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PARABOLIC EQUATIONS IN TIME DEPENDENT DOMAINS

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Dedicated to the memory of Vicent Caselles

Abstract. We show existence and uniqueness results for nonlinear parabolic equations in noncylindrical domains with possible jumps in the time variable.

Contents

1. Introduction

In recent years there has been a renewed interest in problems related with partial differential equations formulated in domains that change in time. This is partly due to the fact that a number of problems in mathematical biology are naturally posed on growing domains (e.g. developing organisms or proliferating cells, see for instance [14, 21, 23]) or domains that evolve in some particular way. Such issues have originated a wide amount of mathematical research, let us mention [6, 15, 16, 30]. To this we should add more classical engineering applications like fluids or gases in settings as channels or pipes with confining walls that may

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be displaced, removed or brought in at will. A sample of different applications of partial differential equations in evolving domains can be found in the recent survey paper [22]. In fact, it is not hard to conjecture that new applications will involve equations in moving domains in the future. Apart from that, partial differential equations posed on non-cylindrical domains are interesting also from the purely mathematical point of view.

This has led to an outburst of works on this subject in the literature that added up to some classical works [5, 11, 13, 24, 25, 35] to the extent that the number of current references is overwhelming. Let us comment on this literature according to the approach, the assumptions on the evolution of the domain where the equation is posed and the types of equations considered. Many authors used semigroup methods to tackle these problems (see for instance [2, 28] and references therein), but other approaches include adding a time viscosity [9], mapping the spacetime domain to a cylindrical domain [4] or use De Giorgi's minimizing movements [7, 20]. As regards time variations of the domain, it is customary to impose some sort of continuity (for instance Lipschitz continuity [31], relaxed to Hölder continuity in [9] and to absolute continuity in [29]), alternatively a monotonicity condition can be used (i.e., expanding domains [7, 20]) or Reinfenberg-type domains can be considered [12]. Concerning the type of equations, most of the works focus on parabolic equations which are assumed to be linear or in divergence form (see however [8, 9, 28, 29] where also other operators are admitted).

In this paper we are interested in well-posedness of parabolic equations in divergence form, in bounded domains that evolve in time. More precisely, we deal with the Cauchy–Dirichlet problem, in a formulation that allows boundary conditions to depend on time.

Let us discuss what are the novelties of this work with respect to the already existing literature. First, we introduce a simple approach to construct solutions, which consists in performing a time slicing of the domain, and then solve a family of approximating equations in cylindrical domains. The simplicity of this approach may allow to use it as a starting point for devising numerical methods for this sort of problems. Despite its simplicity, we are not aware of other works where such a slicing strategy is used. Our approach allows to deal with *nonlinear* equations, which include the parabolic p-Laplacian as a particular case. Also, our slicing technique allows us to deal with very general variations on the domain over time: We only require them to be of bounded variation, allowing for sudden *jumps* (expansions or contractions) of the domain. In particular, we do not impose any constraint on the topology of the evolving domains, which may differ from that of the initial domain. We are also able to prove uniqueness if the set of jumps has a reasonable structure (see Section 5).

To our best knowledge, this generality has not been previously achieved in the literature, except for the case of purely expanding domains [7]. However,

when this paper was close to completion, we learned about the contribution by F. Paronetto in [29]. Although he uses a different method and does not strive for the most general result, his work can be extended to cover quite general operators and boundary conditions.

Possible extensions. Since our main goal is presenting a method to tackle parabolic equations in moving domains, we did not focus on looking for the most general possible result. For instance, for the sake of simplicity we chose to deal only with bounded initial data. We stress that our main idea is to use a time slicing to approximate the original problem by a sequence of problems defined on cylindrical domains. As we do not focus on any particular equation, we chose to use abstract Lions' theory to provide existence for the approximating problems. However, we could also use other theories as starting point to provide existence of approximate solutions. If we are interested in a particular equation (the p-Laplace equation, say) then we will likely be using specific existence results to set up our method, and those will provide a much more accurate framework for the admisible set of initial conditions.

In that line of though, the fact that our present formulation does not allow to deal with degenerate equations, such as the porous media equation and its variants, could appear as a drawback. Again, we argue that suitable modifications of the method here proposed would allow to tackle these problems. In fact, even sticking to Lions' theory, porous media equation and related ones can be treated by making use of the compactness results by Dubinskii [18], carefully adapting our arguments in order to cope with that (see [26], Chapter I, 12). We did not pursue this line here in order to keep the presentation as simple as possible.

We also point out that we cannot deal with operators with linear growth such as the total variation flow or the parabolic minimal surface equation (see [8] for some results in this direction in the one-dimensional case). This is another challenging line to explore. Finally, following the same approach it should be possible to consider similar evolution equations on manifolds evolving in time (see [3] and references therein).

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2. Standing assumptions and main results

Our purpose is to prove existence and uniqueness results for nonlinear parabolic equations with time dependent coefficients in time dependent domains. More precisely, given an open set $\widetilde{\Omega} \subset [0,T] \times I\!R^d$ we shall consider the following problem:

$$\begin{cases} u_t(t,x) = \operatorname{div}(A(t,x,u,\nabla u)) & \text{in } \widetilde{\Omega}, \\ u(0,x) = u_0(x) & \text{in } \Omega(0), \\ u(t,x) = \psi(t,x) & \text{in } \partial_l\widetilde{\Omega} \cup \partial_{-1}\widetilde{\Omega}, \end{cases} \tag{2.1}$$

where we let $\nu^{\widetilde{\Omega}} = (\nu_t, \nu_x)$ be the outer unit normal to $\partial\widetilde{\Omega}$, $\Omega(0)$ is the initial domain defined in Assumption 2.2, and we set

$$\partial_{\pm 1}\widetilde{\Omega} := \{(t,x) \in \partial\widetilde{\Omega} : t > 0, \nu_t = \pm 1\},$$

$$\partial_{+}\widetilde{\Omega} := \{(t,x) \in \partial\widetilde{\Omega} : \nu_t > 0\},$$

$$\partial_{l}\widetilde{\Omega} := \{(t,x) \in \partial\widetilde{\Omega} : |\nu_t| < 1\} = \{(t,x) \in \partial\widetilde{\Omega} : |\nu_x| > 0\}.$$

In order to establish existence and uniqueness of solutions, we shall make suitable assumptions on the flux vector field A, the data u_0, ψ and the domain $\widetilde{\Omega}$.

Assumption 2.1. *The set $\widetilde{\Omega} \subset [0,T] \times I\!R^d$ is a bounded open set with Lipschitz boundary, and we let*

$$\Omega(t) := \{x \in I\!R^d : (t,x) \in \widetilde{\Omega}\} \qquad t \in (0,T).$$

Without loss of generality we may assume that $\Omega(t)$ is an open set for all $t \in (0,T)$.

Notice that $\Omega(t)$ has Lipschitz boundary for a.e. $t \in [0,T]$, and there exist the limits

$$\Omega(t\pm) := \lim_{s \to t\pm} \Omega(s) \qquad \text{for all } t \in [0,T], \tag{2.2}$$

where the limit is taken in the Hausdorff topology.

Assumption 2.2. *The set $\Omega(0) := \Omega(0+)$ is open and has Lipschitz boundary.*

Next we describe our assumptions on the operator A. Let Q_0 be an open set of $I\!R^d$ such that $\cup_{t\in[0,T]}\Omega(t) \subset\subset Q_0$ and let $Q_T := [0,T] \times Q_0$. We shall denote by $\mathcal{M}(Q_T)$ the space of all Radon measures on Q_T.

Assumption 2.3. *The function $A : Q_T \times I\!R \times I\!R^d \to I\!R^d$ is a Carathéodory map satisfying*

$$|A(t,x,z,\xi)| \le c|\xi|^{p-1} + b(t,x), \quad c > 0,\ b \in L^{p'}(Q_T),\ 1 < p < \infty,\ \tfrac{1}{p} + \tfrac{1}{p'} = 1, \tag{2.3}$$

$$A(t,x,z,\xi) \cdot \xi \ge \alpha|\xi|^p - d(t,x), \quad \alpha > 0,\ d \in L^1(Q_T), \tag{2.4}$$

$$(A(t,x,z,\xi) - A(t,x,z,\xi^*)) \cdot (\xi - \xi^*) \geq 0, \tag{2.5}$$

for a.e. $(t,x) \in Q_T$ *, and for all* $z \in \mathbb{R}$, $\xi, \xi^* \in \mathbb{R}^d$. *Moreover, we assume that*

$$|A(t,x,z,\xi) - A(s,y,w,\xi)| \leq (\omega(|t-s| + |x-y|) + C|z-w|)|\xi|^{p-1}, \tag{2.6}$$

where ω *is a modulus of continuity. We assume also that*

$$A(t,x,z,0) = 0 \quad \forall z \in \mathbb{R},\ \text{a.e. } Q_T. \tag{2.7}$$

Note that (2.5) and (2.7) imply that

$$A(t,x,z,\xi) \cdot \xi \geq 0, \quad \text{a.e. in } Q_T, \text{ and for all } z \in \mathbb{R},\ \xi \in \mathbb{R}^d. \tag{2.8}$$

We will consider the problem (2.1) with initial and boundary conditions

$$u(0,x) = u_0(x) \in L^\infty(\Omega(0)), \tag{2.9}$$

$$u(t,x) = \psi(t,x), \qquad (t,x) \in \partial_l\widetilde{\Omega} \cup \partial_{-1}\widetilde{\Omega},\ t > 0. \tag{2.10}$$

Assumption 2.4. *We assume that*

$$\psi \in C((0,T) \times Q_0) \cap L^p(0,T; W_0^{1,p}(Q_0)), \tag{2.11}$$

and

$$\psi_t \in L^1((0,T) \times Q_0) \cap L^{p'}(0,T; W^{-1,p'}(Q_0)). \tag{2.12}$$

Let us now define the space to which the solutions of our problem belong.

