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Signaling in Dynamic Contests: Some Impossibility Results*

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Abstract

General signaling results in dynamic Tullock contests have been missing for long. The reason is the tractability of the problems. In this paper, an uninformed contestant with valuation v_x competes against an informed opponent with valuation, either high v_h or low v_l. We show that; (i) When the hierarchy of valuations is $v_h \geq v_x \geq v_l$, there is no pooling. *Sandbagging* is too costly for the high type. (ii) When the order of valuations is $v_x \geq v_h \geq v_l$, there is no separation if v_h and v_l are close. *Sandbagging* is cheap due to the proximity of valuations. However, if v_h and v_x are close, there is no pooling. First period cost of pooling is high. (iii) For valuations satisfying $v_h \geq v_l \geq v_x$, there is no separation if v_h and v_l are close. *Bluffing* in the first period is cheap for the low valuation type. Conversely, if v_x and v_l are close there is no pooling. *Bluffing* in the first stage is too costly.

JEL: C72, C73, D44, D82.

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1 Introduction

In the last two decades, the theory of contests and tournaments found in economics a field of great development[1], R&D competition (see, e.g. Baye and Hoppe (2003)), rent-seeking (see, e.g. Tullock (1980) and Nitzan (1994)), war, disputes and conflicts resolution (see, e.g. Garfinkel and Skaperdas (2007)), sports competitions (see, e.g. Szymanski (2003)), and so on. However, dynamic contests with incomplete information have experienced little progress (see below for a literature review). The main obstacle has been the low tractability of the problems. For that reason general signaling results in dynamic contests still an open question.

In many economic situations of interest individuals compete for the same good, prize or object. A critical feature is that typically individuals hold different amounts of information about strategic relevant aspects of the competition. Moreover, the same individuals may repeatedly compete between each other.

For example, lobby groups with opposed interest have to renew their influence every time government changes. Heterogeneous individual with different valuations and skills repeatedly dispute a given prize. In procurement, the same firms compete in a regularly basis to win a public or private project.

These individuals, institutions, or firms, differ in many dimensions, which are relevant for the final outcome of the competition. Some aspects are common knowledge, but others do not. A firm may develop a new product or technology that gives a strategic advantage and some lobby or influence groups might have better resources or a higher valuation for a particular issue.

While in a static setting, given his information, each individual should make his best taking into account what the opponent can do, in dynamic settings other incentives arise. In particular, the possibility of credible informs or misleads the opponent through their actions, in order to obtain higher gains or increase their winning likelihood.

This paper focus in two stages repeated Tullock (1980) contests with one-sided incomplete information. The informed contestant has private information about his valuation. The uninformed contestant valuation v_x is public, but he does not know his opponent valuation, which can be high v_h or low v_l.

The informed player might have incentives to reveal or hide his type from the uninformed player. The incentives depend crucially on the valuation hierarchy. When $v_h \geq v_x \geq v_l$ or $v_x \geq v_h \geq v_l$, it is the high type of informed player who has incentives to pool with the low type. In the terminology of Hörner and Sahuguet (2007) this type would like to *sandbag* in the first period in order to relax the uninformed player in the second period.

When $v_h \geq v_l \geq v_x$, it is the low type who has incentives to pool. In this case, he *bluffs* in the first period, pretending to have a high valuation, discouraging the uninformed player, which will provide less effort in the second period.

[1] Konrad (2009) provides a recent and complete survey on the general contests theory. See also Corchón (2007).

We start by developing a general methodology to deal with this kind of problem. As in Münster (2009), we derived the second period equilibrium payoffs conditional on the observed first period effort. Then we compute the first period equilibrium payoffs that depend on the chosen actions.

As pointed out by Münster (2009), Zhang (2008) and Zhang and Wang (2009), when the two stages are modeled with a Tullock lottery, tractability is the major difficulty to compute the perfect Bayesian equilibrium of the game. Moreover, simplifying assumptions that preserve a reasonable level of generality do not help that much. We go around this difficulty by asking what kind of behaviors cannot be part of an equilibrium. The results are a set of impossibility statements, that hold true for any prior belief of the uniformed player.

When the hierarchy of valuations is $v_h \geq v_x \geq v_l$ we show that there is no pooling equilibria for any $\delta \leq 1$. Where δ is a common discount factor. *Sandbagging* is too costly for the high type. The loss in the first period can never compensate the second period gain.

When the hierarchy of valuations is $v_x \geq v_h \geq v_l$, there is no separating equilibria if v_h and v_l are sufficiently close. In this case *sandbagging* becomes cheaper due to the proximity of valuations. In other cases like v_h and v_x sufficiently close, v_l sufficiently small or v_x sufficiently large, there is no pooling equilibrium for any $\delta \leq 1$. In the former two cases, the intuition is again the relative first period high cost of pooling. In the latter scenario, it is the uninformed player who is too strong, to worth any strategic manipulation from a high valuation informed contestant.

When the hierarchy of valuations is $v_h \geq v_l \geq v_x$, we show that there is no separating equilibria if v_h and v_l are sufficiently close. The reason is that *bluffing* is cheap for the low type. On the other hand, if v_x and v_l are sufficient close, there is no pooling equilibria for $\delta \leq 1$. *Bluffing* in the first stage is too costly for the low valuation type. In additionally, we show that if v_x is sufficiently small or v_h is sufficiently large there is no pooling but only for small δ.

The findings presented in this paper are the first general signaling results in repeated Tullock contests. Their importance goes beyond Tullock contests, and the approach developed in the present paper can be helpful in dynamic auctions and other problems with similar structure.

Related literature - In the static setting, it is worth mentioning Malueg and Yates (2004). They look at a two sided contest with incomplete information , they assume a particular specification of the prior distribution that makes the model, particularly tractable and allows to study the effects of correlation on the private valuations.[2]

The literature in signaling in dynamic contests considers two potentially different situation; in the first, the same players repeatedly meet in each period, in the second, at each stage some players are eliminated from the contest. The present paper is in line with the former literature.

[2]Static incomplete information contests are also studied by Hurley and Shogren (1998a,b).

Münster (2009) looks at a two stages contest where each player is uncertainty about the existence of the other player. His model is very particular, but extremely tractable and with interesting results. He shows that a semi-pooling equilibrium in the first stage occurs when the probability of no contestant is below a given cut-off. Otherwise, full separation is the equilibrium. The present paper differs in many aspects, in particular, we model one-sided incomplete information, but we are more general. For the latter reason, we are not able to state for a particular case what equilibria emerges, but we show in which circumstances a specific strategic behavior cannot be an equilibrium.

Hörner and Sahuguet (2007) study two periods general auctions. They show that two types of pooling equilibria can emerge; *sandbagging*, i.e. a high valuation bidder pretends to be weak in order to relax the opponent, and *bluffing*, i.e. a low valuation bidder pretend to be strong to discourage the opponent. In the present paper depending on the valuation hierarchy considered incentives for pooling equilibria of these kinds emerge.[3]

Another connected paper on signaling in auctions is Goeree (2003). In the first stage firms compete in an auction for some technology that confers a second period advantage in their core business. Strategic signaling in the auction stage emerge because bidding high, signals to the opponent how much better they will be on the aftermarket (only the winning bid is observed). He shows that bidding is more aggressive in the second price auction relatively to the first price auction, since bidders do no pay their actual bids. Separating equilibrium exist when the strong bidders' incentives are to overstate their private information, but fail to exist in the inverse case.

The incentives and intuitive arguments of the previous referred contributions are also present and captured in this paper.

Signaling also occurs in elimination contests, where at each stage some players are eliminated from the contest.[4] More closely to the present paper is Zhang and Wang (2009) who study a two stage and two-sided incomplete information elimination contest where both stages are modeled as all-pay auctions. They show that symmetric separating equilibria fail to exist. In a subsequent work, Zhang (2008) studies a problem with a similar structure, where the first stage is an all-pay auction, while the second is a Tullock's (1980) lottery. In this case symmetric separating equilibria exist. The choice of all-pay auction in the first or in both stages is motivated by tractability reasons. These papers highlight

[3] Netzer and Wiermann (2005) also study a dynamic contest with strategic signaling. They assume that each contestant can only choose either high or low effort. Such assumption makes the problem very tractable, but constraints the generality of the results. There are similarities between their dove (hawk) equilibrium and the sandbagging (bluffing) equilibrium of Hörner and Sahuguet (2007).

[4] The literature in elimination contests with complete information is extensive, starting with Rosen (1986). For example, Gradstein and Konrad (1999) discuss when sequential elimination contests lead to a higher effort than simultaneous contests. With incomplete information, Moldovanu and Sela (2006) assume that contestants do not observe the first period effort of their opponents. Rather, ability is inferred from the fact that his still an active contestant, restricting the potential for strategic signaling. Other papers also make specific assumptions about the player behavior or information transmission in order to restrict or simplify signaling effects, see, for example, Amegashie (2006) or Lai and Matros (2006).

the difficulties to derive general results when both periods are modeled with the Tullock function.[5] In the present paper, each stage contest is a Tullock lottery and there is no elimination in the end of the first stage.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 presents the equilibrium payoffs for the general case. Section 4 shows a set of impossibility signaling equilibria for dynamic contests. All proofs are relegated to an appendix.

2 The Model

There are two players. Let X denote the uniformed player and Y the informed player. The uniformed player has a commonly known valuation $v_x \in \mathbb{R}_+$ for the contested object. The informed player can be of two types. He can have a high valuation $v_h \in \mathbb{R}_+$ for the contested object or a low valuation $v_l \in \mathbb{R}_+$, with $v_l \leq v_h$. The common prior probability of a type h in the population is denoted by $p \in (0, 1)$.

The uniformed and informed player efforts are denoted respectively as $x \in \mathbb{R}_+$ and $y \in \mathbb{R}_+$. In the first period, each player chooses simultaneously a non-negative effort; x_1 for the player X, and y_{1h} or y_{1l} for player Y, with $y_{1h} \geq y_{1l}$. The second period efforts are denoted as x_2, and y_{2h} or y_{2l}, respectively. Where y_{2h} is exclusively chosen by a high valuation type, while y_{2l} is the effort corresponding to a low valuation player Y. Further changes of notation will be defined in their due time.

