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POINTWISE ESTIMATES FOR 3-MONOTONE APPROXIMATION

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Abstract. We prove that for a 3-monotone function $F \in C[-1,1]$, one can achieve the pointwise estimates

$$|F(x) - \Psi(x)| \le c\omega_3(F, \rho_n(x)), \quad x \in [-1,1],$$

where $\rho_n(x) := \frac{1}{n^2} + \frac{\sqrt{1-x^2}}{n}$ and c is an absolute constant, both with Ψ, a 3-monotone quadratic spline on the nth Chebyshev partition, and with Ψ, a 3-monotone polynomial of degree $\le n$.

The basis for the construction of these splines and polynomials is the construction of 3-monotone splines, providing appropriate order of pointwise approximation, half of which nodes are prescribed and the other half are free, but "controlled".

1. Introduction and historical background

In recent years there has been much interest and there have been quite a few achievements in questions concerning the degree of approximation of a continuous function f, on a finite interval, which has a certain shape, by algebraic polynomials and by piecewise polynomials possessing the same shape. By shape we mean nonnegativity, monotonicity, convexity and higher order monotonicity (q-monotonicity), and finitely many changes in one of the above shapes in the interval (e.g., f may be nondecreasing and nonincreasing, alternately, or f may be convex and concave, alternately, finitely many times). Estimates on the degree of approximation are either given in the uniform norm, usually involving various moduli of smoothness of f or its derivatives (provided they exist), or are pointwise estimates. Much is known about the degree of positive, monotone and convex approximation and a lot is known on the degree of q-monotone approximation where $q \ge 4$ (mostly negative results), but relatively little is known about the degree of 3-monotone approximation. The interested reader can find details in the recent survey [7].

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We begin with the basic notions and the known results on 3-monotone approximation.

Let $n \in \mathbb{N}$. Throughout the paper $x_j := \cos\frac{j\pi}{n}$, $j = 0, \dots, n$, will denote the Chebyshev knots, and $-1 = x_n < x_{n-1} < \cdots < x_1 < x_0 = 1$, the Chebyshev partition. Set $I_j := [x_j, x_{j-1}]$, $j = 1, \dots, n$, and $|I_j| = x_{j-1} - x_j$. Finally, for $x \in [-1, 1]$, let

$$\rho_n(x) := \frac{1}{n^2} + \frac{\sqrt{1-x^2}}{n}.$$

Let $\mathcal{P}_n$ denote the space of algebraic polynomials of degree $< n$. Denote by $\Delta^3 = \Delta^3[-1, 1]$ the set of 3-monotone continuous functions on $[-1, 1]$, i.e., $f \in \Delta^3$, if $f \in C[-1, 1]$ and f' exists and is convex in $(-1, 1)$. For $f \in \Delta^3$ we denote the degree of 3-monotone polynomial approximation by

$$E_n^{(3)}(f) := \inf_{P_n \in \mathcal{P}_n \cap \Delta^3} \|f - P_n\|,$$

where the norm is the uniform norm on $[-1, 1]$.

It was proved by Beatson [1] (for $k = 1$), Shvedov [12] (for $k = 2$), and Bondarenko [2] (for $k = 3$), that

$$E_n^{(3)}(f) \le c\omega_k(f, 1/n), \quad n \ge N, \tag{1.1}$$

where c is an absolute constant, independent of f and n, and $N = k$ for $k = 1$, 2 and 3, respectively.

We remind the reader that for $g \in C[-1, 1]$ and $k \ge 1$,

$$\omega_k(g, \delta) := \sup_{|h| \le \delta} \|\Delta_h^k(g, \cdot)\|, \quad \delta > 0, \tag{1.2}$$

where

$$\Delta_h^k(g, x) := \begin{cases} \sum_{i=0}^k \binom{k}{i}(-1)^{k-i} g(x + ih), & \text{if } \ [x, x + kh] \subseteq [-1, 1] \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

Furthermore, Shvedov [13] proved that for $k > 4$, (1.1) cannot be had with $c = c(k)$ and $N = N(k)$ (constants which depend on k), and Wu and Zhou [14] proved that for $k > 5$, (1.1) cannot be had even with $c = c(f)$ and $N = N(f)$. Still nothing is known for $k = 4$.

In the case of 3-monotone piecewise polynomial approximation we shall limit ourselves to the uniform and the Chebyshev partition of $[-1, 1]$. The first estimate is due to Konovalov and Leviatan [5], who proved that given $f \in \Delta^3 \cap C^2[-1, 1]$, there exists a quadratic spline $S \in \Delta^3$, with n equidistant nodes in $[-1, 1]$ (the uniform partition), such that

$$\|f - S\| \le \frac{c}{n^2}\omega_1(f'', 1/n), \quad n \ge 1,$$

where c is an absolute constant. This was extended by Prymak [11] who proved for $f \in \Delta^3$, the existence of a piecewise quadratic $S \in \Delta^3$, with n equidistant nodes in $[-1,1]$, such that

$$\|f-S\| \le c\omega_3(f,1/n), \quad n \ge 1.$$

(In fact Prymak [11] has obtained estimates involving the third modulus of smoothness of f for the approximation by 3-monotone piecewise quadratics on an arbitrary partition of $[-1,1]$.)

In 2005 Leviatan and Prymak [10] proved that most of the expected Jackson type norm estimates are valid for 3-monotone piecewise polynomial approximation. Namely, given $f \in \Delta^3 \cap C^r[-1,1]$, where either $r \ge 3$ or $r = 1,2$ and $k = 4-r$, there exist piecewise polynomials $S_1, S_2 \in \Delta^3$, of degree $\le k+r-1$, such that S_1 has n equidistant nodes and S_2 has nodes on the Chebyshev partition, and which satisfy

$$\|f-S_1\| \le \frac{c(k,r)}{n^r}\omega_k(f^{(r)},1/n),$$

and

$$\|f-S_2\| \le \frac{c(k,r)}{n^r}\omega_k^{\varphi}(f^{(r)},1/n),$$

where ω_k^{φ} is the kth Ditzian-Totik (D-T) modulus of smoothness. Namely, for $g \in C[-1,1]$ and $k \ge 1$,

$$\omega_k^{\varphi}(g,\delta) := \sup_{0<h\le\delta} \|\Delta_{h\varphi(\cdot)}^k(g,\cdot)\|, \quad \delta > 0,$$

where Δ_h^k is defined in (1.3) and $\varphi(x) := \sqrt{1-x^2}$, $x \in [-1,1]$.

Recently Dzyubenko, Kopotun and Prymak [4] have closed the gap by proving the only remaining open case, $k = 4$ and $r = 0$, namely, there exist splines $S_1, S_2 \in \Delta^3$, of degree ≥ 3, such that S_1 has n equidistant nodes and S_2 has nodes on the Chebyshev partition, and which satisfy

$$\|f-S_1\| \le c\omega_4(f,1/n),$$

and

$$\|f-S_2\| \le c\omega_4^{\varphi}(f,1/n).$$

Since the purpose of this paper is to establish pointwise estimates involving the third modulus of smoothness for 3-monotone approximation of $f \in \Delta^3$ by both 3-monotone polynomials and quadratic splines on the Chebyshev partition, it is worthwhile mentioning the negative result of Bondarenko and Gilewicz [3], who proved that for $r > 4$, there exists a constant $c = c(r) > 0$, such that for each $n \in \mathbb{N}$, there is an $f = f_n \in \Delta^3 \cap C^r[-1,1]$, $\|f^{(r)}\| \le 1$, such that for every polynomial $P_n \in \mathcal{P}_n \cap \Delta^3$, there is an $x \in [-1,1]$ for which

$$|f_n(x) - P_n(x)| > c\sqrt{n}\rho_n^r(x).$$

Note that while for monotone and convex approximation by polynomials we cannot have estimates involving the third and fourth moduli of smoothness (of the function), respectively, we do have estimates involving higher moduli of the derivatives, provided they exist. The above mentioned negative result shows us that we cannot expect similar results for pointwise 3-monotone approximation, at least not for $r > 4$.

2. The main results

As mentioned above we have the interval $[-1, 1]$ and the Chebyshev partition $-1 = x_n < x_{n-1} < \cdots < x_1 < x_0 = 1$. When we refer to an arbitrary partition of an arbitrary interval $[a, b]$, we will use the notation $a =: \tau_n < \tau_{n-1} < \cdots < \tau_1 < \tau_0 := b$, and we will denote by $\Delta^3[\tau_n, \tau_0]$, the 3-monotone continuous functions on $[\tau_n, \tau_0]$. We will also need the notation $\Delta^2(\tau_n, \tau_0)$, for the set of all convex continuous functions on (τ_n, τ_0).

Theorem 1. *Let $\tau_n < \cdots < \tau_1 < \tau_0$ be given and let $F \in \Delta^3[\tau_n, \tau_0]$ be a function with a derivative $f := F' \in \Delta^2(\tau_n, \tau_0)$. Suppose, that $s \in \Delta^2(\tau_n, \tau_0)$ is a piecewise polynomial of order k (degree $k - 1$) with nodes $\tau_n, \dots, \tau_1, \tau_0$, satisfying*

$$s(\tau_i) = f(\tau_i), \quad i = 0, \dots, n,$$
$$s'(\tau_i+) \ge f'(\tau_i+), \quad i = 1, \dots, n,$$
$$f'(\tau_i-) \ge s'(\tau_i-), \quad i = 0, \dots, n-1.$$

Then, there are at most n additional nodes $\theta_n, \dots, \theta_1$, such that $\tau_n < \theta_n < \tau_{n-1} < \theta_{n-1} < \tau_{n-2} < \cdots < \theta_1 < \tau_0$, and a piecewise polynomial $S \in \Delta^3[\tau_n, \tau_0]$ of order $k + 1$ with the nodes $\tau_n, \theta_n, \tau_{n-1}, \dots, \theta_1, \tau_0$, satisfying

$$\|F - S\|_{C[\tau_i, \tau_{i-1}]} \le 2 \left\| \int_{\tau_i}^{(\cdot)} (f(x) - s(x))\, dx \right\|_{C[\tau_i, \tau_{i-1}]}, \quad i = 1, \dots, n, \tag{2.1}$$

and such that

$$F(\tau_i) = S(\tau_i), \quad i = 0, \dots, n. \tag{2.2}$$

We are now able to state the pointwise estimates for 3-monotone approximation.

We begin with the splines.

Theorem 2. *For each function $F \in \Delta^3$ and every $n \ge 1$, there exists a quadratic spline $S \in \Delta^3$ on the Chebyshev partition $-1 = x_n < \cdots < x_1 < x_0 = 1$, satisfying*

$$|F(x) - S(x)| \le c\, \omega_3(F, \rho_n(x)), \quad x \in [-1, 1], \tag{2.3}$$

where c is an absolute constant.

