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STABLE REPRESENTATIONS OF POSETS

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Abstract. The purpose of this paper is to study stable representations of partially ordered sets (posets) and compare it to the well known theory for quivers. In particular, we prove that every indecomposable representation of a poset of finite representation type is stable with respect to some weight and construct that weight explicitly in terms of the dimension vector. We show that if a poset is primitive then Coxeter transformations preserve stable representations. When the base field is the field of complex numbers we establish the connection between the polystable representations and the unitary χ-representations of posets. This connection explains the similarity of the results obtained in the series of papers.

Introduction

Representation theory of finite dimensional algebras turned into a vast field of study in the last 40-50 years. It was observed that the subject can be approached combinatorially via representations of posets (due to L.A. Nazarova and A.V. Roiter) and representations of quivers (due to P. Gabriel). Despite of certain similarities representations of quivers and posets have significant differences. For instance: the category of representation of given quiver is abelian, while the category of representations of given poset is additive; the global dimension of the category of representations of a given quiver is at most one while it can be arbitrary for the posets; the variety of representations of a fixed dimension of a quiver is affine while it is projective in the case of posets; etc.

The problem of classifying representations of "most" algebras is wild in a sense that it is as difficult as the problem of classifying representations of free algebras, or of any wild quiver (or poset). Nevertheless, one can use geometrical approach (following the ideas of D. Mumford, e.g., [16]) by considering the spaces whose points correspond naturally to isomorphism classes of representations. This is how A. King in [10] defined the moduli spaces of finite dimensional algebras and quivers (we refer to [17] for exhaustive survey of this subject).

In [23] the authors tried to define moduli spaces of posets via moduli spaces of corresponding bound quivers. This rose certain technical problems as, for instance, the category of bound representations of corresponding quiver is "bigger" then the category of representation of underlying poset. One of the goals of the current paper is to develop a general framework to define and study the moduli spaces of posets intrinsically. To be more precise, let $\mathbb{F}$ be a field and

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$\mathcal{S} = \{s_1, \ldots, s_t\}$ be a finite poset. A subspace representation of $\mathcal{S}$ is a tuple $\mathbf{V} = (V_0; V_s)_{s\in\mathcal{S}}$, in which V_0 is a vector space over $\mathbb{F}$ and V_i are its subspaces such that $V_s \subseteq V_t$ if $s \prec t$ (that is, each representation is a homomorphism from $\mathcal{S}$ to the poset of all subspaces of V_0). All subspace representations of $\mathcal{S}$ form an additive category denoted by $\mathfrak{sp}_{\mathcal{S}}$ (see Section 1 for more details). Considering (semi)-stable representations in $\mathfrak{sp}_{\mathcal{S}}$ we adopt the definitions and properties from [18], where (semi)-stable objects in an arbitrary abelian category were studied. Following the ideas of [10] for quivers we approach the classification of representations of posets geometrically. We show that (semi)-stable orbits are connected to the algebraic definition of (semi)-stability in $\mathfrak{sp}_{\mathcal{S}}$ and relate unitary $\boldsymbol{\chi}$-representations of posets (see [13, 14, 19]) to polystable representations of $\mathcal{S}$. Note that application of [10] and [7] requires a special care since the category $\mathfrak{sp}_{\mathcal{S}}$ is not abelian and the variety of all representations of $\mathcal{S}$ of fixed dimension is projective.

The paper is organized as follows. In Section 1 we establish the notation and terminology and prove some preliminary statements. In Section 2 we define an algebraic stability (and costability) in $\mathfrak{sp}_{\mathcal{S}}$, prove the existence of Harder-Narasimhan and Jordan-Hölder filtrations (in $\mathfrak{sp}_{\mathcal{S}}$) and relate stability with costability (under certain assumptions). Section 3 is devoted to the reflection transformations of posets (introduced in [4]). We prove that the corresponding Coxeter transformations preserve stability in the case of primitive posets. Section 4 is devoted to the posets of finite representation type. We prove (Theorem 4.1) that $\mathcal{S}$ is of finite representation type if and only if any indecomposable representation of $\mathcal{S}$ is positively costable, equivalently if and only if any indecomposable representation of $\mathcal{S}$ is positively stable. This theorem is a consequence of Propositions 4.1 and 4.3 which are analogues of the Schofield's characterization of Schurian roots for quivers (see [20, Theorem 6.1]). In Section 5 we relate the introduced concept of stability with the geometric notion and define moduli space of polystable representations of $\mathcal{S}$ with fixed dimension vector. Namely, we consider the embeddings of the projective variety $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ of all representations of $\mathcal{S}$ having the dimension $\boldsymbol{\alpha}$ into a projective space and prove that the set of (semi)-stable points of the $\mathbf{Sl}(\alpha_0)$-action coincides with the set of (semi)-stable representations in the sense of Section 2. In Section 6 we study the moment map of the $\mathbf{Sl}(\alpha_0)$-action on $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ when $\mathbb{F} = \mathbb{C}$. As a consequence of the theorem of Kempf-Ness we obtain that the symplectic quotient of $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ can be identified with the moduli space defined in Section 5. Also we show the that pre-image of 0 of the moment map is the set of $\boldsymbol{\chi}$-representations (defined in [13, 14, 19]).

In Appendix A we prove some additional statements. In Appendix B we describe all exact representations of non-primitive posets of finite representation type, describe their maximal sub-coordinate vectors and state costability condition for each exact representation (this completes the proof of Proposition 4.1). In Appendix B we describe all quite sincere representations of non-primitive posets of finite representation type, describe their maximal sub-dimension vectors and state stability condition for each quite sincere representation (this completes the proof of Proposition 4.3).

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1. Notations and Terminology

We fix a field $\mathbb{F}$ which is assumed algebraically closed in Section 4 and $\mathbb{F} = \mathbb{C}$ in Section 5. A finite poset $\mathcal{S} \equiv (\mathcal{S}, \preceq)$ is given by the set of elements $\{s_1, \ldots, s_n\}$ and a partial order $\preceq$. We assume that elements $s_1, \ldots, s_n$ of $\mathcal{S}$ are enumerated so that $s_i \prec s_j$ implies $i < j$. A poset $\mathcal{S}$ is said to be *primitive* if it is a disjoint union of finite number of linearly ordered posets. Denote by $\mathcal{S}^{op}$ the dual poset $\mathcal{S}^{op} \equiv (\mathcal{S}^{op}, \preceq^\circ)$, in which $a \preceq^\circ b$ if and only if $b \succeq a$ in $\mathcal{S}$. The relation $\preceq$ is uniquely defined by the *incidence matrix* $C_\mathcal{S}$ of $\mathcal{S}$; that is, the integral square $n \times n$ matrix

$$C_\mathcal{S} = [c_{st}]_{s,t\in\mathcal{S}} \in \mathbb{M}_\mathcal{S}(\mathbb{Z}) = \mathbb{M}_n(\mathbb{Z}), \quad \text{with } c_{st} = \begin{cases} 1, & \text{for } s \preceq t, \\ 0, & \text{for } s \npreceq t. \end{cases}$$

It is easy to see that $C_\mathcal{S}$ is invertible, $C_\mathcal{S}^{-1} \in \mathbb{M}_n(\mathbb{Z})$ and that $C_{\mathcal{S}^{op}} = C_\mathcal{S}^{tr}$ (the transpose of $C_\mathcal{S}$). Given a poset $\mathcal{S}$, by $\widehat{\mathcal{S}}$ we understand its enlargement by unique maximal element 0; that is, $\widehat{\mathcal{S}} \equiv (\widehat{\mathcal{S}}, \preceq^0)$ with $\widehat{\mathcal{S}} \setminus \{0\} = \mathcal{S}$ and the order $\preceq^0$ is obvious. The *Tits matrix* $\widehat{C_\mathcal{S}}$ and the *reduced incidence matrix* $C_\mathcal{S}^\circ$ of $\widehat{\mathcal{S}}$ are defined as the following bipartite matrices (we use the notation and terminology from [22]):

$$\widehat{C_\mathcal{S}} = \left[\begin{array}{c|c} 1 & 0 \\ \hline -E & C_\mathcal{S}^{tr} \end{array}\right] \in \mathbb{M}_{\widehat{\mathcal{S}}}(\mathbb{Z}), \qquad C_\mathcal{S}^\circ = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & C_\mathcal{S} \end{array}\right] \in \mathbb{M}_{\widehat{\mathcal{S}}}(\mathbb{Z}),$$

in which E is a $1 \times n$ unit matrix.

A *subspace representation* of $\mathcal{S}$ is a system $\mathbf{V} = (V_0; V_s)_{s\in\mathcal{S}}$ of subspaces V_s of a finite dimensional vector space V_0 such that $V_s \subset V_t$ if $s \prec t$. The vector space V_0 will be called the *ambient* space of $\mathbf{V}$. A *morphism* between two representations $\mathbf{V}$ and $\mathbf{V}'$ is a $\mathbb{F}$-linear map $g : V_0 \to V_0'$ such that $g(V_s) \subset V_s'$ for all s. Denote by $\mathfrak{sp}_\mathcal{S}$ the corresponding additive category of all subspace representations of $\mathcal{S}$.

Interested reader is refereed to [21] where the systematic (up-to-date) exposition of the representation theory of finite posets is given.

The *dimension vector* of $\mathbf{V}$ is a $\mathbb{Z}$-function on $\widehat{\mathcal{S}}$ given by $\mathbf{dim}\ \mathbf{V}(s) = \dim V_s$, that is the dimension vector of $\mathbf{V}$ is an element of $\mathbb{Z}^{\widehat{\mathcal{S}}}$. We say that $\boldsymbol{\alpha} = (\alpha_0; \alpha_s)_{s\in\mathcal{S}} \in \mathbb{Z}^{\widehat{\mathcal{S}}}$ is *admissible* if $\alpha_0 > 0, \alpha_s \geq 0$ and $\alpha_s \leq \alpha_t$ if $s \prec t \in \widehat{\mathcal{S}}$. Clearly $\boldsymbol{\alpha}$ is a dimension vector of some representation of $\mathcal{S}$ iff $\boldsymbol{\alpha}$ is admissible. Fixing an admissible dimension vector $\boldsymbol{\alpha} = (\alpha_0; \alpha_s)_{s\in\mathcal{S}} \in \mathbb{Z}^{\widehat{\mathcal{S}}}$ we consider the following projective variety (see Proposition 6.4 in Appendix A),

$$\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}} = \Big\{ (V_s)_{s\in\mathcal{S}} \in \prod_{s\in\mathcal{S}} \mathrm{Gr}(\alpha_s, \alpha_0) \;\Big|\; V_s \subset V_t \text{ if } s \prec t \Big\}.$$

The group $\mathbf{Gl}(\boldsymbol{\alpha}_0)$ acts on $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ (diagonally) via the base change so that the orbits of this action are in a bijection with the isomorphisms classes of subspace representations of $\mathcal{S}$ with the dimension $\boldsymbol{\alpha}$. In what follows the variety $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ is called *poset variety.*

The *coordinate vector* of $\mathbf{V}$ is a function on $\widehat{\mathcal{S}}$, given by

$$\mathbf{cdn}\ V(s) = \begin{cases} \dim(V_s/\sum_{t\prec s} V_t), & s \neq 0, \\ \dim V_0, & s = 0. \end{cases}$$

Two elements $s_1, s_2 \in \mathcal{S}$ form an *arrow* (denoted by $s_1 \to s_2$) if $s_1 \prec s_2$ and there is no $t \in \mathcal{S}$ such that $s_1 \prec t \prec s_2$. We say that a representation $\mathbf{V}$ is *coordinate* if for any point $s \in \mathcal{S}$ we have that the sum $\sum_{t\to s} V_t$ is direct. One checks that in this case for each $s \in \mathcal{S}$ we have that

$$\mathbf{cdn}\ V(s) = \dim \Big(V_s/\sum_{t\prec s} V_t\Big) = \dim V_s - \sum_{t\to s} \dim V_t,$$

and hence $\mathbf{dim}\,\mathbf{V} = \mathbf{cdn}\,\mathbf{V} \cdot C^{\circ}_{\mathcal{S}}$. It follows from the definition that any representation of a primitive poset is coordinate. Also, any subrepresentation of a coordinate representation is coordinate.

Given $\boldsymbol{\alpha} \in \mathbb{Z}^{\mathcal{S}}$, we define the support of $\boldsymbol{\alpha}$, $\mathbf{supp}\,\boldsymbol{\alpha}$, to be the full subposet of $\mathcal{S}$ of the elements $\{s : \boldsymbol{\alpha}(s) \neq 0\}$. An indecomposable representation $\mathbf{V}$ is called *sincere* (resp. *exact*) if $\mathbf{supp}\,\mathbf{dim}\,\mathbf{V} = \widehat{\mathcal{S}}$ (resp. if $\mathbf{supp}\,\mathbf{cdn}\,\mathbf{V} = \widehat{\mathcal{S}}$).

The following two bilinear forms play a fundamental role in studying the category $\mathfrak{sp}_{\mathcal{S}}$ (cf. [22])

$$d_{\mathcal{S}}, b_{\mathcal{S}} : \mathbb{Z}^{\widehat{\mathcal{S}}} \times \mathbb{Z}^{\widehat{\mathcal{S}}} \to \mathbb{Z},$$

$$d_{\mathcal{S}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \boldsymbol{\alpha} \cdot \widehat{C_{\mathcal{S}}} \cdot \boldsymbol{\beta}^{tr} = \sum_{s\in\mathcal{S}} \alpha_s\beta_s + \sum_{t\prec s\in\mathcal{S}} \alpha_s\beta_t - \alpha_0 \sum_{s\in\mathcal{S}} \beta_s,$$

$$b_{\mathcal{S}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \boldsymbol{\alpha} \cdot C^{-1}_{\widehat{\mathcal{S}}} \cdot \boldsymbol{\beta}^{tr} = \sum_{s\in\mathcal{S}} \alpha_s\beta_s + \sum_{t\prec s\in\mathcal{S}} c^{-}_{st}\alpha_s\beta_t,$$

where c^-_{st} is the (s,t) entry of the matrix $C^{-1}_{\widehat{\mathcal{S}}} \in \mathbb{M}_{\widehat{\mathcal{S}}}(\mathbb{Z})$ inverse to $C_{\widehat{\mathcal{S}}}$. Note that under certain conditions (see Appendix A for details)

$$b_{\mathcal{S}}(\boldsymbol{\alpha},\boldsymbol{\alpha}) = \dim \mathbf{Gl}(\alpha_0) - \dim \mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}.$$

We have

Proposition 1.1. *Let $\mathcal{S}$ be any poset. We have $C^{\circ}_{\mathcal{S}} \cdot C^{-1}_{\widehat{\mathcal{S}}} \cdot C^{\circ\, tr}_{\mathcal{S}} = \widehat{C_{\mathcal{S}^{op}}}$, that is the matrices $C^{-1}_{\widehat{\mathcal{S}}}$ and $\widehat{C_{\mathcal{S}^{op}}}$ are $\mathbb{Z}$-congruent.*

Proof. The matrix $C^{-1}_{\widehat{\mathcal{S}}}$ can be written as the following bipartite matrix

$$C^{-1}_{\widehat{\mathcal{S}}} = \left[\begin{array}{c|c} 1 & 0 \\ \hline -C^{-1}_{\mathcal{S}} \cdot E & C^{-1}_{\mathcal{S}} \end{array}\right].$$

Therefore

$$\begin{aligned} C^{\circ}_{\mathcal{S}} \cdot C^{-1}_{\widehat{\mathcal{S}}} \cdot C^{\circ\, tr}_{\mathcal{S}} &= \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & C_{\mathcal{S}} \end{array}\right] \cdot \left[\begin{array}{c|c} 1 & 0 \\ \hline -C^{-1}_{\mathcal{S}} \cdot E & C^{-1}_{\mathcal{S}} \end{array}\right] \cdot \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & C_{\mathcal{S}} \end{array}\right]^{tr} \\ &= \left[\begin{array}{c|c} 1 & 0 \\ \hline -E & I_n \end{array}\right] \cdot \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & C^{tr}_{\mathcal{S}} \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline -E & C_{\mathcal{S}^*} \end{array}\right] = \widehat{C_{\mathcal{S}^*}}. \end{aligned}$$

□

Corollary 1.1. *If $\mathbf{V},\mathbf{W} \in \mathfrak{sp}_{\mathcal{S}}$ are two coordinate representations hence*

$$b_{\mathcal{S}}(\mathbf{dim}\,\mathbf{V}, \mathbf{dim}\,\mathbf{W}) = d_{\mathcal{S}^*}(\mathbf{cdn}\,\mathbf{V}, \mathbf{cdn}\,\mathbf{W}).$$

Proof. As $\mathbf{V}$ and $\mathbf{W}$ are coordinate representations we have $\mathbf{dim}\,\mathbf{V} = \mathbf{cdn}\,\mathbf{V} \cdot C^{\circ}_{\mathcal{S}}$ and $\mathbf{dim}\,\mathbf{V} = \mathbf{cdn}\,\mathbf{V} \cdot C^{\circ}_{\mathcal{S}}$. Hence, by the previous proposition:

$$\begin{aligned} b_{\mathcal{S}}(\mathbf{dim}\,\mathbf{V}, \mathbf{dim}\,\mathbf{W}) &= \mathbf{dim}\,\mathbf{V} \cdot C^{-1}_{\widehat{\mathcal{S}}} \cdot (\mathbf{dim}\,\mathbf{W})^{tr} \\ &= (\mathbf{cdn}\,\mathbf{V} \cdot C^{\circ}_{\mathcal{S}}) \cdot C^{-1}_{\widehat{\mathcal{S}}} \cdot (\mathbf{cdn}\,\mathbf{W} \cdot C^{\circ}_{\mathcal{S}})^{tr} \\ &= \mathbf{cdn}\,\mathbf{V} \cdot (C^{\circ}_{\mathcal{S}} \cdot C^{-1}_{\widehat{\mathcal{S}}} \cdot C^{\circ\, tr}_{\mathcal{S}}) \cdot (\mathbf{cdn}\,\mathbf{W})^{tr} \\ &= \mathbf{cdn}\,\mathbf{V} \cdot \widehat{C_{\mathcal{S}*}} \cdot (\mathbf{cdn}\,\mathbf{W})^{tr} \\ &= d_{\mathcal{S}^*}(\mathbf{cdn}\,\mathbf{V}, \mathbf{cdn}\,\mathbf{W}). \end{aligned}$$

□

Recall (see [15, Section 2.1] and [22, Section 2]) that given an invertible matrix $A \in \mathbb{M}_n(\mathbb{Z})$ its Coxeter matrix Cox_A is defined as $\mathrm{Cox}_A = -A^{-1} \cdot A^{tr}$. The matrix A is called $\mathbb{Z}$-*regular* if $\mathrm{Cox}_A \in \mathbb{M}_n(\mathbb{Z})$. Given a poset $\mathcal{S}$, by $\mathrm{Cox}_{\mathcal{S}}$ and $\widehat{\mathrm{Cox}}_{\mathcal{S}}$ we denote the Coxeter matrix of $\mathbb{Z}$-regular matrices $C_{\widehat{\mathcal{S}}}$ and $\widehat{C_{\mathcal{S}}}$ respectively. One checks that

$$\mathrm{Cox}_{\mathcal{S}} = \left[\begin{array}{c|c} -1 & -E^{tr} \\ \hline C^{-1}_{\mathcal{S}}E & C^{-1}_{\mathcal{S}}(EE^{tr} - C^{tr}_{\mathcal{S}}) \end{array}\right],$$

and

$$\widehat{\mathrm{Cox}}_{\mathcal{S}} = \left[\begin{array}{c|c} -1 & E^{tr} \\ \hline C^{tr}_{\mathcal{S}}E & C^{tr}_{\mathcal{S}}(EE^{tr} - C^{-1}_{\mathcal{S}}) \end{array}\right],$$

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Given a poset $\mathcal{S}$ with n elements, define the following reflection matrices

$$r^0_{\mathcal{S}} = \left[\begin{array}{c|c} -1 & 0 \\ \hline E & I_n \end{array}\right], \qquad r^*_{\mathcal{S}} = \left[\begin{array}{c|c} 1 & E^{tr} \\ \hline 0 & -I_n \end{array}\right], \qquad r_{\mathcal{S}} = \left[\begin{array}{c|c} -1 & 0 \\ \hline 0 & I_n \end{array}\right],$$

and

$$\widehat{r^0_{\mathcal{S}}} = r_{\mathcal{S}} \cdot r^0_{\mathcal{S}} \cdot r_{\mathcal{S}}, \qquad \widehat{r^*_{\mathcal{S}}} = r_{\mathcal{S}} \cdot r^*_{\mathcal{S}} \cdot r_{\mathcal{S}}.$$

Easy to check that

$$\begin{aligned} \mathrm{Cox}_{\mathcal{S}} &= (C^\circ_{\mathcal{S}})^{-1} \cdot r^0_{\mathcal{S}} \cdot C^\circ_{\mathcal{S}^{op}} \cdot r^*_{\mathcal{S}}, \\ \mathrm{Cox}^{-1}_{\mathcal{S}} &= r^*_{\mathcal{S}} \cdot (C^\circ_{\mathcal{S}^{op}})^{-1} \cdot r^0_{\mathcal{S}} \cdot C^\circ_{\mathcal{S}}, \\ \widehat{\mathrm{Cox}}_{\mathcal{S}} &= (C^\circ_{\mathcal{S}^{op}}) \cdot \widehat{r^0_{\mathcal{S}}} \cdot (C^\circ_{\mathcal{S}})^{-1} \cdot \widehat{r^*_{\mathcal{S}}}, \\ (\widehat{\mathrm{Cox}}_{\mathcal{S}})^{-1} &= (C^\circ_{\mathcal{S}})^{-1} \cdot \widehat{r^*_{\mathcal{S}}} \cdot C^\circ_{\mathcal{S}} \cdot \widehat{r^0_{\mathcal{S}}}. \end{aligned} \tag{1.1}$$

2. Stable representations of posets.

2.1. **Definitions and properties.** The notion of *stability* in an abelian category was defined in [18]. Given an abelian category $\mathfrak{A}$ and a function $\theta\colon K_0(\mathfrak{A}) \to \mathbb{Z}$, an object $X \in \mathfrak{A}$ is called *stable* if $\theta(X) = 0$ and $\theta(Y) < 0$ for any proper sub-object Y of X. Our first aim is to define stable objects in $\mathfrak{sp}_{\mathcal{S}}$. We adopt the definition above as well as the proof of some results from [18] to additive case.