Definition 2.5. *Let* $\mathcal{V}$ *be the closure of* $C_c^1(\widetilde{\Omega})$ *with respect to the norm*

$$||v||_{\mathcal{V}} := \left(\int_{\widetilde{\Omega}} |\nabla v|^p \, dxdt \right)^{1/p}, \quad v \in C_c^1(\widetilde{\Omega}).$$

Notice that functions in $\mathcal{V}$ *do not necessarily have zero trace on* $\partial_{\pm 1}\widetilde{\Omega}$ *or on* $\Omega(0)$.

Our concept of solution will be the following:

Definition 2.6. *We say that a function* $u \in L^1(\widetilde{\Omega})$ *is a weak solution of (2.1) if the following statements hold:*

1. $u - \psi \in \mathcal{V}$ *and* $A(t,x,u,\nabla u) \in L^{p'}(\widetilde{\Omega})$.
2. $u_t \in \mathcal{V}^*$ *(note that this implies that* u *has a trace on* $\partial_{\pm 1}\widetilde{\Omega}$ *and on* $\Omega(0)$*).*
3. $u(0) = u_0$ *a.e. on* $\Omega(0)$ *and* $u = \psi$ *a.e. on every relatively open subset of* $\partial_{-1}\widetilde{\Omega}$.
4. *Let* $\tilde{u} := u\chi_{\widetilde{\Omega}} + \psi\chi_{Q_T \setminus \widetilde{\Omega}}$. *The time derivative* $\tilde{u}_t \in \mathcal{D}'(Q_T)$ *is decomposed uniquely as*

$$\tilde{u}_t = \psi_t \chi_{Q_T \setminus \widetilde{\Omega}} + u_t + u_t^s \qquad \text{with} \qquad u_t^s \ll \mathcal{H}^d_{|\overline{\partial_+\widetilde{\Omega}}}.$$

CRM Preprint Series number 1204

(5) *The following integral formulation*

$$-\int_0^T \int_{\Omega(t)} u\phi_t \, dxdt - \int_{\Omega(0)} u_0\phi(0)\, dx + \int_0^T \int_{\Omega(t)} A(t,x,u,\nabla u)\cdot\nabla\phi \, dxdt = 0 \tag{2.13}$$

holds for all $\phi \in \mathcal{D}([0,T)\times Q_0)$ *with* $\operatorname{supp}\phi \subset\subset \widetilde{\Omega}$.

Let us state the main existence result of this paper:

Theorem 2.7. *Let Assumptions 2.1–2.4 be satisfied. Then there exists a weak solution of* (2.1) *in the sense of Definition 2.6.*

We introduce now some extra constraints on the geometry of the non-cylindrical domain enabling to show uniqueness:

Assumption 2.8. *Let* $\widetilde{\Omega}$ *satisfy the following:*

(1) $\partial_{-1}\widetilde{\Omega} \cap \overline{\partial_{+}\widetilde{\Omega}} = \emptyset$,
(2) *The projection of* $\partial_{+1}\widetilde{\Omega}$ *over* $[0,T]$ *consists on a finite number of times.*

Note that any C^1 bounded domain fulfills the conditions in Assumption 2.8. This is also the case for purely expanding domains with a finite number of jumps in time.

We now state our main uniqueness result.

Theorem 2.9. *Let Assumptions 2.1–2.4 and 2.8 be satisfied. Then the solution of* (2.1) *is unique in the class of weak solutions.*

3. Construction of approximate solutions

Let us divide the interval $[0,T]$ into sub-intervals $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$. The points t_i are chosen so that:

(1) $\Omega(t_i)$ has Lipschitz boundary for all $i \in \{0,\ldots,N-1\}$,
(2) (2.3)–(2.7) hold for a.e. $x \in \Omega(t_i)$ and for all $z \in \mathbb{R}$, $\xi \in \mathbb{R}^d$,
(3) t_i are Lebesgue points of $\psi(t) \in L^p(0,T;W_0^{1,p}(Q_0))$ and $\psi(t_i) \in W^{1,p}(Q_0)$,
(4) t_i are Lebesgue points of the map $t \in [0,T] \to A(t) \in L^1(Q_0\times(-R,R)\times B(0,R))$ for any $R>0$,
(5) $\Delta := \max_{k=0,\ldots,N-1} |t_k - t_{k+1}| \to 0$ as $N\to\infty$.

Let $I_k = [t_k, t_{k+1})$. We iteratively solve the parabolic problem

$$\begin{cases} u^k_t = \operatorname{div}\left(A(t_k,x,u^k,\nabla u^k)\right), & t\in I_k,\ x\in\Omega(t_k) \\ u^k(t,x) = \psi(t,x), & t\in I_k,\ x\in\partial\Omega(t_k) \\ u^k(t_k,x) = \begin{cases} \lim_{t\to t_k-} u^{k-1}(t,x), & x\in\Omega(t_k)\cap\Omega(t_{k-1}) \\ \psi(t_k,x), & x\in\Omega(t_k)\backslash\Omega(t_{k-1}). \end{cases} \end{cases} \tag{3.1}$$

If $t_0 = 0$ we let $u^0(0,x) = u_0(x)$. Notice that the iterative initial condition for $t = t_k$ makes sense thanks to the continuity properties of u^{k-1}, see (3.5).

CRM Preprint Series number 1204

3.1. **Study of the model problem on a time slice.** Let Ω_0 be an open bounded set in $\mathbb{R}^d$ with Lipschitz boundary. Let $A(x,z,\xi)$ be such that (2.3)–(2.7) hold a.e. in $x \in \Omega_0$ and for all $z \in \mathbb{R}$, $\xi \in \mathbb{R}^d$. Let us consider the problem

$$\begin{cases} u_t = \operatorname{div}(A(x,u,\nabla u)) & t \in [0,T],\ x \in \Omega_0, \\ u(t,x) = \psi(t,x) & t \in [0,T],\ x \in \partial\Omega_0, \\ u(0,x) = u_0(x) & x \in \Omega_0. \end{cases} \tag{3.2}$$

where ψ and u_0 satisfy (2.9), (2.11) and (2.12).

Definition 3.1. *We say that a function $u \in L^1((0,T)\times\Omega_0)$ is a weak solution of (3.2) if $u \in L^p(0,T;W^{1,p}(\Omega_0))$, $A(x,u,\nabla u) \in L^{p'}((0,T)\times\Omega_0))$,*

$$-\int_0^T\int_{\Omega_0} u\phi_t\,dxdt - \int_0^T\int_{\Omega_0} u_0\phi(0)\,dx + \int_0^T\int_{\Omega_0} A(x,u,\nabla u)\cdot\nabla\phi\,dxdt = 0 \tag{3.3}$$

holds for all $\phi \in \mathcal{D}([0,T)\times\Omega_0)$, and

$$u(t) - \psi(t) \in W_0^{1,p}(\Omega_0) \qquad \text{a.e. } t \in (0,T).$$

Note that, by (2.3), if $u \in L^p(0,T;W^{1,p}(\Omega_0))$, then $A(x,u,\nabla u) \in L^{p'}((0,T)\times\Omega_0))$.

Proposition 3.2. *Problem* (3.2) *admits a unique weak solution in the sense of Definition 3.1.*

Proof. The proof is a standard application of the theory developed in [26, 27]; we include it for completeness. We consider the auxiliary problem

$$\begin{cases} v_t - \operatorname{div}\left(\widetilde{A}(t,x,v,\nabla v)\right) = -\psi_t & t \in [0,T],\ x \in \Omega_0, \\ v(t,x) = 0 & t \in [0,T],\ x \in \partial\Omega_0, \\ v(0,x) = u_0(x) - \psi(0,x) & x \in \Omega_0. \end{cases} \tag{3.4}$$

Here

$$\widetilde{A}(t,x,z,\xi) := A(x, z+\psi(t,x), \xi + \nabla\psi(t,x)).$$

According to the notation in [26, 27], we let $H = L^2(\Omega_0)$,

$$B = \begin{cases} W_0^{1,p}(\Omega_0) & \text{if } p \geq 2, \\ W_0^{1,p}(\Omega_0) \cap L^2(\Omega_0) & \text{if } 1 < p < 2, \end{cases}$$

and $F = L^p(0,T;B)$, so that B is dense in H and

$$-\operatorname{div}\widetilde{A} : F \to F' = L^{p'}(0,T;B') \quad \text{and} \quad \psi_t \in F',$$

with

$$B' = \begin{cases} W^{-1,p'}(\Omega_0) & \text{if } p \geq 2, \\ W^{-1,p'}(\Omega_0) + L^2(\Omega_0) & \text{if } 1 < p < 2. \end{cases}$$

CRM Preprint Series number 1204

Observe that, by our assumptions on $A(x,z,\xi)$ and ψ, $\widetilde{A}(t,x,z,\xi)$ is a Leray-Lions operator (see [26, 27]). Indeed, the monotonicity requirement is satisfied thanks to (2.5). The coercivity condition follows from (2.4) and Poincare's inequality in a standard way (lower order terms are estimated thanks to (2.3)). Then thanks to Lions' theory there exists some $v \in F$ solving (3.4) in F'. In fact, this solution verifies that

$$v \in L^p(0,T,B) \quad \text{and} \quad v_t \in L^{p'}(0,T;B').$$

Hence $v \in W^{1,p'}(0,T;B')$; thus, the attainment of the initial condition v_0 may be understood in this sense. Notice that $v \in C(0,T;L^2(\Omega_0))$ thanks to Lemma 3.3 below.