For the polynomials we have,

Theorem 3. *For each function $F \in \Delta^3$ and every $n \ge 2$, there exists a polynomial $P_n \in \Delta^3$ of degree $\le n$, satisfying*

$$|F(x) - P_n(x)| \le c\, \omega_3(F, \rho_n(x)), \quad x \in [-1, 1], \tag{2.4}$$

where c is an absolute constant.

The next section is devoted to the proof of Theorem 1. Then we need quite a few lemmas, in Sections 4 and 5, before we are able to prove Theorem 2 in Section 6. Finally, in Section 7, we replace the 3-monotone quadratic spline we construct in Section 6, by a 3-monotone polynomial.

In the sequel c denotes a generic constant which may differ at each occurrence.

3. Splines with controlled nodes

Theorem 1 is an easy consequence of the following lemma, which is a modification of Lemma 1 from [10].

Lemma 1. *Let f, g, f_1, f_2 be continuous functions on $[a,b]$, and such that*

$$f_1(x) \le f(x), g(x) \le f_2(x), \quad x \in [a,b].$$

Then, there are coefficients $\alpha, \alpha_1, \alpha_2 \ge 0$, with $\alpha + \alpha_1 + \alpha_2 = 1$, such that

$$h(x) := \alpha g(x) + \alpha_1 f_1(x) + \alpha_2 f_2(x), \quad x \in [a,b]$$

satisfies

$$\int_a^b f(x)\,dx = \int_a^b h(x)\,dx,$$

$$\left\| \int_a^{(\cdot)} (h(x) - f(x))\,dx \right\|_{C[a,b]} \le 2 \left\| \int_a^{(\cdot)} (g(x) - f(x))\,dx \right\|_{C[a,b]}.$$

Proof. If $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$, then take $h(x) := g(x)$, $x \in [a,b]$, namely, $\alpha = 1$ and $\alpha_1 = \alpha_2 = 0$, and there is nothing to prove. Otherwise, if $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$, then we apply the arguments of proof of Lemma 1 in [10] with g replacing q and f_2 replacing l. The resulting function is the convex combination of g and f_2, namely, $\alpha_1 := 0$, and α, α_2 are defined by the corresponding formula from [10]. On the other hand, if $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$, then we apply similar arguments, which we detail here and which will serve also as a reminder of the proof in [10]. Thus, assume that $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$, and denote

$$\int_a^b \big(g(x) - f(x)\big)\,dx =: A > 0,$$

and

$$\int_a^b \big(f(x) - f_1(x)\big)\,dx =: B \ge 0.$$

Set

$$h(x) := \frac{A f_1(x) + B g(x)}{A + B}.$$

Now,

$$\begin{aligned}\left|\int_a^x \big(h(t)-f(t)\big)\,dt\right| &\le \frac{A}{A+B}\left|\int_a^x \big(f_1(t)-f(t)\big)\,dt\right| + \frac{B}{A+B}\left|\int_a^x \big(g(t)-f(t)\big)\,dt\right| \\ &\le \frac{AB}{A+B} + \frac{B}{A+B}\left\|\int_a^x \big(g(t)-f(t)\big)\,dt\right\|_{C[a,b]} \\ &\le \frac{2B}{A+B}\left\|\int_a^x \big(g(t)-f(t)\big)\,dt\right\|_{C[a,b]} \\ &\le 2\left\|\int_a^x \big(g(t)-f(t)\big)\,dt\right\|_{C[a,b]}.\end{aligned}$$

Finally,

$$\begin{aligned}\int_a^b h(x)\,dx &= \frac{A}{A+B}\int_a^b f_1(x)\,dx + \frac{B}{A+B}\int_a^b g(x)\,dx \\ &= \frac{A}{A+B}(B + \int_a^b f_1(x)\,dx) + \frac{B}{A+B}(-A + \int_a^b g(x)\,dx) \\ &= \int_a^b f(x)\,dx.\end{aligned}$$

This concludes the proof. □

Proof of Theorem 1. Let $i = 1, \dots, n$ be fixed. Put

$$f_1(x) := \max\{f'(\tau_i+)(x-\tau_i)+f(\tau_i), f'(\tau_{i-1}-)(x-\tau_{i-1})+f(\tau_{i-1})\}, \quad x \in [\tau_i, \tau_{i-1}],$$

and

$$f_2(x) := \frac{f(\tau_i)(x-\tau_{i-1})}{\tau_i - \tau_{i-1}} + \frac{f(\tau_{i-1})(x-\tau_i)}{\tau_{i-1}-\tau_i}, \quad x \in [\tau_i, \tau_{i-1}].$$

Then f_2 is a linear function, and f_1 is a piecewise-linear function with one node $\theta_i \in (\tau_i, \tau_{i-1})$. Moreover, the construction of f_1 and f_2 and well known properties of convex functions yield that if $\tilde f$ is a convex function on $[\tau_i, \tau_{i-1}]$, satisfying

$$\tilde f(\tau_i) = f(\tau_i), \quad \tilde f(\tau_{i-1}) = f(\tau_{i-1}), \quad \tilde f'(\tau_i+) \ge f'(\tau_i+) \text{ and } \tilde f'(\tau_{i-1}-) \le f'(\tau_{i-1}-),$$

then

$$f_1(x) \le \tilde f(x) \le f_2(x), \quad x \in [\tau_i, \tau_{i-1}].$$

Hence,

$$f_1(x) \le f(x) \le f_2(x), \quad x \in [\tau_i, \tau_{i-1}],$$

and

$$f_1(x) \le s(x) \le f_2(x), \quad x \in [\tau_i, \tau_{i-1}].$$

By virtue of Lemma 1, we have a function

$$h_i(x) := \alpha s(x) + \alpha_1 f_1(x) + \alpha_2 f_2(x), \quad x \in [\tau_i, \tau_{i-1}],$$

such that $\alpha, \alpha_1, \alpha_2 \geq 0$, and $\alpha + \alpha_1 + \alpha_2 = 1$, which is a convex piecewise polynomial of order k with at most one node $\theta_i \in (\tau_i, \tau_{i-1})$, and satisfies

$$\int_{\tau_i}^{\tau_{i-1}} h_i(x)\,dx = \int_{\tau_i}^{\tau_{i-1}} f(x)\,dx, \tag{3.1}$$

and

$$\left\| \int_{\tau_i}^{(\cdot)} (h_i(x) - f(x))\,dx \right\|_{C[\tau_i,\tau_{i-1}]} \leq 2 \left\| \int_{\tau_i}^{(\cdot)} (s(x) - f(x))\,dx \right\|_{C[\tau_i,\tau_{i-1}]}. \tag{3.2}$$

Note, that the construction of h_i gives

$$h_i(\tau_i) = h_{i+1}(\tau_i), \quad i = 1, \dots, n-1,$$

and

$$h'_{i+1}(\tau_i) \leq f'(\tau_i-) \leq f'(\tau_i+) \leq h'_i(\tau_i), \quad i = 1, \dots, n-1,$$

so that the function

$$h(x) := h_i(x), \quad x \in [\tau_i, \tau_{i-1}], \quad i = 1, \dots, n,$$

is a piecewise polynomial of order k with the nodes $\tau_n, \dots, \tau_0$ and, perhaps, some additional nodes (with at most one node $\theta_i \in (\tau_i, \tau_{i-1})$, $i = 1, \dots, n$), moreover $h \in \Delta^2[\tau_n, \tau_0]$.

Finally let

$$S(x) := \int_{\tau_n}^{x} h(t)\,dt + F(\tau_n), \quad x \in [\tau_n, \tau_0].$$

Then, (2.2) readily follows by (3.1), whence, in turn, (2.1) follows by virtue of (3.2). This completes the proof. □

4. A fundamental lemma

We will need the following relations between the lengths of the various intervals I_j, and between these lengths and $\rho_n(x)$, $x \in I_j$. The following relations are well known (see, e.g., [8, (1.2) and (1.3)]).

$$\rho_n(x) < |I_j| < 5\rho_n(x), \quad x_j \leq x \leq x_{j-1}, \quad j = 1, \dots, n, \tag{4.1}$$

so that, in particular

$$\rho_n(x_{j-1}) < 5\rho_n(x_j), \quad j = 1, \dots, n. \tag{4.2}$$

Also

$$|I_{j\pm1}| < 3|I_j|, \quad j = 1, \dots, n, \tag{4.3}$$

where we put $|I_{n+1}| = |I_0| = 0$, and it is easy to see that for $j > i$,

$$\max\{|I_j|, |I_i|\}\frac{j-i}{3\pi} \leq x_i - x_j. \tag{4.4}$$

Finally, for all $x \in [-1, 1]$ and every $1 \leq j \leq n-1$

$$\rho_n^2(x) \leq c\rho_n(x_j)(\rho_n(x_j) + |x - x_j|) \tag{4.5}$$

and, symmetrically,

$$\rho_n^2(x_j) \le c\rho_n(x)(\rho_n(x) + |x - x_j|), \tag{4.6}$$

where c is an absolute constant.

Lemma 2. *Given numbers $\alpha_j \in [0, 1]$, $j = 1, \dots, M$. If*

$$\sum_{j=1}^{M} \alpha_j \ge S, \tag{4.7}$$

then

$$\sum_{j=1}^{M} j\alpha_j \ge \sum_{j=1}^{S} j. \tag{4.8}$$

Proof. Since $\alpha_j \le 1$, $j = 1, \dots, M$, it follows from (4.7) that for each $K = 1, \dots, S$ that

$$\sum_{j=K}^{M} \alpha_j \ge S - K + 1.$$

Thus, adding these inequalities for $K = 1, \dots, S$, we obtain

$$\alpha_1 + 2\alpha_2 + \cdots + S\alpha_S + S\alpha_{S+1} + \ldots + S\alpha_M \ge \sum_{K=1}^{S}(S - K + 1) = \sum_{j=1}^{S} j,$$

which in turn implies (4.8). □

The next lemma is a fundamental lemma in our construction.

We require the notation

$$x_+^0 = \begin{cases} 1, & x \ge 0, \\ 0, & x < 0. \end{cases}$$

Lemma 3. *Let $F \in C[-1, 1]$, and let the integers D, s and k, such that $n/2 \le s < k \le n$, be given. Assume that*

$$g(x) = \sum_{j=s}^{k} \alpha_j (x - x_j)_+^0,$$

is a step function satisfying

$$0 \le \alpha_j \le C\,\omega_3(F, \rho_n(x_j)), \quad j = s, \dots, k, \tag{4.9}$$

$$g(x_s) = g(x_s) - g(x_k-) > 200\,CD\,\omega_3(F, \rho_n(x_s)), \tag{4.10}$$

$$g(x_l) = g(x_l) - g(x_k-) \le 200\,CD\,\omega_3(F, \rho_n(x_l)), \quad l = s + 1, \dots, k, \tag{4.11}$$

for some constant $C > 0$. Then there exists a nondecreasing polygonal line

$$S(x) = \sum_{j=s}^{k} \frac{\beta_j}{|I_j|}(x - x_j)_+,$$

such that

$$|\beta_j| \le \frac{\alpha_j}{D}, \tag{4.12}$$

$$S(x) = g(x), \quad x \in [-1,1] \setminus [x_k, x_s], \tag{4.13}$$

$$|g(x) - S(x)| \le 402\, CD\, \omega_3(F, \rho_n(x)), \quad x \in [x_k, x_s]. \tag{4.14}$$

Remark 1. Note that in view of (4.9) and (4.10), $F \notin \mathcal{P}_3$.