First we define *proper* sub-objects in $\mathfrak{sp}_{\mathcal{S}}$. A morphism $g\colon \mathbf{U} \to \mathbf{V}$ in $\mathfrak{sp}_{\mathcal{S}}$ is said to be *proper* if, for all $s \in \mathcal{S}$, $g(U_s) = V_s \cap g(U_0)$. Given a representation $\mathbf{V} = (V_0, V_s)_{s\in\mathcal{S}}$ and a subspace $K \subset V_0$, one checks that $\mathbf{V}_K = (K, V_s \cap K)_{s\in\mathcal{S}}$ is the unique subrepresentation of $\mathbf{V}$ with the ambient space K for which the inclusion $\mathbf{V}_K \hookrightarrow \mathbf{V}$ gives a proper monomorphism $\mathbf{V}_K \to \mathbf{V}$. In what follows by *proper subrepresentation* of $\mathbf{V}$ we mean a representation of the form $\mathbf{V}_K$ where K is a proper subspace of the ambient space of $\mathbf{V}$.

Remark 2.1. *Generally, given a representation $\mathbf{V}$ and its subrepresentation $\mathbf{W}$ in $\mathfrak{sp}_{\mathcal{S}}$, the quotient $\mathbf{V}/\mathbf{W}$ does not need to belong to $\mathfrak{sp}_{\mathcal{S}}$. Nevertheless, in the case when $\mathbf{W} = \mathbf{V}_K$ is a proper subrepresentation we have $\mathbf{V}/\mathbf{V}_K \in \mathfrak{sp}_{\mathcal{S}}$ (see Appendix A, Proposition 6.3).*

The map $\mathbf{V} \longmapsto \mathbf{dim}\,\mathbf{V}$ gives rise to an isomorphism between the Grothendieck group $K_0(\mathfrak{sp}_{\mathcal{S}})$ and $\mathbb{Z}^{\widehat{\mathcal{S}}}$. Fixing a form $\boldsymbol{\theta} \in \mathrm{Hom}(\mathbb{Z}^{\widehat{\mathcal{S}}}, \mathbb{Z})$ we say that $\mathbf{V} \in \mathfrak{sp}_{\mathcal{S}}$ is $\boldsymbol{\theta}$-*stable* (resp. $\boldsymbol{\theta}$-*semistable*) if $\boldsymbol{\theta}(\mathbf{dim}\,(\mathbf{V})) = 0$ and

$$\boldsymbol{\theta}(\mathbf{dim}\,(\mathbf{W})) < 0 \qquad (\text{resp. } \leq),$$

for any proper subrepresentation $\mathbf{W}$ of $\mathbf{V}$.

This definition is equivalent to the following. Fixing a basis in $\mathrm{Hom}(\mathbb{Z}^{\widehat{\mathcal{S}}}, \mathbb{Z})$, we will regard $\boldsymbol{\theta}$ as a vector $\boldsymbol{\theta} = (\theta_0; \theta_s)_{s\in\mathcal{S}} \in \mathbb{Z}^{\widehat{\mathcal{S}}}$, so that $\boldsymbol{\theta}(\mathbf{dim}\,\mathbf{V})$ simply means

$\boldsymbol{\theta} \cdot \mathbf{dim}\,\mathbf{V}^{tr}$. Define the $\mu_{\boldsymbol{\theta}}$-*slope* of $\mathbf{V} \in \mathfrak{sp}_{\mathcal{S}}$ as

$$\mu_{\boldsymbol{\theta}}(\mathbf{V}) = \frac{1}{\dim V_0} \sum_{s \in \mathcal{S}} \theta_s \dim V_s.$$

We say that $\mathbf{V} \in \mathfrak{sp}_{\mathcal{S}}$ is $\mu_{\boldsymbol{\theta}}$-*stable* (resp. $\mu_{\boldsymbol{\theta}}$-*semistable*) if

$$\mu_{\boldsymbol{\theta}}(\mathbf{W}) < \mu_{\boldsymbol{\theta}}(\mathbf{V}) \qquad \text{(resp. } \leq)$$

for any proper subrepresentation $\mathbf{W}$ of $\mathbf{V}$.

Proposition 2.1. *Let $0 \to \mathbf{W} \to \mathbf{V} \to \mathbf{U} \to 0$ be an exact sequence of representations and let $\boldsymbol{\theta}$ be a weight. Then the following conditions are equivalent:*

(1) $\mu_{\boldsymbol{\theta}}(\mathbf{W}) \leq \mu_{\boldsymbol{\theta}}(\mathbf{V})$,

(2) $\mu_{\boldsymbol{\theta}}(\mathbf{W}) \leq \mu_{\boldsymbol{\theta}}(\mathbf{U})$,

(3) $\mu_{\boldsymbol{\theta}}(\mathbf{V}) \leq \mu_{\boldsymbol{\theta}}(\mathbf{U})$.

Proof. The proof is similar to the proof of [7, Lemma 2.1] and [8, Lemma 2.6]. □

Proposition 2.2. *Let $\boldsymbol{\theta}$ be any weight. Each representation $\mathbf{V}$ has a unique subspace $K \in V_0$ such that the subrepresentation $\mathbf{W} = \mathbf{V}_K$ satisfies:*

(1) *the value of $\mu_{\boldsymbol{\theta}}(\mathbf{W})$ is maximal among all subrepresentations of $\mathbf{V}$, and*

(2) *$\mathbf{W}$ is maximal among all subrepresentations which have the maximal value $\mu_{\boldsymbol{\theta}}(\mathbf{W})$.*

Proof. Since $\mathfrak{sp}_{\mathcal{S}}$ is noetherian, the existence of a representation $\mathbf{W}$ with (1) and (2) follows. We prove the uniqueness. Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be two non-isomorphic representations satisfying (1) and (2). Consider the following short exact sequence:

$$0 \longrightarrow \mathbf{W}_1 \cap \mathbf{W}_2 \longrightarrow \mathbf{W}_1 \oplus \mathbf{W}_2 \longrightarrow \mathbf{W}_1 + \mathbf{W}_2 \longrightarrow 0.$$

By (1) we get $\mu_{\boldsymbol{\theta}}(\mathbf{W}_1 \cap \mathbf{W}_2) \leq \mu_{\boldsymbol{\theta}}(\mathbf{W}_1) = \mu_{\boldsymbol{\theta}}(\mathbf{W}_2)$ and $\mu_{\boldsymbol{\theta}}(\mathbf{W}_1 + \mathbf{W}_2) = \mu_{\boldsymbol{\theta}}(\mathbf{W}_1) = \mu_{\boldsymbol{\theta}}(\mathbf{W}_2)$. Therefore $\mathbf{W}_1 = \mathbf{W}_1 + \mathbf{W}_2 = \mathbf{W}_2$ by (2). □

Obviously the unique subrepresentation $\mathbf{W}$ from Proposition 2.2 is $\mu_{\boldsymbol{\theta}}$-semistable.

Proposition 2.3 (Harder-Narasimhan filtration)**.** *For any $\mathbf{V} = (V_0; V_s)_{s \in \mathcal{S}} \in \mathfrak{sp}_{\mathcal{S}}$ there is a unique filtration (of vector subspaces)*

$$0 = K^0 \subset K^1 \subset \cdots \subset K^h = V_0,$$

which induces a filtration of $\mathbf{V}$

$$0 = \mathbf{V}^0 \subset \mathbf{V}^1 \subset \cdots \subset \mathbf{V}^h = \mathbf{V},$$

in which $\mathbf{V}^i = \mathbf{V}_{K^i} = (K^i; V_s \cap K^i)_{s \in \mathcal{S}}$, such that:

(1) *$\mathbf{V}^i/\mathbf{V}^{i-1}$ are $\mu_{\boldsymbol{\theta}}$-semistable, and*

(2) *$\mu_{\boldsymbol{\theta}}(\mathbf{V}^i/\mathbf{V}^{i-1}) > \mu_{\boldsymbol{\theta}}(\mathbf{V}^{i+1}/\mathbf{V}^i)$ for all $i = 1, \ldots, h-1$.*

Proof. By Proposition 2.2 there exists a unique $\mathbf{V}^1 \subset \mathbf{V}$ with maximal slope $\mu_{\boldsymbol{\theta}}$. If $\mathbf{V}^1 = \mathbf{V}$ then $\mathbf{V}$ is $\mu_{\boldsymbol{\theta}}$-semistable and we are done. Otherwise we get the first step of the filtration

$$0 \subset \mathbf{V}^1 \subset \mathbf{V}.$$

Now consider $\mathbf{V}/\mathbf{V}^1$. If it is $\mu_{\boldsymbol{\theta}}$-semistable then Proposition 2.1 implies $\mu_{\boldsymbol{\theta}}(\mathbf{V}^1) > \mu_{\boldsymbol{\theta}}(\mathbf{V}/\mathbf{V}^1)$. If $\mathbf{V}/\mathbf{V}^1$ is not $\mu_{\boldsymbol{\theta}}$-semistable then we apply the above procedure to produce a unique linear subspace K^2 ($K^1 \subset K^2 \subset V_0$) with $\mathbf{V}^2/\mathbf{V}^1$ $\mu_{\boldsymbol{\theta}}$-semistable. As $\mu_{\boldsymbol{\theta}}(\mathbf{V}^1) > \mu_{\boldsymbol{\theta}}(\mathbf{V}^2)$, Proposition 2.1 implies that $\mu_{\boldsymbol{\theta}}(\mathbf{V}^1) > \mu_{\boldsymbol{\theta}}(\mathbf{V}^2/\mathbf{V}^1)$. Then by induction we get the desired filtration. The uniqueness of the filtration is clear from the proof. □

Proposition 2.4 (Jordan-Hölder filtration)**.** *For any $\mu_{\boldsymbol{\theta}}$-semistable $\mathbf{V} = (V_0; V_s)_{s\in\mathcal{S}} \in \mathfrak{sp}_{\mathcal{S}}$ there is a filtration (of vector subspaces)*

$$0 = K^0 \subset K^1 \subset \cdots \subset K^h = V_0,$$

which induces a filtration of $\mathbf{V}$

$$0 = \mathbf{V}^0 \subset \mathbf{V}^1 \subset \cdots \subset \mathbf{V}^h = \mathbf{V},$$

in which $\mathbf{V}^i = \mathbf{V}_{K^i} = (K^i; V_s \cap K^i)_{s\in\mathcal{S}}$, such that:

(1) $\mathbf{V}^i/\mathbf{V}^{i-1}$ *are $\mu_{\boldsymbol{\theta}}$-stable, and*
(2) $\mu_{\boldsymbol{\theta}}(\mathbf{V}^i/\mathbf{V}^{i-1}) = \mu_{\boldsymbol{\theta}}(\mathbf{V}^{i+1}/\mathbf{V}^i)$ *for all $i = 1, \dots, h-1$.*

Proof. If $\mathbf{V}$ is stable we are done. Otherwise, let $\mathbf{W}$ be a maximal subspace such that $\mu_{\boldsymbol{\theta}}(\mathbf{V}_W) = \mu_{\boldsymbol{\theta}}(\mathbf{V})$. Then $\mathbf{V}_W$ is $\mu_{\boldsymbol{\theta}}$-semistable. Using Proposition 2.1 we have that $\mathbf{V}/\mathbf{V}_W$ is $\mu_{\boldsymbol{\theta}}$-stable and $\mu_{\boldsymbol{\theta}}(\mathbf{V}/\mathbf{V}_W) = \mu_{\boldsymbol{\theta}}(\mathbf{V})$. Repeating the same procedure for $\mathbf{V}_W$ we get a desired filtration. □

Proposition 2.5. *Suppose that both $\mathbf{V}, \mathbf{V}' \in \mathfrak{sp}_{\mathcal{S}}$ are $\mu_{\boldsymbol{\theta}}$-semistable and $g\colon \mathbf{V} \to \mathbf{V}'$ is a non zero morphism. Then $\mu_{\boldsymbol{\theta}}(\mathbf{V}) \le \mu_{\boldsymbol{\theta}}(\mathbf{V}')$.*

Proof. Consider the proper induced morphism $g'\colon \mathbf{V} \to \operatorname{Im} g$. The kernel $\operatorname{Ker} g'$ is a subrepresentation of $\mathbf{V}$ and we have the following short exact sequence

$$0 \longrightarrow \operatorname{Ker} g' \longrightarrow \mathbf{V} \longrightarrow \operatorname{Im} g \longrightarrow 0.$$

As $\operatorname{Im} g$ is a subrepresentation of $\mathbf{V}'$ we have $\mu_{\boldsymbol{\theta}}(\operatorname{Im} g) \le \mu_{\boldsymbol{\theta}}(\mathbf{V}')$ (since $\mathbf{V}'$ is semistable). Assuming $\mu_{\boldsymbol{\theta}}(\mathbf{V}) > \mu_{\boldsymbol{\theta}}(\mathbf{V}')$ we also have that $\mu_{\boldsymbol{\theta}}(\operatorname{Im} g) < \mu_{\boldsymbol{\theta}}(\mathbf{V})$. Therefore, by Proposition 2.1, $\mu_{\boldsymbol{\theta}}(\operatorname{Ker} g') > \mu_{\boldsymbol{\theta}}(\mathbf{V})$. But this contradicts the $\mu_{\boldsymbol{\theta}}$-semistability of $\mathbf{V}$. □

Corollary 2.1. *If $\mathbf{V}$ is $\mu_{\boldsymbol{\theta}}$-stable then* $\operatorname{End}(\mathbf{V})$ *is a division algebra over $\mathbb{F}$. In particular any stable object is indecomposable. Also if $\mathbb{F}$ is algebraically closed then* $\operatorname{End}(\mathbf{V}) \simeq \mathbb{F}$ *and any stable object is Schurian.*

2.2. **Costability.** In what follows we relate $\boldsymbol{\theta}$-stability with the following notion. Let $\boldsymbol{\theta} \in \mathbb{Z}^{\widehat{\mathcal{S}}}$. We say that $\mathbf{V} \in \mathfrak{sp}_{\mathcal{S}}$ is $\boldsymbol{\theta}$-*costable* (resp. $\boldsymbol{\theta}$-*cosemistable*) if $\boldsymbol{\theta}(\mathbf{cdn}\,(\mathbf{V})) = 0$ and

$$\boldsymbol{\theta}(\mathbf{cdn}\,(\mathbf{W})) < 0 \qquad (\text{resp. } \leq)$$

for any proper subrepresentation $\mathbf{W}$ of $\mathbf{V}$.

Note that in general the function $\mathbf{cdn}$ is not additive (unless $\mathcal{S}$ is primitive), therefore the notion of costability does not posses the properties proven in Section 2.1. Nevertheless, if $\mathbf{V}$ is coordinate then costability is related to stability as the following proposition shows.

Proposition 2.6. *Let* $\mathbf{V} \in \mathfrak{sp}_{\mathcal{S}}$ *be a coordinate representation. Then* $\mathbf{V}$ *is* $\boldsymbol{\theta}$-*stable if and only if* $\mathbf{V}$ *is* $\boldsymbol{\theta}'$-*costable with* $\boldsymbol{\theta}' = \boldsymbol{\theta} \cdot C^{\circ}_{\mathcal{S}^{op}}$.

Proof. If $\mathbf{V}$ is coordinate then for any subrepresentation $\mathbf{W}$ of $\mathbf{V}$ (not necessarily proper) we have

$$\mathbf{dim}\,\mathbf{W} = \mathbf{cdn}\,\mathbf{W} \cdot C^{\circ}_{\mathcal{S}},$$

and hence

$$\begin{aligned}\boldsymbol{\theta}(\mathbf{dim}\,\mathbf{W}) &= \boldsymbol{\theta} \cdot (\mathbf{dim}\,\mathbf{W})^{tr} = \boldsymbol{\theta} \cdot (\mathbf{cdn}\,\mathbf{W} \cdot C^{\circ}_{\mathcal{S}})^{tr} \\ &= \boldsymbol{\theta} \cdot C^{\circ\,tr}_{\mathcal{S}} \cdot (\mathbf{cdn}\,\mathbf{W})^{tr} = \boldsymbol{\theta} \cdot C^{\circ}_{\mathcal{S}^{op}} \cdot (\mathbf{cdn}\,\mathbf{W})^{tr} \\ &= \boldsymbol{\theta}' \cdot (\mathbf{cdn}\,\mathbf{W})^{tr} = \boldsymbol{\theta}'(\mathbf{cdn}\,\mathbf{W}).\end{aligned}$$

Therefore

$$\boldsymbol{\theta}(\mathbf{dim}\,\mathbf{V}) = 0 \quad \text{if and only if} \quad \boldsymbol{\theta}'(\mathbf{cdn}\,\mathbf{V}) = 0,$$

and for any proper subrepresentation W

$$\boldsymbol{\theta}(\mathbf{dim}\,\mathbf{W}) > 0 \quad \text{if and only if} \quad \boldsymbol{\theta}'(\mathbf{cdn}\,\mathbf{W}) > 0.$$

The claim follows. □

Corollary 2.2. *Let* $\mathbf{V} \in \mathfrak{sp}_{\mathcal{S}}$ *be an indecomposable coordinate representation whose endomorphism ring is not a division algebra. Then* $\mathbf{V}$ *can not be costable with respect to some form.*

Proof. If $\mathbf{V}$ is costable then $\mathbf{V}$ is stable by Proposition 2.6 and, therefore its endomorphism ring is a division algebra by Corollary 2.1. This is a contradiction. □

2.3. **Positive stability.** We say that representation $\mathbf{V}$ is *positively stable* (respectively *costable*) if there exists a form $\boldsymbol{\theta} \in \mathbb{Z}^{\widehat{\mathcal{S}}}$ with $\theta_s > 0, s \in \mathcal{S}$ such that $\mathbf{V}$ is $\boldsymbol{\theta}$-stable (respectively costable).

Note that if a representation is $\boldsymbol{\theta}$-stable this does not imply in general that $\theta_s > 0, s \in \mathcal{S}$. For instance, if $\mathbf{V}$ is a general representation of a poset $\mathcal{S}$ with 4 incomparable elements in dimension $\boldsymbol{\alpha} = (2; 1, 1, 1, 1)$, then $\mathbf{V}$ is $(-5; 4, 4, 4, -2)$-stable but the form is not positive.

We need the following extension of stability. Given a $\boldsymbol{\theta}$-stable representation $\mathbf{V}$ of a poset $\mathcal{S}$ and any representation $\widetilde{\mathbf{V}} \in \mathfrak{sp}_{\widetilde{\mathcal{S}}}$ such that $\mathcal{S}$ is a subposet $\widetilde{\mathcal{S}}$ and $\widetilde{\mathbf{V}}\big|_{\mathcal{S}} = \mathbf{V}$, define $\widetilde{\theta}_0 = \theta_0$, $\widetilde{\theta}_s = \theta_s$ if $s \in \mathcal{S}$ and $\widetilde{\theta}_s = 0$ if $s \notin \mathcal{S}$. Obviously $\widetilde{\mathbf{V}}$ is $\widetilde{\boldsymbol{\theta}}$-stable. We prove even a stronger connection.

Proposition 2.7. *Let* $\mathbf{V} \in \mathfrak{sp}_{\mathcal{S}}$ *be a positively stable representation with form* $\boldsymbol{\theta}$*. Any representation* $\widetilde{\mathbf{V}} \in \mathfrak{sp}_{\widetilde{\mathcal{S}}}$*, such that* $\mathcal{S}$ *is a subposet* $\widetilde{\mathcal{S}}$ *and* $\widetilde{\mathbf{V}}\big|_{\mathcal{S}} = \mathbf{V}$ *is positively stable with some form* $\widetilde{\boldsymbol{\theta}}$*.*

Proof. We prove the statement for the case when $\mathcal{S} = \widetilde{\mathcal{S}} \setminus \{\tilde{s}\}$ (the remaining part follows by induction). If $\mathbf{V} = (V_0; V_s)_{s\in\widehat{\mathcal{S}}}$, we view the representation $\widetilde{\mathbf{V}}$ as $\widetilde{\mathbf{V}} = (V_0; V_{\tilde{s}}, V_s)_{s\in\mathcal{S}}$. Let $\mathbf{U}$ be a proper subrepresentation of $\mathbf{V}$ such that $\boldsymbol{\theta}(\mathbf{dim}\,\mathbf{U})$ is maximal. As $\boldsymbol{\theta}(\mathbf{dim}\,\mathbf{U}) > 0$ we have that $\boldsymbol{\theta}(\mathbf{dim}\,\mathbf{U}) \geq 1$. Hence defining $\boldsymbol{\theta}' = (\dim V_{\tilde{s}} \dim V_0 + 1) \cdot \boldsymbol{\theta}$, we have that $\mathbf{V}$ is $\boldsymbol{\theta}'$-stable and for any proper subrepresentation $\mathbf{W}$,

$$\boldsymbol{\theta}'(\mathbf{dim}\,\mathbf{W}) = (\dim V_{\tilde{s}} \dim V_0 + 1) \cdot \boldsymbol{\theta}(\mathbf{dim}\,\mathbf{W}) \geq \dim V_{\tilde{s}} \dim V_0 + 1.$$

Set $\nu = \dim V_{\tilde{s}} \dim V_0 + 1$ and define a form $\widetilde{\theta}$ as follows:

$$\widetilde{\theta}_0 = \nu\theta_0 - \dim V_s, \qquad \widetilde{\theta}_t = \begin{cases} \theta'_t, & t \neq \tilde{s} \\ \dim V_0, & t = \tilde{s}. \end{cases}$$

Then

$$\widetilde{\boldsymbol{\theta}}(\mathbf{dim}\,\widetilde{\mathbf{V}}) = \nu\boldsymbol{\theta}'(\mathbf{dim}\,\mathbf{V}) = 0,$$

and for any proper subrepresentation $\widetilde{\mathbf{W}} = (W_0; W_{\tilde{s}}, W_s)_{s\in\mathcal{S}}$ we have

$$\begin{aligned}\widetilde{\boldsymbol{\theta}}(\mathbf{dim}\,\widetilde{\mathbf{W}}) &= \boldsymbol{\theta}'(\mathbf{dim}\,\mathbf{W}) + \dim V_0 \dim \widetilde{W}_{\tilde{s}} - \dim V_{\tilde{s}} \dim \widetilde{W}_0 \\ &\geq \dim V_{\tilde{s}} \dim V_0 + 1 - \dim V_0 \dim V_{\tilde{s}} > 0.\end{aligned}$$

As $\widetilde{\boldsymbol{\theta}}$ has all positive components except $\widetilde{\theta}_0$, the claim follows. □

Similarly one proves the following

Proposition 2.8. *Let* $\mathbf{V} \in \mathfrak{sp}_{\mathcal{S}}$ *be a positively costable representation with form* $\boldsymbol{\theta}$*. Any representation* $\widetilde{\mathbf{V}} \in \mathfrak{sp}_{\widetilde{\mathcal{S}}}$ *such that* $\mathcal{S}$ *is a subposet* $\widetilde{\mathcal{S}}$*,* $\widetilde{\mathbf{V}}\big|_{\mathcal{S}} = \mathbf{V}$ *and* $\mathbf{cdn}\,\widetilde{\mathbf{V}}\big|_{\mathcal{S}} = \mathbf{cdn}\,\mathbf{V}$ *is positively costable with some form* $\widetilde{\boldsymbol{\theta}}$*.*

3. Reflections and stability

In this section we discuss how the Coxeter transformations for posets defined in [4] act on (semi)stable representations.