The probability that player Y of type $t \in \{h, l\}$ wins the stage $\tau \in \{1, 2\}$ contested object, follows a Tullock (1980) lottery, i.e. $y_{\tau t}/(x_\tau + y_{\tau t})$ when $x_\tau + y_{\tau t} > 0$, where x_τ and $y_{\tau t}$ are player X and Y respective efforts. The expected payoff of player Y of type $t \in \{h, l\}$ at stage $\tau \in \{1, 2\}$ is then

$$\pi_{\tau t} = \frac{y_{\tau t}}{x_\tau + y_{\tau t}} v_t - y_{\tau t}.$$

Analogously, we can write for the player X the respective winning probabilities and payoffs.

Depending on the valuations' hierarchy, i.e. on whether $v_h \geq v_l \geq v_x$, $v_h \geq v_x \geq v_l$ or $v_x \geq v_h \geq v_l$ different incentives arise. In one case player h may have an interest in replicate the player l effort, or vice versa.

Players discount the second period with the common discount factor δ.

In a static contest with asymmetric incomplete information, each type of player Y plays according to his type, and an equilibrium is called Bayesian. The same also happen in the second period of a two stage contest. Because there is no further contest afterwards, conditional on the observed information, player Y rationally chooses an effort compatible with his type.

[5] There are connections with the Tullock contest function and all-pay auction, see Hillman and Riley (1989) and Krishna and Morgan (1997). However, these are two different objects.

An equilibrium of the two stages contest must be perfect Bayesian. The strategy profile and the belief system is such that the strategies are sequentially rational (optimal given the other players' strategies and the belief system) and the belief system is consistent (the probabilities assigned to every reachable node are computed using Bayes's rule).

Information might be different but players' strategic interests are common knowledge.

We now provide a short description of the equilibria and information revelation that can arise in a two period dynamic contest:

In a **full separating** equilibrium one type of player Y must have incentives to separate from the other. In this case, he plays an action that is profitable to him, but cannot be profitable replicated by the opponent. The first period effort is commonly observed before players decide their second period efforts. Information is precise about the identity of player Y. Consequently, in the second stage, player X knows the type of player Y.

In a **perfectly pooling** equilibrium is not profitable for the player with incentives to separate, to do it behind a given effort level. On the other hand, the player with incentives to pool can profitably replicate such effort. Consequently, the observation of the second period action, provides no additional information about the identity of player Y.

In a **semi-separating** equilibrium, the type of player Y that has incentives to separate fully, find it non-profitable behind a given effort. On same time the type with incentives to pool cannot perfectly replicate his effort in a profitable way. However, the latter may find profitable a partial replication, i.e. randomizing his play over the separating effort and his own optimal effort. In the second period, the uniformed player X is expected to hold more precise information about the identity of the opponent. Still not fully informed about the player Y true type.

3 General Results

In this Section, we derive each player payoff function for each stage of the game. The expressions are general and accommodate any potential scenario in two players, two periods, two types setting with one-sided incomplete information.

3.1 The Second Stage

We start from the second and last period of the game. Given his prior p and based on the observation of the effort chosen in the first period, player X forms beliefs about the type of player Y. The Bayesian posteriors about the probability of a type h after observing respectively, high effort y_{1h} or low effort y_{1l} are

$$p_{y_{1h}} \equiv \frac{p\alpha}{p\alpha + (1-p)\,\beta},$$

and

$$p_{y_{1l}} \equiv \frac{p\,(1-\alpha)}{p\,(1-\alpha)+(1-p)\,(1-\beta)}.$$

Where $\alpha \in [0,1]$ denotes the intensity with which a type h chooses effort y_{1h}, and $\beta \in [0,1]$ the intensity with which a type l chooses an effort y_{1h}.

In an equilibrium player Y know that player X effort is optimally with respect to his beliefs and vice versa. The available information and common knowledge of rationality determine players' behavior. In the second period, the informed player Y always plays according to his type. There are no incentives to misrepresent since it is the last period of the relation.

The following Lemma presents the players second period equilibrium payoffs for the general case.

Lemma 1 *Depending on the commonly observed effort $y_{1t} \in \{y_{1l}, y_{1h}\}$, the second period equilibrium payoffs are:*

(i) For the player X

$$\pi_2^{y_{1t}}\left(p_{y_{1t}}\right) = v_x^3 \frac{\left(\left(1-p_{y_t}\right)v_h + p_{y_t}v_l\right)\left(\left(1-p_{y_t}\right)\sqrt{v_h}+p_{y_{1t}}\sqrt{v_l}\right)^2}{\left(\left(1-p_{y_{1t}}\right)v_hv_x + p_{y_{1t}}v_xv_l + v_hv_l\right)^2}.$$

(ii) For the high valuation player Y

$$\pi_{2h}^{y_{1t}}\left(p_{y_{1t}}\right) = \left(v_h \frac{\sqrt{v_h}v_l + \left(1-p_{y_{1t}}\right)v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)}{\left(1-p_{y_{1t}}\right)v_hv_x + p_{y_{1t}}v_xv_l + v_hv_l}\right)^2. \tag{1}$$

(iii) For the low valuation player Y

$$\pi_{2l}^{y_{1t}}\left(p_{y_{1t}}\right) = \left(v_l \frac{\sqrt{v_l}v_h - p_{y_{1t}}v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)}{\left(1-p_{y_{1t}}\right)v_hv_x + p_{y_{1t}}v_xv_l + v_hv_l}\right)^2. \tag{2}$$

The superscript $^{y_{1t}}$ denotes the first period observed effort.

The second stage equilibrium is equivalent to a static Bayesian equilibrium where the prior is replaced by the posterior conditional on the observed effort.

To keep the main text clear, notice that we have not written the second period equilibrium efforts, they can be found on the Proof of Lemma 1. Clearly these actions are not fixed. They depend on the available information and on the players' strategies, through the posteriors that depend on α and β.

For similar reasons, we do not specify the off-equilibrium path posteriors, they are not required for the results presented in the following Sections.

3.2 The First Stage

In the first period depending on the hierarchy of valuation; either player h has incentives toward pooling the player l effort, or vice versa. The payoffs in both cases can be accommodated in a same framework.

Player X knows that there is a probability p of facing a type h and a probability $(1-p)$ of a type l. But he also knows that such does not mean the played action is y_{1h} or y_{1l} respectively. A type h chooses effort y_{1h} with probability α, while a type l chooses effort y_{1h} with probability β. Similarly, effort y_{1l} is played with probability $1-\alpha$ by a h type and with probability of $1-\beta$ by a l type. Consequently, in the first period the relevant information for the player X is the frequency of the different observed actions, i.e.

$$q \equiv \Pr\left(y_{1h}|p,\alpha,\beta\right) = p\alpha + (1-p)\,\beta,$$

and $1-q \equiv \Pr\left(y_{1l}|p,\alpha,\beta\right)$.

Then the player' X first period problem for general p, α and β can be studied knowing only the actions y_{1h} and y_{1l}. Moreover, we can isolate the first period from the next one, because the player X first stage action plays no role in the second period.

The uncertainty faced by player X is known to player Y, then according to his type, he chooses the most convenient effort that maximizes the two periods payoff. Nonetheless, player Y first period equilibrium effort is always conditioned by the uninformed player X behavior.

Lemma 2 *(i) Player X first period equilibrium payoff is*

$$\pi_1(q) = \frac{v_x^3\left((1-q)\sqrt{v_h} + q\sqrt{v_l}\right)^2\left((1-q)\,v_h + qv_l\right)}{\left((1-q)\,v_h v_x + qv_x v_l + v_h v_l\right)^2}.$$

(ii) The high valuation player Y first period equilibrium payoff is

$$\pi_{1h}(\alpha,\beta,q) = \alpha\pi_{1h}^h(q) + (1-\alpha)\,\pi_{1h}^l(q),$$

where

$$\pi_{1h}^h(q) = \left(v_h\frac{\sqrt{v_h}v_l + (1-q)\,v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)}{(1-q)\,v_h v_x + qv_x v_l + v_h v_l}\right)^2, \tag{3}$$

and

$$\begin{aligned}\pi_{1h}^l(q) \quad &= \quad \sqrt{v_h}\sqrt{v_l}\frac{\left(\sqrt{v_l}v_h - qv_x\left(\sqrt{v_h}-\sqrt{v_l}\right)\right)}{\left((1-q)\,v_h v_x + qv_x v_l + v_h v_l\right)^2} \\ &\times\left((1-q)\,v_x\sqrt{v_h}\left(v_h - v_l\right) + qv_x v_l\left(\sqrt{v_h}-\sqrt{v_l}\right) + v_h\sqrt{v_h}v_l\right).\end{aligned} \tag{4}$$

(iii) The low valuation player Y first period equilibrium payoff is

$$\pi_{1l}(\alpha,\beta,q) = \beta\pi_{1l}^h(q) + (1-\beta)\,\pi_{1l}^l(q),$$

where

$$\begin{aligned}\pi_{1l}^h(q) \quad &= \quad \sqrt{v_h}\sqrt{v_l}\frac{\left(\sqrt{v_h}v_l + (1-q)\,v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)\right)}{\left((1-q)\,v_h v_x + qv_x v_l + v_h v_l\right)^2} \\ &\times\left(\sqrt{v_l}v_h v_l - (1-q)\,v_x v_h\left(\sqrt{v_h}-\sqrt{v_l}\right) - qv_x\sqrt{v_l}\left(v_h - v_l\right)\right),\end{aligned} \tag{5}$$

and

$$\pi_{1l}^{l}(q) = \left(v_l \frac{\sqrt{v_l} v_h - q v_x \left(\sqrt{v_h} - \sqrt{v_l} \right)}{(1-q) v_h v_x + q v_x v_l + v_h v_l} \right)^2 . \tag{6}$$

Some of the obtained expressions are lengthy. It's the result of having a general framework that accommodates any potential two players, two periods, and two types setting with one-sided incomplete information.

