \tag{3.22}$$

where C_p is a constant depending only on p and d. Using (3.20), (3.21), and (3.22), we deduce

$$\Big(\sum_{j=0}^{\infty}2^{2j\gamma}\big|\theta_j((-\Delta_0)^{-\gamma/2}f)\big|^2\Big)^{\frac12}\sim(\mathbb{E}\|Tf\|^p)^{1/p}\le C_p\|f\|_p.$$

Replacing $\|f\|_p$ with $(-\Delta_0)^{\gamma/2}f$ yields the desired inequality (3.19).

The inverse inequality

$$\Big\|\Big\{\sum_{j=1}^{\infty}2^{2j\gamma}\big(\theta_j f\big)^2\Big\}^{1/2}\Big\|_p\ge C'(p,\gamma)\|(-\Delta_0)^{\gamma/2}f\|_p \tag{3.23}$$

follows by a duality argumemt. Indeed, let $g\in L_{p'}(\mathbb{S}^{d-1})$ be such that $\|g\|_{p'}=1$ and $\|(-\Delta_0)^{\gamma/2}f\|_p=\int_{\mathbb{S}^{d-1}}\big[(-\Delta_0)^{\gamma/2}f\big]g\,d\sigma(x)$, where $\frac1p+\frac1{p'}=1$. Let $g_1\in W_{p'}^{\gamma}$ be such that $\|g-g_1\|_{p'}\le\frac12$. Then $\|g_1\|_{p'}\le\frac32$ and $\frac12\|(-\Delta_0)^{\gamma/2}f\|_p\le$

$\int_{\mathbb{S}^{d-1}}\big[(-\Delta_0)^{\gamma/2}f\big]g_1\,d\sigma(x)$. Observe that $\int_{\mathbb{S}^{d-1}}\theta_j(F)\theta_i(G)\,d\sigma(x)=0$ for $|i-j|\ge 2$ and any $F,G\in L^1(\mathbb{S}^{d-1})$. It follows that

$$\begin{aligned}
\|(-\Delta_0)^{\gamma/2}f\|_p &\le 2\int_{\mathbb{S}^{d-1}}\big[(-\Delta_0)^{\gamma/2}f\big]g_1\,d\sigma(x)=2\int_{\mathbb{S}^{d-1}}f\big[(-\Delta_0)^{\gamma/2}g_1\big]\,d\sigma(x)\\
&=2\sum_{k=-1}^{1}\sum_{j=\max(-k,0)}^{\infty}\int_{\mathbb{S}^{d-1}}(\theta_j f)\Big[\theta_{j+k}\big((-\Delta_0)^{\gamma/2}g_1\big)\Big]\,d\sigma(x)\\
&\le 6\int_{\mathbb{S}^{d-1}}\Big(\sum_{j=0}^{\infty}|\theta_j f|^2 2^{2j\gamma}\Big)^{1/2}\Big(\sum_{j=0}^{\infty}2^{-2j\gamma}\big|\theta_j\big((-\Delta_0)^{\gamma/2}g_1\big)\big|^2\Big)^{1/2}d\sigma(x)\\
&\le 6\Big\|\Big(\sum_{j=0}^{\infty}|\theta_j f|^2 2^{2j\gamma}\Big)^{1/2}\Big\|_p\Big\|\Big(\sum_{j=0}^{\infty}2^{-2j\gamma}\big|\theta_j\big((-\Delta_0)^{\gamma/2}g_1\big)\big|^2\Big)^{1/2}\Big\|_{p'},
\end{aligned}$$

which, using (3.19) applied to $-\gamma$ and $L_{p'}$, is dominated by

$$\begin{aligned}
C\Big\|\Big(\sum_{j=0}^{\infty}|\theta_j(f)|^2 2^{2j\gamma}\Big)^{1/2}\Big\|_p &\|(-\Delta_0)^{-\gamma/2}(-\Delta_0)^{\gamma/2}g_1\|_{p'}\\
&\le C'\Big\|\Big(\sum_{j=0}^{\infty}|\theta_j(f)|^2 2^{2j\gamma}\Big)^{1/2}\Big\|_p. \quad\square
\end{aligned}$$

4. The sharp Marchaud inequality

This section is devoted to proving the following sharp Marchaud inequality:

THEOREM 4.1. *For $\alpha>0$ and $1<p<\infty$ we have*

$$K_\alpha(f,t)_p\le Ct^\alpha\Big(\int_t^1\frac{K_{\alpha+1}(f,u)_p^q}{u^{q\alpha+1}}\,du\Big)^{\frac1q},\quad q=\min\{p,2\}.$$

From the Stechkin type inverse inequality 3.16, it is easily seen that Theorem 4.1 is essentially equivalent to the following sharp inverse inequality:

COROLLARY 4.2. *For $1<p<\infty$ and $\alpha>0$,*

$$K_\alpha(f,1/n)_p\le C_1(\alpha,p)n^{-\alpha}\Big\{\sum_{k=1}^{n}k^{\alpha q-1}E_k(f)_p^q\Big\}^{1/q},\ q=\min(p,2). \tag{3.24}$$

Recall that for the spherical harmonic expansions, we can define the Stein g-function as follows:

$$\widetilde{g}(f)=\Big(\int_0^1\big|\frac{\partial}{\partial r}P_r(f)\big|^2 r|\log r|\,dr\Big)^{\frac12}, \tag{3.25}$$

where P_rf denotes the Poisson integral of f:

$$P_rf=\sum_{j=0}^{\infty}r^j\,\mathrm{proj}_j\,f,\quad r\in(0,1).$$

The general theorem of Stein (Theorem 2.4 in Chapter 2) then asserts that for $1<p<\infty$ and $f\in L^p(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}}f(x)\,d\sigma(x)=0$,

$$\|f\|_p\approx\|\widetilde{g}(f)\|_p. \tag{3.26}$$

We shall present a proof of Theorem 4.1 that works equally well for orthogonal expansions such that the associated Cesàro means of order δ are positive for some $\delta>0$, from which both (3.26) and the realization result (3.14) will follow.

Recall that we write $F^{(\alpha)}=(-\Delta_0)^{\frac{\alpha}{2}}F$ for $h\in W_1^\alpha$ for the sake of simplicity.

Now we are in a position to prove Theorem 4.1:

PROOF. By the realization result (3.14), it suffices to show that

$$2^{-m\alpha}\|(L_{2^m}(f))^{(\alpha)}\|_p \le C2^{-m\alpha}\bigg(\sum_{j=0}^{m} 2^{j\alpha q}K_{\alpha+1}(f,2^{-j})_p^q\bigg)^{\frac{1}{q}} \tag{3.27}$$

with $q=\min\{p,2\}$. For the rest of the proof, we set $h=(L_{2^m}(f))^{(\alpha)}$ for simplicity. We then claim that for $\delta \ge 0$,

$$\widetilde{g}(h) \le C\bigg(\sum_{j=0}^{\infty} 2^{-j(1+q)}\sum_{k=2^j}^{2^{j+1}-1}|S_k^\delta(h')|^q\bigg)^{\frac{1}{q}}, \quad q=\min\{p,2\}, \tag{3.28}$$

where $\widetilde{g}$ is given by (3.25). For the moment, we take the claim for granted and proceed with the proof.

Indeed, using (3.28) and (3.26), we obtain

$$\|h\|_p \le C\|\widetilde{g}(h)\|_p \le C\bigg(\sum_{j=0}^{\infty} 2^{-j(1+q)}\sum_{k=2^j}^{2^{j+1}-1}\|S_k^\delta(h')\|_p^q\bigg)^{\frac{1}{q}}, \tag{3.29}$$

where the last step uses the Minkovskii inequality for $p>2$. We break the first infinite sum in (3.29) into two parts: $\sum_{j=0}^{m-4}$ and $\sum_{j=m-3}^{\infty}$. Observe that if $2^j \le k \le 2^{j+1}-1$ and $0\le j\le m-4$, then

$$S_k^\delta(h') = L_{2^{j+2}}(S_k^\delta(h')) = S_k^\delta(L_{2^{j+2}}(h')) = S_k^\delta((L_{2^{j+2}}(f))^{(\alpha+1)}). \tag{3.30}$$

It follows that for $\delta > \lambda := \frac{d-2}{2}$,

$$\begin{aligned}
&\bigg(\sum_{j=0}^{m-4} 2^{-j(1+q)}\sum_{k=2^j}^{2^{j+1}-1}\|S_k^\delta(h')\|_p^q\bigg)^{\frac{1}{q}} && (3.31)\\
&\quad\le C\bigg(\sum_{j=0}^{m-4} 2^{-jq}\Big\|(L_{2^{j+2}}(f))^{(\alpha+1)}\Big\|_p^q\bigg)^{\frac{1}{q}}\\
&\quad= C\bigg(\sum_{j=0}^{m-4} 2^{j\alpha q}\Big\|2^{-j(\alpha+1)}(L_{2^{j+2}}(f))^{(\alpha+1)}\Big\|_p^q\bigg)^{\frac{1}{q}}\\
&\quad\le C\bigg(\sum_{j=0}^{m-4} 2^{j\alpha q}K_{\alpha+1}(f,2^{-j})_p^q\bigg)^{\frac{1}{q}}, && (3.32)
\end{aligned}$$

where the second step uses (3.30) and the Cesàro (C,δ)-summability of spherical harmonic expansions for $\delta > \frac{d-2}{2}$, and the last step uses the realization result (3.14). On the other hand, for the sum $\sum_{j\ge m-3}$, using the Cesàro (C,δ)-summability for $\delta>\lambda$, we obtain

$$\begin{aligned}
&\bigg(\sum_{j=m-3}^{\infty} 2^{-j(1+q)}\sum_{k=2^j}^{2^{j+1}-1}\|S_k^\delta(h')\|_p^q\bigg)^{\frac{1}{q}} \le C\bigg(\sum_{j=m-3}^{\infty}2^{-jq}\bigg)^{\frac{1}{q}}\|h'\|_p\\
&\qquad\le C2^{-m}\|h'\|_p = C2^{-m}\|(L_{2^m}(f))^{(\alpha+1)}\|_p \le C2^{m\alpha}K_{\alpha+1}(f,2^{-m})_p, \quad (3.33)
\end{aligned}$$

where the last step uses the realization result (3.14) again. Therefore, combining (3.29), (3.32) with (3.33), we deduce the desired estimate (3.27).

It remains to prove the claim (3.28). We note that

$$\frac{\partial}{\partial r}P_r(h) = \sum_{k=1}^{\infty} kr^{k-1}\operatorname{proj}_k(h) = (1-r)^{\delta+1}\sum_{k=1}^{\infty}\bigg(\sum_{j=1}^{k} jA_{k-j}^\delta \operatorname{proj}_j(h)\bigg)r^{k-1},$$

where we used the identity, $\sum_{k=0}^{\infty} A_k^{\delta} r^k = (1-r)^{-1-\delta}$, in the last step. Thus,

$$\begin{aligned}\Big|\frac{\partial}{\partial r}P_r(h)\Big| &\leq (1-r)^{\delta+1}\sum_{k=1}^{\infty} A_k^{\delta}|S_k^{\delta}(h')|r^{k-1}\\ &\leq C(1-r)^{1+\delta}\sum_{j=0}^{\infty}2^{j\delta}r^{2^j-1}\sum_{k=2^j}^{2^{j+1}-1}|S_k^{\delta}(h')|,\end{aligned}$$

which, using the Cauchy-Schwartz inequality, implies

$$\begin{aligned}\Big|\frac{\partial}{\partial r}P_r(h)\Big|^2 &\leq C(1-r)^{2+2\delta}\Big(\sum_{j=0}^{\infty}2^{j\delta}r^{2^j-1}\Big(\sum_{k=2^j}^{2^{j+1}-1}|S_k^{\delta}(h')|\Big)^2\Big)\Big(\sum_{\ell=0}^{\infty}2^{\ell\delta}r^{2^\ell-1}\Big)\\ &= C\sum_{j=0}^{\infty}\Big[2^{j\delta}\Big(\sum_{k=2^j}^{2^{j+1}-1}|S_k^{\delta}(h')|\Big)^2\sum_{\ell=0}^{\infty}2^{\ell\delta}r^{2^\ell+2^j-2}(1-r)^{2+2\delta}\Big]. \qquad (3.34)\end{aligned}$$

On the other hand, however, since

$$r|\log r|\leq C(1-r),\quad r\in(0,1),$$

and

$$\frac{\Gamma(x+a)}{\Gamma(x)} = x^a + O(x^{a-1}),\qquad \text{as } x\to\infty,\quad a\in\mathbb{R},$$

it follows that

$$\begin{aligned}\sum_{\ell=0}^{\infty}2^{\ell\delta}\int_0^1 r^{2^\ell+2^j-2}(1-r)^{2+2\delta}r|\log r|\,dr &\leq C\sum_{\ell=0}^{\infty}2^{\ell\delta}\int_0^1 r^{2^\ell+2^j-2}(1-r)^{3+2\delta}\,dr\\ &= C\sum_{\ell=0}^{\infty}2^{\ell\delta}\frac{\Gamma(2^\ell+2^j-1)\Gamma(4+2\delta)}{\Gamma(2^\ell+2^j+3+2\delta)}\\ &\leq C\sum_{\ell=0}^{\infty}2^{\ell\delta}(2^\ell+2^j)^{-4-2\delta}\\ &\leq C2^{-j(\delta+4)}.\end{aligned}$$

Thus, using (3.34) and (3.25), we conclude that

$$\widetilde{g}(h)\leq C\Big(\sum_{j=0}^{\infty}2^{-4j}\Big(\sum_{k=2^j}^{2^{j+1}-1}|S_k^{\delta}(h')|\Big)^2\Big)^{\frac{1}{2}}. \qquad (3.35)$$

Now using (3.35) and Hölder's inequality, we obtain for $1<p\leq 2$,

$$|\widetilde{g}(h)|^p\leq C\sum_{j=0}^{\infty}2^{-2jp}\Big(\sum_{k=2^j}^{2^{j+1}-1}|S_k^{\delta}(h')|\Big)^p\leq C\sum_{j=0}^{\infty}2^{-j(p+1)}\sum_{k=2^j}^{2^{j+1}-1}|S_k^{\delta}(h')|^p,$$

whereas for $2<p<\infty$,

$$|\widetilde{g}(h)|^2\leq C\sum_{j=0}^{\infty}2^{-3j}\sum_{k=2^j}^{2^{j+1}-1}|S_k^{\delta}(h')|^2.$$

In either cases, we obtain the desired estimate (3.28), and hence complete the proof. $\square$

5. The sharp Jackson inequality

This section is devoted to showing the following sharp Jackson inequality.