Proof. Note that $x_s \le 0$, so that $\rho_n(x_s) > \rho_n(x_l)$, $l = s+1, \dots, k$, and let u, $s \le u \le k$, be the largest integer such that

$$\omega_3(F, \rho_n(x_u)) \ge \frac{1}{2}\omega_3(F, \rho_n(x_s)). \tag{4.15}$$

Then, by (4.11),

$$\sum_{j=u+1}^{k} \alpha_j \le 200\, CD\, \omega_3(F, \rho_n(x_{u+1})) \le 100\, CD\, \omega_3(F, \rho_n(x_s)).$$

Hence,

$$\sum_{j=s}^{u} \alpha_j \ge 100\, CD\, \omega_3(F, \rho_n(x_s)).$$

Denote by v, $s \le v \le u$, the largest integer such that

$$\sum_{j=v+1}^{u} \alpha_j \ge 65\, CD\, \omega_3(F, \rho_n(x_s)).$$

Put

$$p := \sum_{j=v+1}^{u} \frac{\alpha_j}{|I_j|}, \quad q := \sum_{j=s}^{v} \frac{\alpha_j}{|I_j|}, \quad \Lambda := \frac{p}{q}.$$

Note that $q \ne 0$, since by the definition of v we have

$$\sum_{j=s}^{v+1} > 35CD\omega_3(F, \rho_n(x_s)),$$

and by (4.9), $\alpha_{v+1} \le C\omega_3(F, \rho_n(x_{v+1})) \le C\omega_3(F, \rho_n(x_s))$, whence $\sum_{j=s}^{v} \alpha_j > 0$.

If $\Lambda \le 1$, then we put

$$\hat{\beta}_j := \begin{cases} \frac{-\Lambda \alpha_j}{D}, & j = s, \dots, v, \\ \frac{\alpha_j}{D}, & j = v+1, \dots, u. \end{cases}$$

Otherwise, $\Lambda > 1$, and we set

$$\hat\beta_j := \begin{cases} \frac{-\alpha_j}{D}, & j = s, \dots, v, \\ \frac{\alpha_j}{\Lambda D}, & j = v+1, \dots, u. \end{cases}$$

Denote

$$\beta_j := \hat\beta_j \frac{g(x_s)}{\sum_{i=s}^u \frac{\hat\beta_i}{|I_i|}(x_s - x_i)}, \quad j = s, \dots, u, \tag{4.16}$$

and, finally, put $\beta_j := \hat\beta_j := 0$, for $j = u+1, \dots, k$. We will show that the polygonal line

$$S(x) = \sum_{j=s}^k \frac{\beta_j}{|I_j|}(x - x_j)_+$$

is the required one. To this end, evidently,

$$\sum_{j=t}^k \frac{\hat\beta_j}{|I_j|} \ge 0, \quad t = s+1, \dots, k, \tag{4.17}$$

$$\sum_{j=s}^k \frac{\hat\beta_j}{|I_j|} = 0, \tag{4.18}$$

and

$$|\hat\beta_j| \le \frac{\alpha_j}{D}. \tag{4.19}$$

We will prove that

$$\sum_{j=s}^k \frac{\hat\beta_j}{|I_j|}(x_s - x_j) > g(x_s) = \sum_{j=s}^k \alpha_j. \tag{4.20}$$

Indeed, for $\Lambda \le 1$, by (4.18), (4.4), and Lemma 2, we have

$$\begin{aligned} \sum_{j=s}^k \frac{\hat\beta_j}{|I_j|}(x_s - x_j) &= \sum_{j=s}^u \frac{\hat\beta_j}{|I_j|}(x_v - x_j) \ge \frac{1}{D} \sum_{j=v+1}^u \frac{\alpha_j}{|I_j|}(x_v - x_j) \\ &\ge \frac{1}{3\pi\, D} \sum_{j=v+1}^u \alpha_j (j - v) = \frac{1}{3\pi\, D} \sum_{j=1}^{u-v} j \alpha_j^* \\ &= \frac{C\,\omega_3(F, \rho_n(x_s))}{3\pi\, D} \sum_{j=1}^{u-v} j \frac{\alpha_j^*}{C\,\omega_3(F, \rho_n(x_s))} \ge \frac{C\,\omega_3(F, \rho_n(x_s))}{3\pi\, D} \sum_{j=1}^{65\,D} j \\ &= \frac{65\,D(65\,D+1)C\,\omega_3(F, \rho_n(x_s))}{6\pi D} > (200\,D+1)C\,\omega_3(F, \rho_n(x_s)), \end{aligned}$$

where $\alpha_j^* := \alpha_{j+v}$, and we used the facts that $0 \le \alpha_j^* \le C\,\omega_3(F, \rho_n(x_s))$, and

$$\sum_{j=1}^{u-v} \alpha_j^* = \sum_{j=v+1}^{u} \alpha_j \ge 65\,DC\,\omega_3(F, \rho_n(x_s)).$$

Similarly, if $\Lambda > 1$, then we have

$$\begin{aligned}\sum_{j=s}^{k} \frac{\hat\beta_j}{|I_j|}(x_s - x_j) &= \sum_{j=s}^{u} \frac{\hat\beta_j}{|I_j|}(x_v - x_j)\\ &\ge \frac{1}{D}\sum_{j=s}^{v} \frac{\alpha_j}{|I_j|}(x_j - x_v)\\ &> (200\,D+1)C\,\omega_3(F, \rho_n(x_s)).\end{aligned}$$

On the other hand, the inequalities (4.9) and (4.11) imply

$$g(x_s) \le (200\,D+1)C\,\omega_3(F, \rho_n(x_s)). \tag{4.21}$$

Hence, (4.20) is proven. Now, (4.12) follows by (4.19) and (4.20), and the definition of β_j, (4.16). Also, (4.16) and (4.17) imply that S is non-decreasing, that $S(x_s) = g(x_s)$, and by virtue of (4.18), we get (4.13). Further, since $S(x) = 0$, $x \le x_u$, (4.11) and (4.21) imply (4.14) for $x \in [x_k, x_u]$, where we note that by (4.11), if $x_l \le x < x_{l-1}$, $l = u, \ldots, k$, then

$$g(x) = g(x_l) \le 200\,CD\omega_3(F, \rho_n(x_l)) \le 200\,CD\omega_3(F, \rho_n(x)).$$

Finally, for $x \in [x_u, x_s]$

$$|S(x) - g(x)| \le g(x_s) - S(x_u) = g(x_s) \le (200\,D+1)C\,\omega_3(F, \rho_n(x_s)),$$

by (4.21). Now (4.15) implies (4.14) for $x \in [x_u, x_s]$. This completes the proof. □

Remark 2. Lemma 3 is stated for the interval $[-1, 0]$. The situation is completely symmetric for the interval $[0, 1]$ (one only has to take a mirror image of the conditions, this time with $1 \le k < s \le n/2$). We leave the statement and proof to the reader. (See also Remark 3 below.)

Lemma 3 is the main tool we use in the proof of Lemma 7 below. However, in that proof we may encounter a case where the conditions of Lemma 3 are not satisfied and we need to apply another tool. This is the purpose of the following observation.

Lemma 4. *Let $\lceil n/2 \rceil \le k < n$, and assume that the nonnegative numbers α_j are such that*

$$\sum_{j=s}^{k} \alpha_j \le c\omega_3(F, \rho_n(x_s)), \quad s = \lceil n/2 \rceil, \ldots, k. \tag{4.22}$$

Then for $x_l \le x < x_{l-1}$, $l = \lceil n/2 \rceil, \dots, k-1$, *and for* $-1 \le x < x_{k-1}$, *for* $l = k$, *we have*

$$\sum_{j=\lceil n/2 \rceil}^{l} \alpha_j \Big(\frac{1+x}{1+x_j} \Big)^2 \le \tilde{c}\omega_3(F, \rho_n(x)). \tag{4.23}$$

Proof. Denote $m := \lceil n/2 \rceil$ and let either $x_l \le x < x_{l-1}$, $l = \lceil n/2 \rceil, \dots, k-1$, or $-1 \le x < x_{k-1}$, for $l = k$. We rewrite the left hand side of (4.23) using summation by parts.

$$\sum_{j=m}^{l} \alpha_j \Big(\frac{1+x}{1+x_j} \Big)^2 = \sum_{j=m}^{l} \alpha_j \Big(\frac{1+x}{1+x_m} \Big)^2 + \sum_{s=m+1}^{l} \sum_{j=s}^{l} \alpha_j (1+x)^2 \left[\frac{1}{(1+x_s)^2} - \frac{1}{(1+x_{s-1})^2} \right].$$

By virtue of (4.22) we obtain,

$$\sum_{j=m}^{l} \alpha_j \Big(\frac{1+x}{1+x_j} \Big)^2 \le c\omega_3(F, \rho_n(x_m)) \Big(\frac{1+x}{1+x_m} \Big)^2$$

$$+ c \sum_{s=m+1}^{l} \omega_3(F, \rho_n(x_s)) \left[\Big(\frac{1+x}{1+x_s} \Big)^2 - \Big(\frac{1+x}{1+x_{s-1}} \Big)^2 \right] =: I_1 + I_2.$$

Now, for $m \le s \le l$, it follows that

$$\frac{\rho_n^3(x_s)}{\rho_n^3(x)} \Big(\frac{1+x}{1+x_s} \Big)^{3/2} \le 64, \tag{4.24}$$

where we recall that $x < x_{l-1}$.