Let now $u = v + \psi$. Then u is a weak solution of (3.2) with initial condition $u(0) = u_0$. Clearly $u \in L^p(0,T;B)$, $u_t \in L^{p'}(0,T;B')$ and

$$u \in C(0,T;L^2(\Omega_0)). \tag{3.5}$$

To prove uniqueness let u, v be two different solutions. If A does not depend on u, recalling (2.5) we have that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(u-v,u-v)_H &= \langle u-v,(u-v)_t\rangle_{W_0^{1,p}(\Omega_0)-W^{-1,p'}(\Omega_0)} \\
&= \langle u-v,\operatorname{div}(A(x,\nabla u)-A(x,\nabla v))\rangle_{W_0^{1,p}(\Omega_0)-W^{-1,p'}(\Omega_0)} \\
&= -\langle \nabla u-\nabla v, A(x,\nabla u)-A(x,\nabla v)\rangle_{L^p(\Omega_0)-L^{p'}(\Omega_0)} \\
&\le 0\,.
\end{aligned}$$

Hence $\|u-v\|_2$ is nonincreasing and uniqueness follows.

For the general case, consider $\delta > 0$, let

$$T_\delta(s) = \begin{cases} s & \text{if } -\delta \le s \le \delta, \\ -\delta & \text{if } s < -\delta, \\ \delta & \text{if } s > \delta, \end{cases}$$

and compute as follows:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(u-v,T_\delta(u-v)/\delta)_H \\
&\qquad = -\frac{1}{\delta}\langle \nabla T_\delta(u-v), A(x,u,\nabla u)-A(x,v,\nabla v)\rangle_{L^p(\Omega_0)-L^{p'}(\Omega_0)} \\
&\qquad \le -\frac{1}{\delta}\langle (\nabla u-\nabla v)\chi_{\{|u-v|\le\delta\}}, A(x,u,\nabla v)-A(x,v,\nabla v)\rangle_{L^p(\Omega_0)-L^{p'}(\Omega_0)}.
\end{aligned}$$

CRM Preprint Series number 1204

Then, using (2.6) we get

$$\frac{1}{2}\frac{d}{dt}(u-v, T_\delta(u-v)/\delta)_H \le \frac{C}{\delta}\int_{\{|u-v|\le\delta\}} |u-v||\nabla u-\nabla v||\nabla v|^{p-1}\,dx$$
$$\le C\int_{\{|u-v|\le\delta\}} |\nabla u-\nabla v||\nabla v|^{p-1}\,dx.$$

The term on the far right converges to zero when $\delta\to 0$, since the integrand is in $L^1(\Omega_0)$ and $\nabla(u-v)=0$ a.e. where $u-v=0$. As $T_\delta/\delta(u-v)\to \mathrm{sign}(u-v)$, we get that $\|u-v\|_1$ is non-increasing, and we conclude the proof. □

The following continuity result is standard (see for instance [26, Ch. 2, Rem. 1.2] or [32]).

Lemma 3.3. *Let V be a reflexive Banach space with dual V'. Let H be a Hilbert space that we identify with its dual. Assume that $V\subset H\subset V'$ with the injection $V\subset H$ being dense. Then, $u\in L^p(0,T;V)$ together with $u_t\in L^{p'}(0,T;V')$ imply that there is a representative of u which is continuous from $[0,T]$ to H.*

Since $u^k\in C(t_k,t_{k+1};L^2(\Omega(t_k)))$ we can define the traces

$$u^k(t_k+) := \lim_{t\to t_k+} u^k(t) \qquad u^k(t_{k+1}-) := \lim_{t\to t_{k+1}-} u^k(t),$$

where the limit is taken in $L^2(\Omega(t_k))$.

3.2. **The approximate solutions u^Δ.** We now let

$$\Omega^\Delta := \{(t,x) : t\in[t_k,t_{k+1}),\ x\in\Omega(t_k),\quad k=0,\dots,N-1\}$$
$$= \cup_{k=1,\dots N-1}[t_k,t_{k+1})\times\Omega(t_k).$$

Notice that Ω^Δ does not depend only on $\Delta=\max_{k=0,\dots,N-1}|t_k-t_{k+1}|$, but depends on the entire sequence $\{t_k\}_k$.

Lemma 3.4. *Ω^Δ converges to $\widetilde{\Omega}$ in the Hausdorff sense. As a consequence $\chi_{\Omega^\Delta}\to\chi_{\widetilde{\Omega}}$ strongly in $L^1(Q_T)$ (hence in $L^p(Q_T)$ for all $p<\infty$).*

Proof. The Hausdorff convergence of Ω^Δ to $\widetilde{\Omega}$ can be easily verified when $\widetilde{\Omega}$ is a polyhedron. The claim follows by approximating a generic $\widetilde{\Omega}$ with Lipschitz boundary with polyhedra, in the topology generated by the Hausdorff distance. □

We now glue the solutions $u^k(t,x)$ of (3.1) together and define the approximate solutions

$$u^\Delta(t,x) := \sum_{k=0}^{N-1}\chi_{[t_k,t_{k+1})}(t)u^k(t,x)\chi_{\Omega(t_k)}(x), \tag{3.6}$$

$$\tilde{u}^\Delta(t,x) := \sum_{k=0}^{N-1} \chi_{[t_k,t_{k+1})}(t)(u^k(t,x)\chi_{\Omega(t_k)}(x) + \psi(t,x)\chi_{Q_0\setminus\Omega(t_k)}(x)), \tag{3.7}$$

for $(t,x) \in Q_T$. When we write $u^k(t,x)\chi_{\Omega(t_k)}(x)$ in the above formulae we intend the function which coincides with $u^k(t,x)$ in $\Omega(t_k)$ and it is equal to zero outside $\Omega(t_k)$.

In the sequel we shall prove the compactness of u^Δ and $\tilde{u}^\Delta$ as $\Delta \to 0$.

3.3. **Estimates on u^Δ.** We now derive some estimates on the approximate solutions u^Δ defined in (3.6).

Lemma 3.5. *Assume that $\|\psi\|_\infty, \|u_0\|_\infty \le C$ for some $C > 0$. Then $\|u^\Delta\|_{L^\infty(\Omega^\Delta)} \le C$ for any $t > 0$.*

Proof. It is enough to prove the estimate in $(0,t_1)\times\Omega(0)$. Let $[\cdot]^+$ denote the positive part (resp. $[\cdot]^-$ the negative part) and let $C > 0$. We compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega(0)}([u^\Delta(t)-C]^+)^2\,dx &= \int_{\Omega(0)}[u^\Delta - C]^+ \mathrm{div} A(0,x,u^\Delta,\nabla u^\Delta)\,dx \\ &= -\int_{\Omega(0)} A(0,x,u^\Delta,\nabla u^\Delta)\nabla([u^\Delta - C]^+)\,dx \\ &\quad + \int_{\partial\Omega(0)} A(0,x,u^\Delta,\nabla u^\Delta)\cdot\nu^{\Omega(0)}[\psi - C]^+\,d\mathcal{H}^{d-1}.\end{aligned}$$

The second term above vanishes if $\psi \le C$. As for the first term, note that $\nabla([u^\Delta - C]^+) = \chi_{\{u^\Delta > C\}}\nabla u^\Delta$, so that we can use (2.8) to ensure that it is nonpositive. Hence,

$$\int_{\Omega(0)}([u^\Delta(t)-C]^+)^2\,dx \le \int_{\Omega(0)}([u_0-C]^+)^2\,dx.$$

Thus, if $u_0 \le C$ then $u^\Delta(t) \le C$ too for any $t \in [0,t_1)$. This works in the same way for the time derivative of the integral of $([u^\Delta + C]^-)^2$, with inequalities reversed. If we now choose $C = \max\{\|u_0\|_\infty, \|\psi\|_\infty\}$, we deduce that $\|u^\Delta(t)\|_\infty \le C$. □

Lemma 3.6. *There holds*

$$\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)}|\nabla u^\Delta(t)|^p\,dxdt \le C,$$

for some constant $C > 0$ depending only on $\widetilde{\Omega}$, on ψ and on the structural constants in Assumption 2.3.