THEOREM 5.1. *For $f \in L_p(\mathbb{S}^{d-1})$, $1 < p < \infty$ and $r > 0$, we have*

$$t^r \Big\{ \sum_{1 \le k \le 1/t} k^{sr-1} E_k(f)^s_{L_p(\mathbb{S}^{d-1})} \Big\}^{1/s} \le C\omega^r(f,t)_{L_p(\mathbb{S}^{d-1})}, \quad s = \max(p,2).$$

Using Hardy's inequality and Theorem 2.11, it's easily seen that Theorem 5.1 is essentially equivalent to the following corollary:

COROLLARY 5.2. *For $f \in L_p(\mathbb{S}^{d-1})$, and $1 < p < \infty$,*

$$t^r \Big\{ \int_t^{1/2} \frac{\omega^{r+1}(f,u)^s_{L_p(\mathbb{S}^{d-1})}}{u^{rs+1}} \, du \Big\}^{1/s} \le C\omega^r(f,t)_{L_p(\mathbb{S}^{d-1})}, \quad s = \max(p,2).$$

Our main tool for the proof of Theorem 5.1 is the Littlewood-Paley inequality (3.18).

Proof of Theorem 5.1. By monotonicity of $E_j(f)_p$ in j, it suffices to show that

$$2^{-nr} \Big(\sum_{j=1}^n 2^{jrs} E_{2^j}(f)^s_p \Big)^{1/s} \le CK_r(f, 2^{-n})_p. \tag{3.36}$$

Setting $g_n = L_{2^{n-1}} f$, we have

$$E_{2^n}(f)_p \le \|f - g_n\|_p \le CK_r\big(f, 2^{-nr}\big)_p.$$

As $E_m(f - g_n)_p \le \|f - g_n\|_p$ for all $m \in \mathbb{N}$, we have

$$E_{2^j}(f)_p \le E_{2^j}(f - g_n) + E_{2^j}(g_n)_p \le \|f - g_n\|_p + E_{2^j}(g_n)_p\,.$$

We can now write

$$\begin{aligned} &2^{-nr} \Big(\sum_{j=1}^n 2^{jrs} E_{2^j}(f)^s_p \Big)^{1/s} \\ &\quad \le 2^{-nr} \Big(\sum_{j=1}^n 2^{jrs} E_{2^j}(f - g_n)^s_p \Big)^{1/s} + 2^{-nr} \Big(\sum_{j=1}^n 2^{jrs} E_{2^j}(g_n)^s_p \Big)^{1/s} \\ &\quad \le \frac{2^r}{(2^{rs}-1)^{1/s}} \, \|f - g_n\|_p + 2^{-nr} \Big(\sum_{j=1}^n 2^{jrs} E_{2^j}(g_n)^s_p \Big)^{1/s}. \end{aligned}$$

Therefore, it remains to show that

$$I(n) \equiv 2^{-nr} \Big(\sum_{j=1}^n 2^{jrs} E_{2^j}(g_n)^s_p \Big)^{1/s} \le CK_r\big(f, 2^{-n}\big)_p\,,$$

and using (3.14), it is sufficient to show

$$I(n) \le C2^{-nr} \|(-\Delta_0)^{r/2} g_n\|_p\,,$$

which can be written as

$$\sum_{j=1}^n 2^{jrs} E_{2^j}(g_n)^s_p \le C\|(-\Delta_0)^{r/2} g_n\|^s_p\,.$$

Recalling that $\theta_j(f) = L_{2^{j-1}} f - L_{2^{j-2}} f$, we write for $j < n$

$$\begin{aligned} E_{2^{j+1}}(g_n)_p &\le \|g_n - L_{2^j} g_n\|_p = \|L_{2^n} g_n - L_{2^j} g_n\|_p \\ &= \Big\| \sum_{\ell=j+2}^{n+1} \theta_\ell g_n \Big\|_p. \end{aligned}$$

Applying the Littlewood-Paley inequality given by (3.18) to $f = L_{2^n} g_n - L_{2^j} g_n$, and recalling that $L_{2^i}(L_{2^n} f - L_{2^j} f) = 0$ for $i < j \le n$ and that $\theta_i(L_{2^n} g_n - L_{2^j} g_n) = 0$ for $i > n+1$, we have for $1 < p < \infty$

$$\Big\| L_{2^n} g_n - L_{2^j} g_n \Big\|_p \sim \bigg\| \Big(\sum_{\ell=j+1}^{n+1} (\theta_\ell g_n)^2 \Big)^{1/2} \bigg\|_p .$$

We now have to show

$$\sum_{j=0}^{n+1} 2^{jrs} \bigg\| \Big(\sum_{\ell=j+1}^{n+1} (\theta_\ell g_n)^2 \Big)^{1/2} \bigg\|_p^s \le C \|(-\Delta_0)^{r/2} g_n\|_p^s \tag{3.37}$$

for some C independent of n.

We prove (3.37) separately for $1 < p \le 2$, in which case $s = 2$, and for $2 < p < \infty$ in which case $s = p$. For $1 < p \le 2$ we use $\|f\|_q + \|g\|_q \le \| \, |f| + |g| \, \|_q$ for the quasinorm $\| \ \ \|_q$ when $q \le 1$, and obtain

$$\begin{aligned}
\sum_{j=1}^{n+1} 2^{jr2} \Big\| \sum_{\ell=j+1}^{n+1} (\theta_\ell g_n)^2 \Big\|_{p/2} &\le \Big\| \sum_{j=1}^{n+1} 2^{jr2} \sum_{\ell=j+1}^{n+1} (\theta_\ell g_n)^2 \Big\|_{p/2} \\
&= \Big\| \sum_{\ell=2}^{n+1} (\theta_\ell g_n)^2 \sum_{j=1}^{\ell-1} 2^{jr2} \Big\|_{p/2} \le C_1 \Big\| \sum_{\ell=2}^{n+1} (\theta_\ell g_n)^2 2^{\ell r2} \Big\|_{p/2} \\
&= C_1 \bigg\| \Big(\sum_{\ell=2}^{n+1} (\theta_\ell g_n)^2 2^{\ell r2} \Big)^{1/2} \bigg\|_p^2 ,
\end{aligned}$$

which, using (3.18), is controlled by

$$C \|(-\Delta_0)^{r/2} g_n\|_p^s .$$

This proves (3.37) for the case of $1 < p \le 2$.

To prove (3.37) in the case $2 < p < \infty$ and $s = p$, we use the duality between $L_{p/2}$ and L_q where $q = \frac{p}{p-2} = \left(\frac{p}{2}\right)'$, which implies for $\{b_j(x)\}_{j=1}^{n+1}$ where $b_j(x) \ge 0$ that there exists a sequence $C_j(x) \ge 0$ such that

$$\sum_{j=1}^{n+1} 2^{jrp} C_j(x) b_j(x) = \Big(\sum_{j=1}^{n+1} 2^{jrp} b_j(x)^{p/2} \Big)^{2/p}$$

and $\sum_{j=1}^{n+1} 2^{jrp} C_j(x)^q = 1$. We choose $b_j(x) = \sum_{\ell=j+1}^{n+1} (\theta_\ell g_n)^2$, and hence

$$\begin{aligned}
I(n) &= \int_{\mathbb{S}^{d-1}} \sum_{j=0}^{n+1} 2^{jrp} \Big(\sum_{\ell=j+1}^{n+1} (\theta_\ell g_n)^2 \Big)^{p/2} \\
&= \int_{\mathbb{S}^{d-1}} \Big(\sum_{j=0}^{n+1} 2^{jrp} C_j(x) \sum_{\ell=j+1}^{n+1} (\theta_\ell g_n)^2 \Big)^{p/2} \\
&= \int_{\mathbb{S}^{d-1}} \Big(\sum_{\ell=1}^{n+1} (\theta_\ell g_n)^2 \sum_{j=0}^{\ell-1} 2^{jrp} C_j(x) \Big)^{p/2} .
\end{aligned}$$

Using Hölder's inequality again, we have

$$\begin{aligned}
\sum_{j=1}^{\ell-1} 2^{jrp} C_j(x) &\le \Big\{ \sum_{j=1}^{\ell-1} 2^{jrp} \Big\}^{2/p} \Big\{ \sum_{j=1}^{\ell-1} 2^{jrp} C_j(x)^q \Big\}^{1/q} \\
&\le C 2^{\ell r2} .
\end{aligned}$$

We now have

$$\begin{aligned} I(n) &\leq C \int_{\mathbb{S}^{d-1}} \Big(\sum_{\ell=2}^{n+1} \theta_\ell(g_n)^2 2^{\ell r 2} \Big)^{p/2} \\ &= C \Big\| \Big(\sum_{\ell=2}^{n+1} \theta_\ell(g_n)^2 2^{\ell r 2} \Big)^{1/2} \Big\|_p^p. \end{aligned}$$

Recalling (3.18), we obtain (3.37). □

6. Optimality of the power in the Mauchaud inequality

Examples were given in [**20**] to show the the optimality of the power $s = \max(p, 2)$ in the sharp Jackson inequality (3.36). The examples of [**20**] with slight modifications can also be used to prove the optimality of the power $q = \min(p, 2)$ in the sharp Marchaud inequality (3.24) for $1 < p < 2 + \frac{2}{d-2}$. In this section, we shall show the optimality of the power in the sharp Marchaud inequality for the full range of $2 \leq p < \infty$. More precisely, we shall construct a sequence of functions f_n such that for $2 \leq p < \infty$,

$$K_r(f_n, 2^{-n})_p \geq c 2^{-nr} \Big\{ \sum_{k=1}^{n} 2^{2kr} E_{2^k}(f_n)_p^2 \Big\}^{1/2}, \tag{3.38}$$

with c being a positive constant independent of n. The following proposition plays crucial roles in our construction:

Proposition 6.1. *Let w be an A_∞ weight on $\mathbb{S}^{d-1}$ normalized by $w(\mathbb{S}^{d-1}) = 1$, and let X be a linear subspace of Π_N^d with $\dim X \geq \varepsilon \dim \Pi_N^d$ for some $\varepsilon \in (0, 1)$. Then there exists a function $f \in X$ such that $\|f\|_{p,w} \sim 1$ for all $0 < p \leq \infty$ with the constants of equivalence depending only on ε, d, the A_∞ constant of w, and p when p is small.*

To prove Proposition 6.1, we need the following result of G. G. Lorentz [**54**, Lemma 3.1, p. 410]:

Lemma 6.2. *Given an n-dimensional subspace X of $\mathbb{R}^m$, there exists $x = (x_1, \cdots, x_m) \in X$ such that $\max_{1\leq i\leq m} |x_i| = 1$ and $\#\{i : \ 1 \leq i \leq m, |x_i| = 1\} \geq n$.*

Proof of Proposition 6.1. Let $\Lambda := \{\omega_1, \cdots, \omega_M\}$ be a maximal $\frac{\delta_0}{6N}$-separated subset of $\mathbb{S}^{d-1}$ with δ_0 being the same constant as in Theorem 5.1 in Chapter 1. Then $M \asymp N^{d-1}$, and using Theorems 5.6 and 5.1 of Chapter 1, we have, for all $f \in \Pi_N^d$

$$\|f\|_{p,w} \sim \begin{cases} \Big(\sum_{j=1}^M \lambda_j |f(\omega_j)|^p \Big)^{\frac{1}{p}}, & \text{if } 0 < p < \infty, \\ \max_{1\leq j\leq M} |f(\omega_j)|, & \text{if } p = \infty, \end{cases} \tag{3.39}$$

where $\lambda_j = w(B_j)$, and $B_j = \mathsf{c}(\omega_j, \frac{c}{N})$. Consider the following linear subspace of $\mathbb{R}^M$:

$$\widetilde{X} := \{(f(\omega_1), \cdots, f(\omega_M)) : \ f \in X\}.$$

By (3.39), it follows that $\dim(\widetilde{X}) = \dim X \geq \varepsilon \dim \Pi_N^d \geq c' \varepsilon M$ for some absolute constant $c' \in (0, 1)$. Thus, by Lemma 6.2, there exists a spherical polynomial $f \in X$ with the properties that $\max\limits_{1\leq i\leq M} |f(\omega_i)| = 1$, and the set $I := \{i : \ 1 \leq i \leq M, |f(\omega_i)| = 1\}$ has cardinality $\#I \geq \ \dim\widetilde{X} \geq c' \varepsilon N^{d-1}$.