3.1. **Reflections and Coxeter transformations.** Reflections for posets were defined in [4] using the bimodule language of poset representations. Below we recall this construction. Given $\mathbf{V} = (V_0, V_s)_{s\in\mathcal{S}}$ define $S(\mathbf{V}) = (V_0^*, V_s^\perp)$, where $V_s^\perp = \{\varphi \in V_0^* \mid \varphi(V_s) = 0\}$. Obviously, $S(\mathbf{V}) \in \mathfrak{sp}_{\mathcal{S}^{op}}$, $S^2(\mathbf{V}) \cong \mathbf{V}$ and

$$\mathbf{dim}\, S(\mathbf{V}) = \mathbf{dim}\, \mathbf{V} \cdot r^0_{\mathcal{S}} = (\dim V_0; \dim V_0 - \dim V_s)_{s\in\mathcal{S}}.$$

The second reflection is defined as follows. Given a representation $\mathbf{V} = (V_0; V_s)_{s\in\mathcal{S}} \in \mathfrak{sp}_{\mathcal{S}}$ define by $\mathcal{L}(\mathbf{V})$ the following family of systems of subspaces:

$$\mathcal{L}(\mathbf{V}) = \Big\{(V_0; V_s')_{s\in\mathcal{S}} \mid V_s' \subset V_0,\ V_s' \cong V_s/\sum_{t\prec s} V_t,\ \sum_{t\preceq s} V_t' = V_s\Big\}.$$

Note that having any system of subspaces $(V_0; V_s)_{s\in\mathcal{S}}$ indexed by a poset $\mathcal{S}$, one can form a representation $\Delta_{\mathcal{S}}((V_0; V_s)) = (V_0; \tilde{V}_s)_{s\in\mathcal{S}}$ of $\mathcal{S}$ setting $\tilde{V}_s = \sum_{t\preceq s} V_t$. In particular, it follows that for any $(V_0; V_s')_{s\in\mathcal{S}} \in \mathcal{L}(\mathbf{V})$ we recover $\mathbf{V}$ as $\mathbf{V} = \Delta_{\mathcal{S}}((V_0; V_i'))$.

Now let $(V_0; V_i)_{i\in I}$ be any system of subspaces in V_0 indexed by a finite set I (considered as a poset with trivial partial order), such that the map

$$\begin{aligned} \varphi\colon \bigoplus_{i\in I} V_i &\longrightarrow V_0 \\ (v_i)_{i\in I} &\longmapsto \sum_{i\in I} v_i, \qquad v_i \in V_i \end{aligned}$$

is surjective. Consider the following short exact sequence

$$0 \longrightarrow \ker\varphi \xrightarrow{\psi} \bigoplus\nolimits_{i\in I} V_i \xrightarrow{\varphi} V_0 \longrightarrow 0$$

where $\psi(y) = (p_i(y))_{i\in I}$, $p_i\colon \ker\varphi \to V_i$. Dualizing we get the sequence

$$0 \longrightarrow V_0^* \xrightarrow{\varphi^*} \bigoplus\nolimits_{i\in I} V_i^* \xrightarrow{\psi^*} (\ker\varphi)^* \longrightarrow 0.$$

Denote by $\Gamma((V_0; V_i)_{i\in I})$ the system of subspaces $((\ker\varphi)^*, \mathrm{Im}(p_i^*))_{i\in I}$. The action of Γ on dimensions is the following:

$$\mathbf{dim}\,\Gamma(V_0; V_i)_{i\in I} = \mathbf{dim}\,(V_0; V_i)_{i\in I} \cdot r_I^* = \Big(\sum_{i\in I} \dim V_i - \dim V_0; \dim V_i\Big)_{i\in I}.$$

Denote by E_0 a simple representation of the form $(\mathbb{F}; 0)_{s\in\mathcal{S}}$ in $\mathfrak{sp}_{\mathcal{S}}$. Let $\mathbf{V} \in \mathfrak{sp}_{\mathcal{S}}$ be any representation. Choosing $(V_0; V_s')_{s\in\mathcal{S}} \in \mathcal{L}(\mathbf{V})$ we define $T(\mathbf{V}) \in \mathfrak{sp}_{\mathcal{S}^{op}}$ by

$$T(\mathbf{V}) = \Delta_{\mathcal{S}^{op}}(\Gamma(V_0; V_s')_{s\in\mathcal{S}}).$$

One easily checks that if $\mathbf{V}$ does not contain E_0 as a direct summand then the map φ above is surjective, and therefore $T(\mathbf{V})$ is well-define (in particular, it does not depend on the choice of representative in $\mathcal{L}(\mathbf{V})$) and we have $T^2(\mathbf{V}) \cong \mathbf{V}$. If

$\mathbf{V}$ and $T(\mathbf{V})$ are coordinate (this is always the case when $\mathcal{S}$ is primitive) we have that

$$\mathbf{dim}\, T(\mathbf{V}) = \dim \mathbf{V} \cdot {C^{\circ}_{\mathcal{S}}}^{-1} \cdot r^*_{\mathcal{S}} \cdot C^{\circ}_{\mathcal{S}^{op}}.$$

The compositions (when it makes sense) of reflections S, T will be denoted by

$$\mathrm{F}^+ = S \circ T, \qquad \mathrm{F}^- = T \circ S$$

and called *Coxeter transformations.*

Remark 3.1. *The transformation* F^+ *is related to the Auslander-Reiten translate for poset (cf.* [21, Chapter 11] *and* [2]*).*

Using the formulas (1.1) one checks that the transformations F^+ and F^- act on dimensions of representations (in coordinate cases) as follows:

$$\mathrm{F}^+(\mathbf{dim}\, \mathbf{V}) = \mathbf{dim}\, \mathbf{V} \cdot \mathrm{Cox}_{\mathcal{S}}, \qquad \mathrm{F}^-(\mathbf{dim}\, \mathbf{V}) = \mathbf{dim}\, \mathbf{V} \cdot (\mathrm{Cox}_{\mathcal{S}})^{-1}.$$

3.2. **Stability behaviour of reflections.** First we show that S maps (semi)stable representation into (semi)stable ones.

Lemma 3.1. *Let* $\boldsymbol{\theta} = (\theta_0; \theta_s)_{s\in\mathcal{S}}$ *be a weigth. A representation* $\mathbf{V} = (V_0; V_s)_{s\in\mathcal{S}}$ *is* $\boldsymbol{\theta} = (\theta_0; \theta_s)_{s\in\mathcal{S}}$*-(semi)stable iff the representation* $S(\mathbf{V}) = (V_0^*; V_s^{\perp})_{s\in\mathcal{S}}$ *is* $S(\boldsymbol{\theta})$*-(semi)stable, where*

$$S(\boldsymbol{\theta}) := \boldsymbol{\theta} \cdot \widehat{r^*_{\mathcal{S}}} = \Big(-\sum_{s\in\mathcal{S}} \theta_s - \theta_0; \theta_s \Big)_{s\in\mathcal{S}}.$$

Proof. Notice that

$$\begin{aligned} \boldsymbol{\theta}(\dim \mathbf{V}) &= \sum_{s\in\widehat{\mathcal{S}}} \theta_s \dim V_s, \\ S(\boldsymbol{\theta})(\dim S(\mathbf{V})) &= (-\sum_{s\in\mathcal{S}} \theta_s - \theta_0) \dim V_0 + \sum_{s\in\mathcal{S}} \theta_s (\dim V_0 - \dim V_s) \\ &= -\boldsymbol{\theta}(\dim \mathbf{V}). \end{aligned}$$

Therefore $\boldsymbol{\theta}(\dim \mathbf{V}) = 0$ iff $S(\boldsymbol{\theta})(\dim S(\mathbf{V})) = 0$. Assume that $S(\mathbf{V}) = (V_0^*; V_s^{\perp})_{s\in\mathcal{P}}$ is not $S(\boldsymbol{\theta})$-stable. Therefore, there exists a subspace $M^{\perp}$ such that

$$\sum_{s\in\mathcal{P}} \theta_s \dim(V_s^{\perp} \cap M^{\perp}) + \Big(\sum_{s\in\mathcal{P}} \theta_s + \theta_0 \Big) \dim M^{\perp} \geq 0. \tag{3.1}$$

As $\dim(V_s^{\perp} \cap M^{\perp}) = \dim V_0 - \dim V_s - \dim M + \dim(V_s \cap M)$, from (3.1) we have

$$\sum_{i\in\mathcal{P}} \theta_i (\dim V_0 - \dim V_i - \dim M + \dim(V_i \cap M)) + \Big(\sum_{i\in\mathcal{P}} \theta_i + \theta_0)(\dim V_0 - \dim M \Big) \geq 0.$$

Or, equivalently

$$\sum_{s\in\mathcal{P}} \theta_s \dim(V_s \cap M) + \chi_0 \dim M \geq 0.$$

Hence $(V_0; V_s)_{s\in\mathcal{P}}$ is not $\boldsymbol{\theta}$-stable which is a contradiction. $\square$

Assuming that $\mathcal{S}$ is primitive we prove that the reflection T also maps (semi)stable representation into (semi)stable ones.

Lemma 3.2. *A representation* $\mathbf{V}$ *is* $\boldsymbol{\theta}$*-(semi)stable iff any representation* $\mathbf{W} \in \mathcal{L}(\mathbf{V})$ *is* $\boldsymbol{\theta} \cdot C^{\circ}_{\mathcal{S}^{op}}$*-(semi)stable.*

Proof. The proof is similar to the proof of Proposition 2.6. □

Similarly we have that if a system of subspaces $(V_0; V_s)_{s\in\mathcal{S}}$ indexed by a poset $\mathcal{S}$ (not necessarily a representation) is $\boldsymbol{\theta}$-stable then $\Delta_{\mathcal{S}}((V_0; V_s))$ is $\boldsymbol{\theta} \cdot (C^{\circ}_{\mathcal{S}^{op}})^{-1}$-stable.

Lemma 3.3. *Let* $\boldsymbol{\theta} = (\theta_0; \theta_s)_{s\in\mathcal{S}}$ *be a weight,* $\mathbf{V} = (V_0; V_s)_{s\in\mathcal{S}}$ *a system of subspaces which does not contain* E_0 *as a direct summand. Then* $\mathbf{V}$ *is* $\boldsymbol{\theta} = (\theta_0; \theta_s)_{s\in\mathcal{S}}$*-(semi)stable iff the system* $\Gamma(\mathbf{V})$ *is* $\Gamma(\boldsymbol{\theta})$*-(semi)stable, where*

$$\Gamma(\boldsymbol{\theta}) := \boldsymbol{\theta} \cdot \widehat{r^*_{\mathcal{S}}} = (\theta_0; -\theta_0 - \theta_s)_{s\in\mathcal{S}}.$$

Proof. Suppose that $\mathbf{V}$ is $\boldsymbol{\theta}$-stable and that $\Gamma(\mathbf{V})$ is not $\Gamma(\boldsymbol{\theta})$-stable. Then there exists a subspace $M \in \ker\varphi^*$ such that

$$\frac{\sum_{s\in\mathcal{S}}(\theta_0 - \theta_s)\dim(V_s^* \cap M)}{\dim M} \geq \frac{\sum_{s\in\mathcal{S}} \theta_s \dim(V_s)}{\dim V_0}. \tag{3.2}$$

Consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & M & \longrightarrow & A = \bigoplus_{s\in\mathcal{S}} A_s & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \ker\varphi & \longrightarrow & \bigoplus_{s\in\mathcal{S}} V_s & \xrightarrow{\varphi} & V_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \ker\psi & \longrightarrow & \bigoplus_{s\in\mathcal{S}} V_s \cap K & \xrightarrow{\psi} & K & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We have

$$\begin{aligned}
\dim M &= \dim\ker\varphi - \dim\ker\psi \\
&= \sum_{s\in\mathcal{S}} \dim V_s - \dim V_0 - \Big(\sum_{s\in\mathcal{S}} \dim(V_s \cap K) - \dim K\Big).
\end{aligned}$$

On the other hand

$$\sum_{s\in\mathcal{S}} \dim V_s - \sum_{s\in\mathcal{S}} \dim(V_s \cap K) = \sum_{s\in\mathcal{S}} A_s \leq \sum_{s\in\mathcal{S}} (V_s^* \cap M).$$

Therefore (3.2) reduces to

$$\Big(\sum_{s\in\mathcal{S}}\dim V_s-\dim V_0\Big)\sum_{s\in\mathcal{S}}\theta_s\dim(V_s\cap K)\geq\dim K\sum_{s\in\mathcal{S}}\theta_s\dim V_s\Big(\frac{\sum_{s\in\mathcal{S}}\dim V_s}{\dim V_0}-1\Big),$$

which is equivalent to

$$\frac{\sum_{s\in\mathcal{S}}\theta_s\dim(V_s\cap K)}{\sum_{s\in\mathcal{S}}\theta_s\dim V_s}\geq\frac{\dim K}{\dim V_0}.$$

The last says that $\mathbf{V}$ is not $\boldsymbol{\theta}$-stable which is a contradiction. □

We also have

Proposition 3.1. *Assume that a representation* $\mathbf{V}$ *does not contain* E_0 *as a direct summand. Then* $\mathbf{V}$ *is* $\boldsymbol{\theta}$*-(semi)stable iff a representation* $T(\mathbf{V})$ *is* $T(\boldsymbol{\theta})$*-(semi)stable, where*

$$T(\boldsymbol{\theta})=\boldsymbol{\theta}\cdot C^{\circ}_{\mathcal{S}^{op}}\cdot\widehat{r^*_{\mathcal{S}}}\cdot(C^{\circ}_{\mathcal{S}})^{-1}.$$

Combining this proposition with the previous two lemmas we get

Theorem 3.1. *Assume that* $\mathcal{S}$ *is primitive poset,* $\boldsymbol{\alpha}=(\alpha_0;\alpha_s)_{s\in\mathcal{S}}$ *the dimension vector and* $\boldsymbol{\theta}$ *a weight. A representation* $\mathbf{V}\neq E_0$ *is* $\boldsymbol{\theta}$*-stable (respectively semistable) iff* $\mathrm{F}^+(\mathbf{V})$ *is* $\mathrm{F}^+(\boldsymbol{\theta})$*-stable (respectively semistable), where*

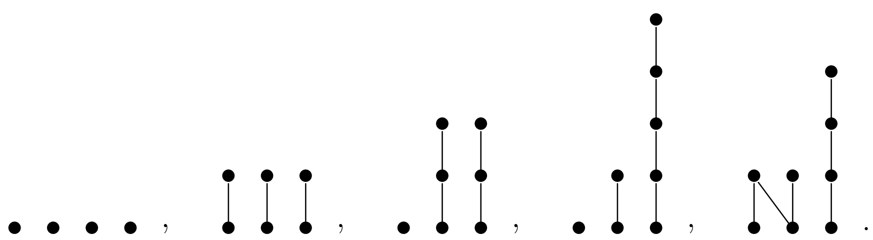
$$\mathrm{F}^+(\boldsymbol{\theta})=\boldsymbol{\theta}\cdot\widehat{\mathrm{Cox}}_{\mathcal{S}}.$$

In a subsequent work we will establish similar statements for reflection T in the case of non-primitive posets.

4. Stability and posets of finite representation type

Recall that M.Kleiner in [11] (see also [21, Theorem 10.1]) showed that a poset $\mathcal{S}$ has only a finite number of non equivalent indecomposable representations (that is, the category $\mathfrak{sp}_{\mathcal{S}}$ is of finite representation type) if and only if it does not contain a full poset whose Hasse diagram is one of the following

(4.1)



Tha posets in list (4.1) we call *critical*. Note that $\mathfrak{sp}_{\mathcal{S}}$ is of tame representation type for each critical poset (see, [21, Chapter 15] for details).

In this section we prove the following

Theorem 4.1. *Let* $\mathcal{S}$ *be a finite poset. The following statements are equivalent.*

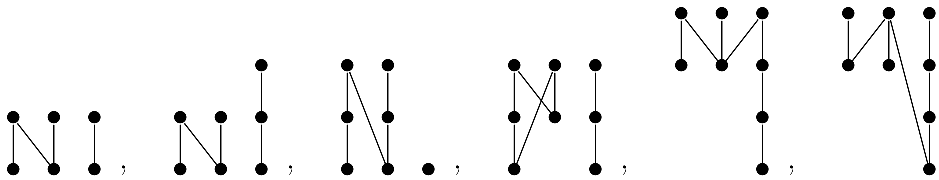
(a) *The category* $\mathfrak{sp}_{\mathcal{S}}$ *is of finite representation type.*

(b) *Any indecomposable representation of $\mathcal{S}$ is positively costable.*
(c) *Any indecomposable representation of $\mathcal{S}$ is positively stable.*

The implication $(c) \Rightarrow (a)$ follows from Proposition 2.5. Indeed, if $\widehat{\mathcal{S}}$ has the infinite representation type then there exist indecomposable representations whose endomorphism ring is not a division algebra. Therefore, they cannot be stable by Corollary 2.1.

The implication $(b) \Rightarrow (a)$ follows from Corollary 2.2 and the fact that any poset of infinite representation type has indecomposable coordinate representations whose endomorphism ring is not a division algebra.

4.1. **Exact posets of finite representation type.** Recall that a poset $\mathcal{S}$ is called *exact* if it admits an exact representation. A complete list of exact posets of finite representation type and their sincere representations was obtained in [11] (see also [21, Chapter 10.7], for corrected list of exact representations). Namely, a non-primitive poset of finite representation type is exact if and only if it has one of the following forms:

(4.2) 

$\mathcal{S}_1$ $\mathcal{S}_2$ $\mathcal{S}_3$ $\mathcal{S}_4$ $\mathcal{S}_5$ $\mathcal{S}_6$

For each non-primitive sincere poset $\mathcal{S}_1, \ldots, \mathcal{S}_6$ we list all its exact representations in the following table.

Poset	*Exact representations*
$\mathcal{S}_1$	$(K^3; K_{123}, K_{1,2,3}; K_1, K_{1,2}; K_3, K_{2,3})$
$\mathcal{S}_2$	1) $(K^3; K_3, K_{1,2,3}; K_{123}, K_{13,2}; K_1, K_{1,2}, K_{1,2,3})$
	2) $(K^4; K_{14}, K_{1,2,4}; K_4, K_{123,4}; K_3, K_{2,3}, K_{1,2,3})$
	3) $(K^4; K_{14}, K_{1,2,4}; K_4, K_{12,23,4}; K_3, K_{2,3}, K_{1,2,3})$
	4) $(K^4; K_{1,24}, K_{1,2,3,4}; K_4, K_{123,4}; K_3, K_{2,3}, K_{1,2,3})$
	5) $(K^4; K_{1,24}, K_{1,2,3,4}; K_4, K_{12,13,4}; K_3, K_{2,3}, K_{1,2,3})$
	6) $(K^5; K_{15,4}, K_{1,2,4,5}; K_5, K_{123,24,5}; K_3, K_{2,3}, K_{1,2,3})$
	7) $(K^5; K_{3,5}, K_{2,3,4,5}; K_{45}, K_{134,24,45}; K_1, K_{1,2}, K_{1,2,3,4})$
	8) $(K^5; K_{1,25}, K_{1,2,3,5}; K_5, K_{13,234,5}; K_4, K_{2,3,4}, K_{1,2,3,4})$
	9) $(K^5; K_{1,25}, K_{1,2,3,5}; K_5, K_{123,24,5}; K_{3,4}, K_{2,3,4}, K_{1,2,3,4})^*$
$\mathcal{S}_3$	$(K^4; K_4, K_{1,4}, K_{1,2,3,4}; K_3, K_{2,3}, K_{1,2,3}; K_{123,24})$
$\mathcal{S}_4$	$(K^4; K_4, K_{3,4}, K_{1,2,3,4}; K_{234}, K_{12,23,4}; K_1, K_{1,2}, K_{1,2,3})$

$\mathcal{S}_5$	$(K^5; K_{125,13}, K_{1,2,3,5}; K_5, K_{1,24,5}; K_4, K_{3,4}, K_{2,3,4}, K_{1,2,3,4,5})$
$\mathcal{S}_6$	$(K^5; K_{1,25}, K_{1,3,25}; K_5, K_{1,2,3,4,5}; K_4, K_{3,4}, K_{2,3,4}, K_{1,2,3,4,5})$

We used the following notation: K^n denotes the vector space over $\mathbb{F}$ with the canonical basis $e_1, \dots, e_n$ and $K_{i_1\dots i_k,\dots,j_1\dots j_m}$ denotes the subspace of K^n generated by the vectors $e_{i_1\dots i_k}, \dots, e_{j_1\dots j_m}$ where

$$e_{i_1\dots i_k} = e_{i_1} + \cdots + e_{i_k}, \dots, e_{j_1\dots j_m} = e_{j_1} + \cdots + e_{j_m}.$$

4.2. **Proof of the implication** $(a) \Rightarrow (b)$**.**

Proposition 4.1. *Suppose that $\mathcal{S}$ has a finite representation type, and $\mathbf{V}$ is a Schurian representation of $\mathbf{V}$. Then*

$$d_{\mathcal{S}^*}(\mathbf{cdn\,W}, \mathbf{cdn\,V}) - d_{\mathcal{S}^*}(\mathbf{cdn\,V}, \mathbf{cdn\,W}) > 0, \tag{4.3}$$

for any proper subrepresentation $\mathbf{W}$ of $\mathbf{V}$.