Proof. Fix k and compute for any $t \in (t_k, t_{k+1})$

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega(t_k)}(u^\Delta-\psi)^2\,dx = -\int_{\Omega(t_k)}\nabla(u^\Delta-\psi)A(t_k,x,u^\Delta,\nabla u^\Delta)\,dx - \int_{\Omega(t_k)}(u^\Delta-\psi)\psi_t\,dx.$$

CRM Preprint Series number 1204

Notice that the last term is well-defined thanks to our assumptions on ψ and to Lemma 3.5. Integrating the former equality on $[t_k, t_{k+1}]$, we obtain

$$\begin{aligned}\frac{1}{2}\int_{\Omega(t_k)}(u^\Delta(t_{k+1})-\psi)^2\,dx &= \frac{1}{2}\int_{\Omega(t_k)}(u^\Delta(t_k)-\psi)^2\,dx\\ &\quad-\int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)}(u^\Delta-\psi)\psi_t\,dxdt\\ &\quad+\int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)}\nabla\psi A(t_k,x,u^\Delta,\nabla u^\Delta)\,dxdt\\ &\quad-\int_0^T\int_{\Omega(t_k)}\nabla u^\Delta A(t_k,x,u^\Delta,\nabla u^\Delta)\,dxdt\\ &=: I+II+III+IV.\end{aligned}$$

Let us now control the last three terms. The second one can be easily estimated as

$$II \le 2\bar{C}\int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)}|\psi_t|\,dxdt, \quad \bar{C}:=\max\{\|\psi\|_\infty,\|u_0\|_\infty\}.$$

Concerning the fourth term, using (2.4) we get

$$IV \le -\int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)}\alpha|\nabla u^\Delta|^p\,dxdt+\int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)}|d(t,x)|\,dxdt.$$

In a similar way, using (2.3) we obtain

$$III \le \int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)}c|\nabla\psi||\nabla u^\Delta|^{p-1}\,dxdt+\int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)}|\nabla\psi|b(t,x)\,dxdt = A+B.$$

Let us estimate A and B. For that we use Young's inequality with weights:

$$a\,b \le \frac{\epsilon^p a^p}{p}+\frac{b^{p'}}{p'\epsilon^{p'}}, \quad \epsilon>0, \text{being } p,p' \text{ given by (2.3)}.$$

Then

$$B \le \frac{1}{p}\|\nabla\psi\|^p_{L^p([t_k,t_{k+1}]\times\Omega(t_k))}+\frac{1}{p'}\|b\|^p_{L^{p'}([t_k,t_{k+1}]\times\Omega(t_k))}$$

and

$$A \le \frac{c\epsilon^p}{p}\|\nabla\psi\|^p_{L^p([t_k,t_{k+1}]\times\Omega(t_k))}+\frac{c}{p'\epsilon^{p'}}\|\nabla u^\Delta\|^p_{L^p([t_k,t_{k+1}]\times\Omega(t_k))}$$

CRM Preprint Series number 1204

for any $\epsilon > 0$. Let us choose ϵ so that $c/(p'\epsilon^{p'}) = \alpha/2$. Collecting all the estimates, we obtain

$$\frac{1}{2}\int_{\Omega(t_k)} (u^\Delta(t_{k+1}) - \psi)^2 \, dx + \frac{\alpha}{2}\int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)} |\nabla u^\Delta|^p \, dxdt$$

$$\leq \frac{1}{2}\int_{\Omega(t_k)} (u^\Delta(t_k) - \psi)^2 \, dx + 2\bar{C}\int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)} |\psi_t| \, dxdt + \frac{c\epsilon^p}{p}\|\nabla\psi\|^p_{L^p([t_k,t_{k+1}]\times\Omega(t_k))}$$

$$+ \frac{1}{p}\|\nabla\psi\|^p_{L^p([t_k,t_{k+1}]\times\Omega(t_k))} + \frac{1}{p'}\|b\|^p_{L^{p'}([t_k,t_{k+1}]\times\Omega(t_k))} + \int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)} |d(t,x)| \, dxdt.$$

By summing up the previous inequality from $k = 0$ to $k = N-1$, we get

$$\frac{1}{2}\int_{\Omega(t_{N-1})} (u^\Delta(t_N) - \psi)^2 \, dx + \frac{\alpha}{2}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)} |\nabla u^\Delta(t)|^p \, dxdt$$

$$\leq \frac{1}{2}\int_{\Omega(0)} (u^\Delta(0) - \psi)^2 \, dx + \frac{1}{p'}\sum_{k=0}^{N-1}\|b\|^p_{L^{p'}([t_k,t_{k+1}]\times\Omega(t_k))}$$

$$+ \frac{1}{p}\left(1 + c\left(\frac{2c}{\alpha p'}\right)^{\frac{p}{p'}}\right)\sum_{k=0}^{N-1}\|\nabla\psi\|^p_{L^p([t_k,t_{k+1}]\times\Omega(t_k))}$$

$$+ 2\bar{C}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)} |\psi_t| \, dxdt + \sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)} |d(t,x)| \, dxdt.$$

On the aid of Lemma 3.5 the thesis follows. □

Recalling the definition of $\tilde{u}^\Delta$ and the assumptions on ψ, from Lemma 3.6 we obtain the following result:

Corollary 3.7. *There exists $C > 0$ depending only on $\widetilde{\Omega}$, on ψ and on the structural constants in Assumption 2.3, such that*

$$\|\tilde{u}^\Delta\|_{L^p(0,T;W^{1,p}(Q_0))} \leq C\,.$$

In particular, the sequence $\{\tilde{u}^\Delta\}$ is weakly relatively compact in $L^p(0,T;W^{1,p}(Q_0))$.

3.4. **Time compactness of $\tilde{u}^\Delta$.** We now show a stronger compactness property of u^Δ. For this aim, we need the following result, proved in [33].

Theorem 3.8. *Let X, B, Y be three Banach spaces such that $X \subset B \subset Y$. Assume that X is compactly embedded in B and*

$$F \text{ is a bounded set in } L^1(0,T;X), \tag{3.8}$$

$$\|\tau_h f - f\|_{L^1(0,T-h;Y)} \to 0 \text{ as } h \to 0, \text{ uniformly for } f \in F. \tag{3.9}$$

Then F is relatively compact in $L^1(0,T;B)$.

Let

$$\psi^\Delta(t,x) := \sum_{k=0}^{N-1} \chi_{[t_k,t_{k+1})}(t)\psi(t,x)\chi_{Q_0\setminus\Omega(t_k)}(x) \;=\; \psi(t,x)\chi_{Q_T\setminus\Omega^\Delta}(t,x),$$

so that we have $\tilde{u}^\Delta(t,x) = u^\Delta(t,x) + \psi^\Delta(t,x)$.

Lemma 3.9. *Let $0<k\le N$ be fixed. Then $u_t^k(t)\chi_{\Omega(t_k)} \in L^{p'}(t_k,t_{k+1},W^{-1,p'}(\Omega(t_k)))$ and the following estimate holds:*

$$\|u_t^k(t)\chi_{\Omega(t_k)}\|_{L^{p'}(t_k,t_{k+1},W^{-1,p'}(\Omega(t_k)))} \le c\|u^\Delta\|^{p-1}_{L^p(t_k,t_{k+1},W^{1,p}(\Omega(t_k)))} + \|b\|_{L^p(t_k,t_{k+1},L^{p'}(\Omega(t_k)))}.$$

Proof. We show the estimate by duality. Let $\phi \in L^p(t_k,t_{k+1},W_0^{1,p}(\Omega(t_k)))$. We compute

$$\langle u_t^k(t)\chi_{\Omega(t_k)},\phi\rangle = -\int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)} A(t_k,x,u^k,\nabla u^k(t))\cdot\nabla\phi\,dxdt.$$

Hence using (2.3)

$$\begin{aligned}|\langle u_t^k\chi_{\Omega(t_k)},\phi\rangle| &\le \int_{t_k}^{t_{k+1}}\int_{\Omega(t_k)} (c|\nabla u^\Delta(t)|^{p-1} + b(t,x))|\nabla\phi|dxdt \\ &\le \int_{t_k}^{t_{k+1}} \left(c\|u^\Delta(t)\|^{p-1}_{W^{1,p}(\Omega(t_k))} + \|b(t)\|_{L^{p'}(\Omega(t_k))}\right)\|\phi\|_{W_0^{1,p}(\Omega(t_k))}\,dt.\end{aligned}$$

The result follows. □

Lemma 3.10. *The sequence $\{\tilde{u}^\Delta\}$ is relatively compact in $L^1_{loc}(\widetilde{\Omega})$.*

Proof. We consider a cylinder $C := [t_1,t_2]\times K \subset\subset \widetilde{\Omega}$. We want to apply Theorem 3.8 with $f = \tilde{u}^\Delta|_C = u^\Delta|_C$, $X = W^{1,p}(K)$, $B = L^1(K)$ and $Y = W^{-1,p'}(K) + L^1(K)$. Here Y is a Banach space equipped with the norm

$$\|y\|_Y := \inf\{\|y_1\|_{W^{-1,p'}(K)} + \|y_2\|_{L^1(K)} :\; y_1+y_2 = y\}.$$

Then $X\subset B\subset Y$ and X is compactly embedded in B.

Notice that, since $C\subset\subset\widetilde{\Omega}$, we have

$$\tilde{u}_t^\Delta|_C = u_t^\Delta|_C = \sum_{k=0}^{N-1}\chi_{[t_k,t_{k+1})}(t)u_t^k(t,x)\chi_{\Omega(t_k)}(x)|_C \quad \text{for } N \text{ large enough.}$$

CRM Preprint Series number 1204

Estimate (3.8) directly follows from Lemma 3.5. In order to prove (3.9), we notice that (with a slight abuse of notation) there holds that

$$\tilde{u}^\Delta(t+h) - \tilde{u}^\Delta(t) = \int_t^{t+h} \tilde{u}_t^\Delta(s)\,ds$$
$$= \int_t^{t+h} \sum_{k=0}^{N-1} \chi_{[t_k,t_{k+1})}(t) u_t^k(s,x)\chi_{\Omega(t_k)}(x)\,ds := u_1^\Delta(t,h).$$

We claim that

$$\int_{t_1}^{t_2-h} \|u_1^\Delta(t,h)\|_{W^{-1,p'}(K)}\,dt \to 0 \qquad \text{as } h \to 0+ \tag{3.10}$$

uniformly in N; this would imply (3.9). To prove it we sum up all the estimates coming from Lemma 3.9 for different values of k in order to cover the cylinder C. We learn that there exist $\tilde{C} > 0$ independent of N and $t \in [t_1, t_2]$ such that $\|u_1(t,h)^\Delta\|_Y \le \tilde{C}h$, which implies (3.10). Hence $\tilde{u}^\Delta$ is strongly compact in $L^1(C)$. Now any compact set in $\widetilde{\Omega}$ can be covered by a finite number of open cylinders. To conclude we take a countable sequence of compact sets embedded in $\widetilde{\Omega}$ whose increasing union exhausts $\widetilde{\Omega}$ and apply a diagonal procedure. □

Corollary 3.11. *There exists a subsequence of $\{\tilde{u}^\Delta\}$ which converges strongly in $L^1(Q_T)$.*

Proof. We can combine Lemma 3.10 with the uniform bound provided by Lemma 3.5 to use Lebesgue's dominated convergence theorem. Note that the functions are constantly equal to ψ outside Ω^Δ and that Lemma 3.4 applies. □

4. Existence of solutions

In this section we prove the existence of weak solutions of (2.1).