Now invoking (3.39), one has $\|f\|_\infty \le C \max_{1\le i\le M} |f(\omega_i)| \le C$, and $\|f\|_{2,w}^2 \sim \sum_{i=1}^M \lambda_i |f(\omega_i)|^2 \ge \sum_{i\in I} \lambda_i$. Also, observe that $\|f\|_{2,w} \le \|f\|_\infty \le C$. Thus, to show the equivalences $\|f\|_{2,w} \sim \|f\|_\infty \sim 1$, we only need to prove that $\sum_{i\in I}\lambda_i \ge c_\varepsilon > 0$.

To this end, we set $E = \cup_{j\in I} B_j$. Since the set $\{\omega_j : \ 1 \le j \le M\}$ is $c/(3N)$-separated, it follows that

$$|E| \ge c \sum_{j\in I} |B_j| \ge c N^{-d+1} \# I \ge c' > 0.$$

Thus, using the A_∞-property of w, we deduce

$$\sum_{j\in I} \lambda_j = \sum_{j\in I} w(B_j) \ge \frac{w(E)}{w(\mathbb{S}^{d-1})} \ge \beta \left(\frac{|E|}{|\mathbb{S}^{d-1}|} \right)^\beta \ge c|E|^\beta \ge c'' > 0,$$

as desired.

Finally, to complete the proof, we note that the equivalence $\|f\|_{p,w} \sim 1$ for all $0 < p \le \infty$ follows directly from the equivalence $\|f\|_{2,w} \sim \|f\|_\infty \sim 1$, Hölder's inequality and the following log-convexity of L^p norms:

$$\|f\|_{r,w} \le \|f\|_{p,w}^\theta \|f\|_{q,w}^{1-\theta}, \ \text{ whenever } 0 < p < r < q \le \infty, \text{ and } \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}. \qquad \square$$

Now we are in a position to construct a sequence of functions f_n with the property (3.38). For each $j \in \mathbb{N}$, let

$$X_j := \bigoplus_{2^{j-1} < k \le 2^j} \mathcal{H}_k^d.$$

Since $\dim \mathcal{H}_k^d \sim k^{d-2}$, it follows that

$$\dim X_j \sim 2^{j(d-1)} \sim \dim \Pi_{2^j}^d.$$

Thus, using Proposition 6.1, we conclude that there exists a spherical polynomial $P_j \in \bigoplus_{2^{j-1}<k\le 2^j} \mathcal{H}_k^d$ such that $\|P_j\|_\infty \sim \|P_j\|_2 \sim 1$ for each $j \in \mathbb{N}$. Let $f_n = \sum_{j=1}^n 2^{-jr} P_j$. Using (3.14), we obtain

$$\begin{aligned} K_r(f_n, 2^{-n})_p &\ge c 2^{-nr} \|(-\Delta_0)^{\frac{r}{2}} f_n\|_p \ge c 2^{-nr} \left\| \sum_{j=0}^n 2^{-jr} (-\Delta_0)^{\frac{r}{2}} P_j \right\|_2 \\ &= c 2^{-nr} \left(\sum_{j=0}^n 2^{-2jr} \|(-\Delta_0)^{\frac{r}{2}} P_j\|_2^2 \right)^{\frac12} \sim 2^{-nr} \left(\sum_{j=0}^n \|P_j\|_2^2 \right)^{\frac12} \sim 2^{-nr}\sqrt{n}. \end{aligned}$$

On the other hand, however,

$$E_{2^j}(f_n)_p \le \| \sum_{k=j+1}^n 2^{-kr} P_k \|_p \le \sum_{k=j+1}^n 2^{-kr} \|P_k\|_p \le c 2^{-nr}.$$

Thus,

$$\begin{aligned} &2^{-nr} \left(\sum_{k=1}^n 2^{2kr} E_{2^k}(f_n)_p^2 \right)^{\frac12} \le c 2^{-nr} \left(\sum_{k=1}^n 2^{2kr} 2^{-2kr} \right)^{\frac12} \le c 2^{-nr} \sqrt{n} \\ &\le c K_r(f_n, 2^{-n})_p, \end{aligned}$$

which is as desired.

7. Notes and further results

1. Inequalities (3.1) and (3.2) were proved by M. Timan [**82**] and by Zygmund [**112**]. They were generalized in several articles (see [**31**], [**80**], [**17**], [**18**] and [**32**]) and described in the texts [**84**, p.338 (12)], [**29**, p. 210] and [**85**, (4.88), p. 191].
2. Most of these results can be found in [**92**] and [**30**]. Formula (3.7) was proved in [**15**].
3. The equivalence (3.10) was proved in [**74**], [**75**], [**92**] and [**9**]. The proofs of (3.14), (3.15) and (3.16) can be found in [**30**].
4. Theorems 3.2 and 5.1 were proved in [**20**] in a more general setting. The proof of the sharp Marchaud inequality (Theorem 4.1) follows along the same line as those of [**17**] and [**18**].
5. Alternative approaches for the sharp Jackson inequalities and the sharp Marchaud inequalities without using the Littlewood-Paley inequalities can be found in [**33**], and [**31**] respectively.
6. The proof of Proposition 6.1 is from [**22**], whereas its idea can be traced back to [**94**].

CHAPTER 4

A transference theorem for the Dunkl transform and its applications

For a family of weight functions invariant under a finite reflection group, we show how weighted L^p multiplier theorems for Dunkl transform on the Euclidean space $\mathbb{R}^d$ can be transferred from the corresponding results for h-harmonic expansions on the unit sphere $\mathbb{S}^d$ of $\mathbb{R}^{d+1}$. The result is then applied to establish a Hörmander type multiplier theorem for the Dunkl transform and to show the convergence of the Bochner -Riesz means of the Dunkl transform of order above the critical index in weighted L^p spaces.

1. Introduction

Let R be a reduced root system in $\mathbb{R}^d$ normalized so that $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in R$. Given a nonzero vector $\alpha \in \mathbb{R}^d$, we denote by σ_α the reflection with respect to the hyperplane perpendicular to α; that is, $\sigma_\alpha x = x - 2(\langle x, \alpha\rangle/\|\alpha\|^2)\alpha$ for all $x \in \mathbb{R}^d$. Let G denote the finite subgroup of the orthogonal group $O(d)$ generated by the reflections σ_α, $\alpha \in R$. Let $\kappa : \ R \to \mathbb{R}_+$ be a nonnegative multiplicity function on R with the property $\kappa(g\alpha) = \kappa(\alpha)$ for all $\alpha \in R$ and $g \in G$. Associated with the reflection group G and the function κ is the weight function h_κ defined by

$$h_\kappa(x) := \prod_{\alpha\in R_+} |\langle x, \alpha\rangle|^{\kappa(\alpha)}, \quad x \in \mathbb{R}^d, \tag{4.1}$$

where R_+ is an arbitrary but fixed positive subsystem of R. The function h_κ is a homogeneous function of degree $\gamma_\kappa := \sum_{\alpha\in R_+} \kappa(\alpha)$, and is invariant under the reflection group G. From now on, we set $\lambda_\kappa = \frac{d-1}{2} + \gamma_\kappa$. Given $1 \le p \le \infty$, we denote by $L^p(\mathbb{R}^d; h_\kappa^2)$ the weighted Lebesgue space endowed with the norm

$$\|f\|_{\kappa,p} := \left(\int_{\mathbb{R}^d} |f(y)|^p h_\kappa^2(y)\, dy \right)^{\frac{1}{p}},$$

with the usual change when $p = \infty$.

The Dunkl transform, a generalization of the classical Fourier transform, is defined, for $f \in L^1(\mathbb{R}^d; h_\kappa^2)$, by

$$\mathcal{F}_\kappa f(x) = c_\kappa \int_{\mathbb{R}^d} f(y) E_\kappa(-ix, y) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d, \tag{4.2}$$

where $c_\kappa = \left(\int_{\mathbb{R}^d} h_\kappa^2(x) e^{-\frac{\|x\|^2}{2}} dx \right)^{-1}$, and $E_\kappa(ix, y) = V_\kappa \left[e^{i\langle x,\cdot\rangle} \right](y)$ is the weighted analogue of the character $e^{i\langle x,y\rangle}$. Here V_κ is the Dunkl intertwining operator associated with the reflection group G and the multiplicity function κ. The Dunkl transform plays the same role as the Fourier transform in classical Fourier analysis, and enjoys properties similar to those of the classical Fourier transform.

In this chapter, we first prove a transference theorem (Theorem 3.1) between the L^p multiplier of h-harmonic expansions on the unit sphere and that of the Dunkl transform. This theorem, combined with the corresponding results on h-harmonic expansions on the unit sphere established in [**24, 26, 25**], is then applied to establish a Hörmander type multiplier theorem for the Dunkl transform (Theorem 4.1), and to show the convergence of the Bochner -Riesz means in the weighted L^p spaces (Theorem 4.4).

This chapter is organized as follows. In Section 2, we describe briefly some known results on Dunkl transform and h-harmonic expansions, which will be needed in later sections. The transference theorem, Theorem 3.1, is proved in Section 3. As applications, we prove Theorems 4.1 and 4.4 in the final section, Section 4.

2. Preliminaries

In this section, we shall present some necessary material on the Dunkl transform and the h-harmonic expansions, most of which can be found in [**38, 51, 72, 73, 86**].

2.1. The Dunkl transform. Let R, R_+, G, κ and h_κ be as defined in Section 1. Let $V_\kappa : C(\mathbb{R}^d) \to C(\mathbb{R}^d)$ be the Dunkl intertwining operator associated with G and h_κ.

The Dunkl transform associated with G and κ is defined by (4.2) with

$$E_\kappa(-ix, y) := V_\kappa[e^{-i\langle x,\cdot\rangle}](y), \quad x, y \in \mathbb{R}^d. \tag{4.3}$$

If $\kappa = 0$ then $V_\kappa = \mathrm{id}$ and the Dunkl transform coincides with the usual Fourier transform, whereas if $d = 1$ and $G = \mathbb{Z}_2$ then it is closely related to the Hankel transform on the real line.

We list some of the known properties of the Dunkl transform in the following lemma.

Lemma 2.1. [**37, 51**]
(i) *If $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ then $\mathcal{F}_\kappa f \in C(\mathbb{R}^d)$ and $\lim_{\|\xi\|\to\infty} \mathcal{F}_\kappa f(\xi) = 0$.*

(ii) *The Dunkl transform $\mathcal{F}_\kappa$ is an isomorphism of the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ onto itself, and $\mathcal{F}_\kappa^2 f(x) = f(-x)$.*

(iii) *The Dunkl transform $\mathcal{F}_\kappa$ on $\mathcal{S}(\mathbb{R}^d)$ extends uniquely to an isometric isomorphism on $L^2(\mathbb{R}^d; h_\kappa^2)$, i.e., $\|f\|_{\kappa,2} = \|\mathcal{F}_\kappa f\|_{\kappa,2}$.*

(iv) *If f and $\mathcal{F}_\kappa f$ are both in $L^1(\mathbb{R}^d; h_\kappa^2)$ then the following inverse formula holds:*

$$f(x) = c_\kappa \int_{\mathbb{R}^d} \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d.$$

(v) *If $f, g \in L^2(\mathbb{R}^d; h_\kappa^2)$ then*

$$\int_{\mathbb{R}^d} \mathcal{F}_\kappa f(x) g(x)\, h_\kappa^2(x)\, dx = \int_{\mathbb{R}^d} f(x) \mathcal{F}_\kappa g(x)\, h_\kappa^2(x)\, dx.$$

(vi) *Given $\varepsilon > 0$, let $f_\varepsilon(x) = \varepsilon^{-2-2\gamma_\kappa} f(\varepsilon^{-1}x)$. Then $\mathcal{F}_\kappa f_\varepsilon(\xi) = \mathcal{F}_\kappa f(\varepsilon\xi)$.*

(vii) *If $f(x) = f_0(\|x\|)$ is radial, then $\mathcal{F}_\kappa f(\xi) = H_{\lambda_\kappa - \frac{1}{2}} f_0(\|\xi\|)$ is again a radial function, where H_α denotes the Hankel transform defined by*

$$H_\alpha g(s) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty g(r) \frac{J_\alpha(rs)}{(rs)^\alpha} r^{2\alpha+1}\, dr,$$

and J_α denotes the Bessel function of the first kind.

Statements (i)-(vi) of Lemma 2.1 above were proved by M. F. E. de Jeu [**51**], and were, in fact, contained in Corollaries 4.7 and 4.22, Theorems 4.26 and 4.20, Lemmas 4.13 and 4.3 (3) of [**51**] , respectively. Statement (vii) of Lemma 2.1 was proved by Dunkl [**37**] and was stated more explicitly in [**86**, Proposition 2.4].