Observe that

$$\frac{1}{(1+x_s)^2} - \frac{1}{(1+x_{s-1})^2} \le c \frac{|I_s|}{(1+x_s)^3},$$

so that by (4.24),

$$\begin{aligned} I_2 &\le c \frac{\omega_3(F, \rho_n(x))}{\rho_n^3(x)} (1+x)^2 \sum_{s=m+1}^{l} \rho_n^3(x_s) \frac{|I_s|}{(1+x_s)^3} \\ &\le c\omega_3(F, \rho_n(x)) \sum_{s=m+1}^{l} \frac{|I_s|}{(1+x_s)^{3/2}} \\ &\le c\omega_3(F, \rho_n(x)) (1+x)^{1/2} \int_{x_l}^{x_m} (1+t)^{-3/2}\, dt \\ &\le c\omega_3(F, \rho_n(x)) \Big(\frac{1+x}{1+x_l} \Big)^{1/2}. \end{aligned} \tag{4.25}$$

Also,

$$I_1 \le c \frac{\rho_n^3(x_m)}{\rho_n^3(x)} \omega_3(F, \rho_n(x)) \Big(\frac{1+x}{1+x_m} \Big)^2 \le c\omega_3(F, \rho_n(x)). \tag{4.26}$$

Combining (4.25) and (4.26), the proof is complete. □

Remark 3. Lemma 4 is stated for the interval $[-1,0]$, but we need it also for the interval $[0,1]$. However, unlike Lemma 3 which was translated, practically verbatim, to $[0,1]$, a similar translation of Lemma 4 to $[0,1]$ is not helpful for the estimates on the approximation by the splines with nodes at the Chebyshev knots, due to the non symmetry of the truncated powers. Rather we will have to modify it. One should note that it is also possible to apply this modification in order to translate Lemma 3 to the interval $[0,1]$. (See details in the last part of the proof of Lemma 7 below.) Nevertheless, we need the translation of Lemma 4 to $[0,1]$ for the estimates on the polynomial approximation, but we defer the statement for further preparations (see Lemma 11).

5. Auxiliary lemmas

We begin with a lemma.

Lemma 5. *Let* $\theta\in(x_j,x_{j-1})$, $N+1\le j\le n-N$. *For any* γ, $|\gamma|<\frac{1}{3}|I_j|$, *there are nonnegative numbers* η_j,μ_j,ν_j *such that*

(5.1)
$$\eta_j(x-x_{j+N})^2+\mu_j(x-x_{j+1})^2+\nu_j(x-x_{j-2})^2=(x-\theta)^2+h_j^2+\gamma(x-x_{j-2}),\quad x\in\mathbb{R},$$

holds with $N\ge1900$ *and* $h_j=7|I_j|$.

Proof. Comparing the coefficients of the various powers of x on both sides of the equation, we observe that (5.1) is equivalent to the system of three linear equations

$$\begin{cases}\eta_j+\mu_j+\nu_j &=1,\\ \eta_jx_{j+N}+\mu_jx_{j+1}+\nu_jx_{j-2} &=\theta-\dfrac{\gamma}{2},\\ \eta_jx_{j+N}^2+\mu_jx_{j+1}^2+\nu_jx_{j-2}^2 &=\theta^2+h_j^2-\gamma x_{j-2}.\end{cases}$$

The solution to the latter is given by

$$\eta_j=\frac{\Delta_{\eta,j}}{\Delta_j},\quad \mu_j=\frac{\Delta_{\mu,j}}{\Delta_j},\quad \nu_j=\frac{\Delta_{\nu,j}}{\Delta_j},$$

where by straightforward computations we have,

$$\Delta:=(x_{j-2}-x_{j+1})(x_{j-2}-x_{j+N})(x_{j+1}-x_{j+N})>0,$$
$$\Delta_{\eta,j}:=(x_{j-2}-x_{j+1})\Big((\theta-x_{j+1})^2+h_j^2-(x_{j-2}-x_{j+1})\Big(\theta+\frac{\gamma}{2}-x_{j+1}\Big)\Big),$$
$$\Delta_{\mu,j}:=(x_{j-2}-x_{j+N})\Big((x_{j-2}-x_{j+N})\Big(\frac{\gamma}{2}+x_{j-2}-\theta\Big)-(\theta-x_{j-2})^2-h_j^2\Big),$$
$$\Delta_{\nu,j}:=(x_{j+1}-x_{j+N})\Big((\theta-x_{j+1})^2+h_j^2-\gamma(x_{j-2}-x_{j+1})$$
$$+(x_{j+1}-x_{j+N})\Big(\theta-\frac{\gamma}{2}-x_{j+1}\Big)\Big).$$

Now,

$$x_{j-2} - x_{j+1} \le 7|I_j|,$$

and

$$x_{j+1} + \frac{|I_j|}{2} \le \theta \pm \frac{\gamma}{2} \le x_{j-2} - \frac{|I_j|}{2}.$$

Hence, we obtain

$$\begin{aligned} h_j^2 - (x_{j-2} - x_{j+1})\Big(\theta + \frac{\gamma}{2} - x_{j+1}\Big) &\ge (7|I_j|)^2 - (x_{j-2} - x_{j+1})^2 \\ &\ge (7|I_j|)^2 - (7|I_j|)^2 = 0, \end{aligned}$$

and as all other terms are nonnegative, we conclude that $\Delta_{\eta,j} > 0$.

Even simpler is the inequality

$$h_j^2 - \gamma(x_{j-2} - x_{j+1}) > 49|I_j|^2 - \frac{1}{3}|I_j| \cdot 7|I_j| > 0,$$

so that, with all other terms being nonnegative, we conclude that $\Delta_{\nu,j} > 0$.

Finally by virtue of (4.3) and (4.4), we obtain

$$(x_{j+1} - x_{j+N})\Big(\frac{\gamma}{2} + x_{j-2} - \theta\Big) > \frac{N-1}{\pi}|I_{j+1}|\frac{|I_j|}{2} \ge \frac{N-1}{6\pi}|I_j|^2.$$

Hence,

$$\begin{aligned} &(x_{j+1} - x_{j+N})\Big(\frac{\gamma}{2} + x_{j-2} - \theta\Big) - (\theta - x_{j+1})^2 - h_j^2 \\ &\qquad > \frac{N-1}{6\pi}|I_j|^2 - 49|I_j|^2 - 49|I_j|^2 = \Big(\frac{N-1}{6\pi} - 98\Big)|I_j|^2 > 0, \end{aligned}$$

since we recall that $N \ge 1900$. Thus, $\Delta_{\mu,j} > 0$, and the proof is complete. □

We also need the following lemma.

Lemma 6. *For any $\theta \in (x_j, x_{j-1})$, $1 \le j \le n-1$, there exists a piecewise quadratic spline $S_\theta \in \Delta^3$ with the Chebyshev knots, such that*

$$S_\theta(x) = (x-\theta)^2, \qquad x \in (x_{j-1}, 1], \tag{5.2}$$

$$|S_\theta(x)| \le c\Big(\frac{1+x}{1+x_j}\Big)^2 |I_j|^2, \qquad x \in [-1, x_j), \tag{5.3}$$

$$|S_\theta(x) - (x-\theta)_+^2| \le c|I_j|^2, \qquad x \in [x_j, x_{j-1}]. \tag{5.4}$$

Proof. For $j = 1$ we take $S_\theta(x) := (x - x_1)_+^2$ and observe that we have a stronger inequality (5.3), namely, $S_\theta(x) = 0$, $x \in [-1, x_1)$.

Otherwise, $1 < j < n$, so take

$$S_\theta(x) := -\eta(1+x)^2 + \mu(x-x_j)_+^2 + \nu(x - x_{j-1})_+^2, \quad x \in [-1, 1],$$

where

$$\eta := \frac{(x_{j-1}-\theta)(\theta-x_j)}{(x_{j-1}+1)(x_j+1)},$$

$$\mu := \frac{(x_{j-1}-\theta)(\theta+1)}{(x_j+1)(x_{j-1}-x_j)},$$

$$\nu := \frac{(\theta-x_j)(\theta+1)}{(x_{j-1}+1)(x_{j-1}-x_j)}.$$

By definition, $\mu, \nu > 0$, so that $S_\theta \in \Delta^3$. Also, straightforward computations yield (5.2). Thus, it is left to show (5.3) and (5.4). To this end, first let $x \in [-1, x_j]$. Then

$$\begin{aligned}|S_\theta(x)| = \eta(1+x)^2 &= \frac{(x_{j-1}-\theta)(\theta-x_j)}{(x_{j-1}+1)(x_j+1)}(1+x)^2 \\ &\le \Big(\frac{1+x}{1+x_j}\Big)^2(x_{j-1}-\theta)(\theta-x_j) \\ &\le \frac{1}{4}\Big(\frac{1+x}{1+x_j}\Big)^2|I_j|^2,\end{aligned}$$

which proves (5.3). Finally, for $x \in (x_j, x_{j-1}]$, we have

$$\begin{aligned}|S_\theta(x)| &= \eta(1+x)^2 + \mu(x-x_j)^2 \\ &\le \frac{1}{4}\Big(\frac{1+x}{1+x_j}\Big)^2|I_j|^2 + \frac{\theta+1}{x_j+1}|I_j|^2 \\ &\le c|I_j|^2,\end{aligned}$$

where for the last inequality we applied (4.4) to conclude that

$$\begin{aligned}\max\Big\{\frac{1+x}{1+x_j}, \frac{1+\theta}{1+x_j}\Big\} &\le \frac{1+x_{j-1}}{1+x_j} \\ &= 1 + \frac{x_{j-1}-x_j}{x_j-x_n} \le 1+3\pi.\end{aligned}$$

This combined with the fact that

$$(x-\theta)_+^2 \le |I_j|^2, \quad x \in (x_j, x_{j-1}],$$

completes the proof of (5.4) and, thus, of the lemma. □

We apply the above lemmas to remove the unwanted θ_j's.

Lemma 7. *Suppose* $\theta_j \in (x_j, x_{j-1})$, $j = N+1, \dots, n-N-1$, *and*

$$\sigma(x) = \sum_{j=N+1}^{n-N-1} q_j (x-\theta_j)_+^2,$$

and assume the coefficients are such that for some $F \in C[-1,1]$,

$$0 \le q_j |I_j|^2 \le c\omega_3(F, \rho_n(x_j)), \quad j = N+1, \dots, n-N-1.$$

Then there is a piecewise quadratic spline $\sigma_1 \in \Delta^3$ *with nodes at the Chebyshev knots, satisfying*

$$|\sigma(x) - \sigma_1(x)| \le \tilde{c}\omega_3(F, \rho_n(x)), \quad x \in [-1,1].$$

Proof. We first point out that by virtue of Lemma 5, for each j, $N+1 \le j \le n-N-1$,

$$\begin{aligned}(x-\theta_j)_+^2 - &\big(\eta_j(x-x_{j+N})_+^2 + \mu_j(x-x_{j+1})_+^2 + \nu_j(x-x_{j-2})_+^2\big) \\ &- h_j^2(x-x_{j-2})_+^0 - \gamma_j(x-x_{j-2})_+ \\ &= \begin{cases} 0, & \text{if} \quad x > x_{j-2}, \text{ or } \quad x \le x_{j+N}, \\ R_j(x), & \text{for} \quad x_{j+N} < x \le x_{j-2}, \end{cases}\end{aligned} \tag{5.5}$$

where γ_j is to be prescribed, and

$$|R_j(x)| \le c|I_j|^2, \quad x_{j+N} < x \le x_{j-2}. \tag{5.6}$$

We split the summation in σ into two parts, the sum of the terms with $\lceil n/2 \rceil + 2 \le j \le n-N-1$, and the rest (which is treated similarly, see the last part of the proof).