The superscript ^{t} refers to the behavior of the corresponding player. For example, $\pi_{1l}^{h}(q)$ denotes the payoff that a low type of player Y obtains when behaves has a high type, while if he behaves according to his type, we write $\pi_{1h}^{h}(q)$.

There are similarities in the structure of Lemma 1 and Lemma 2. For example, expression (1) equals to expression (3) if we replace $p_{y_{1t}}$ for q or vice versa. Similarly, for expression (2) and (6). This is not surprising, in both cases, the player Y is behaving according to his type. The only difference is the amount of information available at each stage.

The reader can find the equilibrium efforts in the Proof of Lemma 2. They depend on the prior and on the specific strategy through q, which depends on α, β and p.

3.3 The Dynamic Contest

Depending on his type being high or low, on the incentives and on the information released from the first period effort choices, player Y wants to find the optimal value of α (when his type is high) or the optimal value of β (when his type is low). The full game expected payoff of the high valuation player Y is

$$\begin{aligned} \pi_h(\alpha, \beta, q, p_{y_h}, p_{y_l}) &= \alpha \pi_{1h}^{h}(q) + (1-\alpha) \pi_{1h}^{l}(q) \\ &\quad + \delta \left(\alpha \pi_{2h}^{y_{1h}}(p_{y_{1h}}) + (1-\alpha) \pi_{2h}^{y_{1l}}(p_{y_{1l}}) \right), \end{aligned}$$

and for the low valuation player Y is

$$\begin{aligned} \pi_l(\alpha, \beta, q, p_{y_h}, p_{y_l}) &= \beta \pi_{1l}^{h}(q) + (1-\beta) \pi_{1l}^{l}(q) \\ &\quad + \delta \left(\beta \pi_{2l}^{y_{1h}}(p_{y_{1h}}) + (1-\beta) \pi_{2l}^{y_{1l}}(p_{y_{1l}}) \right). \end{aligned}$$

When the player h is the one that has incentives to pool the l type effort, in equilibrium $\beta = 0$, i.e. player l plays according to his type. Then the optimal strategy of a player h is determined by the value

$$x^* \equiv \arg\max_x \pi_h(x, 0, q(x), p_{y_h}(x), p_{y_l}(x)).$$

In equilibrium, if $x^* \geq 1$ player h effort must be compatible with his type, i.e. $\alpha = 1$, while if $x^* \leq 0$ he perfectly replicates the l type effort, i.e. $\alpha = 0$. For values $x^* \in (0,1)$ the equilibrium is semi-separating, i.e. $\alpha = x^*$.

The same reasoning applies when the pooling incentives are with player l. In this case we have $\alpha = 1$, because the high type has no incentives other than playing according to his type. However, player l equilibrium effort depends on the value of

$$y^* \equiv \arg\max_y \pi_h \left(0, y, q\left(y\right), p_{y_h}\left(y\right), p_{y_l}\left(y\right)\right).$$

Then, If $y^* \geq 1$ player l optimal effort is to pool player h effort, i.e. $\beta = 1$, while if $y^* \leq 0$ he should play according to his type, i.e. $\beta = 0$. For values $y^* \in (0,1)$ player l randomizes between low and high effort, i.e. $\beta = y^*$.

The problem of finding the optimal value of x^* (or y^*) in a general setting, requires to solve a polynomial of the sixth degree (sextic), for which we do not know a general solution in close form. In fact, tractability has been one of the main obstacles to the development of a signaling theory in dynamic contests. For this reason, general result had been missing for long. Next Section goes around this problem. We follow a negative approach, in the sense that given a valuation hierarchy we ask, what are the potential equilibrium structures that cannot be part of a signaling equilibrium? More specifically, can we say when pooling or separating equilibria fail to exist for a given hierarchy of valuations and discounting?

4 Impossibility Results

In a full separating equilibrium, the observation of the first period action reveals the identity of player Y. In the second stage player X is perfectly informed about the type of player Y. In a pooling equilibrium the type of player Y with incentives to pool, replicates the effort of the opponent. Player X in the second period learns nothing new about the type of player Y. In either case, in the second period, player Y always plays according to his type. The difference is the information available in each scenario, which affects the second stage relative gains of the informed player.

Since player Y second period behavior is known, the question is how he will behave in the first period. Knowing that his effort affect not only his payoff in that period but also the information available in the subsequent stage, and consequently, his payoff in the second period.

A crucial aspect is the hierarchy of valuation among the potential interveniens in the contest. For that reason, the discussion proceeds considering each case in particular.

4.1 Case $v_h \geq v_x \geq v_l$

This hierarchy of valuations is the most commonly found in application. Player X is uncertain on whether player Y has a higher or lower valuation. In this case, it is the high type that has incentives to pool with the low type.

To get a better intuition, suppose a static setting with no uncertainty. Player X would choose a higher effort when facing a type h and a lower effort when facing a l type. Now, suppose that there is uncertainty about the player Y type.

Then the player X optimally chooses an effort in between the two complete information efforts.

The type h of player Y is the one that gains with uncertainty. He can reduce his effort and increase his payoff, because of the uniformed player's X lower effort under uncertainty (relatively to the complete information case). Player l also reduces his effort but obtains a lower payoff because of the uniformed player X high effort under uncertainty (relatively to the complete information case).

In a dynamic setting with two periods, player l would like to destroy the uncertainty by choosing in the first period an effort that player h could not profitably replicate, in this way perfectly signaling to the player X his type. On the other hand, player h would like to preserve the uncertainty by replicating player's l effort (*sandbagging*).

We will show that when $v_h \geq v_x \geq v_l$, the strategy of perfectly replicate player's l effort is not profitable for player h. We depart from the pooling path and then show that playing with probability one the same action as a player l is a strictly dominated strategy for player h.

In the pooling path, in the first period we must have $\alpha = 0$ and $\beta = 0$, and consequently $q = 0$. In the second period low effort y_{1l} is observed with probability one, player X receives no extra information and $p_{y_{1l}} = p$. Player h expected payoffs in the first and second period are respectively $\pi^{l}_{1h}(0)$ and $\pi^{y_{1l}}_{2h}(p)$.

In case of a deviation by player h from y_{1l}, to his type effort y_{1h}, in the first period we have $\alpha = 1$, while β and q remains unchanged (by the definition of Nash equilibrium). An observed high (low) effort perfectly informs the player X that player Y has a high (low) valuation. The second stage posterior changes to $p_{y_{1h}} = 1$ and $p_{y_{1l}} = 0$. Player h expected payoff in the first and second period are $\pi^{h}_{1h}(0)$ and $\pi^{y_{1h}}_{2h}(1)$, respectively.

Separating behavior always return a relatively higher payoff for the h type player in the first period (relatively to pool), i.e. $\pi^{h}_{1h}(0) \geq \pi^{l}_{1h}(0)$. On the other hand, revealing his type to the player X should return a relative loss in the second period, i.e. $\pi^{y_{1l}}_{2h}(p) \geq \pi^{y_{1h}}_{2h}(1)$. When $v_x \geq \sqrt{v_h v_l}$, incentives to deviate from the pooling path are higher when δ is small, because the second period relative loss becomes less important.

However, the inequality $\pi^{y_{1l}}_{2h}(p) \geq \pi^{y_{1h}}_{2h}(1)$ fails when $v_x \leq \sqrt{v_h v_l}$. In this case, there is no relative loss in the second period, pooling must be dominated by the full separating effort for any δ.

The following result formalizes the previous discussion and states when pooling cannot be an equilibrium.

Proposition 3 *In a two players contest with one-sided incomplete information, there are no pooling equilibria if* $v_h \geq v_x \geq v_l$ *and* $\delta \leq 1$*.*

Moreover, if $v_x \leq \sqrt{v_h v_l}$ *there is no pooling equilibria for any* $\delta \geq 0$*.*

The result states that it is too costly for a h type contestant to pretend to be a l type (*sandbagging*) when $v_h \geq v_x \geq v_l$ and $\delta \leq 1$. This is an impossibility result, and typically has a negative connotation. However, contest theory and in particular, the Tullock function has been applied in multiple economic problem, as R&D competition, war and conflicts, sports competitions, and so on, where parties may have the incentive to hide their types and mislead their opponents. Without assuming more structure, i.e. a truthtelling mechanism or a social planner, players tend to behave according to their type.

The previous comment stress the importance of the result of Proposition 3, but is somehow loose in its content, because semi-separation might be payoff superior to full separation. If that is the case, then the h type may successfully mislead the player X with positive probability. In fact, for $\delta \leq 1$ semi-separation occurs for v_x close to v_h and far from v_l, with large p.

The results of Proposition 3 hold for any prior p. This is an important robustness property that we want to stress.

Notice that if $\delta > 1$, the second contest outcome becomes more important, perfectly pooling can emerge as an equilibrium. Clearly, in such case the strength of the incentives will be dictated by the distribution of valuations, in terms of distance, and the values of p and δ.

However, assuming $\delta > 1$, requires a different interpretation than the usual discount factor. In this case δ represents the importance of the second period outcome for both players. In some applications, the outcome of the second contest may be the most important one. Then, strategic signaling in the first contest becomes more likely. Player h has more incentives to sacrifice the first prize in order to gain on the second one.

4.2 Case $v_x \geq v_h \geq v_l$

Proposition 3, relies on the fact that moving from y_{1h} to y_{1l} is too costly for player h. The result is less clear cut and does not generalize in a straightforward way when $v_x \geq v_h \geq v_l$. The reason is that now is less expensive to replicate a low type behavior.

Again, it is the high type that has incentives to pool with the low type, but now player's X valuation for the contest is the highest. In the static setting, uncertainty makes the player X choose an action that is higher (lower) than if he knew that player Y had a low (high) valuation. Player h gains with it, because his chances of winning the contest increase, as well as his payoff. Because of the higher effort of the player X (relatively to the complete information case), the type l reduces his effort and on same time obtains a lower payoff.

