Scientific memory of research stay.

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Project title: **Bifurcations of homoclinic tangency in two-dimensional area-preserving and reversible maps.**

Abstract

Within the project, bifurcations of orbit behavior in area-preserving and reversible maps with a homoclinic tangency were studied. Finitely smooth normal forms for such maps near saddle fixed points were constructed and it was shown that they coincide in the main order with the analytical Birkhoff-Moser normal form. Bifurcations of single-round periodic orbits for two-dimensional symplectic maps close to a map with a quadratic homoclinic tangency were studied. The existence of one- and two-parameter cascades of elliptic periodic orbits was proved.

Introduction.

The systematic study of bifurcations of homoclinic tangency was started by Gavrilov and Shilnikov in [1, 2] in 1972-1973 for the case of two-dimensional diffeomorphisms (three-dimensional flow). They discovered many remarkable dynamical phenomena. In particular, they established that diffeomorphisms with homoclinic tangencies can belong to the boundary of Morse-Smale systems and the transition through this boundary corresponds to the homoclinic omega-explosion. Further, this dynamical phenomena was studied in papers of Newhouse and Palis [3], Palis and Takens [4], Shilnikov and Stenkin [5] etc. Also, in [1, 2] a classification of homoclinic tangencies was done: the corresponding diffeomorphisms were subdivided onto three classes depending on the structure of orbits entirely lying in a small neighbourhood of the homoclinic orbit. One of the fundamental results from [1, 2] is the so-called “theorem on cascade of periodic sinks” that plays an important role in theory of dissipative chaos. This theorem says that, in the dissipative case, for any one parameter family that unfolds generally a quadratic homoclinic tangency, there exists an infinitely sequence of nonintersecting intervals of values of the parameter such that the corresponding diffeomorphism of the family has an asymptotically stable single-round periodic orbit. Various multidimensional extensions of “theorem on cascade of periodic sinks” were established by Newhouse [6], S.Gonchenko [7] and S.Gonchenko, Turaev and Shilnikov [8]. One of fundamental results in homoclinic bifurcations was established by Newhouse [9] who proved the existence of regions of the space of two-dimensional diffeomorphisms where diffeomorphisms with homoclinic tangencies are dense. These regions were called later as Newhouse regions and their existence near any multidimensional system with a homoclinic tangency was proved by S.Gonchenko, Turaev and Shilnikov [10]. Dynamics of systems from Newhouse regions is extremely rich [11, 12] and, in the principal plan, it is impossible to give the complete description of bifurcations of such systems.

However, the most of these results was obtained for the case of general systems. Moreover, one of typical generality conditions relates to that the Jacobians of saddle fixed or periodic points (having homoclinic tangencies) do not equal to 1. Thus, generally speaking, these results can not be applied automatically for conservative and reversible systems which require special considerations. Nevertheless, principal geometric and analytical constructions can be also used for systems with additional structures along, naturally, with special methods. In this direction, rather important results on birth of elliptic periodic points in area-preserving maps under bifurcations of homoclinic tangencies were obtained by Newhouse [13], Mora and Romero [14], S.Gonchenko and Shilnikov [15]. However, more or less complete bifurcation picture, even for single round periodic orbits only, is not known here. It concerns especially to reversible maps.

The main goal of the stay in Research Institute for Applied Mathematics and Cybernetics (Nizhny Novgorod, Russia) was the study of mathematical methods for investigations of dynamics and bifurcations of dynamical systems with complicated orbit behavior, i.e. the systems which possess infinitely many periodic and homoclinic orbits. In the natural sciences, such systems are usually called chaotic. The main problems of this project were groupped around the bifurcation theory of dynamical systems with homoclinic orbits and the theory of Hamiltonian systems, as well of the theory of reversible systems which combine both conservative and dissipative properties. This included the study of global bifurcations in systems with different types of homoclinic and heteroclinic orbits and creation of new methods of the study of symplectic and reversible maps. In particular, it was studied bifurcations of quadratic and cubic homoclinic tangencies for area-preserving and reversible diffeomorphisms to be aimed to obtain results like „theorem on cascade of elliptic points“.

Metodology.

To solve the problems of the project there were used both classical methods of the bifurcation theory of multidimensional systems and quite new methods developed by scientific collectives of Nizhny Novgorod school of nonlinear dynamics (the lider is Prof. L.Shilnikov) and by UB-UPC group on dynamical systems (the liders are Prof. C.Simo and Prof. A.Delshams). The last methods included: methods of construction of finitely smooth normal forms of maps and flows near saddle orbits; rescaling methods of construction of normal forms for return maps; methods of investigation of bifurcations via studying families which contain the so-called omega-moduli (continuous invariants of topological equivalenca on the set of non-wandering orbits) as the governing parameters and other that was published in joint papers of these groups (see [16, 17, 18]).

Results obtained.

Within the project, one of the main topics was the study of homoclinic bifurcations in conservative and reversible systems. In particular, there were considered two-dimensional area-preserving diffeomorphisms having a saddle fixed point whose the stable and unstable invariant manifolds have a quadratic tangency at the points of homoclinic orbit. The main elements of the semi-local dynamics of such diffeomorphisms were analyzed. For this goal, the local and global maps were constructed and their properties were studied. In particular, finitely smooth normal forms of local maps were derived and there was proved that they coincide, in the main order, with the analytical Birkhoff-Moser normal form. Using these normal forms, the boundary volume problem (the Shilnikov's problem) near a conservative saddle fixed point was solved and the corresponding estimates for the solutions and derivatives were obtained.

Using these mathematical tools, bifurcations of single-round periodic orbits (i.e. those ones that have exactly one intersection point with a neighborhood of the homoclinic orbit) were studied for two-dimensional symplectic maps close to a map having a quadratic homoclinic tangency. The corresponding first return maps were derived, bifurcations of their fixed points were studied and bifurcation diagrams for both one-parameter and two-parameter general unfoldings were constructed. By this way, the existence of one- and two-parameter cascades of elliptic single-round periodic orbits was proved. In additional, there were described cases where two-dimensional symplectic maps with quadratic homoclinic tangencies have infinitely many single-round elliptic periodic points. Analogous problems were studied for the case of reversible two-dimensional maps having symmetric quadratic homoclinic tangencies and, in particular, the theorem on cascade of elliptic periodic orbits was proved.

Now, the paper [19] (with co-author S.Gonchenko), collecting these results, is in preparation.

In addition, a series of mathematical articles related to bifurcations of cubic homoclinic tangencies in general (dissipative) case was analyzed and some results for conservative and reversible systems were obtained. For example, conservative cubic Henon maps were derived as normal rescaled forms for the corresponding first return maps.

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