Consider the sum

$$\sum_{j=\lceil n/2 \rceil+2}^{n-N-1} q_j h_j^2 (x-x_{j-2})_+^0 = \sum_{j=\lceil n/2 \rceil}^{n-N-3} q_{j+2} h_{j+2}^2 (x-x_j)_+^0.$$

Our strategy is to apply Lemma 3. Let D be taken so that

$$\frac{h_j^2}{D|I_{j-2}|} < \frac{1}{3}|I_j|, \quad j = \lceil n/2 \rceil, \dots, n-N-3. \tag{5.7}$$

We begin by setting $k_1 = n-N-3$, and we let $s_1 < k_1$ be so that $s_1 \ge \lceil n/2 \rceil$, and the conditions of Lemma 3 are satisfied for $s = s_1$, $k = k_1$, with $\alpha_j := h_{j+2}^2 q_{j+2}$. Note that $\alpha_j \le c\omega_3(F, \rho_n(x_j))$, $\lceil n/2 \rceil \le j \le n-N-3$, for some constant c. Clearly if $s_1 = \lceil n/2 \rceil$, we are done with the construction. Otherwise, set $k_2 := s_1 - 1 > \lceil n/2 \rceil$ and let s_2, $\lceil n/2 \rceil \le s_2 < k_2$, be chosen similarly, with the conditions of Lemma 3 to be satisfied. We proceed like that and let $\lceil n/2 \rceil \le s_m < k_m$ be the last pair to be chosen in this manner. We apply Lemma 3 and Lemma 5, for each pair (s_i, k_i), obtaining a piecewise linear spline

$$S(x) = \sum_{j=s_m}^{n-N-3} \frac{\beta_j}{|I_j|}(x-x_j)_+ = \sum_{j=s_m+2}^{n-N-1} \frac{\beta_{j-2}}{|I_{j-2}|}(x-x_{j-2})_+.$$

Observe that we may choose in (5.5),

$$\gamma_j = -\frac{\beta_{j-2}}{q_j|I_{j-2}|},$$

since the only requirement in Lemma 5 is that $|\gamma_j| < \frac{1}{3}|I_j|$, that is guaranteed by (5.7). Hence,

$$\sum_{j=s_m+2}^{n-N-1} \left(\frac{\beta_{j-2}}{|I_{j-2}|}(x - x_{j-2})_+ + q_j\gamma_j(x - x_{j-2})_+ \right) = 0,$$

and in view of the above construction,

$$\begin{aligned} &\left| \sum_{j=s_m+2}^{n-N-1} q_j h_j^2 (x - x_{j-2})_+^0 + \sum_{j=s_m+2}^{n-N-1} q_j\gamma_j(x - x_{j-2})_+ \right| \\ &= \left| \sum_{j=s_m+2}^{n-N-1} q_j h_j^2 (x - x_{j-2})_+^0 - \sum_{j=s_m+2}^{n-N-1} \frac{\beta_{j-2}}{|I_{j-2}|}(x - x_{j-2})_+ \right| \\ &\le c\omega_3(F, \rho_n(x)), \quad x \in [-1, 1]. \end{aligned} \tag{5.8}$$

Also, given $x_i \le x < x_{i-1}$, $N < i < n - N$, we get by (6.5) and (5.6),

$$\begin{aligned} \sum_{j=s_m+2}^{n-N-1} q_j|R_j(x)| &\le \sum_{j=i-N}^{i+1} q_j|R_j(x)| \\ &\le c \sum_{j=i-N}^{i+1} \omega(F, \rho_n(x_j)) \le c\omega_3(F, \rho_n(x)), \end{aligned} \tag{5.9}$$

where for the last inequality we have applied (4.1) and (4.2).
Thus, letting

$$S_1(x) := \sum_{j=s_m+2}^{n-N-1} q_j \big(\eta_j (x - x_{j+N})_+^2 + \mu_j (x - x_{j+1})_+^2 + \nu_j (x - x_{j-2})_+^2 \big), \tag{5.10}$$

it follows by (5.8) and (5.9) that

$$\left| \sum_{j=s_m+2}^{n-N-1} q_j (x - \theta_j)_+^2 - S_1(x) \right| \le c\omega_3(F, \rho_n(x)), \quad x \in [-1, 1]. \tag{5.11}$$

If it so happens that $s_m = \lceil n/2 \rceil$, then we are done. Otherwise, our process stops, that is, we have an index k (which may even be $k = k_1$), and we cannot find $s < k$ so that the conditions of Lemma 3 are satisfied. Namely, we have the inequalities

$$\sum_{j=s}^{k+2} q_j |I_j|^2 \le CD\,\omega_3(F, \rho_n(x_s)), \quad s = \lceil n/2 \rceil + 2, \dots, k + 2.$$

Then we go back to the original sum $\sum_{j=\lceil n/2\rceil+2}^{k+2} q_j(x-\theta_j)_+^2$, and approximate it using Lemma 6.

To this end, note that by virtue of Lemma 6, for $x_l \le x < x_{l-1}$, $\lceil n/2\rceil \le l \le k+3$, and for $-1 \le x < x_{k+1}$, $l = k+2$,

$$\sum_{j=\lceil n/2\rceil+2}^{k+2} q_j\big|(x-\theta_j)_+^2 - S_{\theta_j}(x)\big| \le \sum_{j=\lceil n/2\rceil+2}^{l} q_j|I_j|^2\Big(\frac{1+x}{1+x_j}\Big)^2$$
$$\le CD\,\omega_3(F,\rho_n(x)), \tag{5.12}$$

where for the last inequality we have applied Lemma 4.

Denoting

$$S_2(x) := \sum_{j=\lceil n/2\rceil+2}^{k+2} q_j S_{\theta_j}(x),$$

and setting

$$S := S_1 + S_2,$$

we conclude that $S \in \Delta^3$, and it follows by (5.11) and (5.12) that,

$$\Big|\sum_{j=\lceil n/2\rceil+2}^{n-N-1} q_j(x-\theta_j)_+^2 - S(x)\Big| \le CD\,\omega_3(F,\rho_n(x)), \quad x\in[-1,1]. \tag{5.13}$$

As mentioned at the beginning of the proof, we construct a similar 3-monotone piecewise quadratic spline with nodes at the Chebyshev knots, approximating

$$\sum_{j=N+1}^{\lceil n/2\rceil+1} q_j(x-\theta_j)_+^2.$$

First we apply the construction of Lemma 3, see Remark 2. However, again we may have an index $k \ge N+1$ such that

$$\sum_{j=k+2}^{s} q_j|I_j|^2 \le CD\,\omega_3(F,\rho_n(x_s)), \quad s = k+2,\dots,\lceil n/2\rceil+1.$$

Thus, we need to apply an analogue of Lemma 4.

To this end, we observe that

$$(x-t)_+^2 = (x-t)^2 - (-x+t)_+^2. \tag{5.14}$$

Hence, substituting $y := -x$ and $\tau_j := -\theta_j$,

$$\sum_{j=k+2}^{\lceil n/2\rceil+1} q_j(x-\theta_j)_+^2 = \sum_{j=k+2}^{\lceil n/2\rceil+1} q_j(x-\theta_j)^2 - \sum_{j=k+2}^{\lceil n/2\rceil+1} q_j(-x+\theta_j)_+^2$$
$$=: P(x) - \sum_{j=k+2}^{\lceil n/2\rceil+1} q_j(y-\tau_j)_+^2.$$

Note that $P(x)$ is a quadratic polynomial. Denote $y_j := -x_j$. Then $\tau_j \in (y_{j-1}, y_j) \subset [-1, 0]$, except for $y_{\lceil n/2\rceil+1}$ and, perhaps, $y_{\lceil n/2\rceil}$ (the latter, only if n is odd), but this requires no significant modification in the proof of Lemma 4. Thus, by Lemmas 4 and 6, there exists a quadratic spline $\hat{S}(y) \in \Delta^3$ such that

$$\Big|\sum_{j=k+2}^{\lceil n/2\rceil+1} q_j(y-\tau_j))_+^2 - \hat{S}(y)\Big| \le CD\,\omega_3(F, \rho_n(y)), \quad y \in [-1, 1],$$

which in turn implies

$$\big|\sum_{j=k+2}^{\lceil n/2\rceil+1} q_j(x-\theta_j)_+^2 - (P(x) - \hat{S}(-x))\big| \le CD\,\omega_3(F, \rho_n(x)), \quad x \in [-1, 1].$$

Finally, we observe that $P(x) - \hat{S}(-x) \in \Delta^3$. This completes the proof. □

6. Quadratic spline with nodes at the Chebyshev knots

We are ready to prove Theorem 2.

Proof of Theorem 2. Given $F \in \Delta^3$, the function $f := F' \in C(-1, 1)$, is convex. Let $s(x)$ denote the piecewise linear interpolant of f on the Chebyshev knots $x_{n-1} < \cdots < x_1$. Then, it readily follows that s is convex and the requirements of Theorem 1 are satisfied in $[x_{n-1}, x_1]$. It was proved in [11, Lemma 3] that

$$\int_{x_i}^{x_{i-1}} |f(t) - s(t)|\,dt \le c\omega_3(F, (x_{i-2} - x_{i+1})/3; [x_{i+1}, x_{i-2}]), \quad 2 \le i \le n-1.$$

Hence, by Theorem 1 (2.1), we obtain a piecewise quadratic $S \in \Delta^3[x_{n-1}, x_1]$ satisfying

$$|F(x) - S(x)| \le c\,\omega_3(F, \rho_n(x)), \quad x \in [x_{n-1}, x_1], \tag{6.1}$$

where we used (4.1), (4.2) and (4.3).

However, note that S may have nodes not only at the Chebyshev knots but, perhaps, also at some $\theta_j \in (x_j, x_{j-1})$, $2 \le j \le n-1$.

We extend the definition of S to the end intervals by

$$S_{|_{[x_1,1]}} := F''(x_1-)(\cdot - x_1)^2 + F'(x_1)(\cdot - x_1) + F(x_1),$$

and

$$S_{|_{[-1,x_{n-1}]}} := F''(x_{n-1}+)(\cdot - x_{n-1})^2 + F'(x_{n-1})(\cdot - x_{n-1}) + F(x_{n-1}).$$

Again, by [11, Lemma 3]

$$\int_{x_1}^{1} |f(t) - S'(t)|\,dt \le c\omega_3(F, (1 - x_3)/3; [x_3, 1]),$$

$$\int_{-1}^{x_{n-1}} |f(t) - S'(t)|\,dt \le c\omega_3(F, (x_{n-3} + 1)/3; [-1, x_{n-3}]),$$

so that, combined with (6.1), we have

$$|F(x) - S(x)| \le c\,\omega_3(F, \rho_n(x)), \quad x \in [-1,1]. \tag{6.2}$$

Clearly, we may write

$$S(x) =: P^*(x) + \sum_{i=1}^{n-1} \alpha_i (x - x_i)_+^2 + \sum_{i=2}^{n-1} q_i (x - \theta_i)_+^2, \quad x \in [-1,1],$$

where P^* is a polynomial of degree ≤ 2, and all $\alpha_i \ge 0$, $1 \le i \le n-1$ and $q_i \ge 0$, $2 \le i \le n-1$.