In the dynamic setting player h might have incentives to pool his effort with type l (sandbagging), in order to obtain a relative gain in the second period. We want to investigate when perfect pooling or full separation cannot be an equilibrium.

Similarly, to the previous Section, perfectly pooling equilibrium, requires in the first stage $\alpha = 0$ and $\beta = 0$, and consequently, $q = 0$. While, in the second stage, a low action y_{1l} is observed with probability one, and player X learns nothing new about the identity of player Y. The equilibrium path posterior is $p_{y_{1l}} = p$. Player h expected payoffs in the first and second period are respectively $\pi_{1h}^{l}(0)$ and $\pi_{2h}^{y_{1l}}(p)$.

In case of a deviation in the first period from y_{1l} to y_{1h}, we have $\alpha = 1$. On same time β and q remain unchanged and equal to 0. Consequently, in the second period, player X observes the action associated with each type, and becomes informed about the identity of the opponent. The posteriors become $p_{y_{1h}} = 1$ and $p_{y_{1l}} = 0$. Player h expected payoff in the first and second stages when deviating are respectively $\pi_{1h}^{h}(p)$ and $\pi_{2h}^{y_{1h}}(1)$.

For a player h, the full separation deviation has associated a relative gain (loss) in the first (second) period, i.e. $\pi_{1h}^{h}(0) \geq \pi_{1h}^{l}(0)$ ($\pi_{2h}^{y_{1l}}(p) \geq \pi_{2h}^{y_{1k}}(1)$). On the contrary to the previous Section, now with $v_x \geq v_h \geq v_l$, these inequalities always hold true. Then, incentives to deviate are higher when δ is small, second period losses becomes less important.

We are also interested in knowing when separation cannot be an equilibrium. In this case, we start assuming a separating path, i.e. $\alpha = 1$, $\beta = 0$ and $q = p$. Consequently, the posteriors become fully informative, i.e. $p_{y_{1h}} = 1$ and $p_{y_{1l}} = 0$. Player h expected payoffs in the first and second period are $\pi_{1h}^{h}(p)$ and $\pi_{2h}^{y_{1h}}(1)$, respectively.

The question is whether a deviation to the perfectly pooling action is profitable or not for player h. In such a case, $\alpha = 0$ and the posterior is not informative, i.e. $p_{y_{1l}} = p$. Player h expected payoffs in the first and second period are $\pi_{1h}^{l}(p)$ and $\pi_{2h}^{y_{1l}}(p)$, respectively.

Player h faces a first (second) period relative loss (gain), when he plays differently than his type in the initial stage, i.e. $\pi_{1h}^{h}(p) \geq \pi_{1h}^{l}(p)$ ($\pi_{2h}^{y_{1l}}(p) \geq \pi_{2h}^{y_{1h}}(1)$). Then incentives to deviate to perfectly pool increase with δ, because the second period gains become more important.

The previous discussion describes in great detail the incentives around each kind of equilibrium, when the valuation hierarchy satisfies $v_x \geq v_h \geq v_l$. The question is then whether by deviating in the first period from a particular equilibrium, can a player h increase his overall payoff?

The following result builds on this intuition and states a set of impossibility results for two players, two types contests with one-sided incomplete information.

Proposition 4 *In a two players contest with one-sided incomplete information, when $v_x \geq v_h \geq v_l$:*

(i) *There is no separating equilibria for any $\delta \geq 0$, when v_h and v_l are sufficiently close to each other.*

(ii) *There is no pooling equilibrium for any $\delta \leq 1$, when v_h and v_x are sufficiently close to each other.*

(iii) *There is no pooling equilibrium for any $\delta \leq 1$, when v_l is sufficiently small.*

(*iv*) *There is no pooling equilibrium for any* $\delta \leq 1$, *when* v_x *is sufficiently large.*

When compared with Proposition 3 the results presented are much less general. However, notice again the independence of the results to the prior value p.

In Part (i) we guarantee the inexistence of a separating equilibrium for values of v_h and v_l not far from each other. The intuition is that the cost of pooling is not too high, since both types are similar, the h type will pool with positive probability the effort of the l type (*sandbagging*).

Similarly Part (ii) guarantees the non-existence of pooling equilibrium for v_h close to v_x. This is the case for $\delta \leq 1$ because the limit cut-off is larger than the unit. As v_h becomes closer to v_x, it goes further way from v_l. Perfect pooling becomes relatively expensive and cannot emerge in equilibrium. Similar reasoning applies to the case where v_l goes to zero.

The connection between the non-separating result of Part (i) and the non-pooling result of Part (ii), implies the existence of semi-separating equilibrium in some interval when we vary $v_h \in [v_l, v_x]$. In the next Section we discuss this issue.

In the last Part, when v_x becomes arbitrary large with respect to the other valuations pooling cannot be an equilibrium. The reason is that the great valuation of the player X causes a discouragement effect on a h player. The latter feels that player X is much more likely to win both periods' contests, then the optimal strategy is to play close to his own type in the first period. Misleading the player X would not lead to a significant decrease on his second period effort.

Notice also that typically separating equilibrium fails to exist for δ sufficiently larger than the unit.

4.3 Case $v_h \geq v_l \geq v_x$

In this valuation hierarchy, both the low and the high type of player Y value more the contested object than player X.

In the static problem with uncertainty, player X chooses an effort that is lower (higher) than the one he would provide if he knew that player Y has low (high) valuation. If the player X would be sure of facing a high valuation type he would provide less effort (relatively to the incomplete information case). There is a discouragement effect, in a sense player X acknowledges player's h higher willingness to win the contest. On the other hand, if the player X would be sure of facing a low valuation type, he would feel that his winning chances are higher, entering in the contest with a greater effort.

Because the player X provides less effort, uncertainty favors the player l which obtain a higher expected payoff in this context. On the other hand, player h has to provide a higher effort to compensate the relatively higher effort of a hopeful player X. Consequently, his payoff is lower than in the perfect information scenario.

In the dynamic setting, in order to benefit from a discouragement effect, player h would like to credibly inform the player X that he is a high valuation individual. On the other hand, player l prefers that the true valuation stays unknown, to benefit from a lower effort of player X. Player l is the one that has incentives to pool on player h. In this case, he *bluffs* pretending a high valuation.

Likewise, in previous Section, we depart from the separating equilibrium and see when a player l can profitably pool on player's h effort. When such is the case, there is no separating equilibrium. On same time, we assume a pooling equilibrium and check if a player l can profitably deviate from it.

In the separating or in the pooling equilibrium path $\alpha = 1$, i.e. player h behaves according to his type. While $\beta = 0$ when the player l effort choice is according to his type, and $\beta = 1$ when l perfectly replicates player's h effort.

A deviation by player l from the separating equilibrium, changes the second stage information structure from the full revealing posteriors $p_{y_{1h}} = 1$ and $p_{y_{1l}} = 0$, to $p_{y_{1h}} = p$ (when the deviation is a perfect pool). A deviation from the first period separating path leads to a relative loss, i.e. $\pi_{1l}^{l}(p) \geq \pi_{1l}^{h}(p)$. Incentives to pool exist if $\pi_{2l}^{y_{1h}}(p) \geq \pi_{2l}^{y_{1l}}(0)$. Since deviating gains are in the second period, larger is δ, larger are the incentives to deviate.

Similarly, a deviation from the pooling equilibrium affects the second period information structure in the opposite order, i.e. from $p_{y_{1h}} = p$, to $p_{y_{1h}} = 1$ and $p_{y_{1l}} = 0$ (when the deviation is full revealing). A deviation from the first period pooling path leads to a relative gain in that stage, i.e. $\pi_{1l}^{l}(1) \geq \pi_{1l}^{h}(1)$, but a second period relatively loss, i.e. $\pi_{2l}^{y_{1h}}(p) \geq \pi_{2l}^{y_{1l}}(0)$. Deviation gains are in the first period, then incentives to deviate are higher when δ is small.

Following the previous discussion, we state, which signaling behaviors cannot be part of a perfect Bayesian equilibrium.

Proposition 5 *In a two players contest with one-sided incomplete information, when* $v_h \geq v_l \geq v_x$:

(i) *There is no separating equilibria for any* $\delta \geq 0$, *when* v_h *and* v_l *are sufficiently close to each other.*

(ii) *There is no pooling equilibria for* $\delta \leq 1$, *when* v_x *and* v_l *are sufficient close to each other.*

(iii) *There is no pooling equilibrium for* $\delta \leq \left(\sqrt{v_h} - \sqrt{v_l}\right)/2\sqrt{v_h}$, *when* v_x *is sufficiently small.*

(iv) *There is no pooling equilibria for* $\delta \leq (v_l + v_x)^2 / v_l (2v_l + v_x)$, *when* v_h *is sufficiently large.*

The results of Part (i) and (ii) are of more general interest and application. Typically, v_h is bounded and v_x is away from zero. Part (i) guarantees the inexistence of a separating equilibrium when the values of v_h and v_l are not far from each other. The intuition relies on the low pooling costs, making *bluffing* possible.

Similarly, the Part (ii) guarantee non-existence of pooling equilibrium for v_x close to v_l. When v_l approaches v_x, the pooling costs increase, because v_l gets further way from v_h. Replicate the effort of a h player becomes more costly for a l player.

These results help us to understand the structure of equilibria for $v_l \in [v_x, v_h]$, as for example the existence of semi-separating equilibria. Notice that by Part (*ii*) at $v_l = v_x$, we must have $\beta_l \in [0,1)$, i.e. no pooling equilibrium. On the other extreme, by Part (*i*) at $v_l = v_h$ there is no separating equilibrium, i.e. $\beta_h \in (0,1]$. By monotonicity there might be a region inside $[v_x, v_h]$ that neither pooling nor separating equilibria are possible, i.e. $\beta \in (0,1)$.