Given $y \in \mathbb{R}^d$, the generalized translation operator $f \to \tau_y f$ is defined on $L^2(\mathbb{R}^d; h_\kappa^2)$ by

$$\mathcal{F}_\kappa(\tau_y f)(\xi) = E_\kappa(-i\xi, y)\mathcal{F}_\kappa f(\xi), \quad \xi \in \mathbb{R}^d.$$

It is known that $\tau_y f(x) = \tau_x f(y)$ for a.e. $x \in \mathbb{R}^d$ and a.e. $y \in \mathbb{R}^d$. In general, the operator τ_y is not positive (see, for instance, [**86**, Proposition 3.10]), and it is still an open problem whether $\tau_y f$ can be extended to a bounded operator on $L^1(\mathbb{R}^d; h_\kappa^2)$. On the other hand, however, it was shown in [**86**, Theorem 3.7] that the generalized translation operator τ_y can be extended to all radial functions in $L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p \le 2$, and $\tau_y : L^p_{\text{rad}}(\mathbb{R}^d; h_\kappa^2) \to L^p(\mathbb{R}^d; h_\kappa^2)$ is a bounded operator, where $L^p_{\text{rad}}(\mathbb{R}^d; h_\kappa^2)$ denotes the space of all radial functions in $L^p(\mathbb{R}^d; h_\kappa^2)$.

The generalized convolution of $f, g \in L^2(\mathbb{R}^d; h_\kappa^2)$ is defined by

$$f *_\kappa g(x) = \int_{\mathbb{R}^d} f(y)\tau_x \widetilde{g}(y) h_\kappa^2(y)\, dy, \tag{4.4}$$

where $\widetilde{g}(y) = g(-y)$. Since τ_y is a bounded operator from $L^p_{\text{rad}}(\mathbb{R}^d; h_\kappa^2)$ to $L^p(\mathbb{R}^d; h_\kappa^2)$ for $1 \le p \le 2$, it follows that the definition of $f *_\kappa g$ can be extended to all $g \in L^p_{\text{rad}}(\mathbb{R}^d; h_\kappa^2)$ and $f \in L^{p'}(\mathbb{R}^d; h_\kappa^2)$ with $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. The generalized convolution satisfies the following basic property

$$\mathcal{F}_\kappa(f *_\kappa g)(\xi) = \mathcal{F}_\kappa f(\xi)\mathcal{F}_\kappa g(\xi). \tag{4.5}$$

2.2. *h*-harmonic expansions. In this subsection, we shall give a brief review of some of the useful results on h-spherical harmonics. Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ denote the unit sphere of $\mathbb{R}^d$ equipped with the usual Lebesgue measure $d\sigma(x)$. For the weight function h_κ given in (4.1), we consider the weighted Lebesgue space $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ of functions on $\mathbb{S}^{d-1}$ endowed with the finite norm

$$\|f\|_{L^p(h_\kappa^2;\mathbb{S}^{d-1})} \equiv \|f\|_{\kappa,p} := \left(\int_{\mathbb{S}^{d-1}} |f(y)|^p h_\kappa^2(y) d\sigma(y)\right)^{1/p}, \qquad 1 \le p < \infty,$$

and for $p = \infty$ we assume that L^∞ is replaced by $C(\mathbb{S}^{d-1})$, the space of continuous functions on $\mathbb{S}^{d-1}$ with the usual uniform norm $\|f\|_\infty$. We shall use the notation $\|\cdot\|_{\kappa,p}$ to denote the weighted norm for functions defined either on $\mathbb{R}^d$ or on $\mathbb{S}^{d-1}$ whenever it causes no confusion.

A homogeneous polynomial is called an h-harmonic if it is orthogonal to all polynomials of lower degree with respect to the inner product of $L^2(h_\kappa^2; \mathbb{S}^{d-1})$. Let $\mathcal{H}_n^d(h_\kappa^2)$ denote the space of all h-harmonics of degree n, and let $\operatorname{proj}_n^\kappa : L^2(h_\kappa^2; \mathbb{S}^{d-1}) \to \mathcal{H}_n^d(h_\kappa^2)$ denote the orthogonal projection operator. The projection $\operatorname{proj}_n^\kappa$ has an integral representation

$$\operatorname{proj}_n^\kappa f(x) := \int_{\mathbb{S}^{d-1}} f(y) P_n^\kappa(x, y) h_\kappa^2(y)\, d\sigma(y),\ x \in \mathbb{S}^{d-1}, \tag{4.6}$$

where $P_n^\kappa(x, y)$ is the reproducing kernel of $\mathcal{H}_n^d(h_\kappa^2)$ which can be written in terms of the interwining operator V_κ as (see [**96**, Theorem 3.2, (3.1)])

$$P_n^\kappa(x, y) = \frac{n + \lambda_k'}{\lambda_\kappa'} V_\kappa \left[C_n^{\lambda_k'}(\langle x, \cdot\rangle)\right](y), \qquad x, y \in \mathbb{S}^{d-1} \tag{4.7}$$

with $\lambda_\kappa' := \lambda_\kappa - \frac{1}{2} = \gamma_\kappa + \frac{d-2}{2}$. Here C_n^λ denotes the standard Gegenbauer polynomial of degree n and index λ as defined in [**81**]. By means of (4.6) and (4.7), the projection $\operatorname{proj}_n^\kappa f$ can be extended to all $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$.

Recall that the following Marcinkiewicz type multiplier theorem was proved in Chapter 2:

THEOREM 2.2. *Let* $\{\mu_j\}_{j=0}^\infty$ *be a sequence of real numbers that satisfies*

(i) $\sup_j |\mu_j| \le c < \infty$,

(ii) $\sup_{j\geq 1} 2^{j(r-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^r \mu_l| \leq c < \infty$, *with r being the smallest integer $\geq \frac{d}{2}+\gamma_\kappa$,*

where $\triangle\mu_l = \mu_l - \mu_{l+1}$ and $\triangle^{j+1}\mu_l = \triangle^j\mu_l - \triangle^j\mu_{l+1}$. Then $\{\mu_j\}$ defines an $L^p(h_\kappa^2;\mathbb{S}^{d-1})$ multiplier for all $1<p<\infty$; that is,

$$\Big\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j^\kappa f \Big\|_{\kappa,p} \leq A_p c \|f\|_{\kappa,p}, \quad 1<p<\infty,$$

where A_p is independent of $\{\mu_j\}$ and f.

For $\delta > -1$, the Cesàro (C,δ) means of the h-harmonic expansion are defined by

$$S_n^\delta(h_\kappa^2; f, x) := (A_n^\delta)^{-1} \sum_{k=0}^{n} A_{n-k}^\delta \operatorname{proj}_k^\kappa f(x), \qquad A_{n-k}^\delta = \binom{n-k+\delta}{n-k}.$$

In the case when $G=\mathbb{Z}_2^d$ and $h_\kappa(x) = \prod_{i=1}^d |\langle x, e_i\rangle|^{\kappa(e_i)}$, the following result was proved in [**26**]:

THEOREM 2.3. *Let $G=\mathbb{Z}_2^d$ and let $1\leq p\leq\infty$ satisfy $|\frac{1}{p}-\frac{1}{2}| \geq \frac{1}{2\sigma_\kappa+2}$ with*

$$\sigma_\kappa := \tfrac{d-2}{2} + \gamma_\kappa - \min_{1\leq i\leq d} \kappa(e_i).$$

Then

$$\sup_{n\in\mathbb{N}} \|S_n^\delta(h_\kappa^2; f)\|_{\kappa,p} \leq c\|f\|_{\kappa,p} \quad \textit{for all } f\in L^p(h_\kappa^2;\mathbb{S}^{d-1})$$

if and only if

$$\delta > \delta_\kappa(p) := \max\left\{(2\sigma_\kappa+1)\left|\frac{1}{p}-\frac{1}{2}\right| - \frac{1}{2}, 0\right\}. \tag{4.8}$$

3. A transference theorem

The main goal in this section is to establish a transference theorem between the L^p multipliers of h-harmonic expansions on the unit sphere $\mathbb{S}^d := \{x\in\mathbb{R}^{d+1}: \|x\|=1\}$ and those of the Dunkl transform in $\mathbb{R}^d$. Let G, R, h_κ be as defined in Section 1. Given $g\in G$, we denote by g' the orthogonal transformation on $\mathbb{R}^{d+1}$ determined uniquely by

$$g'x' = (gx, x_{d+1}) \ \text{ for } \ x' = (x, x_{d+1}) \ \text{ with } x\in\mathbb{R}^d \ \text{ and } x_{d+1}\in\mathbb{R}.$$

Then $G' := \{g' : g\in G\}$ is a finite reflection group on $\mathbb{R}^{d+1}$ with a reduced root system $R' := \{(\alpha,0): \alpha\in R\}$. Let κ' denote the nonnegative multiplicity function defined on R' with the property $\kappa'(\alpha,0)=\kappa(\alpha)$. We denote by $V_{\kappa'}$ the intertwining operator on $C(\mathbb{R}^{d+1})$ associated with the reflection group G' and the multiplicity function κ'. Define the weight function

$$h_{\kappa'}(x, x_{d+1}) := h_\kappa(x) = \prod_{\alpha\in R_+} |\langle x,\alpha\rangle|^{\kappa(\alpha)}, \quad x\in\mathbb{R}^d,\ x_{d+1}\in\mathbb{R}.$$

Recall that $\operatorname{proj}_n^{\kappa'} : L^2(\mathbb{S}^d; h_{\kappa'}^2) \to \mathcal{H}_n^{d+1}(h_{\kappa'}^2)$ denotes the orthogonal projection onto the space of h-harmonics.

Our main result is the following.

THEOREM 3.1. *Let $m: [0,\infty)\to\mathbb{R}$ be a continuous and bounded function, and let U_ε, $\varepsilon>0$, be a family of operators on $L^2(\mathbb{S}^d; h_{\kappa'}^2)$ given by*

$$\operatorname{proj}_n^{\kappa'}\big(U_\varepsilon f\big) = m(\varepsilon n) \operatorname{proj}_n^{\kappa'} f, \quad n=0,1,\cdots. \tag{4.9}$$

Assume that

$$\sup_{\varepsilon>0} \|U_\varepsilon f\|_{L^p(\mathbb{S}^d;h^2_{\kappa'})} \le A\|f\|_{L^p(\mathbb{S}^d;h^2_{\kappa'})}, \quad \forall f \in C(\mathbb{S}^d) \tag{4.10}$$

for some $1 \le p \le \infty$. *Then the function* $m(\|\cdot\|)$ *defines an* $L^p(\mathbb{R}^d;h^2_\kappa)$ *multiplier; that is,*

$$\|T_m f\|_{L^p(\mathbb{R}^d;h^2_\kappa)} \le c_{d,\kappa} A\|f\|_{L^p(\mathbb{R}^d;h^2_\kappa)}, \quad \forall f \in \mathcal{S}(\mathbb{R}^d),$$

where T_m *is an operator initially defined on* $L^2(\mathbb{R}^d;h^2_\kappa)$ *by*

$$\mathcal{F}_\kappa(T_m f)(\xi) = m(\|\xi\|)\mathcal{F}_\kappa f(\xi), \quad f \in L^2(\mathbb{R}^d;h^2_\kappa),\ \xi \in \mathbb{R}^d. \tag{4.11}$$

3.1. Lemmas. The proof of Theorem 3.1 relies on several lemmas.

LEMMA 3.2. *If* $f \in \Pi^{d+1}$ *then for any* $x \in \mathbb{R}^d$ *and* $x_{d+1} \in \mathbb{R}$,

$$V_{\kappa'} f(x, x_{d+1}) = V_\kappa[f(\cdot, x_{d+1})](x) = \int_{\mathbb{R}^d} f(\xi, x_{d+1})\, d\mu^\kappa_x(\xi), \tag{4.12}$$

where $d\mu^\kappa_x$ *is given in (2.4).*

PROOF. Clearly, the second equality in (4.12) follows directly from (2.4). To show the first equality, we set $\widetilde{V}_\kappa f(x, x_{d+1}) = V_\kappa[f(\cdot, x_{d+1})](x)$ for $f \in C(\mathbb{R}^{d+1})$ and $x \in \mathbb{R}^d$. Since $V_{\kappa'}$ is a linear operator uniquely determined by (2.3), it suffices to show that the following conditions are satisfied:

$$\widetilde{V}_\kappa(\mathcal{P}^{d+1}_n) \subset \mathcal{P}^{d+1}_n, \quad \widetilde{V}_\kappa(1) = 1 \quad \text{and} \quad \mathcal{D}_{\kappa',i}\widetilde{V}_\kappa = \widetilde{V}_\kappa\partial_i, \quad 1 \le i \le d+1.$$

Indeed, these conditions can be easily verified using the properties of V_κ in (2.3), and the following identities, which follow directly from (2.2):

$$\mathcal{D}_{\kappa',i}\, g(x, x_{d+1}) = \mathcal{D}_{\kappa,i}\Big[g(\cdot, x_{d+1})\Big](x), \quad 1 \le i \le d,$$
$$\mathcal{D}_{\kappa',d+1}\, g(x, x_{d+1}) = \partial_{d+1} g(x, x_{d+1}), \quad \text{for } g \in \Pi^{d+1},\ x \in \mathbb{R}^d \text{ and } x_{d+1} \in \mathbb{R}.$$

This completes the proof of Lemma 3.2. $\square$

To formulate the next lemma, we define the mapping $\psi : \mathbb{R}^d \to \mathbb{S}^d$ by

$$\psi(x) := \Big(\xi \sin\|x\|, \cos\|x\|\Big) \quad \text{for } x = \|x\|\xi \in \mathbb{R}^d \text{ and } \xi \in \mathbb{S}^{d-1}.$$

Given $N \ge 1$, we denote by $N\mathbb{S}^d := \{x \in \mathbb{R}^{d+1} : \ \|x\| = N\}$ the sphere of radius N in $\mathbb{R}^{d+1}$, and define the mapping $\psi_N : \mathbb{R}^d \to N\mathbb{S}^d$ by

$$\psi_N(x) := N\psi\Big(\frac{x}{N}\Big) = \Big(N\xi \sin\frac{\|x\|}{N}, N\cos\frac{\|x\|}{N}\Big) \tag{4.13}$$

with $x = \|x\|\xi \in \mathbb{R}^d$ and $\xi \in \mathbb{S}^{d-1}$.