Proof. It is clear that it is enough to check the statement in case $\mathcal{S}$ is exact. If $\mathcal{S}$ is primitive the claim follows immediately from Corollary 1.1 and Schofield's characterization of Schurian roots for acyclic quivers [20, Theorem 6.1]. Indeed, in this case any representation of $\mathcal{S}$ corresponds to a representation of an unbound Hasse quiver $Q(\widehat{\mathcal{S}})$ of $\widehat{\mathcal{S}}$. Also, any representation of $\mathcal{S}$ is coordinate. Hence, using Corollary 1.1 and the fact that in this case $d_{\mathcal{S}^{op}}(\mathbf{cdn\,W}, \mathbf{cdn\,V}) = b_{\mathcal{S}}(\mathbf{dim\,W}, \mathbf{dim\,V})$ coincides with the usual Tits form of $Q(\widehat{\mathcal{S}})$, we apply [20, Theorem 5] to prove that (4.3) holds.

Now assume that $\mathcal{S}$ is non-primitive exact. For each representation in Appendix B we completely describe all maximal sub-coordinate dimensions. The statement now follows by direct verification of conditions (4.3). □

By Proposition 4.1 we have that any indecomposable $\mathbf{V} \in \mathfrak{sp}_{\mathcal{S}}$ with coordinate dimension $\boldsymbol{\alpha} = (\alpha_0; \alpha_i)_{i\in\mathcal{S}}$ of a poset of finite representation type is costable with a form $\boldsymbol{\theta} \in \mathbb{Z}^{\widehat{\mathcal{S}}}$ given by

$$\boldsymbol{\theta}(\boldsymbol{\beta}) = d_{\mathcal{S}^*}(\boldsymbol{\beta}, \boldsymbol{\alpha}) - d_{\mathcal{S}^*}(\boldsymbol{\alpha}, \boldsymbol{\beta}).$$

It is straightforward to check that the components of this form are

$$\theta_0 = -\sum_{s\in\mathcal{S}} \alpha_s, \qquad \theta_s = \sum_{s\prec t\in\widehat{\mathcal{S}}} \alpha_t - \sum_{t\prec s\in\widehat{\mathcal{S}}} \alpha_t. \tag{4.4}$$

For instance, a unique exact representation of a poset $\mathcal{S}_1$ is costable with a form $(-6; 4, 1; 5, 2; 4, 2)$.

Now observe that if a representation is exact then for a fixed $s \in \mathcal{S}$ we have

$$\sum_{s\prec t\in\widehat{\mathcal{S}}} \alpha_t > \alpha_0 > \sum_{t\prec s\in\widehat{\mathcal{S}}} \alpha_t.$$

Therefore each $\theta_s > 0, s \in \mathcal{S}$ and any exact representation is positively costable. Now the implication $(a) \Rightarrow (b)$ follows from Proposition 2.8.

4.3. **Proof of the implication** $(a) \Rightarrow (c)$**.** To prove the implication $(a) \Rightarrow (c)$ we show the analogue of Proposition 4.1 for so-called sincere representations and their dimension vectors.

We call a representation $\mathbf{V} = (V_0; V_s)_{s\in\mathcal{S}}$ *sincere* if it is indecomposable, $\mathbf{dim}\,\mathbf{V}$ is sincere and $V_s \neq V_t$ if $s \prec t$ in $\widehat{\mathcal{S}}$. Respectively, $\widehat{\mathcal{S}}$ is called *sincere* if it has at least one sincere representation. The following proposition describes all sincere posets of finite type.

Proposition 4.2. *The set of sincere posets of finite representation type consists of four primitive posets* $(1,1,1)$*,* $(1,2,2)$*,* $(1,2,3)$*,* $(1,2,4)$ *and non-primitive posets* $\mathcal{S}_1, \dots, \mathcal{S}_6$*.*

Proof. Let $\mathcal{S}$ be a poset of finite representation type, $\mathbf{V} \in \mathfrak{sp}_{\mathcal{S}}$ its sincere representation. Precisely one of the following cases occurs:

i) $\mathbf{V}$ is exact representation of $\mathcal{S}$ with $V_s \neq V_0$ for all $s \in \mathcal{S}$.

ii) $\mathbf{V}$ is non-exact representation at some $s \in \mathcal{S}$, therefore $V_s = \sum_{t\prec s} V_t$.

In the first case $\mathcal{S}$ is in list (4.2) of exact posets. In the second case $\mathbf{V}$ generates an indecomposable representation (denoted by $\mathbf{V}^1$) of the reduced poset $\mathcal{S}_s = \mathcal{S} \setminus s$. Obviously, $\mathbf{V}^1$ is a sincere representation of $\mathcal{S}_s$ and therefore it satisfies either 1) or 2) above. Proceeding in this way we eventually obtain an exact representation of some poset $\mathcal{S}_{s_1,\dots,s_k}$ with $V_s \neq V_0,\ s \in \mathcal{S}_{s_1,\dots,s_k}$ from the table (4.2).

Summing up we have the following procedure to describe all sincere posets and their sincere representations:

(1) All exact posets which admit exact and at the same time sincere representation $\mathbf{V}$ (that is, $V_i \neq V_0$) are precisely $(1,1,1)$, $(1,2,2)$, $(1,2,3)$, $(1,2,4)$ and $\mathcal{S}_2$;

(2) Let $\mathcal{S}$ be a sincere poset and $\mathbf{V}$ its sincere representation. Let $\mathcal{I}$ be a subset of $\mathcal{S}$ such that $\sum_{s\in\mathcal{I}} V_s \neq V_0$. Define an extended poset $\mathcal{S}^{\mathcal{I}} = (\mathcal{S} \cup \{\widetilde{s}\}, \prec_{\mathcal{I}})$ with a partial order defined in such a way that its restriction to $\mathcal{S}$ coincides with $\prec$ and $s \prec_{\mathcal{I}} \widetilde{s}$, for all $s \in \mathcal{I}$. Let $V^{\mathcal{I}}$ be a representation of $\mathcal{S}^{\mathcal{I}}$ given by $V_s^{\mathcal{I}} = V_s$ for all $s \in \mathcal{S}$ and $V_{\widetilde{s}}^{\mathcal{I}} = \sum_{s\in I} V_s$. Evidently, $V^{\mathcal{I}}$ is a sincere representation and therefore $\mathcal{S}^{\mathcal{I}}$ is a sincere poset.

The above procedure clearly terminates as the dimensions of V_0 are bounded. Hence inductively we obtain all sincere posets and all their sincere representations. □

Proceeding as in the proof of the previous proposition we obtain the following list of all sincere representations of sincere posets $\mathcal{S}_1, \dots, \mathcal{S}_6$:

Poset	*Sincere representations*
$\mathcal{S}_1$	$(K^3; K_{123}, K_{1,23}; K_1, K_{1,2}; K_3, K_{2,3})$
$\mathcal{S}_2$	$(K^4; K_{123,24}, K_{13,2,4}; K_4, K_{1,4}; K_3, K_{2,3}, K_{1,2,3})$ $(K^4; K_{124,13}, K_{12,13,4}; K_4, K_{1,2,4}; K_3, K_{2,3}, K_{1,2,3})$
$\mathcal{S}_3$	$(K^4; K_4, K_{1,4}, K_{1,3,4}; K_3, K_{2,3}, K_{1,2,3}; K_{123,24})$
$\mathcal{S}_4$	$(K^4; K_4, K_{123,4}, K_{1,23,4}; K_{14}, K_{1,2,4}; K_3, K_{2,3}, K_{1,2,3})$
$\mathcal{S}_5$	$(K^5; K_{15,4}, K_{1,2,4,5}; K_5, K_{123,24,5}; K_3, K_{2,3}, K_{1,2,3}; K_{1,2,3,5})$
$\mathcal{S}_6$	$(K^5; K_5, K_{1,2,5}; K_{134,235}, K_{13,23,4,5}; K_4, K_{3,4}, K_{2,3,4}; K_{1,2,3,4})$

Similarly to Proposition 4.1 one proves the following:

Proposition 4.3. *Let $\mathcal{S}$ be a sincere poset and* $\mathbf{V}$ *its sincere representation. Then* $\mathbf{V}$ *is stable with a form*

$$\boldsymbol{\theta}(\mathbf{W}) = b_{\mathcal{S}}(\mathbf{dim}\,\mathbf{V}, \mathbf{dim}\,\mathbf{W}) - b_{\mathcal{S}}(\mathbf{dim}\,\mathbf{W}, \mathbf{dim}\,\mathbf{V}). \tag{4.5}$$

To prove this proposition we describe the set of maximal subdimensions for all sincere representations of posets of finite representation type and check the stability conditions (4.5). The details are given in Appendix C.

If $\mathcal{S}$ is sincere than it is straighforward to see (see also, [22, Proposition 4.2]) that

$$b_{\mathcal{S}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \boldsymbol{\alpha} \cdot C_{\widehat{\mathcal{S}}}^{-1} \cdot \boldsymbol{\beta}^{tr} = \sum_{s\in\widehat{\mathcal{S}}} \alpha_s\beta_s - \sum_{s\to t\in\widehat{\mathcal{S}}} \alpha_s\beta_t + \sum_{s,t\in\widehat{\mathcal{S}}} r(s,t)\alpha_s\beta_t,$$

in which $r(s,t)$ is the maximal number of $\mathbb{F}$-linear independent minimal commutativity relations with the source s and the terminus t. By Proposition 4.3 we have that any sincere $\mathbf{V} \in \mathfrak{sp}_{\mathcal{S}}$ with dimension $\boldsymbol{\alpha} = (\alpha_0; \alpha_s)_{s\in\mathcal{S}}$ of a poset of finite representation type is stable with a form $\boldsymbol{\theta} \in \mathbb{Z}^{\widehat{\mathcal{S}}}$ given by

$$\boldsymbol{\theta}(\boldsymbol{\beta}) = b_{\mathcal{S}}(\boldsymbol{\beta}, \boldsymbol{\alpha}) - b_{\mathcal{S}}(\boldsymbol{\alpha}, \boldsymbol{\beta}). \tag{4.6}$$

One checks that the components of this form are:

$$\theta_0 = -\sum_{s\to 0\in\widehat{\mathcal{S}}} \alpha_s + \sum_{s\in\mathcal{S}} r(s,0)\alpha_s, \qquad \theta_s = \sum_{s\to t\in\widehat{\mathcal{S}}} \alpha_t - \sum_{t\to s\in\widehat{\mathcal{S}}} \alpha_t - \sum_{t\in\widehat{\mathcal{S}}} r(s,t)\alpha_t. \tag{4.7}$$

For instance, a unique sincere representation of a poset $\mathcal{S}_1$ is stable with a form $(-6; 2, 1; 1, 2; 2, 2)$. By examining each sincere poset we check that the components θ_i are positive (see Appendix C for the details). Therefore, each sincere representation of a poset of finite representation type is positively stable with the form defined by (4.7).

Now let $\mathbf{V}$ be an indecomposable representation of $\widehat{\mathcal{S}}$ of finite representation type. Hence, there is a sincere subposet $\widetilde{\mathcal{I}}$ of $\widehat{\mathcal{S}}$ such that the restriction $\mathbf{V}_{\widetilde{\mathcal{I}}}$

of $\mathbf{V}$ to $\tilde{I}$ is a sincere representation. The representation $\mathbf{V}_{\tilde{\mathcal{I}}}$ is positively stable by considerations above. Then the representation $\mathbf{V}$ is positively stable by Proposition2.7. The implication $(a) \Rightarrow (c)$ follows.

5. Geometric stability

In this section we assume that $\mathbb{F}$ is algebraically closed. Fix the poset $\mathcal{S}$ and the admissible dimension vector $\boldsymbol{\alpha} = (\alpha_0; \alpha_s)_{s\in\mathcal{S}}$. As we mentioned above the variety $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ is projective and the group $\mathbf{Gl}(\boldsymbol{\alpha}_0)$ acts on $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ diagonally. Our goal is to understand the quotient space $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}/\mathbf{Gl}(\boldsymbol{\alpha}_0)$. As usual the main problem is that the quotient space is rarely a projective variety. One possible approach is to construct a "good" quotient is via Geometric Invariant Theory (GIT). We briefly recall this approach, for details we refer to [3] (for general approach), to [10] (where the author constructed the good quotients for representations of quivers) and to [17] (where the author motivated the geometric approach to the classification problem of quiver representations and discussed topological, arithmetic and algebraic methods for the study of moduli spaces).

5.1. **Brief review of GIT quotients.** Let G be a reductive group acting on a projective algebraic variety X. The GIT approach consists of the following steps. First one chooses a linearization of the action, that is, a G-equivariant embedding of X into a projective space $\mathbb{P}^n$ with a linear action of G (via representation of G in $\mathbf{Gl}(n+1)$). An embedding of X to $\mathbb{P}^n$ is defined by choosing a line bundle L over X (which is ample iff the emdedding is closed) and the set of its sections $f_0, \dots, f_n$ (which form a basis in the space of sections $\Gamma(X, L)$). Then one specifies (with respect to L) the sets $X^{ss}(L)$, $X^s(L)$, $X^{us}(L)$ of semi-stable, stable and regular points respectively on X, where

- (i) $x \in X$ is called *semi-stable* if there exist $m > 0$ and $f \in \Gamma(X, L^m)^G$ such that $X_f = \{y \in X \mid f(y) \neq 0\}$ is affine and contains x;
- (ii) $x \in X$ is called *stable* if it is semi-stable, stabilizer of x is finite and G-action on X_f is closed;
- (iii) $x \in X$ is called *unstable* if it is not semi-stable.

The central point GIT is that there exists an algebraic quotient of X by G, denoted by $X//G$, which can be described as the quotient of the open set of $X^{ss}(L)$ of semistable points by the equivalence relation: $x \sim y$ if and only if the orbit closures $\overline{G\cdot x}$ and $\overline{G\cdot y}$ intersects (in $X^{ss}(L)$). Therefore the points of $X//G$ are in one-one correspondence with the closed orbits in $X^{ss}(L)$. Note that in case L is ample then (see [3, Proposition 8.1])

$$X^{ss}(L)//G \cong \mathrm{Proj}\Big(\oplus_{n\geq 0} \Gamma(X, L^{\otimes n})^G \Big), \tag{5.1}$$

and $X^{ss}(L)//G$ is a projective variety. The variety $X^s(L)/G$ is a *geometric quotient*, which parametrizes the stable orbits.

A powerful tool to describe stable points is the Hilbert-Mumford numerical criterion of stability, which is stated in terms of the action to one-parameter subgroups of G. Let $x^* \in \mathbb{F}^{n+1}$ be a representative of $x \in X \subset \mathbb{P}^n$ and $\lambda\colon \mathbb{F}^* \to G$ (regular morphism) be a one-parameter subgroup of G. Then (in appropriate coordinates) it acts by:

$$\lambda(t) \cdot x^* = (t^{m_0} x_0, \dots, t^{m_n} x_n).$$

Set

$$\mu^L(x, \lambda) = \min_t \{m_i : x_i \neq 0\}.$$

The Hilbert-Mumform numerical criterion claims (see [3, Theorem 9.1] for details) that

$$\begin{aligned} x \in X^{ss}(L) &\iff \mu^L(x,\lambda) \leq 0, \\ x \in X^{s}(L) &\iff \mu^L(x,\lambda) < 0, \end{aligned} \tag{5.2}$$

for all one-parameter subgroup of G.

5.2. **Linearization of $\mathbf{Sl}(\boldsymbol{\alpha}_0)$-action.** First note that the orbits of $\mathbf{Gl}(\alpha_0)$-action on $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ are in one-one correspondence with the orbits of $\mathbf{Sl}(\alpha_0)$, so we study the action of $\mathbf{Sl}(\alpha_0)$. Fix a form $\boldsymbol{\theta} = (\theta_s)_{s\in\mathcal{S}} \in \mathbb{Z}^{\widehat{\mathcal{S}}}$ with $\theta_s \geq 0$ for all $s \in \mathcal{S}$. As shown in Proposition 6.4 the variety $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ is closed in the product of Grassmanians $\prod_{s\in\mathcal{S}} \mathrm{Gr}(\alpha_s, \alpha_0)$. Our first aim is to embed the variety $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ into some larger projective space corresponding to linearizing action of $\mathbf{Sl}(\alpha_0)$. We use a slightly modified standard construction (see, for example, [3, Chapter 11] and [12]).

A standard way to embed $\mathrm{Gr}(\alpha_s, \alpha_0)$ into a projective space is via Plucker embedding, that is, for an element $V_i \in \mathrm{Gr}(\alpha_s, \alpha_0)$ we take its basis vectors a_j and wedge them together $a_1 \wedge \cdots \wedge a_{\alpha_s}$ obtaining an element of $\mathbb{P}(\wedge^{\alpha_s}\mathbb{F}^{\alpha_0})$. Then using the Veronese map we embed the projective space $\mathbb{P}(V)$ into the space $\mathbb{P}(\mathrm{Sym}^d(V))$. Respectively, for the product of Grassmanians $\prod_{s\in\mathcal{S}} \mathrm{Gr}(\alpha_s, \alpha_0)$ we have the embedding

$$\prod_{s\in\mathcal{S}} \mathrm{Gr}(\alpha_s, \alpha_0) \hookrightarrow \prod_{s\in\mathcal{S}} \mathbb{P}(\mathrm{Sym}^{\theta_s}(\wedge^{\alpha_s}\mathbb{F}^{\alpha_0})).$$

Using the Segre map $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{(n+1)(m+1)-1}$ we embed the last product into

$$\mathbb{P}\Bigg(\prod_{s\in\mathcal{P}} \mathrm{Sym}^{\theta_s}(\wedge^{d_s}\mathbb{F}^{d_0})\Bigg).$$

Hence, we have the following sequence of inclusions:

$$\mathrm{Gr}(\alpha_s, \alpha_0) \hookrightarrow \mathbb{P}(\wedge^{\alpha_s}\mathbb{F}^{\alpha_0}) \hookrightarrow \mathbb{P}(\mathrm{Sym}^{\theta_s}(\wedge^{\alpha_s}\mathbb{F}^{\alpha_0})).$$

And, therefore we get the following closed embedding of $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$:

$$\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}} \hookrightarrow \mathbb{P}(\wedge^{\alpha_s}\mathbb{F}^{\alpha_0}) \hookrightarrow \mathbb{P}(\mathrm{Sym}^{\theta_s}(\wedge^{\alpha_s}\mathbb{F}^{\alpha_0})).$$

As embedding above is closed, the corresponding line bundle $L_{\boldsymbol{\theta}}$ is ample. Note that $L_{\boldsymbol{\theta}}$ has exactly one $\mathbf{Sl}(\alpha_0)$ linearization, since the center of $\mathbf{Sl}(\alpha_0)$ is 0-dimensional. Our aim is to describe the set of semistable $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-ss}$ and stable $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-s}$ points with respect to $L_{\boldsymbol{\theta}}$ (we adopt the arguments from [3, Theorem 11.1], [8, Theorem 2.2] and [16]).

Theorem 5.1. *Let $\boldsymbol{\theta} = (\theta_s)_{s\in\mathcal{S}} \in \mathbb{Z}_+^{\mathcal{S}}$. Then $\mathbf{V} = (V_0; V_s)_{s\in\mathcal{S}} \in \mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-ss}$ (resp. $\in \mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-s}$) if and only if for any proper subrepresentation $\mathbf{W} \subset \mathbf{V}$ we have $\mu_{\boldsymbol{\theta}}(\mathbf{W}) \leq \mu_{\boldsymbol{\theta}}(\mathbf{V})$ (resp. the strict inequality holds); that is, if and only if $\mathbf{V}$ is $\mu_{\boldsymbol{\theta}}$-semistable (resp. $\mu_{\boldsymbol{\theta}}$-stable).*

Proof. Let $n = \alpha_0 = \dim V_0$ and T be the maximal torus in $\mathbf{Sl}(n)$. Each one-parameter subgroup $\lambda : \mathbb{F}^* \to T$ is conjugated to a diagonal one. Therefore, we assume that

$$\lambda(t) = \operatorname{diag}\{t^{q_1}, \dots, t^{q_n}\},$$

where $q_1 + \cdots + q_n = 0$. Without loss the generality we can assume that $q_1 \geq \cdots \geq q_n$. Also, it is a standard fact that all such groups form a convex set with extreme points $\lambda_r : \mathbb{F}^* \to T$ given by

$$\lambda_r(t) = \operatorname{diag}\{t^{q_1}, \dots, t^{q_n}\},$$

such that $q_1 = \cdots = q_r = n - r$, $q_{r+1} = \cdots = q_n = -r$.