4.1. **Convergence of the approximate solutions.**

Lemma 4.1. *There are functions $\tilde{u}, u$ such that the following statements hold (up to extracting a subsequence) for $N \to \infty$:*

1) $\tilde{u}^\Delta \rightharpoonup \tilde{u}$ *weakly in* $L^p(0,T,W^{1,p}(Q_0)) \cap L^\infty(Q_T)$,
2) $\tilde{u}^\Delta \to \tilde{u}$ *in* $L^1(Q_T)$ *and a.e. in* Q_T,
3) $\psi^\Delta \to \psi\chi_{Q_T\setminus\widetilde{\Omega}}$ *in* $L^1(Q_T)$*and a.e. in* Q_T,
4) $u^\Delta \to u$ *in* $L^1(\widetilde{\Omega})$,
5) $\tilde{u} = u + \psi\chi_{Q_T\setminus\widetilde{\Omega}}$ *and* $u = \tilde{u}\chi_{\widetilde{\Omega}}$.

Proof. The first statement follows from Lemma 3.5 and Corollary 3.7. The second statement follows from Corollary 3.11. To prove the third statement we write

$$\|\psi^\Delta - \psi\chi_{Q_T\setminus\widetilde{\Omega}}\|_{L^1(Q_T)} \le \|\psi\|_{L^\infty(Q_T)}\|\chi_{Q_T\setminus\Omega^\Delta} - \chi_{Q_T\setminus\widetilde{\Omega}}\|_{L^1(Q_T)} \to 0$$

as $N \to \infty$, thanks to Lemma 3.4. It follows that

$$u^\Delta = \tilde{u}^\Delta - \psi^\Delta \to u := \tilde{u} - \psi\chi_{Q_T\setminus\widetilde{\Omega}} \quad \text{in } L^1(\widetilde{\Omega}).$$

Since $\Omega^\Delta \to \widetilde{\Omega}$ by Lemma 3.4, we get that $\tilde{u}^\Delta \to \psi$ a.e. in $Q_T\setminus\widetilde{\Omega}$, so that u is supported on $\widetilde{\Omega}$. □

Recalling Lemma 3.5 it follows that, up to a subsequence, $\tilde{u}^\Delta \to \tilde{u}$ in $L^p(Q_T)$ and $u^\Delta \to u$ in $L^p(\widetilde{\Omega})$, for all $p < \infty$.

We now discuss the convergence of the time derivatives.

Lemma 4.2. *The time derivative $\tilde{u}_t^\Delta$ of $\tilde{u}^\Delta$ can be written as*

(4.1)

$$\begin{aligned}\tilde{u}_t^\Delta = u_t^\Delta + \psi_t^\Delta = &\sum_{k=0}^{N-1} \chi_{[t_k,t_{k+1})(t)} \left[u_t^k(t,x)\chi_{\Omega(t_k)} + \psi_t(t,x)\chi_{Q_0\setminus\Omega(t_k)}(x)\right] \\ &+ \sum_{k=0}^{N-1} \delta_{t_k}(t) \left[\psi(t_k,x) - u^k(t_k-)\right] \chi_{\Omega(t_k-)\setminus\Omega(t_k+)} := B^\Delta + C^\Delta.\end{aligned}$$

Proof. We compute

$$\begin{aligned}u_t^\Delta = &\sum_{k=0}^{N-1} \chi_{[t_k,t_{k+1})}(t) u_t^k(t,x)\chi_{\Omega(t_k)}(x) \\ &+ \sum_{k=1}^{N-1} \delta_{t_k}(t) \left[u^k(t_k+,x)\chi_{\Omega(t_k+)\setminus\Omega(t_k-)}(x) - u^k(t_k-,x)\chi_{\Omega(t_k-)\setminus\Omega(t_k+)}(x)\right]\end{aligned}$$

and

$$\begin{aligned}\psi_t^\Delta = &\sum_{k=0}^{N-1} \chi_{[t_k,t_{k+1})}(t)\psi_t(t,x)\chi_{Q_0\setminus\Omega(t_k)}(x) \\ &+ \sum_{k=1}^{N-1} \delta_{t_k}(t) \left[\psi(t_k+,x)\chi_{(Q_0\setminus\Omega(t_k+))\setminus(Q_0\setminus\Omega(t_k-))}(x)\right. \\ &\qquad \left. -\psi(t_k-,x)\chi_{(Q_0\setminus\Omega(t_k-))\setminus(Q_0\setminus\Omega(t_k+))}(x)\right].\end{aligned}$$

As ψ is continuous $\psi(t_k-,x) = \psi(t_k,x) = u^k(t_k+,x)$ and the thesis follows. □

Notice that the time derivative $\tilde{u}_t^\Delta$ has a singular part which concentrates on the "shrinking parts" of the boundary of Ω^Δ.

Lemma 4.3. *There exists $\widetilde{\Lambda} \in \mathcal{D}'(Q_T)$ such that, up to extraction of a subsequence, $\tilde{u}_t^\Delta \rightharpoonup \widetilde{\Lambda}$ in $\mathcal{D}'(Q_T)$. Moreover, given any cylinder $C := (t_a,t_b) \times$*

CRM Preprint Series number 1204

$K \subset\subset \widetilde{\Omega}$, there holds that $\widetilde{\Lambda}_{|C} \in L^{p'}(0,T;W^{-1,p'}(K))$ and $\tilde{u}_t^\Delta|_C \rightharpoonup \widetilde{\Lambda}_{|C}$ in $L^{p'}(0,T;W^{-1,p'}(K))$. We also have that

$$B^\Delta \to \psi_t \quad \textit{strongly in } L^1(Q_T\backslash\widetilde{\Omega})$$

Finally, there exists $u_t^s \in \mathcal{M}(Q_T)$ such that $u_t^s \ll \mathcal{H}^d_{|\overline{\partial_+\widetilde{\Omega}}}$ and

$$C^\Delta \rightharpoonup u_t^s \quad \textit{weakly in } \mathcal{M}(Q_T).$$

Proof. The first statement on $\widetilde{\Lambda}$ follows for instance using Lemma 3.5. The second statement is a consequence of Lemma 3.9, which provides uniform bounds on the time derivative over cylinders contained in $\widetilde{\Omega}$ as in Lemma 3.10. In order to identify the limit of B^Δ in $Q_T\backslash\widetilde{\Omega}$ we use Lebesgue's dominated convergence theorem together with (2.12) and Lemma 3.4. If we obtain a uniform bound for C^Δ in $\mathcal{M}(Q_T)$ we will get its weak convergence. Note that

$$\left\| \sum_{k=0}^{N-1} \delta_{t_k}(t) \left[\psi(t_k,x) - u^k(t_k-)\right] \chi_{\Omega(t_k-)\backslash\Omega(t_k+)} \right\|_{\mathcal{M}(Q_T)} \leq 2\|\psi\|_{L^\infty(Q_T)} \sum_{k=0}^{N-1} |\Omega(t_k^-)\backslash\Omega(t_k^+)| < \infty$$

as $\widetilde{\Omega}$ is Lipschitz and

$$\sum_{k=0}^{N-1} |\Omega(t_k^-)\backslash\Omega(t_k^+)| \leq \mathcal{H}^d(\partial\widetilde{\Omega}).$$

Finally, note that $C^\Delta \ll \mathcal{H}^d_{\partial_+\widetilde{\Omega}^\Delta}$, where

$$\partial_+\widetilde{\Omega}^\Delta := \bigcup_{k=0}^{N-1} \{t_k\} \times \left(\Omega(t_k^-)\backslash\Omega(t_k^+)\right).$$

By construction $\partial\widetilde{\Omega}^\Delta \to \partial\widetilde{\Omega}$ in Hausdorff sense, and the limit of $\partial_+\widetilde{\Omega}^\Delta$ is contained in $\overline{\partial_+\widetilde{\Omega}}$. □

Corollary 4.4. *Let $\Sigma_{t_1,t_2} := [t_1,t_2] \times K$ such that $\Sigma_{t_1,t_2} \cap \overline{\partial_+\widetilde{\Omega}} = \emptyset$. Then $\tilde{u} \in C(t_1,t_2;L^2(K))$.*

Proof. By previous considerations, we know that $\tilde{u}_t \in L^{p'}(t_1,t_2;W^{-1,p'}(K))$. Let $\phi \in \mathcal{D}(\Sigma_{t_1,t_2})$. Then $\phi\,\tilde{u} \in W_0^{1,p}(K)$ for a.e. $t_1 < t < t_2$. Using Lemma 3.3 we deduce that $\phi\,\tilde{u} \in C(t_1,t_2;L^2(K))$. Being ϕ and K arbitrary, the thesis follows. □

CRM Preprint Series number 1204

Corollary 4.5. *There holds that* $\widetilde{\Lambda}_{|\widetilde{\Omega}} \in \mathcal{V}^*$.