LEMMA 3.3. *If* $f :\ N\mathbb{S}^d \to \mathbb{R}$ *is supported in the set* $\{x \in N\mathbb{S}^d : \arccos(N^{-1}x_{d+1}) \le 1\}$, *then*

$$\int_{\mathbb{S}^d} f(Nx) h^2_{\kappa'}(x)\, d\sigma(x) = N^{-2\lambda_\kappa - 1}\int_{B(0,N)} f\big(\psi_N(x)\big) h^2_\kappa(x) \Big(\frac{\sin(\|x\|/N)}{\|x\|/N}\Big)^{2\lambda_\kappa} dx,$$

where $B(0,N) = \{y \in \mathbb{R}^d :\ \|y\| \le N\}$.

PROOF. First, using the polar coordinate transformation

$$(\xi, \theta) \in \mathbb{S}^{d-1} \times [0,\pi] \to x := (\xi \sin\theta, \cos\theta) \in \mathbb{S}^d,$$

and the fact that $d\sigma(x) = \sin^{d-1}\theta\, d\theta d\sigma(\xi)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{S}^d} f(Nx) h_{\kappa'}^2(x)\, d\sigma(x)\\
&\quad = \int_0^\pi \left[\int_{\mathbb{S}^{d-1}} f(N\xi\sin\theta, N\cos\theta) h_{\kappa'}^2(\xi\sin\theta,\cos\theta)\, d\sigma(\xi)\right](\sin\theta)^{d-1}\, d\theta\\
&\quad = \int_0^1 \int_{\mathbb{S}^{d-1}} f(N\xi\sin\theta, N\cos\theta) h_\kappa^2(\theta\xi)\, d\sigma(\xi)\left(\frac{\sin\theta}{\theta}\right)^{d-1+2\gamma_\kappa} \theta^{d-1}\, d\theta,
\end{aligned}$$

where the last step uses the identity $h_{\kappa'}(y, y_{d+1}) = h_\kappa(y)$, the fact that h_κ^2 is a homogeneous function of degree $2\gamma_\kappa$, and the assumption that f is supported in the set $\{x \in N\mathbb{S}^d : \arccos(N^{-1}x_{d+1}) \le 1\}$. Using the usual spherical coordinate transformation in $\mathbb{R}^d$, the last double integral equals

$$\begin{aligned}
&\int_{\|y\|\le 1} f\left(\frac{Ny\sin\|y\|}{\|y\|}, N\cos\|y\|\right) h_\kappa^2(y)\left(\frac{\sin\|y\|}{\|y\|}\right)^{2\lambda_\kappa} dy\\
&= N^{-d-2\gamma_\kappa}\int_{\|x\|\le N} f\left(N\frac{x}{\|x\|}\sin\frac{\|x\|}{N}, N\cos\frac{\|x\|}{N}\right) h_\kappa^2(x)\left(\frac{\sin(\|x\|/N)}{\|x\|/N}\right)^{2\lambda_\kappa} dx\\
&= N^{-2\lambda_\kappa-1}\int_{B(0,N)} f(\psi_N x) h_\kappa^2(x)\left(\frac{\sin(\|x\|/N)}{\|x\|/N}\right)^{2\lambda_\kappa} dx,
\end{aligned}$$

where the first step uses the homogeneity of the weight h_κ and the change of variables $y = \frac{x}{N}$. This proves the desired formula. $\square$

REMARK 3.4. It is easily seen that the restriction $\psi_N\big|_{B(0,N)}$ of the mapping ψ_N on $B(0, N)$ is a bijection from $B(0, N)$ to $\{x \in N\mathbb{S}^d : \arccos(N^{-1}x_{d+1}) \le 1\}$. Thus, given a function $f : B(0, N) \to \mathbb{R}$, there exists a unique function f_N supported in $\{x \in N\mathbb{S}^d :\ \arccos(N^{-1}x_{d+1}) \le 1\}$ such that

$$f_N(\psi_N x) = f(x), \quad \forall x \in B(0, N). \tag{4.14}$$

On the other hand, using Lemma 3.3, we have

$$\int_{\mathbb{S}^d} f_N(Nx) h_{\kappa'}^2(x)\, d\sigma(x) = N^{-2\lambda_\kappa-1}\int_{B(0,N)} f(x) h_\kappa^2(x)\left(\frac{\sin(\|x\|/N)}{\|x\|/N}\right)^{2\lambda_\kappa} dx. \tag{4.15}$$

The formula (4.15) will play an important role in our proof of Theorem 3.1.

We also need the following observation on a formula of Rösler [**73**] for $\tau_y f(x)$:

LEMMA 3.5. *If $f(x) = f_0(\|x\|)$ is a continuous radial function in $L^2(\mathbb{R}^d; h_\kappa^2)$, then for a.e. $y \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$,*

$$\tau_y f(x) = V_\kappa\Big[f_0\big(\sqrt{\|x\|^2 + \|y\|^2 - 2\|y\|\langle x, \cdot\rangle}\big)\Big]\Big(\frac{y}{\|y\|}\Big). \tag{4.16}$$

Formula (4.16) was first proved in [**73**] under the assumption that f is a radial Schwartz function. Thangavelu and Yuan Xu [**86**, Proposition 3.3] later observed that it also holds for radial functions $f \in L(\mathbb{R}^d; h_\kappa^2)$ with $\mathcal{F}_\kappa f \in L(\mathbb{R}^d; h_\kappa^2)$. Clearly, our assumption in Lemma 3.5 is slightly weaker than that of [**86**, Proposition 3.3].

Lemma 3.5 can be deduced from the result of Rösler [**73**], using a density argument.

PROOF. We first choose a sequence of even, C^∞ functions g_j on $\mathbb{R}$ satisfying

$$\sup_{|t|\le 2^{j+1}} |g_j(t) - f_0(t)| \le 2^{-j}\left(\int_0^{2^j} s^{2\lambda_\kappa}\, ds\right)^{-\frac{1}{2}}.$$

Let φ_j be an even, C^∞ function on $\mathbb{R}$ such that $\chi_{[2^{-j},2^j]}(|t|) \le \varphi_j(t) \le \chi_{[2^{-j-1},2^{j+1}]}(|t|)$, and let $f_j(x) \equiv f_{j,0}(\|x\|) := g_j(\|x\|)\varphi_j(\|x\|)$ for $x \in \mathbb{R}^d$. Then it's easily seen that $\{f_j\}$ is a sequence of radial Schwartz functions on $\mathbb{R}^d$ satisfying

$$\lim_{j\to\infty} \sup_{2^{-j}\le|t|\le 2^j} |f_{j,0}(t) - f_0(t)| = 0 \tag{4.17}$$

and

$$\lim_{j\to\infty} \|f_j - f\|_{\kappa,2} = 0. \tag{4.18}$$

Since each f_j is a radial Schwartz function, by Lemma 1.1 and the already proven case of Lemma 3.5 (see [**73**]), we obtain

$$\tau_y(f_j)(x) = \int_{\|\xi\|\le 1} f_{j,0}(\sqrt{\|x\|^2 + \|y\|^2 - 2\|y\|\langle x,\xi\rangle})d\mu^\kappa_{y/\|y\|}(\xi). \tag{4.19}$$

Next, we fix $y \in \mathbb{R}^d$, and set

$$A_n \equiv A_n(y) := \{x \in \mathbb{R}^d : \quad 2^{-n} \le \Big|\|x\| - \|y\|\Big| \le \|x\| + \|y\| \le 2^n\}$$

for $n \in \mathbb{N}$ and $n \ge n_0(y) := [\log\|y\|/\log 2] + 1$. Since

$$(\|x\| - \|y\|)^2 \le \|x\|^2 + \|y\|^2 - 2\|y\|\langle x,\xi\rangle| \le (\|x\| + \|y\|)^2$$

for all $\|\xi\| \le 1$, it follows by (4.17) that

$$\lim_{j\to\infty} f_{j,0}(\sqrt{\|x\|^2 + \|y\|^2 - 2\|y\|\langle x,\xi\rangle}) = f_0(\sqrt{\|x\|^2 + \|y\|^2 - 2\|y\|\langle x,\xi\rangle})$$

uniformly for $x \in A_n(y)$ and $\|\xi\| \le 1$. This together with (4.19) and Lemma 1.1 implies

$$\begin{aligned}\lim_{j\to\infty} \tau_y(f_j)(x) =& \int_{\|\xi\|\le 1} f_0(\sqrt{\|x\|^2 + \|y\|^2 - 2\|y\|\langle x,\xi\rangle})d\mu^\kappa_{y/\|y\|}(\xi)\\ =&V_\kappa\Big[f_0(\sqrt{\|x\|^2 + \|y\|^2 - 2\|y\|\langle x,\cdot\rangle})\Big]\Big(\frac{y}{\|y\|}\Big)\end{aligned}$$

for every $x \in A_n(y) \setminus \{0\}$ and $n \ge n_0(y)$. On the other hand, however, by (4.18), we have

$$\lim_{j\to\infty} \|\tau_y(f_j) - \tau_y f\|_{\kappa,2} = 0$$

for all $y \in \mathbb{R}^d$. Thus,

$$\tau_y(f)(x) = V_\kappa\Big[f_0(\sqrt{\|x\|^2 + \|y\|^2 - 2\|y\|\langle x,\cdot\rangle})\Big]\Big(\frac{y}{\|y\|}\Big)$$

for a.e. $x \in A_n(y)$ and all $n \ge n_0(y)$. Finally, observing that the set

$$\mathbb{R}^d \setminus \Big(\bigcup_{n=n_0(y)}^\infty A_n(y)\Big) = \{x \in \mathbb{R}^d : \ \|x\| = \|y\|\}$$

has measure zero in $\mathbb{R}^d$, we deduce the desired conclusion. □

REMARK 3.6. By (2.5) and the supporting condition of the measure $d\mu^\kappa_x$, we observe that

$$V_\kappa F(rx) = \int_{\mathbb{R}^d} F(r\xi)\, d\mu^\kappa_x(\xi), \text{for all } F \in C(\mathbb{R}^d),\ x \in \mathbb{R}^d, \text{ and } r > 0. \tag{4.20}$$

Thus, (4.16) can be rewritten more symmetrically as

$$\tau_y f(x) = V_\kappa\Big[f_0\big(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x,\cdot\rangle}\big)\Big](y). \tag{4.21}$$

LEMMA 3.7. *Let $\Phi \in L^1(\mathbb{R}, |x|^{2\lambda_\kappa})$ be an even, bounded function on $\mathbb{R}$, and let T_Φ be an operator $L^2(\mathbb{R}^d; h_\kappa^2) \to L^2(\mathbb{R}^d; h_\kappa^2)$ defined by*

$$\mathcal{F}_\kappa(T_\Phi f)(\xi) := \mathcal{F}_\kappa f(\xi)\Phi(\|\xi\|), \quad f \in L^2(\mathbb{R}^d; h_\kappa^2).$$

Then T_Φ has an integral representation

$$T_\Phi f(x) = \int_{\mathbb{R}^d} f(y)K(x,y)h_\kappa^2(y)\,dy, \quad \textit{for } f \in \mathcal{S}(\mathbb{R}^d) \textit{ and a.e. } x \in \mathbb{R}^d,$$

where

$$\begin{aligned} &K(x,y) \\ =&c\int_0^\infty \Phi(s)V_\kappa\left[\frac{J_{\lambda_\kappa-\frac12}(s\sqrt{\|x\|^2+\|y\|^2-2\langle x,\cdot\rangle})}{(s\sqrt{\|x\|^2+\|y\|^2-2\langle x,\cdot\rangle})^{\lambda_\kappa-\frac12}}\right](y)s^{2\lambda_\kappa}\,ds. \end{aligned} \tag{4.22}$$

Furthermore, $K(x,y) = K(y,x)$ for a.e. $x \in \mathbb{R}^d$ and a.e. $y \in \mathbb{R}^d$.