We proceed to remove the terms involving θ_j, $N+1 \le j \le n-N-1$. This we do by virtue of Lemma 7, by showing that $q_j|I_j|^2 \le c\,\omega_3(F, \rho_n(x_j))$, $N+1 \le j \le n-N-1$. To this end, observe that by (1.3),

$$B_h(x) := \Delta_h^3((\cdot)_+^2, x) \ge 0, \quad x \in [-2,2],$$

and

$$\Delta_h^3((\cdot)_+^2, x) \ge h^2, \quad \text{for} \quad -2h \le x \le -h.$$

For $h = \frac{1}{7}|I_j|$, $N+2 \le j \le n-N-1$, let $x - \theta_j \in [-2h, -h]$. Then it follows that

$$\Delta_h^3(S, x) = \sum_{i=1}^{n-1} \alpha_i B_h(x - x_i) + \sum_{i=2}^{n-1} q_i B_h(x - \theta_i) \ge q_j B_h(x - \theta_j) \ge \frac{q_j}{49}|I_j|^2.$$

On the other hand, by (1.2), for all x such that $x - \theta_j \in [-2h, -h]$,

$$\Delta_h^3(S, x) \le \omega_3(S, (x_{j-2} - x_{j+1}); [x_{j+1}, x_{j-2}]) \le c\,\omega_3(F, \rho_n(x_j)),$$

where we applied (6.2), and (4.1) and (4.3).

Hence, we conclude that

$$q_j|I_j|^2 \le c\,\omega_3(F, \rho_n(x_j)), \quad j = N+1, \dots, n-N-1.$$

Therefore, by virtue of Lemma 7, we have a 3-monotone piecewise quadratic $\bar S$ with nodes at the Chebyshev knots and perhaps additional nodes at θ_i, $1 \le i \le N$ and $n - N \le i \le n$, such that

$$|S(x) - \bar S(x)| \le c\,\omega_3(F, \rho_n(x)),$$

which in turn by (6.2) implies

$$|F(x) - \bar S(x)| \le c\,\omega_3(F, \rho_n(x)), \quad x \in [-1,1]. \tag{6.3}$$

For later purposes, let $\bar S$ be represented by

$$\bar S(x) =: P^*(x) + \sum_{i=n-N}^{n-1} q_i (x - \theta_i)_+^2 + \sum_{i=1}^{n-1} \lambda_i (x - x_i)_+^2 + \sum_{i=2}^{N} q_i (x - \theta_i)_+^2, \quad x \in [-1,1], \tag{6.4}$$

where, evidently, $\lambda_i \ge 0$, $i = N, \dots, n-N-1$, and by the same proof as above (estimating q_i), we conclude that

$$\lambda_i |I_i|^2 \le c\,\omega_3(F, \rho_n(x_i)), \quad i = N, \dots, n-N-1. \tag{6.5}$$

We replace $\bar{S}$ on the intervals $[x_{N+1}, 1]$ and $[-1, x_{n-N-1}]$, by the parabolas $S_1(x) := \frac{1}{2}\bar{S}''(x_{N+1}+)(x - x_{N+1})^2 + \bar{S}'(x_{N+1})(x - x_{N+1}) + \bar{S}(x_{N+1})$ and $S_n(x) := \frac{1}{2}\bar{S}''(x_{n-N-1}-)(x - x_{n-N-1})^2 + \bar{S}'(x_{n-N-1})(x - x_{n-N-1}) + \bar{S}(x_{n-N-1})$, respectively. By virtue of [11, Lemma 3] and (6.3), we obtain

$$|\bar{S}(x) - S_1(x)| \le c\,\omega_3(F, \rho_n(x)), \quad x \in \cup_{i=1}^{N+1} I_i,$$

and

$$|\bar{S}(x) - S_n(x)| \le c\,\omega_3(F, \rho_n(x)), \quad x \in \cup_{i=n-N}^{n} I_i,$$

where, again, we have applied (4.1) and (4.3).

Denote

$$\hat{S}(x) := \begin{cases} \bar{S}(x), & x_{n-N-1} < x < x_{N+1} \\ S_1(x), & x \in \cup_{i=1}^{N+1} I_i, \\ S_n(x), & x \in \cup_{i=n-N}^{n} I_i. \end{cases}$$

Then, $\hat{S} \in \Delta^3$, is piecewise quadratic with nodes only at the Chebyshev knots. Finally, it follows by (6.3) that

$$|F(x) - \hat{S}(x)| \le c\,\omega_3(F, \rho_n(x)), \quad x \in [-1, 1], \tag{6.6}$$

where we applied (4.1). We have proved (6.6) for $n > 2N + 1$. By virtue of Whitney's theorem the quadratic polynomial that interpolates F at $-1, 0, 1$, yields an approximation to F which is bounded by $\omega_3(F, 1)$ (and any quadratic polynomial is automatically in Δ^3). Hence, since $\rho_n(x) \ge \frac{1}{n^2}$, we may extend (6.6) down to $n \ge 1$. This completes our proof. □

For constructing the polynomial approximant in the next section, we need an explicit representation of $\hat{S}$ (surprisingly, it looks asymmetric, but this is due to the asymmetry of the truncated powers $(\cdot - t)_+^2$). This is the purpose of the following lemma.

Lemma 8. *The following representation of $\hat{S}$ is valid.*

$$\hat{S}(x) = P^*(x) + \sum_{i=n-N}^{n-1} q_i (x - \theta_i)^2 + \sum_{i=n-N}^{n-1} \lambda_i (x - x_i)^2 + \sum_{i=N+1}^{n-N-1} \lambda_i (x - x_i)_+^2 =: \tilde{S}(x),$$
$$x \in [-1, 1].$$

Proof. We only have to compare the values of $\hat{S}$ and $\tilde{S}$ near the end points, for both are equal to $\bar{S}$ in $[x_{n-N-1}, x_{N+1}]$. Observe that both $\tilde{S}(x)$ and $S_1(x)$ are quadratic polynomials in $[x_{N+1}, 1]$, that agree up to the second derivative at x_{N+1}, hence identical. Similarly, observe that both $\tilde{S}(x)$ and $S_n(x)$ are quadratic polynomials in $[-1, x_{n-N-1}]$, that agree up to the second derivative at x_{n-N-1}, hence identical. This completes the proof. □

7. Pointwise Polynomial Approximation

We are ready to prove Theorem 3. We begin with some auxiliary lemmas.

Lemma 9. *For every n and $1 \le j \le n-1$, there exist a polynomial $P_j \in \mathcal{P}_{n+1} \cap \Delta^3$ and a number h_j such that $0 \le h_j \le c\rho_n^2(x_j)$, and we have the following estimates.*

$$\left|(x-x_j)_+^2 + h_j(x-x_j)_+^0 - P_j(x)\right| \le c\rho_n^2(x)\left(\frac{|I_j|}{\rho_n(x)+|x-x_j|}\right)^3, \tag{7.1}$$

and

$$\left|(x-x_j)_+^2 + h_j(x-x_j)_+^0 - P_j(x)\right| \le \frac{c\rho_n^{8.5}(x)}{(\rho_n(x)+|x-x_j|)^{6.5}}. \tag{7.2}$$

Proof. By the proof of [2, Lemma 1] there exist $P_j \in \mathcal{P}_{n+1} \cap \Delta^3$ and $|h_j| \le c\rho_n^2(x_j)$, such that

$$\left|(x-x_j)_+^2 + h_j(x-x_j)_+^0 - P_j(x)\right| \le \frac{c\rho_n^{17}(x_j)}{(\rho_n(x_j)+|x-x_j|)^{15}}, \tag{7.3}$$

and in turn (7.1) holds (see there).

Now, by virtue of (4.5),

$$\begin{aligned}(\rho_n(x)+|x-x_j|)^2 &\le 2(\rho_n^2(x)+|x-x_j|^2)\\ &\le c(\rho_n(x_j)(\rho_n(x_j)+|x-x_j|)+|x-x_j|^2)\\ &\le c(\rho_n(x_j)+|x-x_j|)^2,\end{aligned}$$

which, in turn, combined with (4.6), yields

$$\begin{aligned}\frac{\rho_n^{17}(x_j)}{(\rho_n(x_j)+|x-x_j|)^{15}} &\le \frac{c\big(\rho_n(x)(\rho_n(x)+|x-x_j|)\big)^{8.5}}{(\rho_n(x)+|x-x_j|)^{15}}\\ &\le \frac{c\rho_n^{8.5}(x)}{(\rho_n(x)+|x-x_j|)^{6.5}}.\end{aligned}$$

Substituting in (7.3) we obtain (7.2). We are left with having to prove that $h_j \ge 0$. To this end, we note that [2, Lemma 1]) was proved using [6] construction of convex polynomials $\sigma_j \in \mathcal{P}_n$ on $[-1,1]$ such that

$$\sigma_j(-1)=0, \quad \sigma_j(1)=1-x_j, \quad 0 \le \sigma_j'(x) \le 1, \quad x \in [-1,1].$$

(See [6, p. 164-165] for the definition of σ_j and the above properties.)

Hence,

$$\sigma_j(t)-(t-x_j)_+ = \int_{-1}^{t} \sigma_j'(y)dy \ge 0, \quad t \in [-1, x_j],$$

and

$$\sigma_j(t) - (t - x_j)_+ = \int_t^1 (1 - \sigma_j'(y))dy \geq 0, \quad t \in [x_j, 1].$$

Recall that the polynomials P_j and the constants h_j were defined by

$$P_j(x) = 2\int_{-1}^x \sigma_j(t)dt \quad \text{and} \quad h_j = 2\int_{-1}^1 (\sigma_j(t) - (t - x_j)_+)dt.$$

Thus, we immediately conclude that $h_j \geq 0$. This completes the proof. □

Remark 4. Note that Kopotun's [6] construction of σ_j yields polynomials of degree cn. Thus, in order to have the polynomials of degree n, we take the Kopotun construction for $n_1 := [n/c]$. However, in order to avoid unnecessary cumbersome notation, we continue to call it n.