To fix ideas suppose that $\delta \leq 1$, $\beta_l = 0$, and $\beta_h = 1$, and progressively increase v_l on the interval $[v_x, v_h]$, starting from the non-pooling equilibrium of Part (*ii*). As v_l increases there is a point in (v_x, v_h), call it $\underline{v}_l$, where β changes smoothly from 0 to $\beta > 0$. Then as v_l keeps increasing, $\beta > 0$ and increases monotonically until the point $\overline{v}_l$, where the optimal action changes discretely from some $\beta \in (0,1)$ to $\beta = 1$. The previous exercise points out that semi-separating equilibria need not fill all the interval $\beta \in (0,1)$, there is a discontinuity at some point.

A similar analysis extends to the previous Section for v_h varying in $[v_x, v_l]$.

In Part (*iii*) letting $v_x \downarrow 0$ is equivalent to say that player's X valuation is almost irrelevant. Nevertheless, player l may still have incentives to mislead the player X, creating a discouragement effect. However, pooling cannot emerge in equilibrium, if the difference between v_h and v_l is large, i.e. high pooling costs, and the second period is of little importance. It is also worth noticing that $\left(\sqrt{v_h} - \sqrt{v_l}\right)/2\sqrt{v_h} \in (0, 1/2)$. This is the main message of Part (*iii*) of Proposition 5.

Part (*iv*) describes a situation where the player Y of type h has a great relative valuation. A type l of player Y, acknowledges that to replicate the effort of a high valuation type h is extremely costly. Then for a player l is better to play according to his type, in particular, if δ is small. The cut-off threshold is large when v_x is close to v_l. Incentives to pool are lower when the player X has a strong valuation and close to player l.

Notice that since $(v_l + v_x)^2 / v_l (2v_l + v_x) \in (1/2, 4/3)$, we can guarantee the non-existence of pooling equilibrium for any $\delta \leq 1/2$. If in addition $v_l^2 \geq v_x (v_l + v_x)$, we can say that there is no pooling for $\delta \leq 1$.

In Part (*iii*) and (*iv*) of Proposition 5, behaving according to his type is the most intuitive and expected equilibrium. However, for large δ pooling equilibrium cannot be excluded.

Again, we stress the independence of the stated results to the prior value p.

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A Appendix

Proof of Lemma 1. After observing the effort y_{1h} player X updates his beliefs about the identity of player Y. He believes that with probability $p_{y_{1h}}$ player Y has a high valuation. In the second period player X solves

$$\max_{x_2} p_{y_{1h}} \frac{x_2}{x_2 + y_{2h}} v_x + (1 - p_{y_{1h}}) \frac{x_2}{x_2 + y_{2l}} v_x - x_2. \tag{7}$$

Similarly, after observing the low effort y_{1l}, player X posterior is $p_{y_{1l}}$. In the second period player X solves

$$\max_{x_2} p_{y_{1l}} \frac{x_2}{x_2 + y_{2h}} v_x + (1 - p_{y_{1l}}) \frac{x_2}{x_2 + y_{2l}} v_x - x_2. \tag{8}$$

In the second period Y plays according to his type, but constrained by the observed effort, i.e. he knows whether x_2 maximizes (7) or (8). Then in the second period player Y of type $t \in \{h, l\}$ solves

$$\max_{y_{2t}} \frac{y_{2t}}{x_2 + y_{2t}} v_t - y_{2t}. \tag{9}$$

Rearrange the first order conditions from (9), and plug them into the first order conditions obtained from (7) and (8). Solve for x_2 each equality. The solution gives a general expression for the second period player X effort (conditional on the observed effort $y_{1t} \in \{y_{1h}, y_{1l}\}$), i.e.

$$x_2^{y_{1t}} (p_{y_{1t}}) = \frac{v_h v_l v_x^2 \left((1 - p_{y_{1t}}) \sqrt{v_h} + p_{y_{1t}} \sqrt{v_l}\right)^2}{\left((1 - p_{y_{1t}}) v_h v_x + p_{y_{1t}} v_x v_l + v_h v_l\right)^2}.$$

Replace this expression on the first order conditions obtained from (9) for $t \in \{h, l\}$, after some algebra. We obtain the general second period equilibrium effort for the low and high valuation type of player Y (conditional on the observed first period effort y_{1t}), i.e.

$$\begin{aligned} y_{2l}^{y_{1t}} (p_{y_{1t}}) &= \frac{\sqrt{v_h}\sqrt{v_l} v_x v_l \left((1 - p_{y_{1t}}) \sqrt{v_h} + p_{y_{1t}} \sqrt{v_l}\right)}{\left((1 - p_{y_{1t}}) v_h v_x + p_{y_{1t}} v_x v_l + v_h v_l\right)^2} \\ &\quad \times \left(\sqrt{v_l} v_h + p_{y_{1t}} v_x \left(\sqrt{v_l} - \sqrt{v_h}\right)\right), \end{aligned}$$

and

$$y_{2h}^{y_{1t}}\left(p_{y_{1t}}\right) = \frac{\sqrt{v_h}\sqrt{v_l}v_h v_x\left(\left(1-p_{y_{1t}}\right)\sqrt{v_h}+p_{y_{1t}}\sqrt{v_l}\right)}{\left(\left(1-p_{y_{1t}}\right)v_h v_x+p_{y_{1t}}v_x v_l+v_h v_l\right)^2} \times\left(\sqrt{v_h}v_l+\left(1-p_{y_{1t}}\right)v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)\right).$$

Finally, replace the equilibrium efforts $x_2^{y_{1t}}\left(p_{y_{1t}}\right)$, $y_{2l}^{y_{1t}}\left(p_{y_{1t}}\right)$ and $y_{2h}^{y_{1t}}\left(p_{y_{1t}}\right)$ into the objective functions in (7), (8) and (9), to obtain the desired results. ■

Proof of Lemma 2. We are looking for the values of x_1, y_{1h} and y_{1l}, that can be expressed in terms of q and the players' valuations. Player X is uncertainty not only about the type of player Y but also on the frequency in which each type plays each action. Player X solves

$$\max_{x_1} q\frac{x_1}{x_1+y_{1h}}v_x+(1-q)\frac{x_1}{x_1+y_{1l}}v_x-x_1. \tag{10}$$

In equilibrium the high effort choice y_{1h} must be optimal to the player h, similarly the low effort choice y_{1l} must be optimal to player l. Then, depending on his type $t\in\{h,l\}$, player Y solves

$$\max_{y_{1t}}\frac{y_{1t}}{x_1+y_{1t}}v_t-y_{1t}. \tag{11}$$

The first order condition of (10) and the two first order conditions of (11), together form a system of three equations and three unknowns. After some algebra, we obtain the first period efforts as a function of q and the valuations, i.e.

$$x_1(q)=\frac{v_h v_l v_x^2\left((1-q)\sqrt{v_h}+q\sqrt{v_l}\right)^2}{\left((1-q)v_h v_x+q v_l v_x+v_l v_h\right)^2},$$

$$y_{1l}(q)=\frac{\sqrt{v_l}v_l\sqrt{v_h}v_x\left((1-q)\sqrt{v_h}+q\sqrt{v_l}\right)\left(\sqrt{v_l}v_h-q v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)\right)}{\left((1-q)v_h v_x+q v_x v_l+v_h v_l\right)^2},$$

and

$$y_{1h}(q)=\frac{\sqrt{v_h}v_h\sqrt{v_l}v_x\left((1-q)\sqrt{v_h}+q\sqrt{v_l}\right)\left(\sqrt{v_h}v_l+(1-q)v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)\right)}{\left((1-q)v_h v_x+q v_x v_l+v_h v_l\right)^2}.$$

Notice that q is a function of α, β and p. Plug $x_1(q)$, $y_{1l}(q)$ and $y_{1h}(q)$ in player's X objective function (10), we obtain the expression in Part (i).

Player's Y first period payoff depends on whether he is of the h or l type. For player h we have

$$\pi_{1h}(\alpha,\beta,p)=\alpha\pi_{1h}^h(\alpha,\beta,p)+(1-\alpha)\pi_{1h}^l(\alpha,\beta,p),$$

where

$$\pi_{1h}^h(\alpha,\beta,p)\equiv\frac{y_{1h}(q)}{x_1(q)+y_{1h}(q)}v_h-y_{1h}(q), \tag{12}$$

and

$$\pi_{1h}^{l}(\alpha,\beta,p) \equiv \frac{y_{1l}(q)}{x_1(q)+y_{1l}(q)} v_h - y_{1l}(q). \tag{13}$$

For player l we have

$$\pi_{1l}(\alpha,\beta,p) = \beta\pi_{1l}^{h}(\alpha,\beta,p) + (1-\beta)\pi_{1l}^{l}(\alpha,\beta,p),$$

where

$$\pi_{1l}^{h}(\alpha,\beta,p) \equiv \frac{y_{1h}(q)}{x_1(q)+y_{1h}(q)} v_l - y_{1h}(q), \tag{14}$$

and

$$\pi_{1l}^{l}(\alpha,\beta,p) \equiv \frac{y_{1l}(q)}{x_1(q)+y_{1l}(q)} v_l - y_{1l}(q). \tag{15}$$

Now plug $x_1(q)$, $y_{1l}(q)$ and $y_{1h}(q)$ in expressions (12), (13), (14) and (15). To obtain, after some algebraic manipulations, respectively (3), (4), (5) and (6), which depend on q. ■