PROOF. Let $g(x) = H_{\lambda_\kappa-\frac12}\Phi(\|x\|)$, where $x \in \mathbb{R}^d$ and H_α denotes the Hankel transform. Since Φ is an even function in $L^1(\mathbb{R}, |x|^{2\lambda_\kappa}) \cap L^\infty(\mathbb{R})$, it follows by the properties of the Hankel transform that g is a continuous radial function in $L^2(\mathbb{R}^d; h_\kappa^2)$ and $\mathcal{F}_\kappa g(\xi) = \Phi(\|\xi\|)$. Thus, using (4.5), we have

$$T_\Phi f(x) = f *_\kappa g(x) = \int_{\mathbb{R}^d} f(y)\tau_y g(x)h_\kappa^2(y)\,dy$$

for $f \in L^2(\mathbb{R}^d; h_\kappa^2)$. Since g is a continuous radial function in $L^2(\mathbb{R}^d; h_\kappa^2)$, by Lemma 3.5 and Remark 3.6 it follows that

$$\begin{aligned} K(x,y) :&= \tau_y g(x) = V_\kappa\Big[H_{\lambda_\kappa-\frac12}\Phi(\sqrt{\|x\|^2+\|y\|^2+2\langle x,\cdot\rangle})\Big](y) \\ &= c\int_0^\infty \Phi(s)V_\kappa\left[\frac{J_{\lambda_\kappa-\frac12}(s\sqrt{\|x\|^2+\|y\|^2-2\langle x,\cdot\rangle})}{(s\sqrt{\|x\|^2+\|y\|^2-2\langle x,\cdot\rangle})^{\lambda_\kappa-\frac12}}\right](y)s^{2\lambda_\kappa}\,ds, \end{aligned}$$

where the last step uses (2.4), the inequality

$$\left|\Phi(s)\frac{J_{\lambda_\kappa-\frac12}(rs)}{(rs)^{\lambda_\kappa-\frac12}}\right| \le c|\Phi(s)|$$

and Fubini's theorem. This proves the desired equation (4.22). That $K(x,y) = K(y,x)$ follows from the fact that $\tau_x g(y) = \tau_y g(x)$. □

Our final lemma is a well known result for the ultraspherical polynomials:

LEMMA 3.8. [**81**, (8.1.1), p.192] *For $z \in \mathbb{C}$ and $\mu \ge 0$,*

$$\lim_{k\to\infty} k^{1-2\mu}C_k^\mu\left(\cos\frac{z}{k}\right) = \frac{\Gamma(\mu+\frac12)}{\Gamma(2\mu)}\left(\frac{z}{2}\right)^{-\mu+\frac12}J_{\mu-\frac12}(z). \tag{4.23}$$

This formula holds uniformly in every bounded region of the complex z-plane.

3.2. Proof of Theorem 3.1. We first prove the theorem under the additional assumption $|m(t)| \le c_1e^{-c_2t}$ for all $t > 0$ and some $c_1, c_2 > 0$. By Lemma 3.7, the operator T_m has an integral representation

$$T_m f(x) = \int_{\mathbb{R}^d} f(y)K(x,y)h_\kappa^2(y)\,dy,$$

where $K(x,y)$ is given by (4.22) with $\Phi = m$. Thus, it is sufficient to prove that

$$I := \left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)g(x)K(x,y)h_\kappa^2(x)h_\kappa^2(y)\,dxdy\right| \le cA \tag{4.24}$$

whenever $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ and $g \in L^{p'}(\mathbb{R}^d; h_\kappa^2)$ both have compact supports and satisfy $\|f\|_{L^p(\mathbb{R}^d;h_\kappa^2)} = \|g\|_{L^{p'}(\mathbb{R}^d;h_\kappa^2)} = 1$.

To this end, we choose a positive number N to be sufficiently large so that the supports of f and g are both contained in the ball $B(0,N)$. By Remark 3.4, there exist functions f_N and g_N both supported in $\{x \in N\mathbb{S}^d : \ \arccos(N^{-1}x_{d+1}) \le 1\}$ and satisfying

$$f_N\big(\psi_N(x)\big) = f(x), \quad g_N\big(\psi_N(x)\big) = g(x), \quad x \in \mathbb{R}^d, \tag{4.25}$$

where ψ_N is defined by (4.13). It's easily seen from (4.15) that

$$\|f_N(N\cdot)\|_{L^p(\mathbb{S}^d;h^2_{\kappa'})} \le cN^{-\frac{2\lambda_\kappa+1}{p}}, \qquad \|g_N(N\cdot)\|_{L^{p'}(\mathbb{S}^d;h^2_{\kappa'})} \le cN^{-\frac{2\lambda_\kappa+1}{p'}}.$$

Thus, using (4.6), (4.7), (4.9) and the assumption (4.10) with $\varepsilon = \frac{1}{N}$, we obtain

$$\begin{aligned} I_N &:= N^{2\lambda_\kappa+1} \\ &\times \Big|\int_{\mathbb{S}^d}\int_{\mathbb{S}^d}\Big[\sum_{n=0}^\infty m(N^{-1}n)P_n^{\kappa'}(x,y)\Big] f_N(Ny)g_N(Nx)h^2_{\kappa'}(x)h^2_{\kappa'}(y)\,d\sigma(x)\,d\sigma(y)\Big| \\ &\le cA, \end{aligned} \tag{4.26}$$

where $P_n^{\kappa'}(x,y) = \frac{n+\lambda_\kappa}{\lambda_\kappa}V_{\kappa'}[C_n^{\lambda_\kappa}(\langle x,\cdot\rangle)](y)$. Setting

$$H_N(x,y) = N^{-2\lambda_\kappa-1}\sum_{n=0}^\infty m(N^{-1}n)P_n^{\kappa'}\Big(\psi(\frac{x}{N}),\psi(\frac{y}{N})\Big),$$

and invoking (4.25) and Lemma 3.3, we obtain

$$\begin{aligned} I_N = &\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H_N(x,y)f(y)g(x)h^2_\kappa(x)h^2_\kappa(y)\Big(\frac{\sin(\|x\|/N)}{\|x\|/N}\Big)^{2\lambda_\kappa} \\ &\qquad\times\Big(\frac{\sin(\|y\|/N)}{\|y\|/N}\Big)^{2\lambda_\kappa}dx\,dy\Big|. \end{aligned} \tag{4.27}$$

On the other hand, setting

$$b_N(\rho,x,y) = N^{-2\lambda_\kappa-1}\sum_{n=0}^\infty m(\frac{n}{N})P_n^{\kappa'}\Big(\psi(\frac{x}{N}),\psi(\frac{y}{N})\Big)\Big(\int_{\frac{n}{N}}^{\frac{n+1}{N}} t^{2\lambda_\kappa}\,dt\Big)^{-1}\chi_{[\frac{n}{N},\frac{n+1}{N})}(\rho),$$

we have

$$H_N(x,y) = \int_0^\infty b_N(\rho,x,y)\rho^{2\lambda_\kappa}\,d\rho.$$

Hence, by (4.27),

$$\begin{aligned} I_N = &\Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big[\int_0^\infty b_N(\rho,x,y)\rho^{2\lambda_\kappa}\,d\rho\Big]f(y)g(x)h^2_\kappa(x)h^2_\kappa(y) \\ &\qquad\times\Big(\frac{\sin(\|x\|/N)}{\|x\|/N}\Big)^{2\lambda_\kappa}\Big(\frac{\sin(\|y\|/N)}{\|y\|/N}\Big)^{2\lambda_\kappa}dx\,dy\Big|. \end{aligned} \tag{4.28}$$

The key ingredient in our proof is to show that $\lim_{N\to\infty} I_N = cI$, where c is a constant depending only on d and κ. In fact, once this is proven, then the desired estimate (4.24) will follow immediately from (4.26).

To show $\lim_{N\to\infty} I_N = cI$, we make the following two assertions:

Assertion 1. For any $N > 0$ and $x,y \in \mathbb{R}^d$,

$$|b_N(\rho,x,y)| \le ce^{-c_2\rho},$$

where c is independent of x, y and N.

Assertion 2. For any fixed $x,y \in \mathbb{R}^d$ and $\rho > 0$,

$$\lim_{N\to\infty} b_N(\rho,x,y) = cm(\rho)V_\kappa\left[\frac{J_{\lambda_\kappa-\frac12}\big(\rho u(x,y,\cdot)\big)}{\big(\rho u(x,y,\cdot)\big)^{\lambda_\kappa-\frac12}}\right](y), \tag{4.29}$$

where $u(x,y,\xi)=\sqrt{\|x\|^2+\|y\|^2-2\langle x,\xi\rangle}$, and c is a constant depending only on d and κ.

For the moment, we take the above two assertions for granted, and proceed with the proof of Theorem 3.1. By Assertion 1 and Hölder's inequality, we can apply the dominated convergence theorem to the integrals in (4.28), and obtain

$$\lim_{N\to\infty} I_N=\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[\int_0^\infty \lim_{N\to\infty} b_N(\rho,x,y)\rho^{2\lambda_\kappa}\,d\rho\right]f(y)g(x)h_\kappa^2(x)h_\kappa^2(y)\,dx\,dy\right|,$$

which, using Assertion 2, equals

$$=c\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[\int_0^\infty m(\rho)V_\kappa\left[\frac{J_{\lambda_\kappa-\frac12}\big(\rho u(x,y,\cdot)\big)}{\big(\rho u(x,y,\cdot)\big)^{\lambda_\kappa-\frac12}}\right](y)\rho^{2\lambda_\kappa}\,d\rho\right]f(y)g(x)h_\kappa^2(x)h_\kappa^2(y)dx\,dy\right|$$

$$=c\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K(x,y)f(y)g(x)h_\kappa^2(x)h_\kappa^2(y)\,dx\,dy\right|=cI,$$

where the second step uses (4.22). Thus, we have shown the desired relation $\lim_{N\to\infty} I_N=cI$, assuming Assertions 1 and 2.

Now we return to the proofs of Assertions 1 and 2. We start with the proof of Assertion 1. Assume that $\frac{n}{N}\le\rho<\frac{n+1}{N}$ for some $n\in\mathbb{Z}_+$. Then $|m(\frac{n}{N})|\le c_1e^{-c_2\frac{n}{N}}\le ce^{-c_2\rho}$, and $\int_{\frac{n}{N}}^{\frac{n+1}{N}} t^{2\lambda_\kappa}\,dt\ge cN^{-1}\rho^{2\lambda_\kappa}$. Hence,

$$\begin{aligned}|b_N(\rho,x,y)|&=N^{-2\lambda_\kappa-1}\left|m\Big(\frac{n}{N}\Big)P_n^{\kappa'}\Big(\psi\Big(\frac{x}{N}\Big),\psi\Big(\frac{y}{N}\Big)\Big)\right|\left(\int_{\frac{n}{N}}^{\frac{n+1}{N}}t^{2\lambda_\kappa}\,dt\right)^{-1}\\ &\le cN^{-2\lambda_\kappa}\rho^{-2\lambda_\kappa}e^{-c_2\rho}\frac{n+\lambda_\kappa}{\lambda_\kappa}\left|V_{\kappa'}\left[C_n^{\lambda_\kappa}\Big(\Big\langle\psi\Big(\frac{x}{N}\Big),\cdot\Big\rangle\Big)\right]\Big(\psi\Big(\frac{y}{N}\Big)\Big)\right|\\ &\le c(N\rho)^{-2\lambda_\kappa}e^{-c_2\rho}n^{2\lambda_\kappa}\le ce^{-c_2\rho},\end{aligned}$$

where we used (4.7) in the second step, and the positivity of V_κ and the estimate $|C_n^{\lambda_\kappa}(t)|\le cn^{2\lambda_\kappa-1}$ in the third step. This proves Assertion 1.