Suppose that $\mathbf{V} = (V_0; V_s)_{s\in\mathcal{S}}$ is a semistable point. Choose a basis $v_1, \dots, v_n$ of V_0. Set $H_i = \operatorname{span}\{v_1, \dots, v_i\}$, $i = 1, \dots, n$ (in particular, we have $H_n = V_0$ and $H_r = W$). Let K be any subspace of V_0. Then for any integer j, $1 \leq j \leq s = \dim K$, there is a unique integer m_j such that

$$\dim(K \cap H_{m_j}) = j, \quad \dim(K \cap H_{m_j - 1}) = j - 1.$$

Therefore we can represent K (in the basis $e_1, \dots, e_n$) by the matrix A_K of the form

$$A_K = \begin{bmatrix} a_{11} & \dots & a_{1m_1} & 0 & \dots & 0 & 0 & \dots & 0 \\ a_{21} & \dots & \dots & a_{2m_2} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & \dots & \dots & \dots & \dots & \dots & a_{km_k} & \dots & 0 \end{bmatrix}^T,$$

with $a_{jmj} \neq 0$ for all j. Considering the maximal minors of A_K we have that in the Plucker embedding $p_{i_1\dots i_k}(K) = 0$ if $i_j > m_j$ and $p_{m_0\dots m_k}(K) \neq 0$. Also from the matrix representation of K we get

$$p_{i_1\dots i_k}(\lambda(t)K) = t^{q_{i_1}+\cdots+q_{i_k}} p_{i_1\dots i_k}(K).$$

Applying the procedure above to all subspaces V_s in $\mathbf{V}$ we get the numbers $m_1^{(s)}, \dots, m_{\alpha_s}^{(s)}$ for all $s \in \mathcal{S}$ and we have (by thw minimality of the numerical function) that

$$\upmu^{L_{\boldsymbol\theta}}(\mathbf{V},\lambda)=\sum_{s\in\mathcal{S}}\theta_s\sum_{i=1}^{\alpha_i}q_{m_j^{(s)}}.$$

Now, since $\dim(V_s\cap H_j)-\dim(V_s\cap H_{j-1})=0$ if $j\neq m_j^{(s)}$, we rewrite the previous sum as follows:

$$\begin{aligned}\upmu^{L_{\boldsymbol\theta}}(\mathbf{V},\lambda)&=\sum_{s\in\mathcal{S}}\theta_s\sum_{i=1}^{n}q_i\Big(\dim(V_s\cap H_i)-\dim(V_s\cap H_{i-1})\Big)\\&=\sum_{s\in\mathcal{S}}\theta_s\big(\alpha_s q_n+\sum_{i=1}^{n-1}(\dim(V_s\cap H_j)(q_i-q_{i+1})\big)\\&=q_n\sum_{s\in\mathcal{S}}\theta_s\alpha_s+\sum_{j=1}^{n-1}\Big(\sum_{s\in\mathcal{S}}\theta_s\dim(V_s\cap H_j)(q_j-q_{j1})\Big).\end{aligned}$$

Note that $\upmu^{L_{\boldsymbol\theta}}(\mathbf{V},\lambda)$ is linear in $(q_1,\dots,q_n)$. Therefore replacing t λ by a subgroup λ_s we get

$$\upmu^{L_{\boldsymbol\theta}}(\mathbf{V},\lambda_r)=-r\sum_{s\in\mathcal{S}}\theta_s\alpha_s+n\sum_{s\in\mathcal{S}}\theta_s\dim(V_s\cap H_r).$$

By the Hilbert-Mumford numerical criteria (5.2) we have that if $\mathbf{V}$ is semistable (resp. stable) then $\upmu^{L_{\boldsymbol\theta}}(\mathbf{V},\lambda_r)\le 0$ ($\upmu^{L_{\boldsymbol\theta}}(\mathbf{V},\lambda_r)<0$), which is the same as

$$\mu_{\boldsymbol\theta}(\mathbf{W})\le\mu_{\boldsymbol\theta}(\mathbf{V}),\quad(\text{resp }\ <),$$

where $\mathbf{W}=(H_r,V_s\cap H_r)_{s\in\mathcal{S}}$ is a proper subrepresentation of $\mathbf{V}$. Hence $\mathbf{V}$ is $\mu_{\boldsymbol\theta}$-semistable (resp. stable).

Conversely, let $\mathbf{V}$ is $\boldsymbol\theta$-semistable but not semistable with respest to $L_{\boldsymbol\theta}$. Then there exist a one-parameter subgroup λ such that $\upmu^{L_{\boldsymbol\theta}}(\mathbf{V},\lambda)>0$. Hence, there must exist $1\le r\le n-1$ such that $\upmu^{L_{\boldsymbol\theta}}(\mathbf{V},\lambda_r)>0$, which is equivalent to

$$\mu_{\boldsymbol\theta}((H,V_s\cap H)_{s\in\mathcal{S}})>\mu_{\boldsymbol\theta}(\mathbf{V})$$

for some r-dimensional subspace H of V_0. Therefore $\mathbf{V}$ is not $\boldsymbol\theta$-semistable. Contradiction. Similarly one proves the sufficiency of conditions for the strict inequality. □

Corollary 5.1. *If the dimension vector $\boldsymbol\alpha$ satisfies $\boldsymbol\theta(\boldsymbol\beta)\neq 0$ for all $0\neq\boldsymbol\beta<\boldsymbol\alpha$, then*

$$\mathbf{R}_{\mathcal{S},\boldsymbol\alpha}^{\boldsymbol\theta-ss}=\mathbf{R}_{\mathcal{S},\boldsymbol\alpha}^{\boldsymbol\theta-s}. \tag{5.3}$$

Proof. Indeed, in this case each semistable representation is already stable. □

Note that if $\boldsymbol\alpha$ is coprime (that is, $\gcd(\alpha_s : s\in\widehat{\mathcal{S}})=1$) then the equality (5.3) holds for the generic choice of $\boldsymbol\theta$.

5.3. **Polystable representations.** We start by the following proposition (see also [8, Proposition 3.1]).

Proposition 5.1. *Let $\boldsymbol{\theta} = (\theta_s)_{s\in\mathcal{S}}$. Assume that $\mathbf{V} \in \mathfrak{sp}_{\mathcal{S}}$ is $\mu_{\boldsymbol{\theta}}$-semistable and $\mathbf{V} = \oplus_{i=1}^{l}\mathbf{W}_i$ is a direct sum of subrepresentations. Then $\mu_{\boldsymbol{\theta}}(\mathbf{W}_i) = \mu_{\boldsymbol{\theta}}(\mathbf{V})$ and $\mathbf{W}_i$ are $\mu_{\boldsymbol{\theta}}$-semistable.*

Proof. Assume that $\mathbf{V} = \mathbf{W}_1 \oplus \mathbf{W}_2$. Then

$$0 \longrightarrow \mathbf{W}_1 \longrightarrow \mathbf{V} \longrightarrow \mathbf{W}_2 \longrightarrow 0,$$

and

$$0 \longrightarrow \mathbf{W}_2 \longrightarrow \mathbf{V} \longrightarrow \mathbf{W}_1 \longrightarrow 0.$$

As $\mathbf{V}$ is semistable then $\mu_{\boldsymbol{\theta}}(\mathbf{W}_1) \leq \mu_{\boldsymbol{\theta}}(\mathbf{V})$ and $\mu_{\boldsymbol{\theta}}(\mathbf{W}_2) \leq \mu_{\boldsymbol{\theta}}(\mathbf{V})$. By Proposition 2.1 we have that $\mu_{\boldsymbol{\theta}}(\mathbf{W}_1) \geq \mu_{\boldsymbol{\theta}}(\mathbf{V})$ and $\mu_{\boldsymbol{\theta}}(\mathbf{W}_2) \geq \mu_{\boldsymbol{\theta}}(\mathbf{V})$. The statement follows. □

A $\mu_{\boldsymbol{\theta}}$-semistable representation $\mathbf{V}$ will be called *$\mu_{\boldsymbol{\theta}}$-polystable* if it decomposes into a direct sum of finitely many $\mu_{\boldsymbol{\theta}}$-stable subrepresentations. Similarly to [8, Proposition 3.3] one proves the following:

Proposition 5.2. $\mathbf{V}$ *is $\mu_{\boldsymbol{\theta}}$-polystable if and only if the orbit of $\mathbf{V}$ in $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-ss}$ is closed.*

As a conseguence of this proposition we have that $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-ss}//\mathbf{Sl}(\alpha_0)$ parametrizes $\mu_{\boldsymbol{\theta}}$-polystable representations. Denote by $\mathfrak{sp}_{\mathcal{S}}^{\boldsymbol{\theta}-ps}$ the additive subcategory of $\mathfrak{sp}_{\mathcal{S}}^{\boldsymbol{\theta}-ps}$ consisting of $\mu_{\boldsymbol{\theta}}$-polystable representations. Then $\mathfrak{sp}_{\mathcal{S}}^{\boldsymbol{\theta}-ps}$ is semisimple, where $\mu_{\boldsymbol{\theta}}$-stable representations are precisely the simple objects.

5.4. **Moduli space of representations of posets.** Let $\boldsymbol{\theta} = (\theta_s)_{s\in\mathcal{S}} \in \mathbb{Z}^{\widehat{\mathcal{S}}}$ and fix an admissible dimension vector $\boldsymbol{\alpha} \in \mathbb{Z}^{\widehat{\mathcal{S}}}$. We will make the following identification:

$$\mathcal{M}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-ss} = \mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-ss}//\mathbf{Sl}(\alpha_0), \qquad \mathcal{M}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-s} = \mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-s}/\mathbf{Sl}(\alpha_0).$$

Corollary 5.2. *By* (5.1)*, the variety $\mathcal{M}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-ss}$ is projective and by Proposition 5.2 it parametrizes the isomorphisms classes of $\mu_{\boldsymbol{\theta}}$-polystable representations of $\mathcal{S}$ of dimension vector $\boldsymbol{\alpha}$.*

Corollary 5.3. *The variety $\mathcal{M}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-s}$ is open in $\mathcal{M}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-ss}$ and parametrizes the isomorphisms classes of $\mu_{\boldsymbol{\theta}}$-stable representations of $\mathcal{S}$ of dimension vector $\boldsymbol{\alpha}$.*

If $\mathcal{M}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-s}$ is non-empty we have that

$$\begin{aligned}\dim \mathcal{M}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-s} &= \dim \mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}} - \dim \mathbf{Sl}(\alpha_0)\\ &= \dim \mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}} - \alpha_0^2 + 1.\end{aligned}$$

By Proposition 6.5 in Appendix A we have that if $\mathcal{S}$ is primitive or n-point extension of primitive poset (see Appendix A for the definition) then $\dim \mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}} = \alpha_0^2 - b_{\mathcal{S}}(\boldsymbol{\alpha},\boldsymbol{\alpha})$. Therefore, in these cases we have

$$\dim \mathcal{M}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-s} = 1 - b_{\mathcal{S}}(\boldsymbol{\alpha},\boldsymbol{\alpha}). \tag{5.4}$$

Note that this dimension formula is a direct analogue of the dimension formula for moduli space of $\mu_{\boldsymbol{\theta}}$-stable representations of quiver Q which given in terms of the quadratic form associated with Q (see, for example, [17, Section 3.5]).

5.5. **Moduli spaces and Coxeter functors.** Assume that $\mathcal{S}$ is primitive and $\boldsymbol{\alpha} \neq (1;0)_{s\in\mathcal{S}}$. Due to Theorem 3.1 Coxeter transformation F^+ (defined in Section 3) gives rise to a map between moduli spaces:

$$\mathrm{F}^+\colon \mathcal{M}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-ss} \longrightarrow \mathcal{M}_{\mathcal{S},\mathrm{F}^+(\boldsymbol{\alpha})}^{\mathrm{F}^+(\boldsymbol{\theta})-ss}.$$

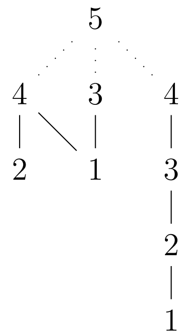
Applying $(\mathrm{F}^+)^n$, in certain cases (for instance when $\mathcal{S}$ is of finite representation type, or when $\boldsymbol{\alpha}$ is preprojective) we are able to obtain the information about $\mathcal{M}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-ss}$ knowing it in simpler cases (e.g., one-dimensional cases). We believe that a more careful study of these maps deserves further attention.

5.6. **Examples.** First assume that $\mathcal{S}$ is a poset of finite representation type. By Theorem 4.1 we have that if $\boldsymbol{\alpha}$ is an admissible indecomposable dimension then both sets $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-ss}$ and $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-s}$ are non-empty. Therefore $\mathcal{M}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-ss}$ and $\mathcal{M}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-s}$ are non-empty as well. As $\mathcal{S}$ is of finite representation type, the orbit of indecomposable $\mathbf{V}$ with dimension $\boldsymbol{\alpha}$ is dense in $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ therefore $\mathcal{M}_{\mathcal{S},\boldsymbol{\alpha}}^{\boldsymbol{\theta}-s}$ consists of one point.

Now assume that the poset $\mathcal{S}$ is one of the critical poset from list (4.1). Consider dimension vector $\boldsymbol{\alpha}_{\mathcal{S}}$ which a minimal imaginary root of form $b_{\mathcal{S}}$ (that is, minimal $\boldsymbol{\alpha}_{\mathcal{S}}$ so that $b_{\mathcal{S}}(\boldsymbol{\alpha}_{\mathcal{S}},\boldsymbol{\alpha}_{\mathcal{S}}) = 0$):

$$\begin{aligned}
\boldsymbol{\alpha}_{(1,1,1,1)} &= (2;1,1,1,1);\\
\boldsymbol{\alpha}_{(2,2,2)} &= (3;1,2,1,2,1,2);\\
\boldsymbol{\alpha}_{(1,3,3)} &= (4;2,1,2,3,1,2,3);\\
\boldsymbol{\alpha}_{(1,2,5)} &= (6;3,2,4,1,2,3,4,5);\\
\boldsymbol{\alpha}_{(N,4)} &= (5;2,4,1,3,1,2,3,4).
\end{aligned}$$

For instance, for unique non-primitive critical poset $(N,4)$ we have



It is straightforward to see that both $\mathbf{R}^{\boldsymbol{\theta}-ss}_{\mathcal{S},\boldsymbol{\alpha}}$ and $\mathbf{R}^{\boldsymbol{\theta}-s}_{\mathcal{S},\boldsymbol{\alpha}}$ are non-empty for the choice of $\boldsymbol{\theta}$ given by the formulas (4.6). Therefore the moduli spaces $\mathcal{M}^{\boldsymbol{\theta}-ss}_{\mathcal{S},\boldsymbol{\alpha}}$ and $\mathcal{M}^{\boldsymbol{\theta}-s}_{\mathcal{S},\boldsymbol{\alpha}}$ are non-empty as well. The poset $\mathcal{S}$ is primitive or 1-point extension of a primitive poset, therefore equality (5.4) holds and we have $\dim \mathcal{M}^{\boldsymbol{\theta}-s}_{\mathcal{S},\boldsymbol{\alpha}} = 1$ (as $b_{\mathcal{S}}(\boldsymbol{\alpha},\boldsymbol{\alpha}) = 0$).

6. Moment map and unitary representation of posets

6.1. **Unitary representation of posets.** We assume that $\mathbb{F} = \mathbb{C}$. By a *unitary representation* of $\mathcal{S}$ we mean a subspace representation $\mathbf{U} = (U_0, U_s)_{s\in\mathcal{S}}$ in which the ambient U_0 is a unitary space. Two unitary representations $\mathbf{U} = (U_0, U_s)$ and $\mathbf{U}' = (U_0', U_s')$ of $\mathcal{S}$ are unitarily equivalent if there exists a unitary bijection $\varphi\colon U_0 \to U_0'$ such that $\varphi(U_s) = U_s'$ for all $i \in \mathcal{S}$. Result of [5] gives a complete classification of indecomposable systems of two unitary subspaces (which is already a non finite problem). In [1] the authors classified the posets which have finite, tame and wild unitary type. Note that the problem of classifying of unitary representations is wild even for the poset $\mathcal{S} = \{s_1, s_2, s_3 \mid s_1 \prec s_2\}$. It turned out that the classification becomes possible for a broader class of posets if one imposes additional conditions on unitary representations (cf. [13, 14, 19]).

We say that a unitary representation $\mathbf{U} = (U_0; U_s)_{s\in\mathcal{S}}$ is a representation of weight $\boldsymbol{\chi} = (\chi_s)_{s\in\mathcal{S}} \in \mathbb{Z}_+^{\widehat{\mathcal{S}}}$ (or $\boldsymbol{\chi}$-*representation*) if

$$\sum_{s\in\mathcal{S}} \chi_s P_{U_s} = \chi_0 I, \tag{6.1}$$

where P_M denote the orthogonal projection of U_0 onto subspace M, and $\chi_0 \in \mathbb{Q}$ is determined by the trace identity of (6.1). All $\boldsymbol{\chi}$-representations of $\mathcal{P}$ form an additive category denoted by $\mathfrak{usp}_{\mathcal{S},\boldsymbol{\chi}}$.

There is an obvious (forgetfull) functor $\mathfrak{F}\colon \mathfrak{usp}_{\mathcal{S},\boldsymbol{\chi}} \to \mathfrak{sp}_{\mathcal{S}}$ which relates to $\boldsymbol{\chi}$-representation $\mathbf{U} = (U_0, U_s)_{s\in\mathcal{S}}$ the underlying system of vector spaces (forgetting the inner product). We prove the following (see also [19, Lemma 5])

Proposition 6.1. *Let* $\mathbf{U} = (U_0; U_s)_{s\in\mathcal{S}} \in \mathfrak{usp}_{\mathcal{S},\boldsymbol{\chi}}$ *be* $\boldsymbol{\chi}$*-representation. Then* $\mathfrak{F}(\mathbf{U})$ *is* $\mu_{\boldsymbol{\theta}}$*-polystable with* $\boldsymbol{\theta} = (\chi_s)_{s\in\mathcal{S}}$.

Proof. First suppose that $\mathbf{U}$ is indecomposable. Equating the traces of both sides in (6.1), we get $\sum_{s\in\mathcal{S}} \chi_s \dim U_s = \chi_0 \dim U$. If M is any proper subspace of U then $\sum_{s\in\mathcal{S}} \chi_s P_{U_s} P_M = \chi_0 P_M$. Therefore $\chi_0 = \mu_{\boldsymbol{\theta}}(\mathbf{U})$. Equating the traces of both sides in the last equality we get

$$\sum_{s\in\mathcal{S}} \chi_s \mathrm{tr}(P_{U_s} P_M) = \chi_0 \dim M.$$

It follows from [5] that $\mathrm{tr}(P_{M_1\cap M_2}) \le \mathrm{tr}(P_{M_1} P_{M_2})$ for each two subspaces M_1 and M_2, and so

$$\sum_{s\in\mathcal{S}} \chi_s \mathrm{tr}(P_{U_s\cap M}) \le \sum_{i\in\mathcal{S}} \chi_i \mathrm{tr}(P_{U_s} P_M) = \chi_0 \dim M.$$

It remains to prove that the last inequality is strict. Indeed, assuming that $\mathrm{tr}(P_{U_s\cap M}) = \mathrm{tr}(P_{U_s} P_M)$ for all s, we obtain that each P_{U_s} commutes with P_M. Hence, the subspace M is invariant with respect to the projections P_{U_i} and the representation $\mathbf{U}$ is decomposable. This contradicts the assumption. Therefore, $\mu_{\boldsymbol{\theta}}(\mathbf{W}) < \mu_{\boldsymbol{\theta}}(\mathbf{U})$ for any proper subrepresentation $\mathbf{W}$, and $\mathbf{U}$ is $\mu_{\boldsymbol{\theta}}$-stable.

Now, if $\mathbf{U}$ is a decomposable $\boldsymbol{\chi}$-representation, we get that $\mathbf{U}$ is $\mu_{\boldsymbol{\theta}}$-semistable. Then, proceeding as in Proposition 5.1 one proves that $\mathbf{U}$ is $\mu_{\boldsymbol{\theta}}$-polystable. □

Having a subspace representation $\mathbf{V} = (V_0; V_s)_{s\in\mathcal{S}}$ we say that it is $\boldsymbol{\chi}$-*unitarizable* if there is an inner product on V_0 such that $\mathbf{V}$ is a $\boldsymbol{\chi}$-representation with respect to this product.

It was shown in [19] that $\mathfrak{usp}_{\mathcal{S},\boldsymbol{\chi}}$ has a finite number of unitarily non equivalent indecomposable representations for each weight $\boldsymbol{\chi}$ if and only if $\mathcal{S}$ is of finite representation type; that is, if and only if $\mathcal{S}$ contains one of the Kleiner's critical posets. There are other similarities between $\boldsymbol{\chi}$-representations and usual representations of poset (see, for example, [23] and the references therein). In this section we explain these similarities via the Kempf-Ness theorem (which establishes the homeomorphism between GIT and symplectic quotients) and by constructing the functorial connection between the categories $\mathfrak{usp}_{\mathcal{S},\boldsymbol{\chi}}$ and $\mathfrak{sp}_{\mathcal{S}}$.