Proof. Let $\phi \in C_c^1(\widetilde{\Omega})$. Thanks to Lemma 3.9 we have that

$$|\langle \widetilde{\Lambda}, \phi \rangle| \leq ||\phi||_\nu \left(c\|\tilde{u}\|^{p-1}_{L^p(0,T,W^{1,p}(Q_0))} + \|b\|_{L^p(0,T,L^{p'}(Q_0))} \right).$$

Our claim follows by a duality argument. □

Corollary 4.6. *Let us denote the limit of* (4.1) *as defined by Lemma 4.3 by* $\tilde{u}_t$. *Then we have the following decomposition as measures:*

$$\tilde{u}_t := \psi_t + u_t^r + u_t^s,$$

where

- u_t^r *is supported in* $\widetilde{\Omega}$ *and* $u_t^r \in \mathcal{V}^*$,
- u_t^s *is supported in* $\overline{\partial_+ \widetilde{\Omega}}$ *and* $u_t^s \ll \mathcal{H}^d_{\overline{|\partial_+ \widetilde{\Omega}}}$.

4.2. **Recovery of the limit equation.** Our next aim is identifying the limit equation. Let us define

$$A^\Delta(t,x) := \sum_{k=0}^{N-1} \chi_{[t_k,t_{k+1})}(t) A(t_k, x, u^k, \nabla u^k) \chi_{\Omega(t_k)}(x).$$

Lemma 4.7. *There exists a function* $\bar{A} \in L^{p'}(Q_T)^d$ *such that* $A^\Delta \rightharpoonup \bar{A}$ *in* $L^{p'}(Q_T)^d$ *as* $N \to \infty$, *up to a subsequence. Moreover* $\bar{A}$ *is supported in* $\widetilde{\Omega}$.

Proof. This follows directly from (2.3) and Lemma 3.6. □

To identify $\bar{A}$ we will require a number of auxiliary results.

Lemma 4.8. *Let* ϕ *be smooth and such that supp* $\phi \subset \Omega^\Delta \cap \widetilde{\Omega}$. *Given* $\tau > 0$ *we define*

$$\rho^\tau := \frac{1}{\tau} \int_{t-\tau}^t ((\phi(t) - \phi(s))u(s)\, ds$$

(we set $\rho^\tau := 0$ *when the previous formula does not make sense), being* u *the function defined in Lemma 4.1. Then* $\rho^\tau \in \mathcal{V}$ *for any* $\tau > 0$ *and* $\rho^\tau \to 0$ *in* $\mathcal{V}$ *as* $\tau \to 0$.

Proof. Since $\operatorname{supp} \rho^\tau \subset \Omega^\Delta \cap \widetilde{\Omega}$ for small τ, we can approximate ρ^τ in the norm of $\mathcal{V}$ by functions in $C_c^1(\widetilde{\Omega})$ convolving with a mollifying sequence, so that $\rho^\tau \in \mathcal{V}$.

Let now $K \subset \mathbb{R}^d$ be an open set such that $K \subset \Omega^\Delta(t)$ a.e. $t \in (t_a, t_b)$ for some values $0 \leq t_a < t_b \leq T$. Thanks to [17, Ch. 2, Th. 9], we get that $\rho^\tau \to 0$ in $L^p(t_a, t_b; W^{1,p}(K))$ as $\tau \to 0$. Covering $\operatorname{supp} \phi$ with a finite collection of cylinders of the form $(t_a, t_b) \times K$ yields the desired result. □

Lemma 4.9. *Let* ϕ *be smooth and such that supp* $\phi \subset \Omega^\Delta \cap \widetilde{\Omega}$. *Then*

$$\limsup_{N\to\infty} \int_0^T \int_{\Omega^\Delta(t)} A^\Delta \cdot \nabla u^\Delta \phi \, dxdt \leq \int_0^T \int_{\Omega(t)} \bar{A} \cdot \nabla u \, \phi \, dxdt. \tag{4.2}$$

Proof. Let $\tau > 0$ and define

$$u^\tau(t) = \frac{1}{\tau}\int_{t-\tau}^t u(s)\,ds.$$

By multiplying the equation for u^Δ by $(u^\Delta - u^\tau)\phi$ and integrating by parts we get

$$\begin{aligned}\int_0^T\int_{\Omega^\Delta(t)}(u^\Delta-u^\tau)u_t^\Delta\phi\,dxdt &= \int_0^T\int_{\Omega^\Delta(t)}\left(u^\Delta-u^\tau\right)\mathrm{div}A^\Delta\phi\,dxdt\\ &= -\int_0^T\int_{\Omega^\Delta(t)}A^\Delta\cdot\nabla u^\Delta\phi\,dxdt\\ &\quad -\int_0^T\int_{\Omega^\Delta(t)}A^\Delta\cdot\nabla\phi\,u^\Delta\,dxdt\\ &\quad +\int_0^T\int_{\Omega^\Delta(t)}A^\Delta\cdot\nabla u^\tau\phi\,dxdt\\ &\quad +\int_0^T\int_{\Omega^\Delta(t)}A^\Delta\cdot\nabla\phi\,u^\tau\,dxdt\\ &:= I+II+III+IV.\end{aligned}$$

Let us elaborate on the left hand side of the previous equality. We compute

$$\begin{aligned}\int_0^T\int_{\Omega^\Delta(t)}u^\Delta u_t^\Delta\phi\,dxdt &= \int_0^T\int_{\Omega^\Delta(t)}\phi\frac{\partial}{\partial t}\left[\frac{(u^\Delta(t))^2}{2}\right]dxdt\\ &= -\int_0^T\int_{\Omega^\Delta(t)}\frac{(u^\Delta(t))^2}{2}\phi_t\,dxdt \to -\int_0^T\int_{\Omega(t)}\frac{u^2}{2}\phi_t\,dxdt\end{aligned}$$

as $N\to\infty$, thanks to Lemma 4.1. Next, we have that

$$\begin{aligned}-\int_0^T\int_{\Omega^\Delta(t)}u^\tau u_t^\Delta\phi\,dxdt &= -\int_0^T\int_{\Omega^\Delta(t)}u_t^\Delta\frac{\phi}{\tau}\int_{t-\tau}^t u(s)\,ds\,dxdt\\ &= -\int_0^T\int_{\Omega^\Delta(t)}u_t^\Delta\left\{(\phi\,u)^\tau+\frac{1}{\tau}\int_{t-\tau}^t((\phi(t)-\phi(s))u(s)ds\right\}dxdt\\ &= \int_0^T\int_{\Omega^\Delta(t)}(\phi u)_t^\tau u^\Delta\,dxdt-\int_0^T\int_{\Omega^\Delta(t)}\rho^\tau u_t^\Delta\,dxdt\end{aligned}$$

$$= \int_0^T \int_{\Omega^\Delta(t)} \frac{\phi(t)u(t) - \phi(t-\tau)u(t-\tau)}{\tau} u^\Delta \, dxdt$$
$$- \int_0^T \int_{\Omega^\Delta(t)} \rho^\tau u_t^\Delta \, dxdt$$
$$=: A + B.$$

Thanks to our assumptions on ϕ and (4.1) we have that

$$B = - \int_0^T \int_{\Omega^\Delta(t)} \rho^\tau \sum_{k=0}^{N-1} \chi_{[t_k,t_{k+1})} u_t^k \chi_{\Omega(t_k)} \, dxdt$$

for τ small enough. We then pass to the limit in B by Lebesgue's dominated convergence theorem. Indeed, if τ is small enough Lemma 4.3 enables to get a.e. convergence of the integrand, domination follows as the duality product is uniformly bounded. To deal with the limit of A as $N \to \infty$ we may use Lemma 4.1(4) together with the fact that the incremental ratio is essentially bounded (after Lemma 3.5). Gathering all the previous and letting $N \to \infty$, we find that

$$- \int_0^T \int_{\Omega^\Delta(t)} u^\tau u_t^\Delta \phi \, dxdt \to$$
$$\int_0^T \int_{\Omega(t)} \frac{\phi(t)u(t) - \phi(t-\tau)u(t-\tau)}{\tau} u \, dxdt - \int_{Q_T} \rho^\tau \widetilde{\Lambda} \, dxdt,$$

which is bounded from below by

$$\int_0^T \int_{\Omega(t)} \frac{\phi(t) - \phi(t-\tau)}{\tau} \frac{u^2(t)}{2} \, dxdt - \int_{Q_T} \rho^\tau \widetilde{\Lambda} \, dxdt.$$

Letting $\tau \to 0+$ and using Lemma 4.8, we obtain

$$\int_0^T \int_{\Omega(t)} \phi_t \frac{u^2(t)}{2} \, dxdt,$$

so that $\liminf_{\tau\to 0+} \liminf_{N\to\infty} (I + II + II + IV) \geq 0$.