Proof of Proposition 3. When $v_h \geq v_x \geq v_l$ player h is the one that has incentives to pool with player l. A perfectly pooling behavior implies; in the first period, $\alpha = 0$, $\beta = 0$ and $q = 0$, the second period equilibrium path posterior is $p_{y_{1l}} = p$. ($p_{y_{1h}}$ is off-the-equilibrium path) Player h pooling payoff is,

$$\pi_h(0,0,0,p_{y_{1h}},p) = \pi_{1h}^{l}(0) + \delta\pi_{2h}^{y_{1l}}(p),$$

with

$$\pi_{1h}^{l}(0) = \frac{v_l(v_x v_h - v_x v_l + v_h v_l)}{(v_x+v_l)^2}, \tag{16}$$

given by (4) in Lemma 2, and

$$\pi_{2h}^{y_{1l}}(p) = \left(v_h \frac{\sqrt{v_h}v_l + (1-p)v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)}{(1-p)v_h v_x + p v_l v_x + v_l v_h}\right)^2, \tag{17}$$

given by (1) in Lemma 1. If the player h deviates in the first period, to the full separating effort y_{1h}, we have $\alpha = 1$. The value q remains unchanged and equal to 0. The posteriors change to $p_{y_{1h}} = 1$ and $p_{y_{1l}} = 0$. Player h payoff associated with a full separating deviation is

$$\pi_h(1,0,0,1,0) = \pi_{1h}^{h}(0) + \delta\pi_{2h}^{y_{1h}}(1),$$

with

$$\pi_{1h}^{h}(0) = \frac{\left(\sqrt{v_h}v_l + v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)\right)^2}{(v_x+v_l)^2}, \tag{18}$$

given by (3) in Lemma 2, and

$$\pi_{2h}^{y_{1h}}(1) = \frac{v_h^3}{(v_x+v_h)^2}, \tag{19}$$

given by (1) in Lemma 1. We have $\pi_{1h}^{h}(0) \geq \pi_{1h}^{l}(0)$, i.e. separating behavior should return a higher payoff in the first period. While, in the second period we should expect the deviating player h to obtain a relative loss from revealing his type to the player X, i.e. $\pi_{2h}^{y_{1l}}(p) \geq \pi_{2h}^{y_{1h}}(1)$. However, the latter inequality fails, for any non-negative $v_x \leq \sqrt{v_h v_l}$. In this case, since there is no relative loss in the second period, the pooling equilibrium is dominated by the full separating effort for all $\delta \geq 0$.

Outside this case, incentives to deviate are higher when δ is small. Then for $0 \leq \delta \leq \delta^*$, there is no pooling equilibrium. The value of δ^* is given by

$$\begin{aligned} \delta^* &\equiv \frac{\pi_{1h}^{h}(0) - \pi_{1h}^{l}(0)}{\pi_{2h}^{y_{1l}}(p) - \pi_{2h}^{y_h}(1)} \\ &= \frac{v_x\left(\sqrt{v_h} - \sqrt{v_l}\right)^2}{v_h^2\left(v_x + v_l\right)^2\left(\left(\frac{\sqrt{v_h}v_l + (1-p)v_x\left(\sqrt{v_h} - \sqrt{v_l}\right)}{(1-p)v_h v_x + p v_l v_x + v_l v_h}\right)^2 - \frac{v_h}{(v_x + v_h)^2}\right)}. \end{aligned} \tag{20}$$

The expression for δ^* is general and gives a cut-off discount factor value below which deviations from pooling to perfect separation are profitable. Since p can take any value in $(0,1)$ and the relation $v_h \geq v_x \geq v_l$ may appear in different forms, it is hard to establish general conclusions. We will look at the asymptotics of expression (20). The highest and lowest incentives for deviating must be at the extreme values. Our goal is to show when pooling cannot be an equilibrium.

The function δ^* has a vertical asymptote at $v_x = \sqrt{v_h v_l}$, i.e. $\delta^* \to -\infty$ when $v_x \uparrow \sqrt{v_h v_l}$, and $\delta^* \to \infty$ when $v_x \downarrow \sqrt{v_h v_l}$, being monotonically decreasing in $v_x \in \left(\sqrt{v_h v_l}, v_h\right]$. Then the highest incentives must be when v_x and v_h are close to each other. For completeness let's look at all cases.

The limit of δ^* when $v_l \uparrow v_x$ or $v_x \downarrow v_l$ must be negative since $v_x \leq \sqrt{v_h v_x}$. In this case the pooling equilibrium is dominated by the full separating action, because there is no relative loss in the second period. Then for any $\delta \geq 0$ there is no pooling equilibrium.

Now let's look at the case when (20) reaches the minimum value in the interval $v_x \in \left(\sqrt{v_h v_l}, v_h\right]$, i.e. when $v_h \downarrow v_x$ or $v_x \uparrow v_h$ to obtain

$$\delta^* \to \frac{\left(\sqrt{v_x} - \sqrt{v_l}\right)^2}{\left(v_x + v_l\right)\left(\left(\frac{v_l + (1-p)\sqrt{v_x}\left(\sqrt{v_x} - \sqrt{v_l}\right)}{(1-p)v_x + p v_l + v_l}\right)^2 - \frac{1}{4}\right)}$$

where $v_x = v_h$. This limit expression has a vertical asymptote at $p = 1$ which does not depend on the values of v_x and v_l; it goes to ∞ when $p \uparrow 1$ and to $-\infty$ when $p \downarrow 1$. Moreover, it is monotonically increasing in $p \in (0,1)$. Then its lowest value on the interval $\left(\sqrt{v_h v_l}, v_h\right]$ must be obtained at $p \downarrow 0$, and equals to

$$\frac{4\left(v_x + v_l\right)}{2(v_x + v_l) + \left(\sqrt{v_x} + \sqrt{v_l}\right)^2},$$

which is always larger than one, since $\left(\sqrt{v_x}-\sqrt{v_l}\right)^2 \geq 0$. Then we conclude that for any $p \in (0,1)$ there is no perfectly pooling equilibrium for $\delta \leq 1$.

The limit of δ^* when $v_l \downarrow 0$ gives $\left(v_x+v_h\right)^2 / v_x\left(v_x+2v_h\right)$ which is always larger than one for any $v_x \leq v_h$. Analogously, we could show that the vertical asymptote of (20) moves to $v_x = 0$, and (20) is now monotonically decreasing in $v_x \in (0, v_h]$. Then pooling cannot be an equilibrium for $\delta \leq 1$.

Finally take the limit of δ^* when $v_h \uparrow \infty$. In this case we have again $\pi_{2h}^{y_h}(p) \leq \pi_{2h}^{y_h}(1)$ because $v_x < \infty$. The denominator of (20) becomes negative, there is no relative loss in the second period, the pooling equilibrium is dominated by full separation. Then, for any $\delta \geq 0$ there is no pooling equilibrium. ∎

Proof of Proposition 4. When $v_x \geq v_h \geq v_l$ player h is the one that has incentives to pool on the player l effort. We start by checking whether there are profitable (full separating) deviations from the pooling path. Perfectly pooling behavior implies, in the first period, $\alpha = 0$, $\beta = 0$ and $q = 0$. The equilibrium path posterior is $p_{y_{1l}} = p$. Player h pooling payoff is

$$\pi_h\left(0,0,0,p_{y_{1h}},p\right) = \pi_{1h}^l(0) + \delta\pi_{2h}^{y_{1l}}(p),$$

with $\pi_{1h}^l(0)$ given by (16) and $\pi_{2h}^{y_{1l}}(p)$ given by (17).

If h deviates in the first period to the separating effort y_{1h}, we have $\alpha = 1$, while q remains unchanged and equal to 0. The posteriors become $p_{y_{1h}} = 1$ and $p_{y_{1l}} = 0$. Player h payoff associated with a deviation to the full separating effort is

$$\pi_h(1,0,0,1,0) = \pi_{1h}^h(0) + \delta\pi_{2h}^{y_{1h}}(1),$$

with $\pi_{1h}^h(0)$ given by (18) and $\pi_{1h}^h(0)$ given by (19).

We have $\pi_{1h}^h(0) \geq \pi_{1h}^l(0)$, i.e. separating behavior returns a higher payoff in the first period. While in the second period the deviating player h obtains a relative loss from revealing his type to the player X, i.e. $\pi_{2h}^{y_{1l}}(p) \geq \pi_{2h}^{y_{1k}}(1)$. Since $v_x \geq v_h \geq v_l$, on contrary to the Proof of Proposition 3, these inequalities always hold true.

Incentives to deviate are higher when δ is small. Then if $\delta \leq \delta^*$, there is no pooling equilibrium. The value of δ^* is given by (20).

Contrary to the Proof of Proposition 3, the expression δ^* is not always monotonic in $v_h \in [v_l, v_x]$. Moreover, p can take any value in $(0,1)$, and $v_x \geq v_h \geq v_l$ may appear in different forms. We look at asymptotic impossibility results that hold for any p. Expression (20) is strictly positive. Its lowest value occurs always at an extreme point, but not its maximum.

(i_a) Let's start by taking the limits of δ^* when $v_l \uparrow v_h$ and $v_h \downarrow v_l$ to obtain the value 0. Then, independently of the prior p, deviations to the full separation effort are not profitable for any feasible δ, i.e. no pooling only if $\delta \leq 0$.

(ii_a) Now, take the limits of δ^* when $v_h \uparrow v_x$ or $v_x \downarrow v_h$ to obtain

$$\delta^* \to \frac{\left(\sqrt{v_x}-\sqrt{v_l}\right)^2}{\left(v_x+v_l\right)^2\left(\left(\frac{v_l+(1-p)\sqrt{v_x}\left(\sqrt{v_x}-\sqrt{v_l}\right)}{(1-p)v_x+pv_l+v_l}\right)^2-\frac{1}{4}\right)}, \tag{21}$$

where $v_x = v_h$. This limit expression has a vertical asymptote at $p = 1$ which does not depend on the values of v_x and v_l; it goes to ∞ when $p \uparrow 1$ and to $-\infty$ when $p \downarrow 1$. Moreover, (21) is monotonically increasing in $p \in (0, 1)$. Then its lowest value on this interval must be obtained at $p \downarrow 0$, and equals to

$$\frac{4\left(v_x + v_l\right)}{2(v_x + v_l) + \left(\sqrt{v_x} - \sqrt{v_l}\right)^2},$$

which is always larger than one for any positive v_x and v_l. Then for any $p \in (0, 1)$ and v_h and v_x are sufficiently close, there is no perfectly pooling equilibrium for $\delta \leq 1$.