6.2. **Moment map, symplectic reduction and the Kempf-Ness theorem.** We briefly recall the idea behind the symplectic quotients and the Kempf-Ness theorem. Suppose again that G is a complex reductive group acting linearly on a smooth complex projective variety $X \subset \mathbb{P}^n$. Apart from taking GIT quotient (as in Section 5.1) one can alternatively consider the so-called symplectic quotient. As G is a complex reductive group, it is equal to the complexification of its maximal compact subgroup K (by $\mathfrak{k}$ we denote the corresponding Lie algebra of K). Complex projective space $\mathbb{P}^n$ has a natural Kähler structure given by the Fubini-Study form, therefore X is symplectic with symplectic form ω. Assuming that K acts unitarily, there is a moment map for this action

$$\Phi\colon X \longrightarrow \mathfrak{k}^*,$$

which satisfies:

(1) Φ is K-equivariant with respect to the action of K on X and to the coaction of K on $\mathfrak{k}^*$; that is, the following holds

$$\Phi(g\cdot p) = g\Phi(p)g^{-1}, \quad p \in M,\ g \in K;$$

(2) Φ lifts the infinitesimal action, in the sense that, for all $A \in \mathfrak{k}^*$ we have

$$d\Phi^A = \omega(A_X, --)$$

where $\Phi^A\colon X \to \mathbb{R}$ is the map given by $x \mapsto \Phi(x)\cdot A$, and the infinitesimal action $\mathfrak{k} \to Vect(X)$ is given by $A \to A_X$ with

$$A_{X,x} = \frac{d}{dt}\exp(tA)\cdot x\Big|_{t=0}.$$

Theorem 6.1 (Kempf-Ness theorem, [9])**.** *There is an inclusion $\Phi^{-1}(0) \subset X^{ss}(L)$ which induces a homeomorphism between the symplectic reduction and the GIT quotient*

$$\Phi^{-1}(0)/K \cong X^{ss}(L)//G.$$

6.3. **χ-unitarizable representations via moment map.** Let $\boldsymbol{\theta} = (\theta_s)_{s\in\mathcal{S}} \in \mathbb{Z}^{\mathcal{S}}$ be a weight with positive components, $\boldsymbol{\alpha} = (\alpha_0, \alpha_s)_{s\in\mathcal{S}} \in \mathbb{Z}^{\widehat{\mathcal{S}}}$ an admissible dimension, and $\mathbf{V} = (V_0; V_s) \in \mathfrak{sp}_{\mathcal{S}}$ a representation with $\mathbf{dim}\,\mathbf{V} = \boldsymbol{\alpha}$. We regard $\mathbf{V}$ as a point in $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}(L_{\boldsymbol{\theta}})$ after the embedding of $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ into the projective space as in Section 5.2. It is easy to check that the moment map of $\mathbf{Sl}(\alpha_0)$-action on $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}(L)$ has a form

$$\begin{aligned}\Phi\colon \mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}(L) &\to \mathfrak{su}(\alpha_0)^*,\\ \mathbf{V} = (V_0, V_s)_{s\in\mathcal{S}} &\mapsto \sum_{s\in\mathcal{S}} \theta_s A_s A_s^* - \mu_{\boldsymbol{\theta}}(\mathbf{V})I,\end{aligned}$$

where A_i is an isometry which embeds V_i into $\mathbb{C}^{\alpha_0}$. Considering $\Phi^{-1}(0)$ we get

$$\Phi^{-1}(0) = \left\{ (P_s)_{s\in\mathcal{P}} \in (M_{\alpha_0}(\mathbb{C}))_{s\in\mathcal{S}} \;\middle|\; \begin{array}{c} P_s = P_s^* = P_s^2,\ \mathrm{rank}(P_s) = \alpha_s, \\ P_sP_t = P_sP_t = P_s, \quad s \prec t, \\ \sum_{s\in\mathcal{S}} \theta_s P_s = \mu_{\boldsymbol{\theta}}(\boldsymbol{\alpha})I \end{array} \right\},$$

therefore $\Phi^{-1}(0)$ is a set of objects $\mathbf{U}$ in $\mathfrak{usp}_{\mathcal{S},\boldsymbol{\theta}}$ with dimension $\boldsymbol{\alpha}$. If $\mathbf{U}$ is $\boldsymbol{\chi}$-representation then $\mathfrak{F}(\mathbf{U})$ is $\mu_{\boldsymbol{\chi}}$-polystable by Proposition 6.1. Therefore, the functor $\mathfrak{F}(\cdot)$ yields a natural map $\phi : \Phi^{-1}(0) \to \mathbf{R}^{\boldsymbol{\theta}-ss}_{\mathcal{S},\boldsymbol{\alpha}}$. As a consequence of the Kempf-Ness theorem we have

Theorem 6.2. *Let $\mathcal{S}$ be a poset, $\boldsymbol{\alpha} = (\alpha_0; \alpha_s)_{s\in\mathcal{S}}$ a dimension vector and $\boldsymbol{\theta} = (\theta_0; \theta_s)_{s\in\mathcal{S}}$ a form such that $\boldsymbol{\theta}(\boldsymbol{\alpha}) = 0$. The map $\phi : \Phi^{-1}(0) \to \mathbf{R}^{\boldsymbol{\theta}-ss}_{\mathcal{S},\boldsymbol{\alpha}}$ induces a bijection:*

$$\Phi^{-1}(0)/U(\alpha_0) \simeq \mathcal{M}^{\boldsymbol{\theta}-ss}_{\mathcal{S},\boldsymbol{\alpha}}.$$

We immediately have

Corollary 6.1. *A representation* $\mathbf{V} = (V_0; V_s)_{s\in\mathcal{S}}$ *is* $\boldsymbol{\chi} = (\chi_s)_{s\in\mathcal{S}}$*-unitarizable if and only if* $\mathbf{V}$ *is* $\mu_{\boldsymbol{\chi}}$*-polystable.*

Applying Theorem 4.1 we have

Corollary 6.2. *Any indecomposable representation* $\mathbf{V}$ *of a poset of finite representation type is* $\boldsymbol{\chi}$*-unitarizable, where* $\boldsymbol{\chi}$ *is constructed by formulas* (4.7) *with respect to the dimension of* $\mathbf{V}$*.*

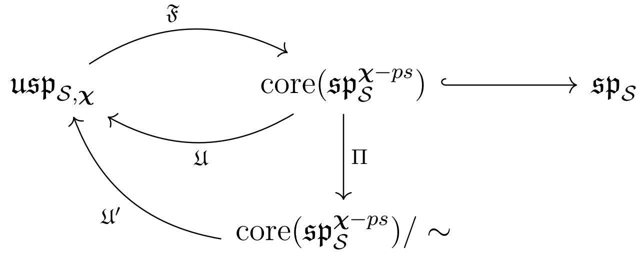
6.4. **Relation between the categories $\mathfrak{usp}_{\mathcal{S},\boldsymbol{\chi}}$ and $\mathfrak{sp}_{\mathcal{S}}$.** Recall that $\operatorname{core}(\mathfrak{sp}_{\mathcal{S}})$ of $\mathfrak{sp}_{\mathcal{S}}$ is a maximal sub-groupoid of $\mathfrak{sp}_{\mathcal{S}}$: the subcategory consisting of the same objects as in $\mathfrak{sp}_{\mathcal{S}}$, in which morphisms are the isomorphisms in $\mathfrak{sp}_{\mathcal{S}}$. Note that $\mathfrak{sp}_{\mathcal{S}}$ and $\operatorname{core}(\mathfrak{sp}_{\mathcal{S}})$ are the "same" from the classification point of view.

As we mentioned above there is a forgetful functor $\mathfrak{F}\colon \mathfrak{usp}_{\mathcal{S},\boldsymbol{\chi}} \to \mathfrak{sp}_{\mathcal{S}}$. It follows from Corollary 6.1 that the image of $\mathfrak{F}$ (on objects) coincides with the objects of $\operatorname{core}(\mathfrak{sp}_{\mathcal{S}}^{\boldsymbol{\chi}-ps})$. Now we construct the functor in opposite direction. Let $\mathbf{V} = (V_0; V_s) \in \mathfrak{sp}_{\mathcal{S}}^{\boldsymbol{\chi}-ps}$. There is unique inner product in $\mathbf{V}_0$ which makes $\mathbf{V}$ into a $\boldsymbol{\chi}$-representation. Denote the resulting $\boldsymbol{\chi}$-representation by $\mathfrak{U}(\mathbf{V}) \in \mathfrak{usp}_{\mathcal{S},\boldsymbol{\chi}}$. Given an invertible morphism $g\colon \mathbf{V} \to \mathbf{W}$ define $\mathfrak{U}(g) = \varphi$, where $g = \varphi A$ is a right polar decomposition of g (φ is a unitary map and A is positive definite). As g is invertible, the right polar decomposition is unique and hence $\mathfrak{U}(g)$ is well-defined. Also, one checks that it is a morphism between $\mathfrak{U}(\mathbf{V})$ and $\mathfrak{U}(\mathbf{W})$ (see [19, Theorem 3]). One can easily see that $\mathfrak{U}$ preserves the composition of morphisms and therefore yields a functor.

Consider the following relation on morphisms in $\operatorname{core}(\mathfrak{sp}_{\mathcal{S}})$. Given two morphisms $g_1, g_2\colon \mathbf{V} \to \mathbf{W}$ we say that $g_1 \sim g_2$ if $\varphi_1 = \varphi_2$ in right polar decompositions $g_1 = \varphi_1 A_1$ and $g_2 = \varphi_2 A_2$ with respect to some inner product in V_0 and W_0. One can show (the proof is left to the reader) that the relation $\sim$ does not depend on the choice of inner product and in fact is an equivalence relation on morphisms in $\operatorname{core}(\mathfrak{sp}_{\mathcal{S}})$. By $\operatorname{core}(\mathfrak{sp}_{\mathcal{S}})/\sim$ we denote the corresponding quotient category and by $\Pi\colon \operatorname{core}(\mathfrak{sp}_{\mathcal{S}}) \to \operatorname{core}(\mathfrak{sp}_{\mathcal{S}})/\sim$ the quotient functor. By construction if follows that $\mathfrak{U}$ factors as $\mathfrak{U}'\circ\Pi$ (as the unitary parts in polar decomposition of morphisms $g_1 \sim g_2$ are the same).

Proposition 6.2. *Functors* $\Pi \circ \mathfrak{F}$ *and* $\mathfrak{U}'$ *establish an isomorphism between the categories* $\mathfrak{usp}_{\mathcal{S},\boldsymbol{\chi}}$ *and* $\operatorname{core}(\mathfrak{sp}_{\mathcal{S}}^{\boldsymbol{\chi}-ps})/\sim$*.*

Summing up the constructions above we have the following



In particular one shows that $\mathfrak{usp}_{\mathcal{S},\boldsymbol{\chi}}$ has finitely many indecomposable objects for any $\boldsymbol{\chi}$ iff $\mathfrak{sp}_{\mathcal{S}}$ has finitely many indecomposable representations (up to an isomorphism), which reproves [19, Theorem 1].

Appendix A. Some additional statements.

Proposition 6.3. *Let $\mathbf{V} = (V_0; V_s)_{s\in\mathcal{S}}$ and $K \subset V_0$ then for the induced representation $\mathbf{V}_K = (K; V_s \cap K)_{s\in\mathcal{S}}$ we have $\mathbf{V}/\mathbf{V}_K \in \mathfrak{sp}_{\mathcal{S}}$.*

Proof. Let $s, t \in \mathcal{S}$ and $s \prec t$. It is enough to show that $V_s/(V_s\cap K) \subset V_t/(V_t\cap K)$. Define the following map

$$\rho : V_s/(V_s \cap K) \to V_t/(V_t \cap K),$$

by $\rho(x + V_s \cap K) = x + V_t \cap K$, for any $x \in V_s$. Clearly, ρ is linear. If $x_1 + V_s \cap K = x_2 + V_s \cap K$ then $x_1 - x_2 \in V_s \cap K$ and $x_1 - x_2 \in V_t \cap K$, therefore $\rho(x_1+V_s\cap K) = \rho(x_2+V_s\cap K)$ and ρ is well-defined. We have $\ker\rho = 0$, therefore ρ is an inclusion. □

Proposition 6.4. *Let $\boldsymbol{\alpha} = (\alpha_0; \alpha_s)_{s\in\mathcal{S}}$ be a dimension vector. Variety $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ is a Zariski closed, irreducible subset of $\prod_{s\in\mathcal{S}} \mathrm{Gr}(\alpha_s, \alpha_0)$.*

Proof. Given the elements $s_1, \dots, s_m \in \mathcal{S}$ denote by $\mathcal{S}(s_1, \dots, s_m)$ a subposet of $\mathcal{S}$ consisting of these elements. We will use the same letter $\boldsymbol{\alpha}$ (abusing the notation) to denote the restriction of the dimension vector on the subposet of $\mathcal{S}$. Clearly, $\mathbf{R}_{\mathcal{S}(s),\boldsymbol{\alpha}} = \mathrm{Gr}(\alpha_s, \alpha_0)$. Given two incomparable points s_1 and s_2 we have that $\mathbf{R}_{\mathcal{S}(s_1,s_2),\boldsymbol{\alpha}} = \mathrm{Gr}(\alpha_{s_1}, \alpha_0) \times \mathrm{Gr}(\alpha_{s_2}, \alpha_0)$. If $s_1 \prec s_2$ then $\mathbf{R}_{\mathcal{S}(s_1,s_2),\boldsymbol{\alpha}}$ is a flag of two subspaces and therefore it is a Zariski closed in $\mathrm{Gr}(\alpha_{s_1}, \alpha_0) \times \mathrm{Gr}(\alpha_{s_2}, \alpha_0)$.

Now, for any two $s_1, s_2 \in \mathcal{S}$ let π_{s_1,s_2} be the restriction to $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ of the projection

$$\prod_{s\in\mathcal{S}} \mathrm{Gr}(\alpha_s, \alpha_0) \twoheadrightarrow \mathrm{Gr}(\alpha_{s_1}, \alpha_0) \times \mathrm{Gr}(\alpha_{s_2}, \alpha_0).$$

Then we have

$$\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}} = \bigcap_{(s_1,s_2)\in\mathcal{S}\times\mathcal{S}} \pi^{-1}_{s_1,s_2}\big(\mathbf{R}_{\mathcal{S}(s_1,s_2),\boldsymbol{\alpha}}\big).$$

Hence $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ is Zariski closed.

Now we prove that it is irreducible. We proceed by induction on the height of $\mathcal{S}$. If $\mathcal{S}$ is of height 1, then $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}} = \prod_{s\in\mathcal{S}} \mathrm{Gr}(\alpha_s, \alpha_0)$ is irreducible as a product of irreducible varieties. Now suppose that $\mathcal{S}$ has an arbitrary height greater than 1. By $\mathcal{S}_M$ denote the set of all minimal elements in $\mathcal{S}$. Denote by $\mathcal{P}_s$ a principal filter in $\mathcal{S}$ generated by s (that is, $\mathcal{P}_s = \{t \in \mathcal{S} \mid s \preceq t\}$). Given two elements $s_1, s_2 \in \mathcal{S}_M$ we say that they are equivalent if the intersection between $\mathcal{P}_{s_1}$ and $\mathcal{P}_{s_2}$ is non-empty. This defines the equivalence relation on $\mathcal{S}_M$. By $\mathcal{X}$ denote the set of equivalence classes, and by $[s]$ the equivalence class of $s \in \mathcal{S}_M$ in $\mathcal{X}$. For any $x \in \mathcal{X}$ define

$$\mathcal{O}_x = \bigcup_{s\in\mathcal{S}_M,[s]=x} \mathcal{P}_s \setminus \{s\}.$$

Mention that each $\mathcal{O}_x$ has height strictly less that the height of $\mathcal{S}$. Consider the following natural projection:

$$f : \mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}} \twoheadrightarrow \prod_{s\in\mathcal{S}_M} \mathrm{Gr}(\alpha_s, \alpha_0).$$

Now one checks that the fibers of f are isomorphic to

$$\prod_{x\in\mathcal{X}} \mathbf{R}_{\mathcal{O}_x,\widetilde{\boldsymbol{\alpha}}^{(x)}},$$

in which $\widetilde{\boldsymbol{\alpha}}^{(x)} = (\widetilde{\alpha}_0^{(x)}, \widetilde{\alpha}_t^{(x)})_{t\in\mathcal{O}_x}$ is an admissible dimension vector given by

$$\widetilde{\alpha}_0^{(x)} = \alpha_0 - \sum_{s\in\mathcal{S}_M,[s]=x} \alpha_s, \qquad \widetilde{\alpha}_t^{(x)} = \alpha_t - \sum_{s\in\mathcal{S}_M,[s]=x} \alpha_s, \quad t \in \mathcal{O}_x.$$

Now the induction pass follows by [6, Theorem 11.14], as each $\mathbf{R}_{\mathcal{O}_x,\widetilde{\boldsymbol{\alpha}}^{(x)}}$ is irreducible by induction assumption. □

In fact, the proof of the previous proposition gives a method to compute the dimension of $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$. We will do this in two special cases. We say that a poset $\mathcal{S}$ is a *n-point extension of a primitive poset*, if there exist n incomparable minimal elements $s_1, \dots, s_n \in \mathcal{S}$ such that $\mathcal{S} \setminus \{s_1, \dots, s_n\}$ is a primitive poset.

Proposition 6.5. *Let $\mathcal{S}$ be a primitive poset or n-point extension of a primitive poset, and $\boldsymbol{\alpha} = (\alpha_0; \alpha_s)_{s\in\mathcal{S}}$ be a dimension vector. We have that*

$$\dim \mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}} = \alpha_0^2 - b_{\mathcal{S}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}).$$

Proof. In case $\mathcal{S}$ is primitive the proof is straightforward, as in this case $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ splits as a product of flags of subspaces. In case $\mathcal{S}$ is n-point extension of a primitive poset, projecting the variety $\mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}}$ as in the previous proposition for the set of its minimal elements we reduce the statement to a primitive case. The details are left to the reader. □

Conjecture. *For any poset $\mathcal{S}$ and an admissible dimension vector $\boldsymbol{\alpha} \in \mathbb{Z}^{\widehat{\mathcal{S}}}$ we have:*

$$\dim \mathbf{R}_{\mathcal{S},\boldsymbol{\alpha}} = \alpha_0^2 - b_{\mathcal{S}}(\boldsymbol{\alpha}, \boldsymbol{\alpha}).$$

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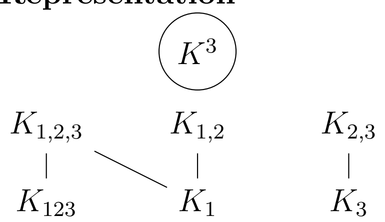
Appendix B. Exact representations of finite representation type, their maximal subcoordinate vectors and costability condition

For each non-primitive posets $\mathcal{S}_1, \dots, \mathcal{S}_6$ of finite representation type (given in Section 3.1) we list its exact representations, costability condition (calculated by formulas (4.4)) and maximal subcoordinate vectors. By K^n we denote the $\mathbb{F}$ vector space with canonical basis $e_1, \dots, e_n$ and $K_{i_1 \dots i_k, \dots, j_1 \dots j_m}$ denotes the subspace of K^n generated by the vectors $e_{i_1 \dots i_k}, \dots, e_{j_1 \dots j_m}$ in which

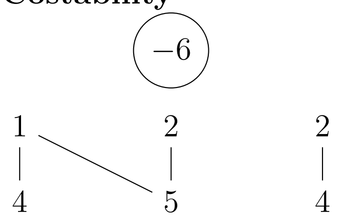
$$e_{i_1 \dots i_k} = e_{i_1} + \dots + e_{i_k}, \dots e_{j_1 \dots j_m} = e_{j_1} + \dots + e_{j_m}.$$

Poset $\mathcal{S}_1$.

Representation



Costability

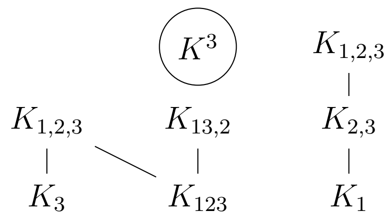


Maximal subcoordinate vectors

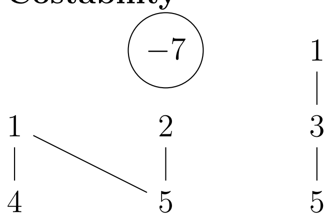
(1; 0, 0; 1, 0; 0, 0), (1; 0, 1; 0, 0; 1, 0), (1; 0, 1; 0, 1; 0, 1), (1; 1, 0; 0, 0; 0, 0), (2; 0, 1; 1, 0; 1, 0), (2; 0, 1; 1, 1; 0, 1), (2; 0, 2; 0, 1; 1, 1), (2; 1, 0; 1, 0; 0, 1), (2; 1, 1; 0, 1; 0, 1), (2; 1, 1; 0, 1; 1, 0),

Poset $\mathcal{S}_2$.

Representation



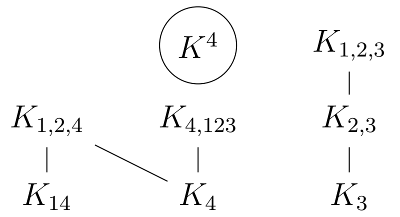
Costability



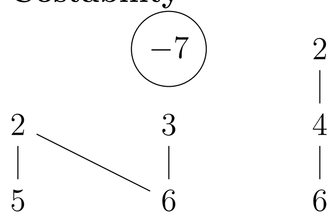
Maximal subcoordinate vectors

(1; 0, 0; 1, 0; 0, 0, 1), (1; 0, 1; 0, 0; 1, 0, 0), (1; 0, 1; 0, 1; 0, 0, 1), (1; 0, 1; 0, 1; 0, 1, 0), (1; 1, 0; 0, 0; 0, 0, 1), (2; 0, 1; 1, 0; 1, 0, 1), (2; 0, 1; 1, 1; 0, 1, 1), (2; 0, 2; 0, 1; 1, 1, 0), (2; 1, 0; 1, 0; 0, 1, 1), (2; 1, 1; 0, 1; 0, 1, 1), (2; 1, 1; 0, 1; 1, 0, 1).