Next, we show Assertion 2. A straightforward calculation shows that for $\frac{n}{N}\le\rho\le\frac{n+1}{N}$ and $\rho>0$,

$$\left(\int_{\frac{n}{N}}^{\frac{n+1}{N}}t^{2\lambda_\kappa}\,dt\right)^{-1}=\frac{N}{\rho^{2\lambda_\kappa}}(1+o_\rho(1)),\quad\text{as } N\to\infty.$$

This implies that for $\frac{n}{N}\le\rho\le\frac{n+1}{N}$ and $\rho>0$,

$$\begin{aligned}b_N(\rho,x,y)&=m(\rho)\frac{n^{2\lambda_\kappa}}{(N\rho)^{2\lambda_\kappa}}n^{-2\lambda_\kappa}P_n^{\kappa'}\Big(\psi\Big(\frac{x}{N}\Big),\psi\Big(\frac{y}{N}\Big)\Big)(1+o_\rho(1))\\ &=cm(\rho)n^{-2\lambda_\kappa+1}V_{\kappa'}\left[C_n^{\lambda_\kappa}\Big(\Big\langle\psi\Big(\frac{x}{N}\Big),\cdot\Big\rangle\Big)\right]\Big(\frac{y}{\|y\|}\sin\frac{\|y\|}{N},\cos\frac{\|y\|}{N}\Big)+o_\rho(1),\end{aligned}$$

where we used the continuity of m in the first step, and the estimate $n^{-2\lambda_\kappa}\left|P_n^{\kappa'}\Big(\psi(\frac{x}{N}),\psi(\frac{y}{N})\Big)\right|\le c$, as well as the fact that $\lim_{N\to\infty}\dfrac{n^{2\lambda_\kappa}}{(N\rho)^{2\lambda_\kappa}}=1$ in the last step (see [**25**]). Thus, using Lemma 3.2 and (4.20), we obtain

$$\begin{aligned}b_N(\rho,x,y)&=cm(\rho)n^{-2\lambda_\kappa+1}\int_{\mathbb{R}^d}C_n^{\lambda_\kappa}\Big(\frac{1}{\|x\|}\sin\frac{\|x\|}{N}\sum_{j=1}^d x_j\xi_j+\cos\frac{\|y\|}{N}\cos\frac{\|x\|}{N}\Big)\\ &\quad\times d\mu^\kappa_{\frac{y}{\|y\|}\sin\frac{\|y\|}{N}}(\xi)+o_\rho(1)\\ &=cm(\rho)n^{-2\lambda_\kappa+1}\int_{\|\xi\|\le\|y\|}C_n^{\lambda_\kappa}\Big(\cos\theta_N(x,y,\xi)\Big)d\mu_y^\kappa(\xi)+o_\rho(1),\end{aligned}\tag{4.30}$$

where $\theta_N(x, y, \xi) \in [0, \pi]$ satisfies

$$\cos\theta_N(x, y, \xi) = \left(\frac{1}{\|x\|\|y\|}\sum_{j=1}^d x_j\xi_j\right)\sin\frac{\|x\|}{N}\sin\frac{\|y\|}{N} + \cos\frac{\|x\|}{N}\cos\frac{\|y\|}{N},$$

Since

$$\begin{aligned}\cos\theta_N(x, y, \xi) &= 1 - \frac{1}{2N^2}\left(\|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d x_j\xi_j\right) + O_{\|x\|,\|y\|}(N^{-4})\\ &= 1 - \frac{1}{2N^2}u(x, y, \xi)^2 + O_{\|x\|,\|y\|}(N^{-4}),\end{aligned}$$

it follows that

$$\begin{aligned}\theta_N(x, y, \xi) &= 2\arcsin\left(\frac{1}{2N}\sqrt{u(x, y, \xi)^2 + O_{\|x\|,\|y\|}(N^{-2})}\right)\\ &= \frac{1}{N}\sqrt{u(x, y, \xi)^2 + O_{\|x\|,\|y\|}(N^{-2})} + O_{\|x\|,\|y\|}(N^{-2})\\ &= \frac{\rho u(x, y, \xi) + o_{\|x\|,\|y\|,\rho}(1)}{n},\end{aligned}$$

where the last step uses the uniform continuity of the function $t \in [0, M] \to \sqrt{t}$ for any $M > 0$, and the relation $\lim_{N\to\infty}\frac{n}{N\rho} = 1$.

Thus, by (4.30) and (4.23), we have

$$\begin{aligned}&\lim_{N\to\infty} b_N(\rho, x, y)\\ &= cm(\rho)\lim_{N\to\infty}\int_{\|\xi\|\le\|y\|} n^{-2\lambda_\kappa+1}C_n^{\lambda_\kappa}\left(\cos\frac{\rho u(x, y, \xi) + o_{x,y,\rho}(1)}{n}\right)d\mu_y^\kappa(\xi)\\ &= cm(\rho)\int_{\|\xi\|\le\|y\|}(\rho u(x, y, \xi))^{-\lambda_\kappa+\frac12}J_{\lambda_\kappa-\frac12}(\rho u(x, y, \xi))\,d\mu_y^\kappa(\xi)\\ &= cm(\rho)V_\kappa\Big[(\rho u(x, y, \cdot))^{-\lambda_\kappa+\frac12}J_{\lambda_\kappa-\frac12}(\rho u(x, y, \cdot))\Big](y),\end{aligned}$$

where we used the fact that $\|C_n^{\lambda_\kappa}\|_\infty \le cn^{2\lambda_\kappa-1}$, the bounded convergence theorem and (4.23) in the last step. This proves Assertion 2.

In summary, we have shown the theorem with the additional assumption $|m(t)| \le c_1e^{-c_2t}$.

Finally, we prove that the conclusion of Theorem 3.1 remains true without the additional assumption $|m(t)| \le c_1e^{-c_2t}$. To this end, let $m_\delta(t) = m(t)e^{-\delta t}$ for $\delta > 0$, and define $T_{m_\delta}:\ L^2(\mathbb{R}^d, h_\kappa^2) \to L^2(\mathbb{R}^d; h_\kappa^2)$ by

$$\mathcal{F}_\kappa(T_{m_\delta}f)(\xi) = m_\delta(\xi)\mathcal{F}_\kappa f(\xi), \quad f \in L^2(\mathbb{R}^d; h_\kappa^2).$$

It is known (see [**38**, p. 191]) that given any $\varepsilon > 0$, $f \mapsto \sum_{n=0}^\infty e^{-n\varepsilon}\operatorname{proj}_n^{\widetilde{\kappa}} f$ is a positive operator on $L^p(\mathbb{S}^d; h_{\widetilde{\kappa}}^2)$ that satisfies

$$\sup_{\varepsilon>0}\Big\|\sum_{n=0}^\infty e^{-n\varepsilon}\operatorname{proj}_n^{\widetilde{\kappa}} f\Big\|_{L^p(\mathbb{S}^d;h_{\widetilde{\kappa}}^2)} \le \|f\|_{L^p(\mathbb{S}^d;h_{\widetilde{\kappa}}^2)}.$$

Indeed, this follows from [**?**, Theorem 4.2] and the fact that V_κ is positive, which was proved in [**72**]. Thus, applying Theorem 3.1 for the already proven case, we have

$$\sup_{\delta>0}\Big\|T_{m_\delta}f\Big\|_{L^p(\mathbb{R}^d;h_\kappa^2)} \le cA\|f\|_{L^p(\mathbb{R}^d;h_\kappa^2)}. \tag{4.31}$$

On the other hand, from the definition we can decompose the operator T_{m_δ} as

$$T_{m_\delta}f = P_\delta(Tf), \tag{4.32}$$

where $\mathcal{F}_\kappa(Tf)(\xi) = m(\|\xi\|)\mathcal{F}_\kappa f(\xi)$ and

$$\mathcal{F}_\kappa(P_\delta f)(\xi) = e^{-\delta\|\xi\|}\mathcal{F}_\kappa f(\xi).$$

The function $P_\delta f$ is called the Poisson integral of f, and it can be expressed as a generalized convolution (see [**26**])

$$P_\delta f(x) := (f *_\kappa P_\delta)(x)$$

with

$$P_\delta(x) := 2^{\gamma_\kappa+\frac{d}{2}}\frac{\Gamma(\gamma_\kappa+\frac{d+1}{2})}{\sqrt{\pi}}\frac{\delta}{(\delta^2+\|x\|^2)^{\gamma_\kappa+\frac{d+1}{2}}}.$$

It was shown in [**26**, Theorem 6.2] that

$$\lim_{\delta\to 0+} P_\delta f(x) = f(x), \quad \text{a.e. } x\in\mathbb{R}^d$$

for any $f\in L^q(\mathbb{R}^d; h_\kappa^2)$ with $1\le q<\infty$. Since m is bounded, $Tf\in L^2(\mathbb{R}^d;h_\kappa^2)$ for $f\in L^2(\mathbb{R}^d;h_\kappa^2)$. Thus, for any $f\in\mathcal{S}$, using (4.32),

$$\lim_{\delta\to 0+} T_{m_\delta} f(x) = \lim_{\delta\to 0+} P_\delta(Tf)(x) = Tf(x), \quad \text{a.e. } x\in\mathbb{R}^d, \tag{4.33}$$

which combined with (4.31) and the Fatou theorem implies the desired estimate

$$\|Tf\|_{L^p(\mathbb{R}^d;h_\kappa^2)} \le cA\|f\|_{L^p(\mathbb{R}^d;h_\kappa^2)}.$$

This completes the proof of the theorem.

4. Applications

4.1. Hörmander's multiplier theorem and the Littlewood-Paley inequality. As a first application of Theorem 3.1, we shall prove the following Hörmander type multiplier theorem:

THEOREM 4.1. *Let $m:\ (0,\infty)\to\mathbb{R}$ be a bounded function satisfying $\|m\|_\infty\le A$ and Hörmander's condition*

$$\frac{1}{R}\int_R^{2R} |m^{(r)}(t)|\,dt \le AR^{-r}, \quad \textit{for all } R>0, \tag{4.34}$$

where r is the smallest integer $\ge \lambda_\kappa+1$. Let T_m be an operator on $L^2(\mathbb{R}^d;h_\kappa^2)$ defined by

$$\mathcal{F}_\kappa(T_m f)(\xi) = m(\|\xi\|)\mathcal{F}_\kappa f(\xi), \quad \xi\in\mathbb{R}^d.$$

Then

$$\|T_m f\|_{\kappa,p} \le C_p A\|f\|_{\kappa,p}$$

for all $1<p<\infty$ and $f\in\mathcal{S}(\mathbb{R}^d)$.

PROOF. Let $\mu_\ell = m(\ell\varepsilon)$ for $\varepsilon>0$ and $\ell=0,1,\cdots$. Then

$$\begin{aligned}|\triangle^r\mu_\ell| &= \varepsilon^r\Big|\int_{[0,1]^r} m^{(r)}\big(\varepsilon t_1+\cdots+\varepsilon t_r+\varepsilon\ell\big)dt_1\cdots dt_r\Big| \\ &\le \int_{[0,\varepsilon]^r}|m^{(r)}\big(t_1+\cdots+t_r+\varepsilon\ell\big)|dt_1\cdots dt_r \le \varepsilon^{r-1}\int_{\varepsilon\ell}^{\varepsilon(r+\ell)}|m^{(r)}(t)|\,dt.\end{aligned}$$

This implies that for $2^j \geq r$

$$\begin{aligned} 2^{j(r-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^r \mu_l| &\leq 2^{j(r-1)} \varepsilon^{r-1} \sum_{l=2^j}^{2^{j+1}} \int_{\varepsilon\ell}^{\varepsilon(r+\ell)} |m^{(r)}(t)|\, dt \\ &\leq (r-1) 2^{j(r-1)} \varepsilon^{r-1} \int_{2^j\varepsilon}^{\varepsilon(2^{j+1}+r)} |m^{(r)}(t)|\, dt \\ &\leq 2^{j(r-1)} (r-1) \varepsilon^{r-1} \int_{2^j\varepsilon}^{2^{j+2}\varepsilon} |m^{(r)}(t)|\, dt \leq c_r A, \end{aligned}$$

where the last step uses (4.34). On the other hand, however, for $2^j \leq r$, we have

$$2^{j(r-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^r \mu_l| \leq c_r \max_j |\mu_j| \leq c_r A.$$

Thus, using Theorem 2.2, we deduce

$$\sup_{\varepsilon>0} \Big\| \sum_{n=0}^{\infty} m(\varepsilon n) \operatorname{proj}_n^{\kappa'} f \Big\|_{L^p(\mathbb{S}^d; h_{\kappa'}^2)} \leq c \|f\|_{L^p(\mathbb{S}^d; h_{\kappa'}^2)}.$$

The desired conclusion then follows by Theorem 3.1. ☐

REMARK 4.2. Hörmander's condition is normally stated in the following form

$$\left(\frac{1}{R} \int_R^{2R} |m^{(r)}(t)|^2\, dt \right)^{\frac{1}{2}} \leq A R^{-r}, \quad \text{for all } R > 0. \tag{4.35}$$

See, for instance, [**46**, Theorem 5.2.7]. Clearly, the condition (4.34) in Theorem 4.1 is weaker than (4.35). On the other hand, however, Theorem 4.1 is applicable only to radial multiplier $m(\|\cdot\|)$.

COROLLARY 4.3. *Let Φ be an even C^∞–function that is supported in the set $\{x \in \mathbb{R} : \ \frac{9}{10} \leq |x| \leq \frac{21}{10}\}$ and satisfies either*

$$\sum_{j\in\mathbb{Z}} \Phi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R} \setminus \{0\},$$

or

$$\sum_{j\in\mathbb{Z}} |\Phi(2^{-j}\xi)|^2 = 1, \quad \xi \in \mathbb{R} \setminus \{0\}.$$

Let $\triangle_j$ be an operator defined by

$$\mathcal{F}_\kappa(\triangle_j f)(\xi) = \Phi(2^{-j}\|\xi\|) \mathcal{F}_\kappa f(\xi), \quad \xi \in \mathbb{R}^d.$$

Then we have

$$\|f\|_{\kappa,p} \sim_{\kappa,p} \left\| \Big(\sum_{j\in\mathbb{Z}} |\triangle_j f|^2 \Big)^{\frac{1}{2}} \right\|_{\kappa,p}$$

holds for all $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ and $1 < p < \infty$.