(iii_a) The limit of δ^* when $v_l \downarrow 0$ gives the value $\left(v_x + v_h\right)^2 / v_x \left(v_x + 2v_h\right)$ which is always larger than one. More specifically, there is no pooling equilibrium for $\delta \leq 1$ when v_l is close to zero 0.

(iv_a) Finally let's take the limit of δ^* for $v_x \uparrow \infty$ to obtain

$$\delta^* \to \frac{\left(\left(1 - p\right) v_h + p v_l\right)^2}{\left(1 - p\right)^2 v_h^2},$$

which is always larger than one for any $p \in (0, 1)$. Then, there is no pooling equilibrium for $\delta \leq 1$.

Now, we look for profitable (perfectly pooling) deviations from the full separating path. The separating path implies, in the first period, $\alpha = 1$, $\beta = 0$ and $q = p$. The second period posteriors are $p_{y_{1h}} = 1$ and $p_{y_{1l}} = 0$. Player h separating equilibrium payoff is

$$\pi_h\left(1, 0, p, 1, 0\right) = \pi_{1h}^h\left(p\right) + \delta \pi_{2h}^{y_{1h}}\left(1\right),$$

with

$$\pi_{1h}^h\left(p\right) = \left(v_h \frac{\sqrt{v_h} v_l + \left(1 - p\right) v_x \left(\sqrt{v_h} - \sqrt{v_l}\right)}{\left(1 - p\right) v_h v_x + p v_x v_l + v_h v_l}\right)^2$$

given by (3) in Lemma 2, and

$$\pi_{2h}^{y_{1h}}\left(1\right) = \frac{v_h^3}{\left(v_x + v_h\right)^2}$$

given by (1) in Lemma 1. If h deviates, by pooling on l first period effort y_{1l}, we have $\alpha = 0$. The deviation path posterior is now $p_{y_{1l}} = p$. Then player's h associated payoff is

$$\pi_h\left(0, 0, p, p_{y_h}, p\right) = \pi_{1h}^l\left(p\right) + \delta \pi_{2h}^{y_{1l}}\left(p\right),$$

with

$$\begin{aligned}
\pi_{1h}^l\left(p\right) \quad &= \quad \sqrt{v_h}\sqrt{v_l} \frac{\left(\sqrt{v_l} v_h - p v_x \left(\sqrt{v_h} - \sqrt{v_l}\right)\right)}{\left(\left(1 - p\right) v_h v_x + p v_x v_l + v_h v_l\right)^2} \\
&\quad \times \left(\left(1 - p\right) v_x \sqrt{v_h} \left(v_h - v_l\right) + p v_x v_l \left(\sqrt{v_h} - \sqrt{v_l}\right) + v_h \sqrt{v_h} v_l\right).
\end{aligned}$$

given by (4) in Lemma 2, and

$$\pi_{2h}^{y_{1l}}(p) = \left(v_h \frac{\sqrt{v_h} v_l + (1-p) v_x \left(\sqrt{v_h} - \sqrt{v_l} \right)}{(1-p) v_h v_x + p v_x v_l + v_h v_l} \right)^2$$

given by (1) in Lemma 1.

Notice that we have; $\pi_{1h}(p) = \pi_{2h}^{y_{1l}}(p)$, and $\pi_{1h}^{h}(p) \geq \pi_{1h}^{l}(p)$, i.e. a relative loss in the first stage from deviating, and $\pi_{2h}^{y_{1l}}(p) > \pi_{2h}^{y_{1h}}(1)$, i.e. a relative gain in the second period due to the first period deviation. The incentives to deviate (pooling) increase with δ. We want to know when separating equilibria fails to exist, i.e. when

$$\begin{aligned} \delta \quad \geq \quad \delta^* &\equiv \frac{\pi_{1h}^{h}(p) - \pi_{1h}^{l}(p)}{\pi_{2h}^{y_{1l}}(p) - \pi_{2h}^{y_{1h}}(1)} \\ &= \frac{1 - \frac{\sqrt{v_h}\sqrt{v_l}\left(\sqrt{v_l}v_h - p v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)\right)\left((1-p)v_x\sqrt{v_h}(v_h-v_l)+p v_x v_l\left(\sqrt{v_h}-\sqrt{v_l}\right)+v_h\sqrt{v_h}v_l\right)}{v_h^2\left(\sqrt{v_h}v_l+(1-p)v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)\right)^2}}{1 - \frac{v_h\left((1-p)v_h v_x + p v_l v_x + v_l v_h\right)^2}{(v_x+v_h)^2\left(\sqrt{v_h}v_l+(1-p)v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)\right)^2}} \end{aligned} \tag{22}$$

Again, we look at extreme values of Expressions (22).

(i_b) The limits of δ^* when $v_l \uparrow v_h$ or $v_h \downarrow v_l$ takes the value 0 in both cases. Together with (ii), independently of the prior p, full separation cannot be an equilibrium for any $\delta \geq 0$.

(ii_b) Now, take the limit of δ^* when $v_h \uparrow v_x$ to obtain

$$\delta^* \to \frac{1 - \frac{\sqrt{v_l}\left(\sqrt{v_l} - p\left(\sqrt{v_x}-\sqrt{v_l}\right)\right)\left((1-p)\sqrt{v_x}(v_x-v_l)+p v_l\left(\sqrt{v_x}-\sqrt{v_l}\right)+\sqrt{v_x}v_l\right)}{\sqrt{v_x}\left(v_l+(1-p)\sqrt{v_x}\left(\sqrt{v_x}-\sqrt{v_l}\right)\right)^2}}{1 - \frac{\left((1-p)v_x + p v_l + v_l\right)^2}{4\left(v_l+(1-p)\sqrt{v_x}\left(\sqrt{v_x}-\sqrt{v_l}\right)\right)^2}}.$$

Similarly, to the limit found in (21), the previous expression has a vertical asymptote at $p = 1$ which does not depend on the values of v_x and v_l, and it is monotonically increasing in $p \in (0,1)$. Then its lowest value on this interval is obtained at $p \downarrow 0$, and it is equal to

$$\frac{4(v_x + v_l)}{2(v_x + v_l) + \left(\sqrt{v_x} - \sqrt{v_l}\right)^2},$$

which is always larger than one. Then for any $p \in (0,1)$ there is no separating equilibrium for $\delta \geq \delta^* \geq 1$.

(iii_b) The limit of δ^* when $v_l \downarrow 0$ gives again the value $(v_x + v_h)^2 / v_x (v_x + 2v_h)$ which is always larger than one. Then there is no separating equilibrium for $\delta \geq \delta^* \geq 1$.

(iv_b) Finally, take the limit of δ^* for $v_x \uparrow \infty$ to obtain

$$\delta^* \to \frac{\left((1-p)\sqrt{v_h} + p\sqrt{v_l}\right)\left((1-p) v_h + p v_l\right)}{(1-p)^2 \sqrt{v_h} v_h},$$

which is always larger than one for any $p \in (0,1)$. There is no separating equilibrium for $\delta \geq \delta^* \geq 1$.

Some of the limits obtained in the first and second parts of the proof are similar. Since the general cutoffs (20) and (22) are different, the proof would not be complete without looking at all cases. Placing together the asymptotic inequalities, the results stated in (i), (ii), (iii) and (iv) obtain. ■

Proof of Proposition 5. When $v_h \geq v_l \geq v_x$ player l is the one that has incentives to pool with a player h effort. In a separating equilibrium, in the first period we have $\alpha = 1$, $\beta = 0$ and $q = p$. Because the first period actions are full revealing the posteriors are $p_{y_{1h}} = 1$ and $p_{y_{1l}} = 0$. Player l payoff is

$$\pi_l(1,0,p,1,0) = \pi_{1l}^l(p) + \delta \pi_{2l}^{y_{1l}}(0),$$

with

$$\pi_{1l}^l(p) = \left(v_l \frac{\sqrt{v_l} v_h - p v_x \left(\sqrt{v_h} - \sqrt{v_l}\right)}{(1-p) v_h v_x + p v_l v_x + v_l v_h} \right)^2,$$

given by (6) in Lemma 2, and

$$\pi_{2l}^{y_{1l}}(0) = \frac{v_l^3}{(v_x + v_l)^2},$$

given by (2) in Lemma 1. In case of a deviation to the perfectly pooling effort, in the first stage, we have $\beta = 1$, while $\alpha = 1$ and $q = p$ stay unchanged. In the second period we have the posterior $p_{y_{1h}} = p$. Player l payoff is

$$\pi_l(1,1,p,p,p_{y_{1l}}) = \pi_{1l}^h(p) + \delta \pi_{2l}^{y_{1h}}(p),$$

with

$$\begin{aligned} \pi_{1l}^h(p) &= \sqrt{v_h}\sqrt{v_l} \frac{\left(\sqrt{v_h} v_l + (1-p) v_x \left(\sqrt{v_h} - \sqrt{v_l}\right)\right)}{\left((1-p) v_h v_x + p v_l v_x + v_l v_h\right)^2} \\ &\quad \times \left(\sqrt{v_l} v_h v_l - (1-p) v_x v_h \left(\sqrt{v_h} - \sqrt{v_l}\right) - p v_x \sqrt{v_l}\left(v_h - v_l\right)\right), \end{aligned}$$

given by (5) in Lemma 2, and

$$\pi_{2l}^{y_{1h}}(p) = \left(v_l \frac{\sqrt{v_l} v_h - p v_x \left(\sqrt{v_h} - \sqrt{v_l}\right)}{(1-p) v_h v_x + p v_l v_x + v_l v_h} \right)^2,$$

given by (2) in Lemma 1.