Representation



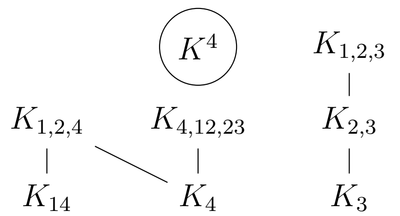
Costability



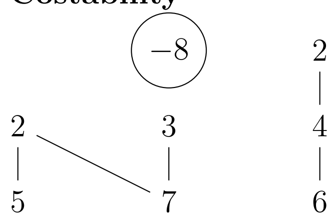
Maximal subcoordinate vectors

$(1;0,0;0,0;1,0,0)$, $(1;0,0;0,1;0,0,1)$, $(1;0,0;1,0;0,0,0)$, $(1;0,1;0,0;0,0,1)$,
$(1;0,1;0,0;0,1,0)$, $(1;1,0;0,0;0,0,0)$, $(2;0,0;1,0;1,0,0)$, $(2;0,0;1,1;0,0,1)$,
$(2;0,1;0,0;1,1,0)$, $(2;0,1;0,1;0,1,1)$, $(2;0,1;0,1;1,0,1)$, $(2;0,1;1,0;0,1,0)$,
$(2;0,2;0,0;0,1,1)$, $(2;1,0;0,0;1,0,0)$, $(2;1,0;0,1;0,0,1)$, $(2;1,0;1,0;0,0,1)$,
$(2;1,1;0,0;0,1,0)$, $(3;0,1;1,0;1,1,0)$, $(3;0,1;1,1;0,1,1)$, $(3;0,1;1,1;1,0,1)$,
$(3;0,2;0,1;1,1,1)$, $(3;1,0;1,0;1,0,1)$, $(3;1,0;1,1;0,1,1)$, $(3;1,1;0,1;0,1,1)$,
$(3;1,1;0,1;1,0,1)$, $(3;1,1;0,1;1,1,0)$, $(3;1,1;1,0;0,1,1)$.

Representation



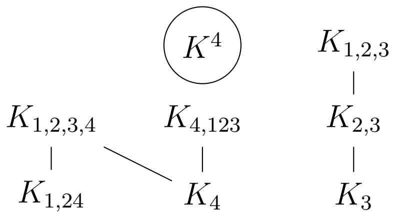
Costability



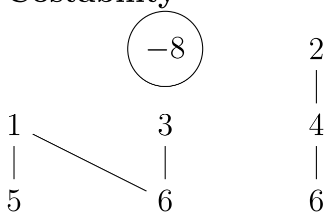
Maximal subcoordinate vectors

$(1;0,0;0,0;1,0,0)$, $(1;0,0;0,1;0,1,0)$, $(1;0,0;1,0;0,0,0)$, $(1;0,1;0,0;0,1,0)$,
$(1;0,1;0,1;0,0,1)$, $(1;1,0;0,0;0,0,0)$, $(2;0,0;1,0;1,0,0)$, $(2;0,0;1,1;0,1,0)$,
$(2;0,1;0,1;1,0,1)$, $(2;0,1;0,1;1,1,0)$, $(2;0,1;0,2;0,1,1)$, $(2;0,1;1,0;0,1,0)$,
$(2;0,1;1,1;0,0,1)$, $(2;0,2;0,1;0,1,1)$, $(2;1,0;0,1;1,0,0)$, $(2;1,0;1,0;0,0,1)$,
$(2;1,1;0,1;0,0,1)$, $(2;1,1;0,1;0,1,0)$, $(3;0,1;1,1;1,0,1)$, $(3;0,1;1,1;1,1,0)$,
$(3;0,1;1,2;0,1,1)$, $(3;0,2;0,2;1,1,1)$, $(3;1,0;1,1;1,0,1)$, $(3;1,1;0,2;0,1,1)$,
$(3;1,1;0,2;1,0,1)$, $(3;1,1;0,2;1,1,0)$, $(3;1,1;1,1;0,1,1)$.

Representation



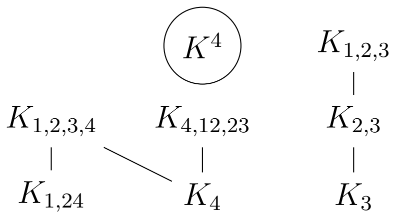
Costability



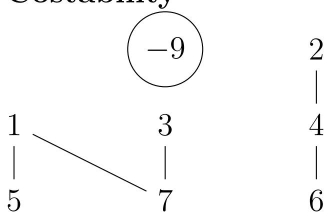
Maximal subcoordinate vectors

$(1;0,0;1,0;0,0,0)$, $(1;0,1;0,0;0,1,0)$, $(1;0,1;0,0;1,0,0)$, $(1;0,1;0,1;0,0,1)$, $(1;1,0;0,0;0,0,1)$, $(2;0,1;1,0;1,0,0)$, $(2;0,1;1,1;0,0,1)$, $(2;0,2;0,0;1,1,0)$, $(2;0,2;0,1;0,1,1)$, $(2;0,2;0,1;1,0,1)$, $(2;1,0;1,0;0,0,1)$, $(2;1,0;1,0;0,1,0)$, $(2;1,1;0,0;1,0,1)$, $(2;1,1;0,1;0,1,1)$, $(2;2,0;0,0;0,0,1)$, $(3;1,1;1,0;1,1,0)$, $(3;1,1;1,1;0,1,1)$, $(3;1,1;1,1;1,0,1)$, $(3;1,2;0,1;1,1,1)$, $(3;2,0;1,0;0,1,1)$, $(3;2,1;0,1;0,1,1)$, $(3;2,1;0,1;1,0,1)$.

Representation



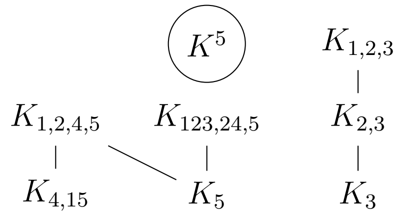
Costability



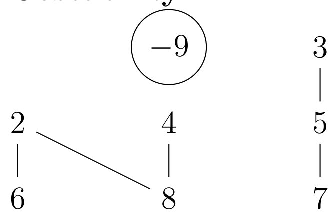
Maximal subcoordinate vectors

$(1;0,0;1,0;0,0,0)$, $(1;0,1;0,0;1,0,0)$, $(1;0,1;0,1;0,0,1)$, $(1;0,1;0,1;0,1,0)$, $(1;1,0;0,0;0,0,1)$, $(2;0,1;1,0;1,0,0)$, $(2;0,1;1,1;0,1,0)$, $(2;0,2;0,1;1,0,1)$, $(2;0,2;0,1;1,1,0)$, $(2;0,2;0,2;0,1,1)$, $(2;1,0;1,0;0,1,0)$, $(2;1,0;1,1;0,0,1)$, $(2;1,1;0,1;0,1,1)$, $(2;1,1;0,1;1,0,1)$, $(2;2,0;0,1;0,0,1)$, $(3;1,1;1,1;1,0,1)$, $(3;1,1;1,1;1,1,0)$, $(3;1,1;1,2;0,1,1)$, $(3;1,2;0,2;1,1,1)$, $(3;2,0;1,1;0,1,1)$, $(3;2,1;0,2;0,1,1)$, $(3;2,1;0,2;1,0,1)$.

Representation



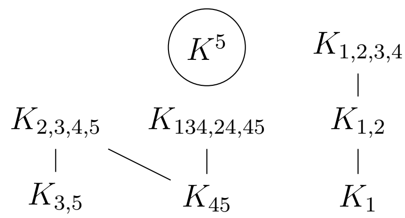
Costability



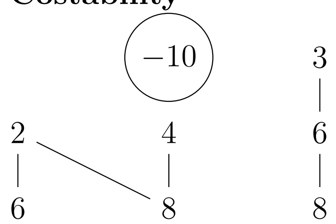
Maximal subcoordinate vectors

$(1;0,0;0,0;1,0,0)$, $(1;0,0;0,1;0,0,1)$, $(1;0,0;1,0;0,0,0)$, $(1;0,1;0,0;0,0,1)$,
$(1;0,1;0,0;0,1,0)$, $(1;0,1;0,1;0,0,0)$, $(1;1,0;0,0;0,0,0)$, $(2;0,0;1,0;1,0,0)$,
$(2;0,0;1,1;0,0,1)$, $(2;0,1;0,0;1,1,0)$, $(2;0,1;0,1;0,1,1)$, $(2;0,1;0,1;1,0,1)$,
$(2;0,1;0,2;0,0,1)$, $(2;0,1;1,0;0,1,0)$, $(2;0,1;1,1;0,0,0)$, $(2;0,2;0,0;0,1,1)$,
$(2;0,2;0,1;0,0,1)$, $(2;1,0;0,0;1,0,0)$, $(2;1,0;0,1;0,0,1)$, $(2;1,0;1,0;0,0,1)$,
$(2;1,1;0,0;0,0,1)$, $(2;1,1;0,1;0,1,0)$, $(2;2,0;0,0;0,0,0)$, $(3;0,1;1,0;1,1,0)$,
$(3;0,1;1,1;0,1,1)$, $(3;0,1;1,1;1,0,1)$, $(3;0,1;1,2;0,0,1)$, $(3;0,2;0,1;1,1,1)$,
$(3;0,2;0,2;0,1,1)$, $(3;0,2;0,2;1,0,1)$, $(3;1,0;1,0;1,0,1)$, $(3;1,0;1,1;0,1,1)$,
$(3;1,1;0,1;1,0,1)$, $(3;1,1;0,1;1,1,0)$, $(3;1,1;0,2;0,1,1)$, $(3;1,1;1,0;0,1,1)$,
$(3;1,1;1,1;0,0,1)$, $(3;1,1;1,1;0,1,0)$, $(3;1,2;0,1;0,1,1)$, $(3;2,0;0,1;0,0,1)$,
$(3;2,0;0,1;1,0,0)$, $(3;2,0;1,0;0,0,1)$, $(3;2,1;0,1;0,1,0)$, $(4;1,1;1,1;1,1,1)$,
$(4;1,1;1,2;0,1,1)$, $(4;1,1;1,2;1,0,1)$, $(4;1,2;0,2;1,1,1)$, $(4;2,0;1,1;1,0,1)$,
$(4;2,1;0,2;0,1,1)$, $(4;2,1;0,2;1,0,1)$, $(4;2,1;0,2;1,1,0)$, $(4;2,1;1,1;0,1,1)$.

Representation



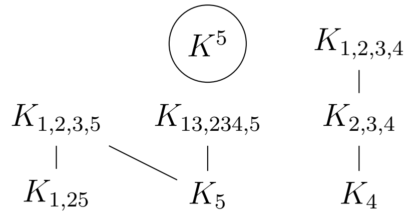
Costability



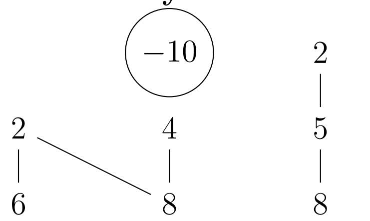
Maximal subcoordinate vectors

$(1;0,0;0,0;1,0,0)$, $(1;0,0;1,0;0,0,0)$, $(1;0,1;0,0;0,1,0)$, $(1;0,1;0,1;0,0,1)$,
$(1;1,0;0,0;0,0,1)$, $(2;0,0;1,0;1,0,0)$, $(2;0,1;0,0;1,1,0)$, $(2;0,1;0,1;1,0,1)$,
$(2;0,1;0,2;0,0,2)$, $(2;0,1;1,0;0,1,0)$, $(2;0,1;1,1;0,0,1)$, $(2;0,2;0,1;0,1,1)$,
$(2;1,0;0,0;1,0,1)$, $(2;1,0;1,0;0,0,1)$, $(2;1,1;0,0;0,1,1)$, $(2;1,1;0,1;0,0,2)$,
$(2;1,1;0,1;0,1,0)$, $(2;2,0;0,0;0,0,1)$, $(3;0,1;1,0;1,1,0)$, $(3;0,1;1,1;1,0,1)$,
$(3;0,1;1,2;0,0,2)$, $(3;0,2;0,1;1,1,1)$, $(3;0,2;0,2;0,1,2)$, $(3;0,2;0,2;1,0,2)$,
$(3;1,0;1,1;1,0,1)$, $(3;1,1;0,1;1,0,2)$, $(3;1,1;0,1;1,1,1)$, $(3;1,1;0,2;0,1,2)$,
$(3;1,1;1,1;0,0,2)$, $(3;1,1;1,1;0,1,1)$, $(3;1,2;0,1;0,1,2)$, $(3;2,0;0,1;1,0,1)$,
$(3;2,0;1,0;0,0,2)$, $(3;2,1;0,1;0,0,2)$, $(3;2,1;0,1;0,1,1)$, $(4;1,1;1,1;1,1,1)$,
$(4;1,1;1,2;0,1,2)$, $(4;1,1;1,2;1,0,2)$, $(4;1,2;0,2;1,1,2)$, $(4;2,0;1,1;1,0,2)$,
$(4;2,1;0,2;0,1,2)$, $(4;2,1;0,2;1,0,2)$, $(4;2,1;0,2;1,1,1)$, $(4;2,1;1,1;0,1,2)$.

Representation



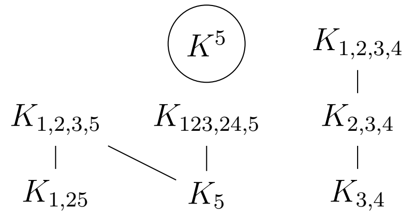
Costability



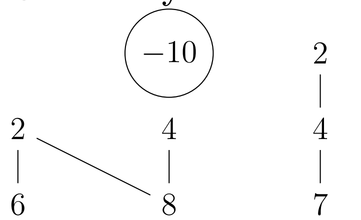
Maximal subcoordinate vectors

$(1;0,0;0,0;1,0,0)$, $(1;0,0;0,1;0,1,0)$, $(1;0,0;1,0;0,0,0)$, $(1;0,1;0,0;0,1,0)$,
$(1;0,1;0,1;0,0,1)$, $(1;1,0;0,0;0,0,1)$, $(2;0,0;1,0;1,0,0)$, $(2;0,0;1,1;0,1,0)$,
$(2;0,1;0,1;0,2,0)$, $(2;0,1;0,1;1,0,1)$, $(2;0,1;0,1;1,1,0)$, $(2;0,1;0,2;0,1,1)$,
$(2;0,1;1,0;0,1,0)$, $(2;0,1;1,1;0,0,1)$, $(2;0,2;0,0;0,2,0)$, $(2;0,2;0,1;0,1,1)$,
$(2;1,0;0,0;1,0,1)$, $(2;1,0;1,0;0,0,1)$, $(2;1,0;1,0;0,1,0)$, $(2;1,1;0,1;0,1,1)$,
$(2;2,0;0,0;0,0,1)$, $(3;0,1;1,1;0,2,0)$, $(3;0,1;1,1;1,0,1)$, $(3;0,1;1,1;1,1,0)$,
$(3;0,1;1,2;0,1,1)$, $(3;0,2;0,1;1,2,0)$, $(3;0,2;0,2;0,2,1)$, $(3;0,2;0,2;1,1,1)$,
$(3;1,0;1,0;1,0,1)$, $(3;1,0;1,0;1,1,0)$, $(3;1,0;1,1;0,2,0)$, $(3;1,1;0,1;1,1,1)$,
$(3;1,1;0,2;0,2,1)$, $(3;1,1;1,0;0,2,0)$, $(3;1,1;1,1;0,1,1)$, $(3;1,2;0,1;0,2,1)$,
$(3;2,0;0,1;1,0,1)$, $(3;2,0;1,0;0,1,1)$, $(3;2,1;0,1;0,1,1)$, $(4;1,1;1,1;1,2,0)$,
$(4;1,1;1,2;0,2,1)$, $(4;1,1;1,2;1,1,1)$, $(4;1,2;0,2;1,2,1)$, $(4;2,0;1,1;1,1,1)$,
$(4;2,1;0,2;0,2,1)$, $(4;2,1;0,2;1,1,1)$, $(4;2,1;1,1;0,2,1)$.

Representation



Costability

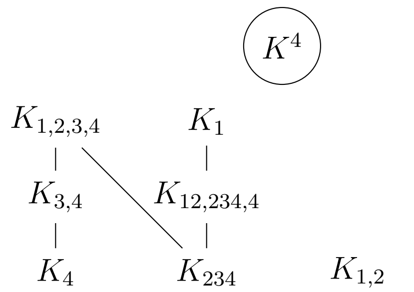


Maximal subcoordinate vectors

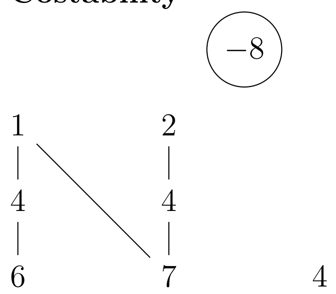
$(1;0,0;0,1;0,1,0)$, $(1;0,0;1,0;0,0,0)$, $(1;0,1;0,0;0,1,0)$, $(1;0,1;0,0;1,0,0)$, $(1;0,1;0,1;0,0,1)$, $(1;1,0;0,0;0,0,1)$, $(2;0,0;1,1;0,1,0)$, $(2;0,1;0,0;2,0,0)$, $(2;0,1;0,1;1,1,0)$, $(2;0,1;0,2;0,1,1)$, $(2;0,1;1,0;1,0,0)$, $(2;0,1;1,1;0,0,1)$, $(2;0,2;0,0;1,1,0)$, $(2;0,2;0,1;0,1,1)$, $(2;0,2;0,1;1,0,1)$, $(2;1,0;0,1;1,0,0)$, $(2;1,0;1,0;0,0,1)$, $(2;1,0;1,0;0,1,0)$, $(2;1,1;0,0;1,0,1)$, $(2;1,1;0,1;0,1,1)$, $(2;2,0;0,0;0,0,1)$, $(3;0,1;1,0;2,0,0)$, $(3;0,1;1,1;1,1,0)$, $(3;0,1;1,2;0,1,1)$, $(3;0,2;0,1;2,0,1)$, $(3;0,2;0,1;2,1,0)$, $(3;0,2;0,2;1,1,1)$, $(3;1,0;1,1;1,1,0)$, $(3;1,1;0,1;2,0,1)$, $(3;1,1;0,2;1,1,1)$, $(3;1,1;1,0;1,1,0)$, $(3;1,1;1,1;0,1,1)$, $(3;1,1;1,1;1,0,1)$, $(3;1,2;0,1;1,1,1)$, $(3;2,0;1,0;0,1,1)$, $(3;2,1;0,1;0,1,1)$, $(3;2,1;0,1;1,0,1)$, $(4;1,1;1,1;2,0,1)$, $(4;1,1;1,1;2,1,0)$, $(4;1,1;1,2;1,1,1)$, $(4;1,2;0,2;2,1,1)$, $(4;2,1;0,2;1,1,1)$, $(4;2,1;0,2;2,0,1)$, $(4;2,1;1,1;1,1,1)$.

Poset $\mathcal{S}_3$.

Representation



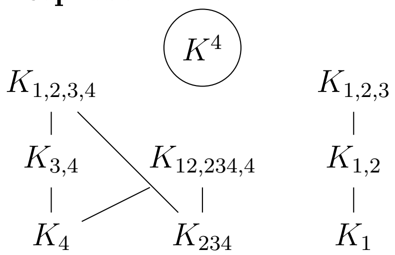
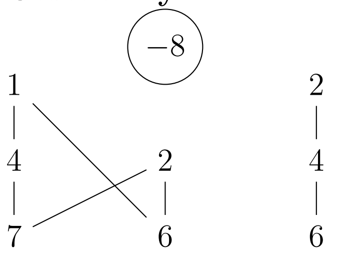
Costability



Maximal subcoordinate vectors

$(1;0,0,0;1,0,0;0)$, $(1;0,0,1;0,0,1;1)$, $(1;0,0,1;0,1,0;0)$, $(1;0,1,0;0,0,1;0)$, $(1;1,0,0;0,0,0;0)$, $(2;0,0,1;1,0,1;1)$, $(2;0,0,1;1,1,0;0)$, $(2;0,0,2;0,0,1;2)$, $(2;0,0,2;0,1,1;1)$, $(2;0,1,0;1,0,1;0)$, $(2;0,1,1;0,1,1;1)$, $(2;1,0,0;1,0,0;0)$, $(2;1,0,1;0,0,1;1)$, $(2;1,0,1;0,1,0;1)$, $(2;1,1,0;0,0,1;0)$, $(3;0,1,1;1,0,1;2)$, $(3;0,1,1;1,1,1;1)$, $(3;0,1,2;0,1,1;2)$, $(3;1,0,1;1,0,1;1)$, $(3;1,0,1;1,1,0;1)$, $(3;1,0,2;0,1,1;2)$, $(3;1,1,0;1,0,1;1)$, $(3;1,1,1;0,1,1;1)$.

Poset $\mathcal{S}_4$.

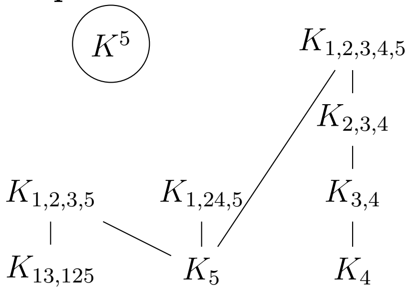
Representation	**Costability**
	

Maximal subcoordinate vectors

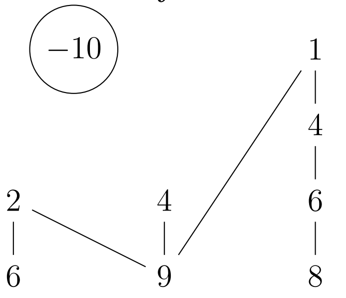
$(1;0,0,0;1,0;0,0,0)$, $(1;0,0,1;0,0;1,0,0)$, $(1;0,0,1;0,1;0,1,0)$, $(1;0,1,0;0,0;0,0,1)$, $(1;1,0,0;0,0;0,0,0)$, $(2;0,0,1;1,0;1,0,0)$, $(2;0,0,1;1,1;0,1,0)$, $(2;0,0,2;0,1;1,1,0)$, $(2;0,1,0;1,0;0,0,1)$, $(2;0,1,0;1,0;0,1,0)$, $(2;0,1,1;0,1;0,1,1)$, $(2;0,1,1;0,1;1,0,1)$, $(2;1,0,0;1,0;0,0,1)$, $(2;1,0,1;0,0;1,0,0)$, $(2;1,0,1;0,1;0,1,0)$, $(2;1,1,0;0,0;0,0,1)$, $(3;0,1,1;1,1;0,1,1)$, $(3;0,1,1;1,1;1,0,1)$, $(3;0,1,1;1,1;1,1,0)$, $(3;0,1,2;0,2;1,1,1)$, $(3;1,0,1;1,0;1,0,1)$, $(3;1,0,1;1,1;0,1,1)$, $(3;1,0,2;0,1;1,1,0)$, $(3;1,1,0;1,0;0,1,1)$, $(3;1,1,1;0,1;0,1,1)$, $(3;1,1,1;0,1;1,0,1)$.