PROOF. Corollary 4.3 follows directly from Theorem 4.1. Since the proof is quite standard (see, for instance, [**78**]), we omit the details. ☐

4.2. The Bochner-Riesz means. Given $\delta > -1$, the Bochner-Riesz means of order δ for the Dunkl transform are defined by

$$B_R^\delta(h_\kappa^2; f)(x) = c\int_{\|y\|\le R}\left(1-\frac{\|y\|^2}{R^2}\right)^\delta \mathcal{F}_\kappa f(y) E_\kappa(ix,y) h_\kappa^2(y)\,dy, \quad R>0. \tag{4.36}$$

Convergence of the Bochner -Riesz means in the setting of Dunkl transform was studied recently by Thangavelu and Yuan Xu [**86**, Theorem 5.5], who proved that if $\delta > \lambda_\kappa := \frac{d-1}{2} + \gamma_\kappa$ and $1 \le p \le \infty$ then

$$\sup_{R>0} \|B_R^\delta(h_\kappa^2; f)\|_{\kappa,p} \le c\|f\|_{\kappa,p}. \tag{4.37}$$

Our next result concerns the critical indices for the validity of (4.37) in the case of $G = \mathbb{Z}_2^d$:

THEOREM 4.4. *Suppose that $G = \mathbb{Z}_2^d$, $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p \le \infty$, and $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{2\lambda_\kappa+2}$. Then* (4.37) *holds if and only if*

$$\delta > \delta_\kappa(p) := \max\left\{(2\lambda_\kappa+1)\Big|\frac{1}{p}-\frac{1}{2}\Big| - \frac{1}{2}, 0\right\}. \tag{4.38}$$

It should be pointed out that the result of [**86**, Theorem 5.5] is applicable to the case of a general finite reflection group G, while Theorem 4.4 above applies to the case of $\mathbb{Z}_2^d$ only.

PROOF. We start with the proof of the sufficiency. Assume that $\kappa := (\kappa_1, \cdots, \kappa_d)$ and $h_\kappa(x) := \prod_{j=1}^d |x_j|^{\kappa_j}$. Let $\kappa' = (\kappa, 0)$ and $h_{\kappa'}(x, x_{d+1}) = h_\kappa(x)$ for $x \in \mathbb{R}^d$ and $x_{d+1} \in \mathbb{R}$. Set $m(t) = (1-t^2)_+^\delta$. By the equivalence of the Riesz and the Cesàro summability methods of order $\delta \ge 0$ (see [**45**]), we deduce from Theorem 2.3

$$\sup_{\varepsilon>0}\Big\|\sum_{n=0}^\infty m(\varepsilon n)\operatorname{proj}_n^{\kappa'} f\Big\|_{L^p(\mathbb{S}^d; h_{\kappa'}^2)} \le c\|f\|_{L^p(\mathbb{S}^d; h_{\kappa'}^2)}$$

whenever $|\frac{1}{p} - \frac{1}{2}| \ge \frac{1}{2\sigma_{\kappa'}+2}$ and $\delta > \delta_{\kappa'}(p)$, where $\sigma_{\kappa'} = \lambda_\kappa$ and $\delta_{\kappa'}(p) = \delta_\kappa(p)$. Thus, invoking Theorem 3.1, we conclude that for $\delta > \delta_\kappa(p)$,

$$\|B_1^\delta(h_\kappa^2; f)\|_{\kappa,p} \le c\|f\|_{\kappa,p}.$$

The estimate (4.37) then follows by dilation. This proves the sufficiency.

The necessity part of the theorem follows from the corresponding result for the Hankel transform. To see this, let $f(x) = f_0(\|x\|)$ be a radial function in $L^p(\mathbb{R}^d, h_\kappa^2)$. Using (4.36) and Lemma 2.1 (vii), we have

$$B_R^\delta(h_\kappa^2; f)(x) = \int_0^R \left(1-\frac{r^2}{R^2}\right)^\delta H_{\lambda_\kappa - \frac{1}{2}} f_0(r) r^{2\lambda_\kappa}\left[\int_{\mathbb{S}^{d-1}} E_\kappa(ix, ry') h_\kappa^2(y')\,d\sigma(y')\right] dr.$$

However, by [**86**, Proposition 2.3] applied to $n = 0$ and $g = 1$, we have

$$\int_{\mathbb{S}^{d-1}} E_\kappa(ix, ry') h_\kappa^2(y')\,d\sigma(y') = c\left(\frac{r\|x\|}{2}\right)^{-\lambda_\kappa+\frac{1}{2}} J_{\lambda_\kappa-\frac{1}{2}}(r\|x\|).$$

It follows that

$$\begin{aligned} B_R^\delta(h_\kappa^2; f)(x) &= c\int_0^R \left(1-\frac{r^2}{R^2}\right)^\delta H_{\lambda_\kappa-\frac{1}{2}} f_0(r)\left(\frac{r\|x\|}{2}\right)^{-\lambda_\kappa+\frac{1}{2}} J_{\lambda_\kappa-\frac{1}{2}}(r\|x\|) r^{2\lambda_\kappa}\,dr \\ &= c\widetilde{B}_R^\delta f_0(\|x\|), \end{aligned}$$

where $\widetilde{B}_R^\delta$ denotes the Bockner-Riesz mean of order δ for the Hankel transform $H_{\lambda_\kappa-\frac{1}{2}}$. However, it is known (see [**93**]) that $\widetilde{B}_R^\delta$, $0<\delta<\lambda_\kappa$, is bounded on $L^p((0,\infty),t^{2\lambda_\kappa})$ if and only if

$$\frac{2\lambda_\kappa+1}{\lambda_\kappa+\delta+1}<p<\frac{2\lambda_\kappa+1}{\lambda_\kappa-\delta}. \tag{4.39}$$

Thus, to complete the proof of the necessity part of the theorem, by (4.39), we just need to observe that if $f(x)=f_0(\|x\|)$ is a radial function in $L^p(\mathbb{R}^d;h_\kappa^2)$, then

$$\|f\|_{\kappa,p}=c\|f_0\|_{L^p(\mathbb{R};|x|^{2\lambda_\kappa})}. \qquad \square$$

5. Notes and further results

1. The Dunkl transform plays the same role as the Fourier transform in classical Fourier analysis. Many important properties of the Dunkl transform were proved in [**51**]). Properties on the generalized translation operator and the generalized convolution can be found in [**86**].
2. Several important results in classical Fourier analysis have been extended to the setting of Dunkl transform by Thangavelu and Yuan Xu [**87, 86**]. The problem, however, turns out to be rather difficult in general. One of the difficulties comes from the fact that the generalized translation operator τ_y, which plays the role of the usual translation $f\to f(\cdot-y)$, is not positive in general (see, for instance, [**86**, Proposition 3.10]). In fact, even the L^p boundedness of τ_y is not established in general (see [**87, 86**]).
3. A very useful explicit formula for the intertwining operator V_κ was obtained by Dunkl [**36**] in the case of $G=\mathbb{Z}_2$, and was later extended to the more general case of $G=\mathbb{Z}_2^d$, ($d\in\mathbb{N}$) by Xuan Xu [**96**]. The positivity of V_κ as well as many important properties of V_κ were proved by Rösler [**72**].
4. In the unweighted case, for the classical Fourier transform, Theorem 4.4 is well known, and in fact, it follows from the following Tomas-Stein restriction theorem (see, for instance, [**46**, Section 10.4]):

$$\|\widehat{f}\|_{L^2(\mathbb{S}^{d-1})}\le C_p\|f\|_{L^p(\mathbb{R}^d)},\quad 1\le p\le\frac{2d+2}{d+3}, \tag{4.40}$$

where $\widehat{f}$ denotes the usual Fourier transform of f. In the weighted case, while estimates similar to (4.40) can be proved for the Dunkl transform $\mathcal{F}_\kappa f$ (see [**53**, Theorem 4.1]), they do not seem to be enough for the proof of Theorem 4.4. A similar fact was indicated in [**26**] for the case of the Cesàro means for h-harmonic expansions on the unit sphere, where global estimates for the projection operators have to be replaced with more delicate local estimates, which are significantly more difficult to prove than the corresponding global estimates.
5. In the case of ordinary spherical harmonics (i.e., $\kappa=0$), Theorem 3.1 is due to Bonami and Clerc [**5**, Theorem 1.1].
6. The main results of this chapter were proved in [**23**]. The idea of the proof of Theorem 3.1 can be traced back to [**5**].

APPENDIX A

Jacobi and related orthogonal polynomials

A.1. Jacobi polynomials

For parameters $\alpha, \beta > -1$, the Jacobi weight function is defined by

$$w_{\alpha,\beta}(x) = c_{\alpha,\beta}(1-x)^{\alpha}(1+x)^{\beta}, \quad -1 < x < 1.$$

where the normalization constant is given by

$$c_{\alpha,\beta}^{-1} := \int_{-1}^{1} w_{\alpha,\beta}(x)dx = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

For $n \geq 0$, the Jacobi polynomials are defined by

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right) \\ &= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right). \end{aligned} \tag{A.1}$$

They are orthogonal polynomials with respect to $w_{\alpha,\beta}$: For $n, m \geq 0$,

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)w_{\alpha,\beta}(x)dx = \frac{(\alpha+1)_n(\beta+1)_n(\alpha+\beta+n+1)}{n!(\alpha+\beta+2)_n(\alpha+\beta+2n+1)}\delta_{n,m}. \tag{A.2}$$

Some properties of Jacobi polynomials are listed below

1. The leading coefficient is $k_n = \dfrac{(n+\alpha+\beta+1)_n}{2^n n!}$;
2. $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ and $P_n^{(\alpha,\beta)}(1) = (\alpha+1)_n/n!$;
3. $P_n^{(\alpha,\beta)}(x)$ satisfies the differential equation

$$(1-x^2)y'' - (\alpha - \beta + (\alpha+\beta+2)x)y' + n(n+\alpha+\beta+1)y = 0.$$

4. $\dfrac{d}{dx}P_n^{(\alpha,\beta)}(x) = \dfrac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x)$.
5. three-term relation

$$\begin{aligned} P_{n+1}^{(\alpha,\beta)}(x) = &\frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}xP_n^{(\alpha,\beta)}(x) \\ &+ \frac{(2n+\alpha+\beta+1)(\alpha^2-\beta^2)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}P_n^{(\alpha,\beta)}(x) \\ &- \frac{(\alpha+n)(\beta+n)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}P_{n-1}^{(\alpha,\beta)}(x). \end{aligned}$$

They also have the following additional properties:

$$\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{A.3}$$

$$P_n^{(\alpha+1,\beta)}(x) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)} \times \sum_{j=0}^{n} \frac{(2j+\alpha+\beta+1)\Gamma(j+\alpha+\beta+1)}{\Gamma(j+\beta+1)} P_j^{(\alpha,\beta)}(x), \tag{A.4}$$

and for $\alpha, \beta > -\frac{1}{2}$,

$$\left|P_n^{(\alpha,\beta)}(\cos\theta)\right| \le c_{\alpha,\beta} n^{-\frac{1}{2}}(n^{-1}+\theta)^{-\alpha-\frac{1}{2}}(n^{-1}+\pi-\theta)^{-\beta-\frac{1}{2}}. \tag{A.5}$$

A.2. Gegenbauer polynomials

For $\lambda > -1/2$, the Gengenbauer weight function is defined by,

$$w_\lambda(x) := c_\lambda(1-x^2)^{\lambda-1/2}, \quad c_\lambda = c_{\lambda-1/2,\lambda-1/2}, \quad -1 < x < 1,$$

a special case of the Jacobi weight. The Gegenbauer polynomials are defined by

$$C_n^\lambda(x) = \frac{\left(2\lambda\right)_n}{\left(\lambda+\frac{1}{2}\right)_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x). \tag{A.6}$$

Their orthogonal relation is given by

$$\int_{-1}^{1} C_n^\lambda(x) C_m^\lambda(x) w_\lambda(x)dx = \frac{\lambda(2\lambda)_n}{(n+\lambda)n!}\delta_{n,m}. \tag{A.7}$$

They satisfy the following properties:

1. The leading coefficient is $\dfrac{(\lambda)_n 2^n}{n!}$.
2. $C_n^\lambda(1) = \dfrac{(2\lambda)_n}{n!}$.
3. The three-term relation:

$$C_{n+1}^\lambda(x) = \frac{2(n+\lambda)}{n+1} x C_n^\lambda(x) - \frac{n+2\lambda-1}{n+1} C_{n-1}^\lambda(x).$$

They also satisfy the following additional properties

$$\frac{d}{dx} C_n^\lambda(x) = 2\lambda C_{n-1}^{\lambda+1}(x), \tag{A.8}$$

$$(n+\lambda) C_n^\lambda(x) = \lambda\Big(C_{n+1}^\lambda(x) - C_{n-1}^\lambda(x)\Big). \tag{A.9}$$

For $n \ge 0$ and $\lambda > 0$,

$$C_n^\lambda(x) = \frac{(\lambda)_n 2^n}{n!} x^n {}_2F_1\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ 1-n-\lambda \end{matrix}; \frac{1}{x^2}\right). \tag{A.10}$$

For $|x| \le 1$ and $|r| < 1$

$$\frac{1-r^2}{(1-2xr+r^2)^{\lambda+1}} = \sum_{n=0}^{\infty} \frac{n+\lambda}{\lambda} C_n^\lambda(x) r^n. \tag{A.11}$$

The special case $\lambda = 0$ is the Chebyshev polynomial of the first kind, denoted by $T_n(x)$, and satisfies

$$\lim_{\lambda\to 0} \frac{1}{\lambda} C_n^\lambda(x) = T_n(x) = \cos n\theta, \quad x = \cos\theta.$$

The case $\lambda = 1$ is the Chebyshev polynomial of the second kind, denoted by $U_n(x)$

$$U_n(x) = C_n^1(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta.$$

The case $\lambda = \frac{1}{2}$ is the Legendre polynomial, often denoted by

$$P_n(x) = C_n^{1/2}(x)$$

which are orthogonal for dx on $-1 \leq x \leq 1$.

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