A deviation from the first period separating path leads to a relative loss, i.e. $\pi_{1l}^l(p) \geq \pi_{1l}^h(p)$. Incentives to pool exist when $\pi_{2l}^{y_{1h}}(p) \geq \pi_{2l}^{y_{1l}}(0)$. Both inequalities hold true for $v_h \geq v_l \geq v_x$. The cut-off discount factor δ^* is non-negative and is given by

$$\begin{aligned}
\delta^* &\equiv \frac{\pi_{1l}^{l}(p) - \pi_{1l}^{h}(p)}{\pi_{2l}^{y_{1h}}(p) - \pi_{2l}^{y_{1l}}(0)} \\
&= \frac{1 - \frac{\sqrt{v_h}\sqrt{v_l}\left(\sqrt{v_h}v_l + (1-p)v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)\right)\left(\sqrt{v_l}v_h v_l - (1-p)v_x v_h\left(\sqrt{v_h}-\sqrt{v_l}\right) - p v_x\sqrt{v_l}(v_h - v_l)\right)}{v_l^2\left(\sqrt{v_l}v_h - p v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)\right)^2}}{1 - \frac{v_l((1-p)v_h v_x + p v_l v_x + v_l v_h)^2}{(v_x+v_l)^2\left(\sqrt{v_l}v_h - p v_x\left(\sqrt{v_h}-\sqrt{v_l}\right)\right)^2}}.
\end{aligned}$$

When $\delta \geq \delta^*$ there in no separating equilibrium. This expression has nice properties; it is monotonically decreasing in $v_l \in [v_x, v_h]$ and monotonically increasing for $v_x \in [0, v_l]$ and $v_h \in [v_l, \infty)$.

(i_a) The limits of δ^* when $v_l \uparrow v_h$ or $v_h \downarrow v_l$ takes the value 0, i.e. when v_l and v_h are close to each other, there is no separating equilibrium for all $\delta \geq 0$.

(ii_a) The limit of δ^* when $v_x \uparrow v_l$ or $v_l \downarrow v_x$, takes the value

$$\delta^* \to \frac{1 - \frac{\sqrt{v_h}\sqrt{v_x}\left(\sqrt{v_h} + (1-p)\left(\sqrt{v_h}-\sqrt{v_x}\right)\right)\left(\sqrt{v_x}v_h - (1-p)v_h\left(\sqrt{v_h}-\sqrt{v_x}\right) - p\sqrt{v_x}(v_h - v_x)\right)}{v_x\left(v_h - p\sqrt{v_x}\left(\sqrt{v_h}-\sqrt{v_x}\right)\right)^2}}{1 - \frac{((1-p)v_h + p v_x + v_h)^2}{4\left(v_h - p\sqrt{v_x}\left(\sqrt{v_h}-\sqrt{v_x}\right)\right)^2}},$$

where $v_x = v_l$. We are interested to know when this limit is smaller than one, because in that situation, there will be values of $\delta \geq \lim \delta^* \in [0,1]$ where separation is impossible. Solving for p, we obtain two roots

$$p = \frac{2\sqrt{v_h}\left(3v_h - \sqrt{v_h}\sqrt{v_x} - v_x \pm \sqrt{v_h^2 + v_x^2 - v_h v_x}\right)}{4\sqrt{v_h}v_h - 3v_h\sqrt{v_x} - 2\sqrt{v_h}v_x + \sqrt{v_x}v_x}.$$

Their shape is similar, but one root is smaller than the other. For fix v_x both roots decrease monotonically with v_h. With maximum when $v_h \downarrow v_x$ and minimum when $v_h \uparrow \infty$. The limit when $v_h \uparrow \infty$ converges to 1 and 2 for the smaller and the larger root respectively. Then if v_x and v_l are close to each other, there is no separating equilibrium for $\delta \geq 1$.

(iii_a) Take the limit of δ^* when $v_x \downarrow 0$ to obtain

$$\delta^* \to \frac{v_h(1-p) - v_l p - (1-2p)\sqrt{v_h v_l}}{2p\sqrt{v_h v_l}},$$

This expression decreases monotonically with $p \in (0,1)$. Then when $p \downarrow 0$ we have a maximum value $\delta^* \to \infty$, and for $p \uparrow 1$ we obtain the minimum value $\delta^* \to \left(\sqrt{v_h} - \sqrt{v_l}\right)/2\sqrt{v_h} \in (0,1)$. Then there is no separating equilibrium for $\delta \geq \infty$, i.e. we can say nothing about the non-existence of separating equilibrium.

(iv_a) Finally, when $v_h \uparrow \infty$, we have $\delta^* \to \infty$. Again, we cannot conclude anything about the non-existence of a separating equilibrium.

Now, suppose that there is a pooling equilibrium, then in the first stage we have $\alpha = 1$, $\beta = 1$ and $q = 1$. The first period actions provide no information. The posterior is $p_{y_{1h}} = p$. Player l payoff is

$$\pi_l\left(1, 1, 1, p, p_{y_{1l}}\right) = \pi_{1l}^{h}\left(1\right) + \delta \pi_{2l}^{y_{1h}}\left(p\right),$$

with

$$\pi_{1l}^{h}\left(1\right) = \frac{v_h\left(v_h v_l - v_x v_h + v_x v_l\right)}{\left(v_x + v_h\right)^2},$$

given by (5) in Lemma 2, and

$$\pi_{2l}^{y_{1h}}\left(p\right) = \left(v_l \frac{\sqrt{v_l} v_h + p v_x\left(\sqrt{v_l} - \sqrt{v_h}\right)}{\left(1-p\right) v_h v_x + p v_l v_x + v_l v_h}\right)^2,$$

given by (2) in Lemma 1. In case of a deviation to the full separating action, in the first period we have $\beta = 0$, while $\alpha = 1$ and $q = 1$ remain the same. In the second period, we have the posteriors $p_{y_{1h}} = 1$ and $p_{y_{1l}} = 0$. Player l payoff is

$$\pi_l\left(1, 0, 1, 1, 0\right) = \pi_{1l}^{l}\left(1\right) + \delta \pi_{2l}^{y_{1l}}\left(0\right),$$

with

$$\pi_{1l}^{l}\left(1\right) = \frac{\left(\sqrt{v_l} v_h - \sqrt{v_h} v_x + \sqrt{v_l} v_x\right)^2}{\left(v_x + v_h\right)^2},$$

given by (6) in Lemma 2, and

$$\pi_{2l}^{y_{1l}}\left(0\right) = \frac{v_l^3}{\left(v_x + v_l\right)^2},$$

given by (2) in Lemma 1. A deviation from the first period pooling path leads to a relative gain in the first period, i.e. $\pi_{1l}^{l}\left(1\right) \geq \pi_{1l}^{h}\left(1\right)$. However, the second period pooling payoff must be relatively larger, i.e. $\pi_{2l}^{y_{1h}}\left(p\right) \geq \pi_{2l}^{y_{1l}}\left(0\right)$. Both inequalities are always true. The cut-off discount factor δ^* is non-negative and is given by

$$\begin{aligned}
\delta^* &\equiv \frac{\pi_{1l}^{l}\left(1\right) - \pi_{1l}^{h}\left(1\right)}{\pi_{2l}^{y_h}\left(p\right) - \pi_{2l}^{y_l}\left(0\right)} \\
&= \frac{\left(\sqrt{v_l} v_h - \sqrt{v_h} v_x + \sqrt{v_l} v_x\right)^2 - v_h\left(v_h v_l - v_x v_h + v_x v_l\right)}{v_l^2\left(v_x + v_h\right)^2\left(\left(\frac{\sqrt{v_l} v_h + p v_x\left(\sqrt{v_l} - \sqrt{v_h}\right)}{(1-p) v_h v_x + p v_l v_x + v_l v_h}\right)^2 - \frac{v_l}{\left(v_x + v_l\right)^2}\right)}.
\end{aligned}$$

When $\delta \leq \delta^*$ there in no pooling equilibrium. δ^* is monotonically decreasing in $v_l \in [v_x, v_h]$ and monotonically increasing for $v_x \in [0, v_l]$ and $v_h \in [v_l, \infty)$.

(i_b) The limits of δ^* when $v_l \uparrow v_h$ or $v_h \downarrow v_l$ takes the value 0. Then for v_l and v_h close to each other there is no perfectly pooling for all $\delta \leq 0$. With (i_a), we conclude that for any $\delta \geq 0$ there is no separating equilibrium.

(ii_b) The limit of δ^* when $v_x \uparrow v_l$ or $v_l \downarrow v_x$, takes the value

$$\delta^* \to \frac{\left(v_h - \sqrt{v_h}\sqrt{v_x} + v_x\right)^2 - v_h v_x}{\left(v_x + v_h\right)^2 \left(\left(\frac{v_h + p\sqrt{v_x}\left(\sqrt{v_x} - \sqrt{v_h}\right)}{(1-p)v_h + p v_x + v_h}\right)^2 - \frac{1}{4}\right)},$$

where $v_x = v_l$. We can equate this limit to one and then solve for p to obtain

$$p = \frac{8v_h}{5v_h - 3v_l \pm \sqrt{\left(v_h + v_x\right)\left(5v_h - 8\sqrt{v_h}\sqrt{v_x} + 5v_x\right)}}.$$

Both roots decrease monotonically with v_h. The smaller and the larger root limits when $v_h \uparrow \infty$ are respectively $8/\left(5+\sqrt{5}\right) > 1$ and $8/\left(5-\sqrt{5}\right)$. Then we conclude, for $\delta \leq 1$ there in no pooling equilibrium.

(iii_b) Take the limit of δ^* when $v_x \downarrow 0$ to obtain

$$\delta^* \to \frac{\left(\sqrt{v_h} - \sqrt{v_l}\right)}{2p\sqrt{v_h}},$$

This expression depends on the prior $p \in (0,1)$, and decreases monotonically with p. Then at $p \downarrow 0$ we have the maximum value $\delta^* \to \infty$, and for $p \uparrow 1$ we obtain the minimum value $\delta^* \to \left(\sqrt{v_h} - \sqrt{v_l}\right)/2\sqrt{v_h} \in (0,1)$. Independently of the value that p can take, when v_x approach zero there is no pooling for $\delta \leq \left(\sqrt{v_h} - \sqrt{v_l}\right)/2\sqrt{v_h}$. ■