We are now ready to compute the limit of $I + II + III + IV$ when $N \to \infty$. First, we find out that

$$II \to - \int_0^T \int_{\Omega(t)} \bar{A} \nabla \phi \, u \, dxdt$$

using Lemmas 4.1(4) and 4.7. We also have

$$III \to \int_0^T \int_{\Omega(t)} \bar{A} \nabla u^\tau \phi \, dxdt$$

CRM Preprint Series number 1204

as $N \to \infty$ (clearly $\nabla u^\tau \in L^p(Q_T)^d$). Note that $\nabla u^\tau = (\nabla u)^\tau \to \nabla u$ in $L^p_{loc}(\widetilde{\Omega})^d$, as in the proof of Lemma 4.8. Taking limit $\tau \to 0$, the above integral converges to

$$\int_0^T \int_{\Omega(t)} \bar{A}\nabla u\, \phi\, dxdt.$$

Finally, arguing as before we get that

$$IV \to \int_0^T \int_{\Omega(t)} \bar{A}\nabla\phi\, u^\tau\, dxdt$$

as $N \to \infty$, which converges to

$$\int_0^T \int_{\Omega(t)} \bar{A}\nabla\phi\, u\, dxdt,$$

after taking the limit $\tau \to 0$. Hence

$$\limsup_{N\to\infty} \int_0^T \int_{\Omega(t)} A^\Delta \nabla u^\Delta \phi\, dxdt \le \int_0^T \int_{\Omega(t)} \bar{A}\phi\nabla u\, dxdt,$$

and the result follows. □

Lemma 4.10. *There holds $\bar{A}(t,x) = A(t,x,u,\nabla u)$ a.e. in $\widetilde{\Omega}$.*

Proof. Let $0 \le \phi \in C^1_0(Q_T)$ with $\operatorname{supp}\phi \subset \Omega^\Delta \cap \widetilde{\Omega}$, and let $g \in C^1(\overline{Q_T})$. Thanks to the monotonicity assumption (2.5), we have

$$\int_0^T \sum_{k=1}^{N-1} \int_{\Omega(t_k)} (A(t_k,x,u^\Delta,\nabla u^\Delta) - A(t_k,x,u^\Delta,\nabla g))(\nabla u^\Delta(t) - \nabla g)\phi\, dxdt \ge 0.$$

From Lemma 4.9 we get

$$\limsup_{N\to\infty} \int_0^T \sum_{k=1}^{N-1} \int_{\Omega(t_k)} A(t_k,x,u^\Delta,\nabla u^\Delta)\nabla u^\Delta \phi\, dxdt \le \int_0^T \int_{\Omega(t)} \bar{A}\nabla u\phi\, dxdt.$$

We now show that

(4.3)
$$\int_0^T \sum_{k=1}^{N-1} \int_{\Omega(t_k)} A(t_k,x,u^\Delta,\nabla g)\nabla u^\Delta \phi\, dxdt \to \int_0^T \int_{\Omega(t)} A(t,x,u,\nabla g)\nabla u\, \phi\, dxdt,$$

as $N \to \infty$. Indeed, recalling (2.6) we have

$$\left| A(t,x,u,\nabla g) - \sum_{k=1}^{N-1} A(t_k,x,u^\Delta,\nabla g)\chi_{[t_k,t_{k+1})} \right| \le \sum_{k=1}^{N-1} \chi_{[t_k,t_{k+1})} \big(w(|t-t_k|) + C|u(t,x) - u^\Delta(t,x)| \big) |\nabla g|^{p-1}.$$

Note that the right-hand side above converges to zero a.e. in $\widetilde{\Omega}$ and also in $L^p(\widetilde{\Omega})$ for all $p < \infty$ as $N \to \infty$. On the other hand, $\nabla u^\Delta \rightharpoonup \nabla u$ weakly in $L^p_{loc}(\widetilde{\Omega})^d$ thanks to Lemma 4.1, which yields (4.3). In a similar way we show that

$$\int_0^T \sum_{k=1}^{N-1} \int_{\Omega(t_k)} A(t_k, x, u^\Delta, \nabla g)\nabla g\, \phi\, dxdt \to \int_0^T \int_{\Omega(t)} A(t, x, u, \nabla g)\nabla g\, \phi\, dxdt.$$

Finally we obtain that

$$\int_0^T \sum_{k=1}^{N-1} \int_{\Omega(t_k)} A(t_k, x, u^\Delta, \nabla u^\Delta)\nabla g\, \phi\, dxdt \to \int_0^T \int_{\Omega(t)} \bar{A}\nabla g\, \phi\, dxdt$$

thanks to Lemma 4.7. Summing up, we obtain

$$\int_0^T \int_{\Omega(t)} (\bar{A} - A(t, x, u, \nabla g))(\nabla u(t) - \nabla g)\phi\, dxdt \geq 0.$$

This implies that $\bar{A} = A(t, x, u, \nabla u)$ for a.e. $(t, x) \in \operatorname{supp}\phi$, by means of Minty–Browder's method (see for instance [19], Chap. 9.1). □

4.3. **Recovery of boundary and initial conditions.**

Proposition 4.11. *The function u defined in Lemma 4.1 is a weak solution of problem 2.1 in the sense of Definition 2.6. Furthermore, $u(t) \to u_0$ a.e. as $t \to 0$*

Proof. Let $\phi \in C_0^\infty((0, T) \times Q_0)$ with $\operatorname{supp}\phi \subset \Omega^\Delta \cap \widetilde{\Omega}$. We fix a value of $k \in \{1, \dots, N-1\}$ and test the approximating problem in $[t_k, t) \times \Omega(t_k)$ with $t < t_{k+1}$. That is,

$$\int_{\Omega(t_k)} u^\Delta(t)\phi(t)\, dx + \int_{t_k}^t \int_{\Omega(t_k)} A^\Delta \cdot \nabla\phi\, dxds = \int_{\Omega(t_k)} u^\Delta(t_k)\phi(t_k)\, dx + \int_{t_k}^t \int_{\Omega(t_k)} u^\Delta(s)\phi_s\, dxds$$

for any $t \in [t_k, t_{k+1})$. By adding these contributions from 0 to $t \in (t_j, t_{j+1}], j \in \{1, \dots, N-1\}$ we get

$$\begin{aligned} &\int_{\Omega^\Delta(t)} u^\Delta(t)\phi(t)\, dx + \int_0^t \int_{\Omega^\Delta(s)} A^\Delta \cdot \nabla\phi\, dxds \\ &\quad = \int_{\Omega(0)} u_0\phi(0)\, dx + \int_0^t \int_{\Omega^\Delta(s)} u^\Delta(s)\phi_s\, dxds \\ &\qquad + \sum_{k=1}^{j} \left(\int_{\Omega(t_k+)} u^\Delta(t_k+)\phi(t_k)\, dx - \int_{\Omega(t_k-)} u^\Delta(t_k-)\phi(t_k)\, dx \right). \end{aligned} \tag{4.4}$$

CRM Preprint Series number 1204

Since $\operatorname{supp}\phi \subset \Omega^\Delta$, we also have

$$\int_{\Omega(t_k+)} u^\Delta(t_k+)\phi(t_k)\,dx - \int_{\Omega(t_k-)} u^\Delta(t_k-)\phi(t_k)\,dx$$
$$= \int_{\Omega(t_k+)\setminus\Omega(t_k-)} \psi(t_k)\phi(t_k)\,dx - \int_{\Omega(t_k-)\setminus\Omega(t_k+)} u^\Delta(t_k-)\phi(t_k)\,dx = 0.$$

Thanks to Lemma 4.1(4), u^Δ converges strongly to u in $L^1(\operatorname{supp}\phi)$. Hence we can pass to the limit in (4.4) and obtain

$$\int_{\Omega(t)} u(t)\phi(t)\,dx + \int_0^t \int_{\Omega(s)} A(t,x,u,\nabla u)\cdot\nabla\phi\,dxds$$
$$= \int_{\Omega(0)} u_0\phi(0)\,dx + \int_0^t \int_{\Omega(s)} u(s)\phi_s\,dxds$$

for a.e. $0 < t \le T$, which holds for any $\phi \in C_0^\infty((0,T)\times Q_0)$ with $\operatorname{supp}\phi \subset\subset \widetilde{\Omega}$. This can be stated as

$$u_t = \operatorname{div} A(t,x,u,\nabla u) \quad \text{in } \mathcal{D}'(\widetilde{\Omega}).$$

Furthermore, since $\tilde{u} \in L^p(0,T;W^{1,p}(Q_0))$ and $\tilde{u} = \psi$ a.e. $Q_T\setminus\widetilde{\Omega}$, we get that $u(t) - \psi(t) \in W_0^{1,p}(\Omega(t))$ for almost any $t \in (0,T)$. Hence we also recover the boundary conditions at $\partial_l\widetilde{\Omega}$ in the limit.

Let us deal next with the initial condition. Note that for t small enough we have

$$\int_{\Omega(t)} u(t)\phi(t)\,dx = \int_{\Omega(0)} u_0\phi(0)\,dx + C(\phi)t,$$

here we use that we assume condition *4* on the time slicing (and specifically on $t_0 = 0$) as specified at the beginning of Section 3. Hence

$$\lim_{t\to 0}\int_{\Omega(t)} u(t)\phi(t)\,dx = \int_{\Omega(0)} u_0\phi(0)\,dx.$$

Now let $K \subset\subset \Omega(0)$ such that $\tilde{u} \in C(0,t_1,L^2(K))$ for some $t_1 > 0$ (which exists as $\widetilde{\Omega}$ is Lipschitz). Then $u(t)$ converges in $L^2(K)$ to some $\bar{u}_0$ as $t \to 0$. This limit $\bar{u}_0$ must agree with the distributional limit u_0 over K. Hence $u(t) \to u_0$ in $L^2_{loc}(\Omega(0))$ as $t \to 0$. In particular we get a.e. convergence to the initial condition. Note that this works in the same way for any relatively open subset of $\partial_{-1}\widetilde{\Omega}$.