Remark 5. Since we will have to use often the inequalities (7.1) and (7.2), we introduce a single notation for both right hand sides. Thus, denote

$$A_{n,j}(x) := \min\left\{\rho_n^2(x)\left(\frac{|I_j|}{\rho_n(x) + |x - x_j|}\right)^3, \frac{\rho_n^{8.5}(x)}{(\rho_n(x) + |x - x_j|)^{6.5}}\right\}. \tag{7.4}$$

Lemma 10. *There is an N such that for every $n > 2N + 1$ and $N \leq j \leq n/2$, there exists a polynomial $Q_j \in \mathcal{P}_{n+1} \cap \Delta^3$ such that the following inequalities hold.*

$$\left|(x - x_j)_+^2 - Q_j(x)\right| \leq cA_{n,j}(x), \quad x \in [-1, x_j], \tag{7.5}$$

and

$$\left|(x - x_j)_+^2 - Q_j(x)\right| \leq c\left(\frac{1 - x}{1 - x_j}\right)^2 |I_j|^2 + cA_{n,j}(x), \quad x \in (x_j, 1]. \tag{7.6}$$

Proof. Fix $N > 0$ large enough, to be prescribed, and let $b := \kappa n^{-2} := \max_{N \leq j \leq n/2} \frac{16h_j}{1 - x_j} < 1$, where $\kappa = \kappa(n) = O(1)$. Set

$$Q_j(x) := \gamma_j P_j\left(\frac{x - \xi_j}{1 + b}\right), \quad N \leq j \leq n/2,$$

where the polynomials P_j are given in Lemma 9, and γ_j and ξ_j are determined by the conditions $T_j(1) = T_j'(1) = 0$, where

$$T_j(x) := \gamma_j\left(\frac{x - \xi_j}{1 + b} - x_j\right)^2 + \gamma_j h_j - (x - x_j)^2.$$

The conditions $T_j(1) = T_j'(1) = 0$ are equivalent to the following system of two equations:

$$\begin{cases} \frac{\gamma_j}{(1+b)^2}(1 - x_j - \xi_j - bx_j)^2 + \gamma_j h_j & = (1 - x_j)^2 \\ \frac{\gamma_j}{(1+b)^2}(1 - x_j - \xi_j - bx_j) & = 1 - x_j. \end{cases}$$

Eliminating γ_j, we obtain a quadratic equation for ξ_j,

$$\xi_j^2 + (2bx_j + x_j - 1)\xi_j + (1 + b)^2 h_j + bx_j(bx_j + x_j - 1) = 0.$$

For $N>0$ sufficiently large, the discriminant of the above equation, $(1-x_j)^2-4(1+b)^2h_j$, is positive, and we take ξ_j to be the solution

$$\xi_j := -bx_j + \frac{(1-x_j)-\sqrt{(1-x_j)^2-4(1+b)^2h_j}}{2}.$$

Then straightforward computations yield

$$\gamma_j = \frac{2(1+b)^2}{1+\sqrt{1-\frac{4(1+b)^2h_j}{(1-x_j)^2}}},$$

and since $h_j\ge 0$, this implies that $(1+b)^2\le\gamma_j\le 2(1+b)^2\le 8$, and

$$0\le \frac{\gamma_j}{(1+b)^2}-1\le\frac{4(1+b)^2h_j}{(1-x_j)^2}. \tag{7.7}$$

Also, since $h_j\ge 0$, we have by the definition of b,

$$0\le\xi_j+bx_j=\frac{4(1+b)^2h_j}{(1-x_j)+\sqrt{(1-x_j)^2-4(1+b)^2h_j}}\le\frac{16h_j}{1-x_j}\le b.$$

Hence, $|\xi_j|<b$.

Set $x' := (x-\xi_j)/(1+b)$. Then

$$-1\le x'\le 1, \quad \text{and} \quad |x-x'|\le 2\kappa n^{-2}, \quad x\in[-1,1], \tag{7.8}$$

so that $c_1\rho_n(x)\le\rho_n(x')\le c_2\rho_n(x)$, and

$$\rho_n(x)+|x-x_j|\le c\rho_n(x')+|x'-x_j|+cn^{-2}\le c(\rho_n(x')+|x'-x_j|). \tag{7.9}$$

Fix j, $N\le j\le n/2$. If $x\in[-1,1]$ is such that $x'\in[-1,x_j]$, then by Lemma 9 and (7.9), we obtain

$$\left|P_j\left(\frac{x-\xi_j}{1+b}\right)\right|\le cA_{n,j}(x). \tag{7.10}$$

If $x'\in(x_j,1]$ and $x\in[-1,x_j]$, then $0\le x_j-x'\le x-x'\le 2\kappa n^{-2}\le c|I_j|$, $0\le h_j\le c|I_j|^2$, and $A_{n,j}(x)\ge c|I_j|^2$. Hence, by Lemma 9 and (7.9),

$$\left|P_j\left(\frac{x-\xi_j}{1+b}\right)-\left(\frac{x-\xi_j}{1+b}-x_j\right)^2-h_j\right|\le\left|P_j\left(\frac{x-\xi_j}{1+b}\right)\right|+(x'-x_j)^2+h_j\le cA_{n,j}(x).$$

Hence, together with (7.10), we obtain (7.5).

In order to prove (7.6), fix $x\in(x_j,1]$. If $x'\in[-1,x_j]$, then $(x-x_j)^2\le c|I_j|^2\le cA_{n,j}(x)$. Thus, by (7.10),

$$|(x-x_j)^2+Q_j(x)|\le(x-x_j)^2+|Q_j(x)|\le cA_{n,j}(x),$$

and (7.6) is proved.

Otherwise, $x' \in (x_j, 1]$. Then by Lemma 9 and (7.9), we obtain

$$\left|P_j\left(\frac{x-\xi_j}{1+b}\right) - \left(\frac{x-\xi_j}{1+b} - x_j\right)^2 - h_j\right| \le cA_{n,j}(x),$$

and so

$$|(x-x_j)_+^2 - Q_j(x)| < \left|(x-x_j)_+^2 - \gamma_j\left(\frac{x-\xi_j}{1+b} - x_j\right)^2 - \gamma_j h_j\right| + cA_{n,j}(x), \quad x \in (x_j, 1]. \tag{7.11}$$

Now by virtue of (7.7),

$$\left|(x-x_j)^2 - \gamma_j\left(\frac{x-\xi_j}{1+b} - x_j\right)^2 - \gamma_j h_j\right| = |T_j(x)|$$
$$= \left|1 - \frac{\gamma_j}{(1+b)^2}\right|(1-x)^2 < c\left(\frac{1-x}{1-x_j}\right)^2 |I_j|^2,$$

hence together with (7.11), we obtain (7.6). This completes our proof. □

We are ready to state the mirror of Lemma 4.

Lemma 11. *Let $N \le k < \lceil n/2 \rceil$, and assume that the nonnegative numbers α_j are such that*

$$\sum_{j=k}^{s} \alpha_j \le c\omega_3(F, \rho_n(x_s)), \quad s = k, \dots, \lceil n/2 \rceil. \tag{7.12}$$

Then for $x_{l+1} < x \le x_l$, $l = k+1, \dots, \lceil n/2 \rceil$, and for $x_{k+1} < x \le 1$, for $l = k$, we have

$$\sum_{j=l}^{\lceil n/2 \rceil} \alpha_j \left(\frac{1-x}{1-x_j}\right)^2 \le \tilde{c}\omega_3(F, \rho_n(x)). \tag{7.13}$$

Proof. The proof is a repetition of the proof of Lemma 4. We only need to observe that, instead of (4.24), we have for all $l \le s \le \lceil n/2 \rceil$,

$$\frac{\rho_n^3(x_s)}{\rho_n^3(x)} \left(\frac{1-x}{1-x_s}\right)^{3/2} \le 64,$$

for $x > x_{l+1}$. □

We quote a lemma resembling what was done in Lemma 5.

Lemma 12. [2, Lemma 4] *With N sufficiently large, let $n > 2N+1$. Set*

$$r_j := \frac{1}{26}\big((x_j - x_{j-N})^2 + (x_j - x_{j+N})^2\big),$$

and put $D = 20N^2$.

If $|b_j| < \frac{r_j}{D|I_j|}$*, then the linear system of equations*

$$\begin{cases} \eta_j + \mu_i + \nu_j & = 1 \\ 2\eta_j(x_j - x_{j-N}) + 2\nu_j(x_j - x_{j+N}) & = b_j \\ \eta_j\big((x_j - x_{j-N})^2 + h_{j-N}\big) + \mu_j h_j + \nu_j\big((x_j - x_{j+N})^2 + h_{j+N}\big) & = r_j, \end{cases}$$

has a unique solution (η_j, μ_j, ν_j)*, satisfying* $\eta_j \geq 0$*,* $\mu_j \geq 0$ *and* $\nu_j \geq 0$*.*

Remark 6. How big N is depends on the quantities h_j, $1 \leq j \leq n-1$ of Lemma 9, so that it is an absolute constant since the Q_j's defining the h_j's are fixed (see [2, Lemma 1]), so it is independent of F. Since we depend in our proof below on the quadratic spline of Theorem 2, we take $N \geq 1900$.

We are ready to prove Theorem 3.

Proof of Theorem 3. Recall that

$$\tilde{S}(x) = \tilde{P}(x) + \sum_{j=N+1}^{n-N-1} \lambda_j (x - x_j)_+^2, \tag{7.14}$$

where $\tilde{P} := P^* + \sum_{i=n-N}^{n-1} q_i(\cdot - \theta_i)^2 + \sum_{i=n-N}^{n-1} \lambda_i(\cdot - x_i)^2$, is a quadratic polynomial, and that it satisfies

$$|F(x) - \tilde{S}(x)| \leq c\,\omega_3(F, \rho_n(x)). \tag{7.15}$$

Also, by (6.5) and (4.1),

$$\lambda_j \leq c\,\omega_3(F, |I_j|)|I_j|^{-2}, \quad j = N+1, \dots, n-N-1. \tag{7.16}$$

Let b_j, $N+1 \leq j \leq n-N-1$, satisfying the requirements of Lemma 12, to be prescribed. For the triples (η_j, μ_j, ν_j), of nonnegative numbers that add up to 1, guaranteed by Lemma 12, we define

$$R_n := \tilde{P} + \sum_{j=N+1}^{n-N-1} \lambda_j(\eta_j P_{j-N} + \mu_j P_j + \nu_j P_{j+N}),$$

where the polynomials P_j are from Lemma 9. Then it follows that

$$R_n^{(3)}(x) \geq 0, \quad x \in [-1, 1].$$

We will prove that

$$|\tilde{S}(x) - R_n(x)| \leq c\,\omega_3(F, \rho_n(x)), \tag{7.17}$$

which combined with (7.15) yields the required estimate, proving Theorem 3 for $n > 2N + 1$.