Poset $\mathcal{S}_5$.

Representation



Costability

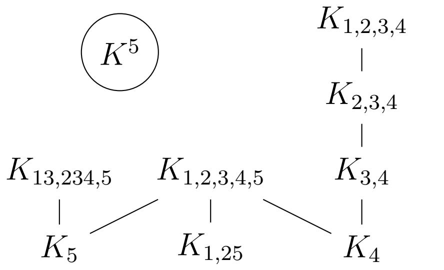


Maximal subcoordinate vectors

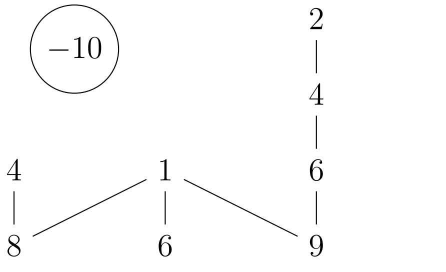
$(1;0,0;0,0;1,0,0,0)$, $(1;0,0;0,1;0,0,1,0)$, $(1;0,0;1,0;0,0,0,0)$, $(1;0,1;0,0;0,0,1,0)$, $(1;0,1;0,0;0,1,0,0)$, $(1;0,1;0,1;0,0,0,1)$, $(1;1,0;0,0;0,0,0,1)$, $(2;0,0;1,0;1,0,0,0)$, $(2;0,0;1,1;0,0,1,0)$, $(2;0,1;0,0;1,1,0,0)$, $(2;0,1;0,1;0,1,1,0)$, $(2;0,1;0,1;1,0,0,1)$, $(2;0,1;0,1;1,0,1,0)$, $(2;0,1;0,2;0,0,1,1)$, $(2;0,1;1,0;0,0,1,0)$, $(2;0,1;1,0;0,1,0,0)$, $(2;0,1;1,1;0,0,0,1)$, $(2;0,2;0,0;0,1,1,0)$, $(2;0,2;0,1;0,0,1,1)$, $(2;1,0;0,1;1,0,0,1)$, $(2;1,0;1,0;0,0,0,1)$, $(2;1,1;0,1;0,0,0,2)$, $(2;1,1;0,1;0,0,1,1)$, $(2;1,1;0,1;0,1,0,1)$, $(2;2,0;0,0;0,0,0,2)$, $(3;0,1;1,0;1,1,0,0)$, $(3;0,1;1,1;0,1,1,0)$, $(3;0,1;1,1;1,0,0,1)$, $(3;0,1;1,1;1,0,1,0)$, $(3;0,1;1,2;0,0,1,1)$, $(3;0,2;0,1;1,1,1,0)$, $(3;0,2;0,2;1,0,1,1)$, $(3;1,0;1,1;1,0,0,1)$, $(3;1,1;0,1;1,1,0,1)$, $(3;1,1;0,2;0,0,1,2)$, $(3;1,1;0,2;0,1,1,1)$, $(3;1,1;0,2;1,0,0,2)$, $(3;1,1;0,2;1,0,1,1)$, $(3;1,1;1,0;0,1,1,0)$, $(3;1,1;1,1;0,0,1,1)$, $(3;1,1;1,1;0,1,0,1)$, $(3;1,2;0,1;0,1,1,1)$, $(3;2,0;0,1;1,0,0,2)$, $(3;2,0;1,0;0,0,1,1)$, $(3;2,1;0,1;0,0,1,2)$, $(3;2,1;0,1;0,1,0,2)$, $(4;1,1;1,1;1,1,0,1)$, $(4;1,1;1,1;1,1,1,0)$, $(4;1,1;1,2;0,1,1,1)$, $(4;1,1;1,2;1,0,1,1)$, $(4;1,2;0,2;1,1,1,1)$, $(4;2,0;1,1;1,0,1,1)$, $(4;2,1;0,2;0,1,1,2)$, $(4;2,1;0,2;1,0,1,2)$, $(4;2,1;0,2;1,1,0,2)$, $(4;2,1;1,1;0,1,1,1)$.

Poset $\mathcal{S}_6$.

Representation



Costability



Maximal subcoordinate vectors

$(1;0,0;0,0;1,0,0,0)$, $(1;0,0;0,1;0,1,0,0)$, $(1;0,0;1,0;0,0,0,1)$,
$(1;0,1;0,1;0,0,0,1)$, $(1;0,1;0,1;0,0,1,0)$, $(1;1,0;0,0;0,0,0,0)$,
$(2;0,0;0,1;1,1,0,0)$, $(2;0,0;1,0;1,0,0,1)$, $(2;0,0;2,0;0,0,0,1)$,
$(2;0,1;0,1;1,0,0,1)$, $(2;0,1;0,1;1,0,1,0)$, $(2;0,1;0,2;0,1,1,0)$,
$(2;0,1;1,1;0,0,1,1)$, $(2;0,1;1,1;0,1,0,1)$, $(2;0,2;0,2;0,0,1,1)$,
$(2;1,0;0,0;1,0,0,0)$, $(2;1,0;0,1;0,1,0,0)$, $(2;1,0;1,0;0,0,0,1)$,
$(2;1,0;1,0;0,0,1,0)$, $(2;1,1;0,1;0,0,0,1)$, $(2;1,1;0,1;0,0,1,0)$,
$(3;0,1;0,2;1,1,1,0)$, $(3;0,1;1,1;1,0,1,1)$, $(3;0,1;1,1;1,1,0,1)$,
$(3;0,1;2,0;1,0,0,1)$, $(3;0,1;2,1;0,0,1,1)$, $(3;0,1;2,1;0,1,0,1)$,
$(3;0,2;0,2;1,0,1,1)$, $(3;0,2;0,3;0,1,1,1)$, $(3;0,2;1,2;0,1,1,1)$,
$(3;1,0;0,1;1,1,0,0)$, $(3;1,0;1,0;1,0,0,1)$, $(3;1,0;1,0;1,0,1,0)$,
$(3;1,0;2,0;0,0,1,1)$, $(3;1,1;0,1;1,0,0,1)$, $(3;1,1;0,1;1,0,1,0)$,
$(3;1,1;0,2;0,1,1,0)$, $(3;1,1;1,1;0,0,1,1)$, $(3;1,1;1,1;0,1,0,1)$,
$(3;1,1;1,1;0,1,1,0)$, $(3;1,2;0,2;0,0,1,1)$, $(4;0,2;1,2;1,1,1,1)$,
$(4;0,2;2,1;1,0,1,1)$, $(4;0,2;2,1;1,1,0,1)$, $(4;0,2;2,2;0,1,1,1)$,
$(4;1,1;1,1;1,1,0,1)$, $(4;1,1;1,1;1,1,1,0)$, $(4;1,1;2,0;1,0,1,1)$,
$(4;1,1;2,1;0,1,1,1)$, $(4;1,2;1,1;1,0,1,1)$, $(4;1,2;1,2;0,1,1,1)$.

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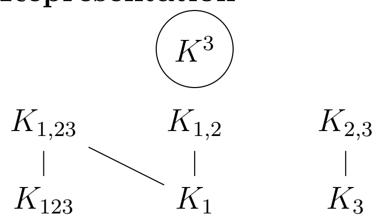
Appendix C. Sincere representations of finite representation type, their maximal subdimension vectors and stability conditions.

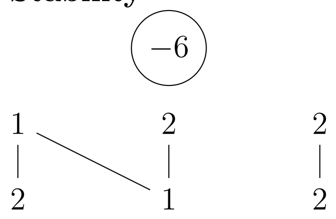
For each non-primitive posets $\mathcal{S}_1, \ldots, \mathcal{S}_6$ of finite representation type (given in Section 3.1) we list its sincere representations, stability condition (calculated by formulas (4.7)) and maximal subdimension vectors. We use the same notation as in Appendix B.

Poset $\mathcal{S}_1$.

Representation



Stability

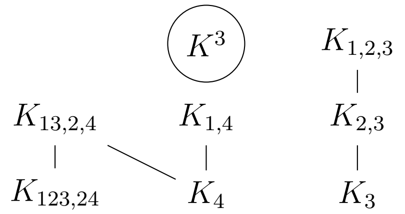


Maximal subdimension vectors

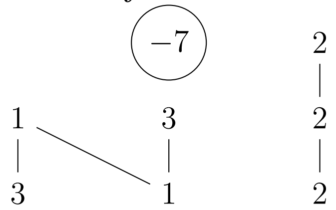
(1; 0, 0; 0, 0; 1, 1), (1; 0, 0; 0, 1; 0, 1), (1; 0, 1; 0, 0; 0, 1), (1; 0, 1; 1, 1; 0, 0), (1; 1, 1; 0, 0; 0, 0), (2; 0, 1; 0, 1; 1, 2), (2; 0, 1; 1, 1; 1, 1), (2; 0, 1; 1, 2; 0, 1), (2; 1, 1; 0, 1; 1, 1), (2; 1, 2; 1, 1; 0, 1)

Poset $\mathcal{S}_2$.

Representation



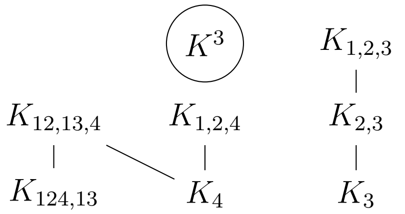
Stability



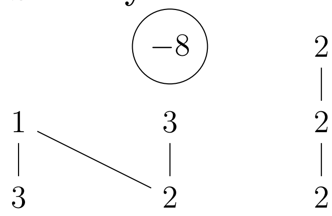
Maximal subdimension vectors

(1; 0, 0; 0, 0; 1, 1, 1), (1; 0, 0; 0, 1; 0, 0, 1), (1; 0, 1; 0, 0; 0, 1, 1), (1; 0, 1; 1, 1; 0, 0, 0), (1; 1, 1; 0, 0; 0, 0, 1), (2; 0, 1; 0, 0; 1, 2, 2), (2; 0, 1; 0, 1; 1, 1, 2), (2; 0, 1; 1, 1; 1, 1, 1), (2; 0, 1; 1, 2; 0, 0, 1), (2; 1, 1; 0, 0; 1, 1, 2), (2; 1, 1; 0, 1; 0, 1, 2), (2; 1, 2; 0, 0; 0, 1, 2), (2; 1, 2; 1, 1; 0, 1, 1), (2; 2, 2; 0, 0; 0, 0, 1), (3; 1, 2; 0, 1; 1, 2, 3), (3; 1, 2; 1, 1; 1, 2, 2), (3; 1, 2; 1, 2; 1, 1, 2), (3; 2, 2; 0, 1; 1, 1, 2), (3; 2, 3; 1, 1; 0, 1, 2).

Representation



Stability

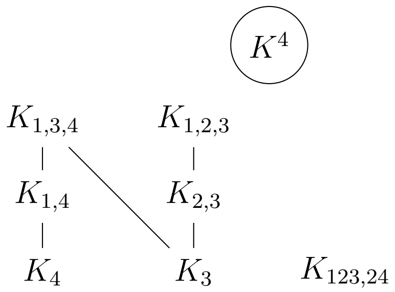


Maximal subdimension vectors

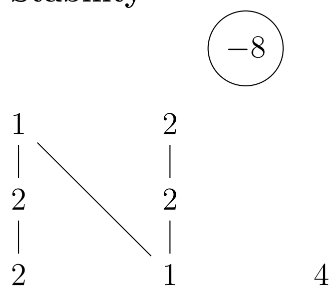
(1; 0, 0; 0, 0; 1, 1, 1), (1; 0, 0; 0, 1; 0, 1, 1), (1; 0, 1; 0, 1; 0, 0, 1), (1; 0, 1; 1, 1; 0, 0, 0), (1; 1, 1; 0, 0; 0, 0, 1), (1; 1, 1; 0, 1; 0, 0, 0), (2; 0, 1; 0, 1; 1, 2, 2), (2; 0, 1; 0, 2; 0, 1, 2), (2; 0, 1; 1, 1; 1, 1, 1), (2; 0, 1; 1, 2; 0, 1, 1), (2; 1, 1; 0, 1; 1, 1, 2), (2; 1, 1; 0, 2; 0, 1, 1), (2; 1, 2; 0, 1; 0, 1, 2), (2; 1, 2; 1, 2; 0, 0, 1), (2; 2, 2; 0, 1; 0, 0, 1), (3; 1, 2; 0, 2; 1, 2, 3), (3; 1, 2; 1, 2; 1, 2, 2), (3; 1, 2; 1, 3; 0, 1, 2), (3; 2, 2; 0, 2; 1, 1, 2), (3; 2, 3; 1, 2; 0, 1, 2).

Poset $\mathcal{S}_3$.

Representation



Stability

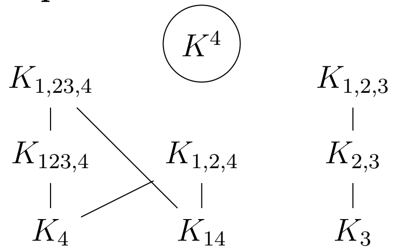


Maximal subdimension vectors

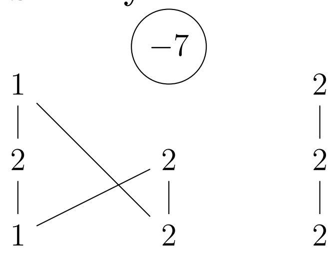
(1; 0, 0, 0; 0, 0, 1; 1), (1; 0, 0, 1; 1, 1, 1; 0), (1; 0, 1, 1; 0, 0, 1; 0), (1; 1, 1, 1; 0, 0, 0; 0), (2; 0, 0, 1; 0, 0, 1; 2), (2; 0, 0, 1; 1, 1, 2; 1), (2; 0, 0, 1; 1, 2, 2; 0), (2; 0, 1, 1; 0, 1, 2; 1), (2; 0, 1, 2; 1, 1, 2; 0), (2; 1, 1, 1; 0, 1, 1; 1), (2; 1, 1, 2; 1, 1, 1; 0), (2; 1, 2, 2; 0, 0, 1; 0), (3; 0, 1, 2; 1, 1, 2; 2), (3; 0, 1, 2; 1, 2, 3; 1), (3; 1, 1, 2; 0, 1, 2; 2), (3; 1, 1, 2; 1, 2, 2; 1), (3; 1, 2, 3; 1, 1, 2; 1).

Poset $\mathcal{S}_4$.

Representation



Stability

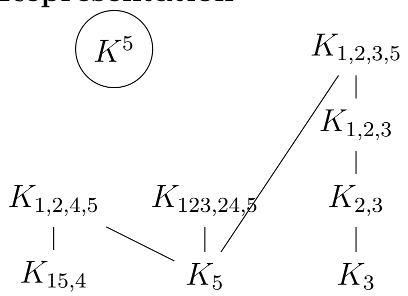


Maximal subdimension vectors

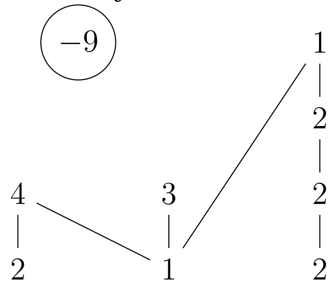
$(1;0,0,0;0,0;1,1,1)$, $(1;0,0,0;0,1;0,1,1)$, $(1;0,0,1;0,0;0,1,1)$, $(1;0,0,1;0,1;0,0,1)$, $(1;0,0,1;1,1;0,0,0)$, $(1;0,1,1;0,0;0,0,1)$, $(1;1,1,1;0,1;0,0,0)$, $(2;0,0,1;0,1;1,2,2)$, $(2;0,0,1;0,2;0,1,2)$, $(2;0,0,1;1,1;1,1,1)$, $(2;0,0,1;1,2;0,1,1)$, $(2;0,1,1;0,1;1,1,2)$, $(2;0,1,2;0,1;0,1,2)$, $(2;0,1,2;1,1;0,1,1)$, $(2;1,1,1;0,1;1,1,1)$, $(2;1,1,1;0,2;0,1,1)$, $(2;1,1,2;0,1;0,1,1)$, $(2;1,1,2;1,2;0,0,1)$, $(2;1,2,2;0,1;0,0,1)$, $(3;0,1,2;0,2;1,2,3)$, $(3;0,1,2;1,2;1,2,2)$, $(3;1,1,2;0,2;1,2,2)$, $(3;1,1,2;1,2;1,1,2)$, $(3;1,1,2;1,3;0,1,2)$, $(3;1,2,2;0,2;1,1,2)$, $(3;1,2,3;1,2;0,1,2)$.

Poset $\mathcal{S}_5$.

Representation



Stability

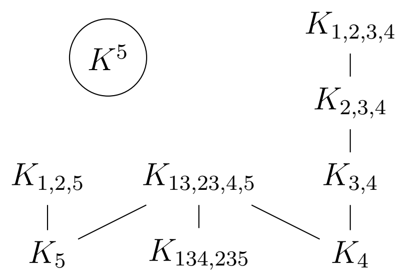


Maximal subdimension vectors

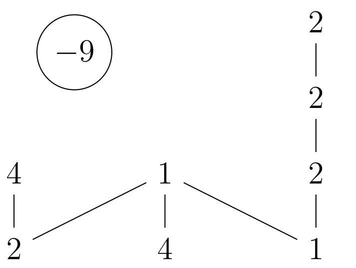
(1; 0, 0; 0, 0; 1, 1, 1, 1), (1; 0, 0; 0, 1; 0, 0, 1, 1), (1; 0, 1; 0, 0; 0, 1, 1, 1),
(1; 0, 1; 1, 1; 0, 0, 0, 1), (1; 1, 1; 0, 0; 0, 0, 0, 1), (2; 0, 1; 0, 0; 1, 2, 2, 2),
(2; 0, 1; 0, 1; 1, 1, 2, 2), (2; 0, 1; 1, 1; 1, 1, 1, 2), (2; 0, 1; 1, 2; 0, 0, 1, 2),
(2; 0, 2; 0, 0; 0, 1, 2, 2), (2; 0, 2; 1, 1; 0, 1, 1, 2), (2; 0, 2; 1, 2; 0, 0, 0, 1),
(2; 1, 1; 0, 0; 1, 1, 1, 2), (2; 1, 2; 0, 0; 0, 1, 1, 2), (2; 1, 2; 0, 1; 0, 1, 1, 1),
(2; 1, 2; 1, 1; 0, 0, 1, 2), (2; 2, 2; 0, 0; 0, 0, 0, 1), (3; 0, 2; 0, 1; 1, 2, 3, 3),
(3; 0, 2; 1, 1; 1, 2, 2, 3), (3; 0, 2; 1, 2; 1, 1, 2, 3), (3; 0, 2; 1, 3; 0, 0, 1, 2),
(3; 1, 2; 0, 1; 1, 2, 2, 3), (3; 1, 2; 1, 1; 1, 1, 2, 3), (3; 1, 2; 1, 2; 0, 1, 2, 3),
(3; 1, 3; 1, 1; 0, 1, 2, 3), (3; 1, 3; 1, 2; 0, 1, 1, 2), (3; 2, 2; 0, 1; 1, 1, 1, 2),
(3; 2, 3; 0, 1; 0, 1, 1, 2), (3; 2, 3; 1, 1; 0, 0, 1, 2), (4; 1, 3; 1, 2; 1, 2, 3, 4),
(4; 1, 3; 1, 3; 1, 1, 2, 3), (4; 2, 3; 0, 2; 1, 2, 2, 3), (4; 2, 3; 1, 2; 1, 1, 2, 3),
(4; 2, 4; 1, 2; 0, 1, 2, 3).

Poset $\mathcal{S}_6$.

Representation



Stability



Maximal subdimension vectors

(1; 0, 0; 0, 1; 1, 1, 1, 1), (1; 0, 0; 1, 1; 0, 0, 0, 1), (1; 0, 1; 0, 0; 0, 0, 1, 1),
(1; 1, 1; 0, 1; 0, 0, 0, 0), (2; 0, 0; 0, 1; 1, 2, 2, 2), (2; 0, 0; 0, 2; 1, 1, 2, 2),
(2; 0, 0; 1, 2; 1, 1, 1, 2), (2; 0, 0; 2, 2; 0, 0, 0, 1), (2; 0, 1; 0, 1; 1, 1, 2, 2),
(2; 0, 1; 0, 2; 0, 0, 1, 2), (2; 0, 1; 1, 1; 0, 1, 1, 2), (2; 0, 2; 0, 1; 0, 0, 1, 2),
(2; 1, 1; 0, 2; 1, 1, 1, 1), (2; 1, 1; 1, 2; 0, 0, 1, 1), (2; 1, 2; 0, 1; 0, 0, 1, 1),
(3; 0, 1; 0, 2; 1, 2, 3, 3), (3; 0, 1; 1, 2; 1, 2, 2, 3), (3; 0, 1; 1, 3; 1, 1, 2, 3),
(3; 0, 1; 2, 3; 1, 1, 1, 2), (3; 0, 2; 0, 2; 1, 1, 2, 3), (3; 0, 2; 1, 2; 0, 1, 2, 3),
(3; 1, 1; 0, 2; 1, 2, 2, 2), (3; 1, 1; 1, 3; 1, 1, 2, 2), (3; 1, 1; 2, 3; 0, 0, 1, 2),
(3; 1, 2; 0, 2; 1, 1, 2, 2), (3; 1, 2; 1, 2; 0, 1, 2, 2), (3; 1, 2; 1, 3; 0, 0, 1, 2),
(3; 1, 3; 0, 2; 0, 0, 1, 2), (4; 0, 2; 1, 3; 1, 2, 3, 4), (4; 0, 2; 2, 3; 1, 2, 2, 3),
(4; 1, 2; 1, 3; 1, 2, 3, 3), (4; 1, 2; 2, 4; 1, 1, 2, 3), (4; 1, 3; 1, 3; 1, 1, 2, 3).

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