Finally we justify that $u - \psi \in \mathcal{V}$. Once we have shown that the boundary conditions on $\partial_l\widetilde{\Omega}$ are fulfilled, it is easy to construct a sequence η_n belonging to $C_c^1(\widetilde{\Omega})$ and satisfying $\|(u-\psi)-\eta_n\|_{\mathcal{V}} \to 0$ as $n \to \infty$. For instance, we may consider $G \in C^1(\mathbb{R})$ such that $|G(t)| \le |t|$, $G(t) = 0$ if $|t| \le 1$ and $G(t) = t$ if $|t| \ge 2$. We also consider ρ_n to be a standard mollifying sequence. Then $\eta_n = G(n\rho_n * (u-\psi))/n$ has the desired properties. □

The argument above also shows that, given a cylinder $[t_1, t_2] \times K \subset\subset \widetilde{\Omega}$, the map $t \mapsto u_{|K}$ is L^2-continuous in $[t_1, t_2]$. As a consequence, if we fix $t > 0$ then $u(s) \to u(t)$ as $s \to t$ a.e. in $\Omega(t)$. In this sense, we can claim that $t \mapsto u(t) \in C(0, T, L^2(\Omega(t)))$.

5. Uniqueness of solutions

Proof of Theorem 2.9. Let $\tilde{u}_1, \tilde{u}_2$ be two solutions of (2.1) with the same initial datum. Let $\epsilon > 0$ and $q_\epsilon \geq 0$ be a smooth convex approximation of the absolute value such that $q_\epsilon(0) = q_\epsilon'(0) = 0$ and $\operatorname{supp} q_\epsilon'' \subset [-\epsilon/2, \epsilon/2]$. Then we have that $q_\epsilon'(\tilde{u}_1 - \tilde{u}_2) \in L^p(0, T, W_0^{1,p}(Q_0))$.

Let us consider first that $\partial_{+1}\widetilde{\Omega} = \emptyset$, we will remove this restriction later on. We pick $\{\phi_n\}_n \in \mathcal{D}(Q_T)$ such that $\phi_n \to q_\epsilon'(\tilde{u}_1 - \tilde{u}_2)$ strongly in $L^p(0, T, W_0^{1,p}(Q_0))$ and $\operatorname{supp} \phi_n \subset \widetilde{\Omega}$. Note that the pairing

$$\langle (\tilde{u}_1 - \tilde{u}_2)_t, \phi_n \rangle_{\mathcal{V}^* - \mathcal{V}}$$

makes sense and is bounded independently of n. Then we substitute ϕ_n in (2.13). On one hand, when $n \to \infty$ we get

$$\int_0^T \int_{Q_0} \phi_n (\tilde{u}_1 - \tilde{u}_2)_t \, dxdt \to \int_0^T \int_{Q_0} q_\epsilon'(\tilde{u}_1 - \tilde{u}_2)(u_1 - u_2)_t \, dxdt.$$

On the other hand, integrating by parts and using (2.3),

$$\begin{aligned}
&\int_0^T \int_{Q_0} \phi_n (\tilde{u}_1 - \tilde{u}_2)_t \, dxdt \\
&\quad = -\int_0^T \int_{Q_0} \nabla \phi_n \left(A(t, x, \tilde{u}_1, \nabla \tilde{u}_1) - A(t, x, \tilde{u}_2, \nabla \tilde{u}_2) \right) dxdt \\
&\quad \to -\int_0^T \int_{Q_0} \nabla q_\epsilon'(\tilde{u}_1 - \tilde{u}_2) \left(A(t, x, \tilde{u}_1, \nabla \tilde{u}_1) - A(t, x, \tilde{u}_2, \nabla \tilde{u}_2) \right) dxdt \text{ as } n \to \infty.
\end{aligned}$$

This term can be rewritten as

$$\begin{aligned}
&-\int_0^T \int_{Q_0} q_\epsilon''(\tilde{u}_1 - \tilde{u}_2) \nabla (\tilde{u}_1 - \tilde{u}_2) [A(t, x, \tilde{u}_1, \nabla \tilde{u}_1) - A(t, x, \tilde{u}_2, \nabla \tilde{u}_2)] \, dxdt \\
&\quad = -\int_0^T \int_{Q_0} q_\epsilon''(\tilde{u}_1 - \tilde{u}_2) \nabla (\tilde{u}_1 - \tilde{u}_2) [A(t, x, \tilde{u}_1, \nabla \tilde{u}_1) - A(t, x, \tilde{u}_1, \nabla \tilde{u}_2)] \, dxdt \\
&\qquad -\int_0^T \int_{Q_0} q_\epsilon''(\tilde{u}_1 - \tilde{u}_2) \nabla (\tilde{u}_1 - \tilde{u}_2) [A(t, x, \tilde{u}_1, \nabla \tilde{u}_2) - A(t, x, \tilde{u}_2, \nabla \tilde{u}_2)] \, dxdt.
\end{aligned}$$

CRM Preprint Series number 1204

The first term above is less or equal than zero due to (2.5), hence we can neglect it. As regards the second term, we use (2.6) to write

$$
\begin{aligned}
&-\int_0^T \int_{Q_0} q_\epsilon''(\tilde{u}_1-\tilde{u}_2)\nabla(\tilde{u}_1-\tilde{u}_2)[A(t,x,\tilde{u}_1,\nabla\tilde{u}_2)-A(t,x,\tilde{u}_2,\nabla\tilde{u}_2)]\,dxdt\\
&\leq \int_0^T \int_{Q_0} q_\epsilon''(\tilde{u}_1-\tilde{u}_2)|\nabla(\tilde{u}_1-\tilde{u}_2)|C|\nabla\tilde{u}_2|^{p-1}|\tilde{u}_1-\tilde{u}_2|\,dxdt\\
&\leq C\|q_\epsilon''\|_\infty \epsilon \int_0^T \int_{Q_0} |\nabla(\tilde{u}_1-\tilde{u}_2)||\nabla\tilde{u}_2|^{p-1}\,dxdt = O(\epsilon).
\end{aligned}
$$

Now we notice the following decomposition of the boundary measure

$$
(\tilde{u}_1-\tilde{u}_2)_t^s = F(t,x)\mathcal{H}^{d-1}_{|\partial\Omega(t)}\otimes dt + B_{1,2} \quad \text{with} \quad B_{1,2}\perp \mathcal{H}^{d-1}_{|\partial\Omega(t)}\otimes dt.
$$

We also observe that:

(1) $B_{1,2}$ is concentrated in $\partial_{+1}\widetilde{\Omega}$.
(2) The set $S=\{(t,x)\in\partial_l\widetilde{\Omega}/\nu_t=+1\}$ verifies $\left(\mathcal{H}^{d-1}_{|\partial\Omega(t)}\otimes dt\right)(S)=0$.
(3) The density $F(t,x)$ is supported in $\overline{\partial_+\widetilde{\Omega}}$, which by Assumption 2.8 does not intersect $\partial_{-1}\widetilde{\Omega}$.

Since $\partial_{+1}\widetilde{\Omega}=\emptyset$, it follows that

$$
\int_0^T\int_{Q_0} q_\epsilon'(\tilde{u}_1-\tilde{u}_2)(\tilde{u}_1-\tilde{u}_2)_t^s\,dxdt = \int_0^T\left(\int_{\partial\Omega(t)} F(t,x)q_\epsilon'(\tilde{u}_1-\tilde{u}_2)\,d\mathcal{H}^{d-1}\right)dt = 0,
$$

so that

$$
\begin{aligned}
\int_0^T \frac{d}{dt}\int_{Q_0} q_\epsilon(\tilde{u}_1-\tilde{u}_2)\,dxdt &= \int_0^T\int_{Q_0} q_\epsilon'(\tilde{u}_1-\tilde{u}_2)(u_1-u_2)_t\,dxdt\\
&\quad + \int_0^T\int_{Q_0} q_\epsilon'(\tilde{u}_1-\tilde{u}_2)(\tilde{u}_1-\tilde{u}_2)_t^s \leq O(\epsilon),
\end{aligned}
$$

which in turn implies

$$
\int_{Q_0} q_\epsilon(\tilde{u}_1-\tilde{u}_2)(T)\,dx \leq \int_{Q_0} q_\epsilon(\tilde{u}_1-\tilde{u}_2)(0)\,dx + O(\epsilon)
$$

for any $T>0$ and any $\epsilon>0$. By letting $\epsilon\to 0$, we obtain the uniqueness result.

Finally, if $\partial_{+1}\widetilde{\Omega}$ is not empty, we consider first $t_1 := \inf\{t\in(0,T)/(t,x)\in\partial_{+1}\widetilde{\Omega}\}$ and repeat the previous argument on $(0,t_1)$. This shows uniqueness over $[0,t_1)\times Q_0$. Now we compute the traces of both $\tilde{u}_1$ and $\tilde{u}_2$ as $t\nearrow t_1$ to see that they coincide. Therefore we can consider any two solutions launched by $\tilde{u}_1(t_1)$ over (t_1,T) and argue as before that they will agree in $[t_1,t_2)$, being $t_2 := \inf\{t\in(t_1,T)/(t,x)\in\partial_{+1}\widetilde{\Omega}\}$. We repeat this procedure until we reach the

whole time interval $(0, T)$ (which we do in a finite number of steps thanks to Assumption 2.8). □

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