To this end, we follow (part of) the proof of [2, Theorem 1] and set

$$\eta_j P_{j-N} + \mu_j P_j + \nu_j P_{j+N} - (\cdot - x_j)_+^2 =: v_j + t_j + u_j,$$

where

$$\begin{aligned} v_j &:= \eta_j \big(P_{j-N} - h_{j-N}(\cdot - x_{j-N})_+^0 - (\cdot - x_{j-N})_+^2 \big) \\ &\quad + \mu_j \big(P_j - h_j(\cdot - x_j)_+^0 - (\cdot - x_j)_+^2 \big) \\ &\quad + \nu_j \big(P_{j+N} - h_{j+N}(\cdot - x_{j+N})_+^0 - (\cdot - x_{j+N})_+^2 \big), \end{aligned}$$

$$\begin{aligned} t_j &:= \eta_j \big((\cdot - x_{j-N})_+^2 - 2(x_j - x_{j-N})(\cdot - x_j)_+ - (x_j - x_{j-N})^2 (\cdot - x_j)_+^0 - (\cdot - x_j)_+^2 \big) \\ &\quad + \nu_j \big((\cdot - x_{j+N})_+^2 - 2(x_j - x_{j+N})(\cdot - x_j)_+ \|! - (x_j - x_{j+N})^2 (\cdot - x_j)_+^0 - (\cdot - x_j)_+^2 \big) \\ &\quad + \eta_j h_{j-N} \big((\cdot - x_{j-N})_+^0 - (\cdot - x_j)_+^0 \big) + \nu_j h_{j+N} \big((\cdot - x_{j+N})_+^0 - (\cdot - x_j)_+^0 \big), \end{aligned}$$

and

$$u_j := r_j (\cdot - x_j)_+^0 + b_j (\cdot - x_j)_+,$$

where r_j and b_j are from Lemma 12.

In order to derive the estimate for $v_j(x)$, we have by virtue of Lemma 9,

$$|v_j(x)| \le c \big(A_{n,j-N}(x) + A_{n,j}(x) + A_{n,j+N}(x) \big). \tag{7.18}$$

Fix $x \in [-1,1]$ and separate the sum

$$\sum_{j=N+1}^{n-N-1} \lambda_j |v_j(x)| = \sum_{j:\rho_n(x) \le |I_j|} \lambda_j |v_j(x)| + \sum_{j:\rho_n(x) > |I_j|} \lambda_j |v_j(x)| =: \sum{}' + \sum{}''.$$

For j that satisfy $|I_j| \ge \rho_n(x)$, we have by (7.16),

$$\begin{aligned} \lambda_j &\le c\, \omega_3(F, |I_j|) |I_j|^{-2} \le c \left(\frac{|I_j|}{\rho_n(x)} + 1 \right)^3 |I_j|^{-2} \omega_3(F, \rho_n(x)) \\ &= c \frac{\omega_3(F, \rho_n(x))}{\rho_n^3(x)} \left(1 + \frac{\rho_n(x)}{|I_j|} \right)^3 |I_j| \\ &\le c \frac{\omega_3(F, \rho_n(x))}{\rho_n^3(x)} |I_j|. \end{aligned}$$

Also, in view of (4.3),

$$\lambda_j \le c \frac{\omega_3(F, \rho_n(x))}{\rho_n^3(x)} \min\{ |I_{j-N}|, |I_{j+N}| \}.$$

Therefore, by (7.18)

$$\begin{aligned}\sum{}' &\le \sum{}' c\lambda_j\rho_n^{8.5}(x)\bigg(\frac{1}{(\rho_n(x)+|x-x_{j-N}|)^{6.5}}\\
&\quad+\frac{1}{(\rho_n(x)+|x-x_j|)^{6.5}}+\frac{1}{(\rho_n(x)+|x-x_{j+N}|)^{6.5}}\bigg)\\
&\le c\,\omega_3(F,\rho_n(x))\rho_n^{5.5}(x)\sum_{j=1}^{n-1}\frac{|I_j|}{(\rho_n(x)+|x-x_j|)^{6.5}}\\
&\le c\,\omega_3(F,\rho_n(x))\rho_n^{5.5}(x)\int_{\rho_n(x)}^{\infty}\frac{du}{u^{6.5}}<c\,\omega_3(F,\rho_n(x)).\end{aligned}\tag{7.19}$$

For the other sum, note that if $|I_j|<\rho_n(x)$, then $\omega_3(F,|I_j|)<c\omega_3(F,\rho_n(x))$. Also by (4.3), $|I_j|\le c\min\{|I_{j-N}|,|I_{j+N}|\}$. Hence, together with (7.18) and (7.16) we obtain,

$$\begin{aligned}\sum{}'' &\le c\omega_3(F,\rho_n(x))\rho_n^2(x)\sum{}''\bigg[\frac{|I_{j-N}|}{(\rho_n(x)+|x-x_{j-N}|)^3}\\
&\quad+\frac{|I_j|}{(\rho_n(x)+|x-x_j|)^3}+\frac{|I_{j+N}|}{(\rho_n(x)+|x-x_{j+N}|)^3}\bigg]\\
&\le c\omega_3(F,\rho_n(x))\rho_n^2(x)\sum_{j=1}^{n-1}\frac{|I_j|}{(\rho_n(x)+|x-x_j|)^3}\\
&\le c\omega_3(F,\rho_n(x))\rho_n^2(x)\int_{\rho_n(x)}^{\infty}\frac{du}{u^3}<c\omega_3(F,\rho_n(x)).\end{aligned}\tag{7.20}$$

Thus, combining (7.19) and (7.20), we obtain

$$\sum_{j=N+1}^{n-N-1}\lambda_j|v_j(x)|<c\omega_3(F,\rho_n(x)).\tag{7.21}$$

At the same time the support of the function t_j is contained in $[x_{j+N},x_{j-N}]$, so that for $x\in I_i$, $1\le i\le n$,

$$\begin{aligned}\sum_{j=N+1}^{n-N-1}\lambda_j|t_j(x)| &= \sum_{\max\{N+1,i-N\}}^{\min\{n-N,i+N-1\}}\lambda_j|t_j(x)|\\
&\le c\sum_{\max\{N+1,i-N\}}^{\min\{n-N-1,i+N-1\}}\omega_3(F,\rho_n(x_j))\le c\omega_3(F,\rho_n(x)),\end{aligned}\tag{7.22}$$

since $|t_j(x)|\le c\,|I_j|^2$ and we applied (7.16), and by (4.1) and (4.2),

$$\omega_3(F,\rho_n(x_j))\le c\omega_3(F,\rho_n(x)),\quad \max\{N+1,i-N\}\le j\le\min\{n-N-1,i+N-1\}.$$

Finally, we estimate $\sum_{j=N+1}^{n-N-1} \lambda_j u_j$. To this end, we have the piecewise constant

$$\sum_{j=N+1}^{n-N-1} \alpha_j (x - x_j)_+^0,$$

with $0 \le \alpha_j := \lambda_j r_j \le c\omega_3(F, \rho_n(x_j))$ (see (7.16)). We repeat what we have done in the proof of Lemma 7. We deal separately with the summation on $j \ge \lceil \frac{n}{2} \rceil$ and with the rest. We begin with $k_1 = n - N - 1$ and (if possible) find $s_1 < k_1$ such that the conditions of Lemma 3 are satisfied for the pair (s_1, k_1). Then we take $k_2 = s_1 - 1$ and find $s_2 < k_2$ with similar properties. If after a few steps, we arrive at $\lceil \frac{n}{2} \rceil = s_m < k_m$, with the pair (s_m, k_m) satisfying the conditions of Lemma 3, we are done. Otherwise, the process stops, that is, we have an index $k \le n - N - 1$ (which again may be $k = k_1$), such that

$$\sum_{j=s}^{k} \alpha_j \le CD\,\omega_3(F, \rho_n(x_s)), \quad s = \lceil n/2 \rceil, \dots, k. \tag{7.23}$$

We go through a similar process for the other summation, that is, for $j \le \lceil n/2 \rceil$. Again, this process may end with $s'_{m'} = \lceil n/2 \rceil$, in which case we are done, or we may have an index $k' \ge N + 1$ such that,

$$\sum_{j=k'}^{s'} \alpha_j \le CD\,\omega_3(F, \rho_n(x_{s'})), \quad s' = k', \dots, \lceil n/2 \rceil. \tag{7.24}$$

For the sums of the former type we obtain by the same proof as of Lemma 7, non decreasing piecewise linear functions

$$\sum_{j=k_1}^{s_m} \beta_j (x - x_j)_+ \quad \text{and} \quad \sum_{j=s'_{m'}}^{k'_1} \beta_j (x - x_j)_+,$$

such that

$$\left| \left\{ \sum_{j=N+1}^{s'_{m'}} + \sum_{j=s_m}^{n-N-1} \right\} \left(\alpha_j (x - x_j)_+^0 - \beta_j (x - x_j)_+ \right) \right| \le cD\,\omega_3(F, \rho_n(x)), \quad x \in [-1, 1],$$

with $|\beta_j| < \frac{\alpha_j}{D|I_j|}$.

Thus, with b_j defined by $\lambda_j b_j := \beta_j$, $j = k_1, \dots, s_m$ and $j = s'_{m'}, \dots, k'_1$,

$$\left| \left\{ \sum_{j=N+1}^{s'_{m'}} + \sum_{j=s_m}^{n-N-1} \right\} \lambda_j u_j(x) \right| \le cD\,\omega_3(F, \rho_n(x)), \quad x \in [-1, 1]. \tag{7.25}$$

Now, we have to deal with the remaining elements, that is, these in (7.23) and (7.24). We go back to the basic representation (7.14) and replace the truncated powers $(x - x_j)_+^2$, $j = k', \dots, \lceil n/2 \rceil$, by the polynomials of Lemma 10. One should note that unlike the spline case (see Lemma 6), the polynomials do not coincide with the truncated powers $(x-x_j)_+^2$ on $[-1,x_j]$, and that our estimates on $(x_j,1]$ do not involve only the terms $\big((1-x)^2/(1-x_j)^2\big)|I_j|^2$, but also the terms $A_{n,j}(x)$. The sum of the terms $A_{n,j}(x)$ is dealt with by the same proof for the v_j's (see (7.19) and (7.20)), and we estimate the sum of the terms $\big((1-x)^2/(1-x_j)^2\big)|I_j|^2$, as in the proof of Lemma 7. Hence, we obtain for this sum, the required estimate by Lemma 11. Finally, we apply (5.14) to move the truncated powers $(x-x_j)_+^2$, $j=\lceil n/2\rceil,\dots,k$ to the interval $[0,1]$, and similarly obtain the approximating polynomials and the required estimates as explained above. So, we summarize that

$$\left|\left\{\sum_{j=\lceil n/2\rceil}^{k}+\sum_{j=k'}^{\lceil n/2\rceil}\right\}u_j(x)\right|\le c\omega_3(F,\rho_n(x)),\quad x\in[-1,1]. \tag{7.26}$$

Combining (7.21), (7.22), (7.25) and (7.26), we obtain (7.17). We complete the proof for $2\le n\le 2N+1$, by taking the interpolating quadratic we took in the proof of Theorem 2. This completes the proof. □

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